13.1.1(a) Consider arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in S, \forall n \in \mathbb{N}$. There are only finitely many intervals, so at least one of them contains infinitely many terms of the sequence. Let I be this interval. We will consider the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ for which the k-th term is the k-th element of (x_n) that is in I. As (x_{n_k}) is a sequence with elements in the compact interval I, there is a convergent subsequence $(x_{n_{k_i}})$ that converges to $c \in I$. This subsequence is a subsequence of (x_n) and converges to an element of S, so S is sequentially compact.
(b) Infinite unions of compact intervals are not always compact. For example,

$$\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$$

is a union of compact intervals, but is not sequentially compact, because the sequence $x_n = n$ has no convergent subsequences. The statement is still not true if we impose boundedness. For example, $(0, 1]$ is not sequentially compact (why?) but $(0, 1] = \cup_{n=1}^{\infty}[\frac{1}{n}, 1]$.

13.2.2(a) $f(x) = \frac{1}{x} - 1$ for $x \in (0, 1]$ and $f(0) = 14$ is one such function. Its domain is $[0, 1]$, its range is $[0, \infty)$. The function f is not continuous.
(b) No function satisfying (a) can be continuous. The domain is a compact interval, so all continuous functions with this domain are bounded. As this function is not bounded, it cannot hope to be continuous.

13.3.2 Write the polynomial as $p(x) = a_n x^n + \ldots + a_0$. We will show that $p(x)$ has a minimum if $a_n > 0$. If $a_n > 0$ from a previous homework we know that $\lim_{x\to\pm\infty} p(x) = \infty$. In particular, this implies that there is $M > 0$ such that for $x < -M$ and $x > M$, $p(x) \geq a_0$. As $[-M, M]$ is a compact interval and $p(x)$ is continuous, $p(x)$ has a minimum on $[-M, M]$, call this minimum $\underline{m}$. For every $x \in \mathbb{R}$, if $x \notin [-M, M]$ then $x > a_0 = p(0) \geq \underline{m}$, and if $x \in [-M, M]$ then $x \geq \underline{m}$, so p has a minimum at $\underline{m}$. The same sort of argument shows that if $a_n < 0$ that $p(x)$ has a maximum on $(-\infty, \infty)$.

13.4.1 (a) Let I be a compact interval. As its image is a compact interval, there is a, b such that $f(I) = [a, b]$. By the definition of image, as a and b are in the image of f, there is $x \in I$ such that $f(x) = a$ and $y \in I$ such that $f(y) = b$. Therefore f has a maximum at y and a minimum at x.
(b) Notice that the image is an interval, so if $I = [\alpha, \beta]$, then $f(\alpha), f(\beta)$ is in the image, and so is every element in between them. Hence for every c between $f(\alpha)$ and $f(\beta)$, there is $z \in I$ such that $f(z) = c$, and thus f has the IVT.

13.4.2 Consider the function $f(x) = \sin(\frac{1}{x}), x \neq 0$ and $f(0) = 0$. This is not a continuous function, as there is a sequence (x_n) converging to 0 such that $f(x_n) = 1$ for all n. It does satisfy the property of the previous question. If $[a, b]$ is a compact interval, either it does not contain 0, and f is a continuous function on $[a, b]$, and hence has image a compact interval, or it contains 0, and then the image of f on $[a, b]$ is $[0, 1]$.

13.5.2 Let p be a period of f, a continuous, periodic function defined on $\mathbb{R}$. As f is continuous, by uniform continuity on compact intervals, f is uniformly continuous on $[-p, p]$. This implies that given $\epsilon > 0$, there exists $\delta > 0$, which we may assume is also smaller than $p/2$, such that if $x, y \in [-p, p]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. We will show that in fact, for any $x, y \in \mathbb{R}$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. If $x, y \in \mathbb{R}$ satisfy that $|x - y| < p/2$ then either $np < x, y < (n + 1)p$ for some unique n, or $x < mp < y$ for some unique m. In the first case, $|x - y| = |(x - np) - (y - np)| < \delta$, and $x - np, y - np \in [0, p]$. Then $|f(x) - f(y)| = |f(x - np) - f(y - np)| < \epsilon$. In the second case $x - mp \in [-p, 0]$ and $y - mp \in [0, p]$. As $|x - y| = |(x - mp) - (y - mp)| < \delta$, then $|f(x) - f(y)| = |f(x - mp) - f(y - mp)| < \epsilon$.

13.5.3 Given $\epsilon > 0$ we want to find $\delta > 0$ such that if $x, y \in I \cup J$ and $|x - y| < \delta$ then $|f(x) - f(y)| < 2\epsilon$.
As f is uniformly continuous on I, there is a δ_1 such that for $|x - y| < \delta_1$, $|f(x) - f(y)| < \epsilon$ as long as $x, y \in I$. Similarly, there is a δ_2 such that this condition holds as long as $x, y \in J$. Let $\delta = \min(\delta_1, \delta_2)$. Certainly this δ still works for I and J. Now suppose that $x \in I$ and $x \notin J$, $y \in J$ and $y \notin I$, and $|x - y| < \delta$. WLOG, $x < y$. As $I \cap J$ and I and J are intervals, there is $z \in I \cap J$ and $x < z < y$. Therefore $|x - z| < \delta$, and $x, z \in I$ and $|y - z| < \delta$ and $y, z \in J$, so we can apply uniform continuity separately. Hence $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)| \leq 2\epsilon$.

13.5.4 There are several ways to show that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$. One way is to apply the last problem and show that it is uniformly continuous on $[0, 1]$ and $[1, \infty)$. It is uniformly continuous on $[0, 1]$ because it is a continuous function on a compact interval. Now we prove from the definition that it is uniformly continuous on $[1, \infty)$. Given $\epsilon > 0$, for $\delta = \epsilon$, if $|x - y| \leq \delta$ then

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x} + \sqrt{y}} \leq |x - y| < \epsilon$$

The first equality comes from thinking of $x - y$ as a difference of squares. The second comes from the fact that $x, y \geq 1$. Therefore $\sqrt{(x)}$ is uniformly continuous on $[1, \infty)$, and the desired result follows.

13.5.5 As $\ln(x)$ is continuous at 1, given $\epsilon > 0$, there is $\delta > 0$ such that if $|x - 1| < \delta$, then $|\ln(x) - \ln(1)| < \epsilon$. For $x, y \in \mathbb{R}$, such that $|x - y| < \delta$ the $|x/y - 1| < \delta/x \leq \delta$ (as $x \geq 1$). Therefore $|\ln(x) - \ln(y)| = |\ln(x/y)| = |\ln(x/y) - \ln(1)| < \epsilon$. Hence $\ln(x)$ is uniformly continuous on $[1, \infty)$.

13.5.6 (a)For $\epsilon > 0$, let $\delta = \epsilon/K$. Then if $|x - y| < \delta$,

$$|f(y) - f(x)| \leq K|y - x| < K\epsilon/K = \epsilon$$

so f is uniformly continuous.

(b)$\sqrt{(x)}$ is uniformly continuous, but does not satisfy this condition (called Lipschitz continuity). We will show that for every K there is x, y pair such that the slope of the secant between x and y is greater than K: Let $x = 0, y = \frac{1}{4K^2}$

$$\frac{|\sqrt{x} - \sqrt{y}|}{|x - y|} = \frac{\sqrt{y}}{y} = \frac{\sqrt{1}}{\sqrt{y}} = 2K > K$$