Teddy Einstein
Math 3110

HW4 Solutions

Problem 1: 6-1
Select $a, b \in \mathbb{R}$ and let $x_0 = a$, $x_1 = b$. Then continue the sequence by letting each new term be the average of the preceding two:

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}, \qquad n \geq 2$$

i. Prove x_n is Cauchy:

Proof. Claim: for $n \geq 2$, $x_n - x_{n-1} = 2^{-(n-1)}(b-a)$. For $n = 2$: $|x_n - x_{n-1}| = \frac{1}{2}(x_1 - x_0)$ and the claim holds. Suppose the claim holds for $2 \leq n \leq k$. Then $x_{k+1} - x_k = \frac{1}{2}(x_k - x_{k-1}) = \frac{1}{2} \cdot \frac{1}{2^{-(k-1)}}(b-a)$, so the claim holds by induction.

Without loss of generality, assume $m < n$. By the triangle inequality:

$$|x_n - x_m| \leq \sum_{i=m+1}^{n} |x_i - x_{i-1}| \leq \sum_{i=m+1}^{n} \frac{1}{2^{i-1}}|b-a|$$

Observe that $\sum_{i=m+1}^{n} \frac{1}{2^{i-1}} = \frac{1}{2^{-m-1}} \sum_{i=0}^{n-m-1} 2^{-i} \leq \frac{1}{2^m} \to 0$ as $m \to \infty$. Thus given $\epsilon > 0$ there exists N such that if $m > N$, $\frac{1}{2^m} < \frac{\epsilon}{|a-b|}$, so then if $m > N$:

$$|x_n - x_m| < \epsilon,$$

so x_n is Cauchy. □

ii. Find $\lim x_n$ in terms of a, b.

Solution: By the preceding part

$$x_n = x_0 + (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) = x_0 + \sum_{i=1}^{n} \frac{1}{2^{-(i-1)}}(b-a)$$

which by the geometric sum formula:

$$= a + \frac{2}{3}(1 - 2^{-n})(b-a) \to \frac{1}{3}a + \frac{2}{3}b$$

as $n \to \infty$ (by the fact that $(1 - 2^{-n}) \to 1$ by linearity of limits for sequences).

Problem 2: 6-2
Let $S \subseteq \mathbb{R}$ be bounded.

i. Prove that there exists a sequence in S that converges to $\sup S$.

Proof. $\sup S$ exists because S is bounded. Let $n \in \mathbb{N}$. Suppose toward a contradiction that every $x \in S$ has the property that $x < \sup S - \frac{1}{n}$. Then $\sup S - \frac{1}{n}$ is an upper bound for S which is strictly less than $\sup S$ contradicting the definition of sup.

Hence there exists $a_n \in S$ such that $\sup S - \frac{1}{n} \leq a_n \leq \sup S$ (where the upper bound follows from the definition of sup). Since $\frac{1}{n} \to 0$, the squeeze theorem implies $a_n \to \sup S$. □

ii. Let $A, B \subseteq \mathbb{R}$ which are bounded. Show $\sup(A+B) = \sup A + \sup B$.

Proof. $A + B$ is trivially bounded, so $\sup(A+B)$ exists. Let $a \in A$ and $b \in B$ so that $a \leq \sup A$ and $b \leq \sup B$. Thus $a + b \leq \sup A + \sup B$. Since our choice of a, b was arbitrary, every element of $A + B$ is at most $\sup A + \sup B$, so $\sup(A+B) \leq \sup(A) + \sup(B)$.

Suppose x is an upper bound of $A+B$. There exist sequences $a_n \to \sup A$ and $b_n \to \sup B$ in A, B respectively. Thus given $\epsilon > 0$, there exists N such that if $n > N$, $\sup A - a_n \leq \frac{\epsilon}{2}$ and $\sup B - b_n \leq \frac{\epsilon}{2}$. Thus $\sup A + \sup B < a_n + b_n + \epsilon \leq x + \epsilon$. Since ϵ can be arbitrarily small, we conclude that $x \geq \sup A + \sup B$, so $\sup A + \sup B$ is the least upper bound of $A + B$. □

Problem 3: 6-4

Let (x_n, y_n) be a sequence in a bounded rectangle $AxB = R \subseteq \mathbb{R}^2$. Prove that (x_n, y_n) has a convergent subsequence.

Proof. Since x_n is bounded, by the Bolzano-Weierstra theorem, x_n has a convergent subsequence $x_{n_i} \to x$ for some $x \in A$. The sequence y_{n_i} is a subsequence of y_n and is hence bounded, so y_{n_i} has a convergent subsequence $y_{n_{i_j}} \to y$ for some $y \in B$. Hence $(x_{n_{i_j}}, y_{n_{i_j}}) \to (x, y) \in R$ is a convergent subsequence of (x_n, y_n) (observe $|(x_n, y_n) - (x, y)| = \sqrt{(x_{n_{i_j}} - x)^2 + (y_{n_{i_j}} - y)^2}$ which can be made arbitrarily small). □

Problem 4: 6-6

Let a_n be a bounded sequence in $\mathbb{R}$.

i. Show that lim inf and lim sup are well defined. In other words, given a sequence a_n, define T_n to be the nth tail and let $\overline{b_n} = \sup T_n$ and $\underline{b_n} = \inf T_n$, prove that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \overline{b_n}$ and $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \underline{b_n}$ converge.

Proof. Observe that $T_{n+1} \subseteq T_n$, so every upper bound of T_n is an upper bound of T_{n+1}. Hence $\sup T_n \geq \sup T_{n+1}$, so $\overline{b_n}$ is a non-increasing sequence. Further, $T_n = \sup_{i\geq n} a_i$, so $T_n \geq a_i$ for all $i \geq n$; in particular, $T_n \geq a_n$. Thus if a_n is bounded below, then so is T_n. Hence by the completeness property, $\overline{b_n}$ converges and $\limsup_{n\to\infty} a_n$ is well defined. By a similar argument $\liminf_{n\to\infty} a_n$ is well defined. □

ii. Let $x_n = \frac{1}{n} + (-1)^n$ Compute $\liminf x_n$ and $\limsup x_n$.

Solution: Since $0 \le \frac{1}{n} \le 1$ is decreasing, if N is even, $\frac{1}{N} + 1 \ge \frac{1}{n} + (-1)^n$ for all $n > N$. Hence $\sup T_N = 1 + \frac{1}{N}$, so $\limsup x_n = 1$.

On the other hand, observe that $-1 < x_n$ for all $n \in \mathbb{N}$, but for any $N \in \mathbb{N}$, since $\frac{1}{n} \to 0$, given $\epsilon > 0$ there exists $n > N$ such that $x_n < -1 + \epsilon$, so any lower bound of a tail of x_n is at most -1. Hence $\liminf_{n\to\infty} x_n = -1$.

iii. Prove $\liminf a_n \le \limsup a_n$.

Proof. Observe that $\sup T_n \ge a \ge \inf T_n$ for all $a \in T_n$, so because limits of convergent sequences preserve order, $\limsup a_n \ge \liminf a_n$. □

iv. Prove that $\lim_{n\to\infty} a_n$ exists if and only if $\limsup a_n = \liminf a_n$.

Proof. Assume $L := \lim_{n\to\infty} a_n$ exists. Then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - L| < \epsilon$ so for $n > N$:

$$L - \epsilon < a_n < L + \epsilon$$

so $L+\epsilon$ is an upper bound on T_n whenever $n > N$, so $\sup T_n \le L+\epsilon$. Similarly, $\inf T_n \ge L+\epsilon$. Thus:

$$L - \epsilon \le \inf T_n \le \sup T_n \le L + \epsilon \Rightarrow L - \epsilon \le \liminf a_n \le \limsup a_n \le L + \epsilon$$

since ϵ can be made arbitrarily small, $\limsup a_n = \liminf a_n = L$.

Conversely suppose $\limsup a_n = \liminf a_n$. Then by recycling arguments from before, $\inf T_n \le a_n \le \sup T_n$, so by the squeeze theorem, $a_n \to \limsup a_n$. □

Problem 5: 6-7

Let S be the set of cluster points of a bounded sequence a_n in $\mathbb{R}$. Prove that $\limsup a_n = \max S$ and $\liminf a_n = \min S$.

Proof. By the preceding problem $M := \limsup a_n$ exists and the sequence $\sup T_n \to M$. Hence there exists N in naturals such that for $n > N$ $|\sup T_n - M| < \frac{\epsilon}{2}$; choose such an n. By problem $6-2$, there exists a sequence of points $t_j \to \sup T_n$ such that $t_j = a_i$ for some $i \ge n$ because T_n is a subsequence of a_n. Thus there exists $N' \in \mathbb{N}$ such that $N' > N$ and for all $j > N'$, $|t_j - \sup T_n| < \frac{\epsilon}{2}$. Therefore by the triangle inequality:

$$|t_j - \sup T_n| < \epsilon$$

Let $\delta > 0$ be given. Suppose toward a contradiction that there are only finitely many i such that $|a_i - M| < \delta$. Then there exists α such that for all $n > \alpha$, $|a_n - M| \ge \delta$. However, the argument in the preceding paragraph shows that there exists $A > \alpha$ such that there is a $t \in T_A$ such that $|t - M| < \frac{\delta}{2} < \delta$, so we have a contradiction, and $M \in S$.

Let $p \in S$. Then by the Cluster point theorem, there exists a subsequence of a_n, a_{n_i}, which converges to p. However, $\sup T_{n_i} \geq a_{n_i}$ by previous arguments. Thus by comparison of sequences, $\limsup a_n \geq \lim_{n_i \to \infty} a_{n_i} = p$ because subsequences of convergent sequences converge to the same limit. Hence $\limsup a_n = \max S$. The argument for $\liminf a_n = \min S$ is similar. □

Problem 6: 7.1.1

Evaluate the following series:

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+2)} \qquad \sum_{n=1}^{\infty} \frac{1}{n(n+k)}$$

Solution: By partial fraction decomposition

$$\frac{1}{n^2-1} = \frac{1}{2(n-1)} - \frac{1}{2(n+1)}$$

So the mth partial sum of the first series is:

$$s_m := \frac{1}{2} + \frac{1}{4} - \frac{1}{2(m+1)}$$

which converges to (provide more detail) $\frac{3}{4}$.

By partial fraction decomposition:

$$\frac{(-1)^n}{n(n+2)} = \frac{(-1)^{n-1}}{2}\left(\frac{1}{n} - \frac{1}{(n+2)}\right)$$

The $2m$th partial sum of this series is:

$$s'_{2m} := \frac{1}{2} - \frac{1}{4} + \frac{1}{2m+2} - \frac{1}{2m+1}$$

which converges to $\frac{1}{4}$. Similarly show that $s'_{(2m+1)}$ converges to $\frac{1}{4}$ and use this to argue that s'_m converges to $\frac{1}{4}$.

Finally:

$$\frac{1}{n(n+k)} = \frac{1}{k}\left(\frac{1}{n} - \frac{1}{(n+k)}\right)$$

which has partial sums for $m > k$:

$$s''_m = \frac{1}{k}\left(1 + \frac{1}{2} + \ldots + \frac{1}{k}\right) - \frac{1}{kn} - \frac{1}{kn-1} - \ldots - \frac{1}{kn-k}$$

by the tail convergence theorem, it follows that the final sum evaluates to $\frac{1}{k}(1 + \frac{1}{2} + \ldots + \frac{1}{k})$.

Problem 7: 7.1.2

Translate the Cauchy criterion for sequences into a Cauchy criterion for series:

Solution: A series $\sum_{i=0}^{\infty} a_i$ converges if and only if given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > m > N$:

$$\left| \sum_{i=m}^{n} a_i \right| < \epsilon$$

— that is, if s_i is the sequence of partial sums of the series, $|s_m - s_n| < \epsilon$ so that s_i is a Cauchy sequence.

Problem 8: 7.2.2

Prove that if $a_n \geq 0$ and $\sum a_n$ converges, then so does $\sum a_n^2$.

Proof. Since $\sum a_n$ converges, $a_n \to 0$ as $n \to \infty$. Hence a_n is bounded above by some M. Therefore:

$$\sum_{n=0}^{k} a_n^2 \leq M \sum_{n=0}^{k} a_n \leq \sum_{n=0}^{\infty} a_n$$

so that the sequence of partial sums of $\sum_{n=0}^{\infty} a_n^2$ is non-decreasing (b/c $a_n^2 > 0$) and bounded above. Thus $\sum a_n$ converges because its sequence of partial sums converges by the completeness axiom. □

Problem 9: 7.3.1

Prove the infinite triangle inequality for an absolutely convergent series by using the finite triangle inequality, then prove it by mimicking the proof for the finite triange inequality.

Proof. Let $\sum_{i=0}^{\infty} a_i$ be an absolutely convergent series.

Let s_k be the kth partial sum of the series; then,

$$|s_k| \leq \sum_{i=0}^{k} |a_i|$$

by the finite triangle inequality. Since s_k converges because $\sum_{i=0}^{\infty} a_i$ is absolutely convergent, $|s_k|$ converges as well to $|\sum_{i=0}^{\infty} a_i|$, so by sequence comparison, the result holds. □

Proof. Adopt the same notation as in the first proof:

$$-\sum_{i=0}^{k} |a_i| \leq \sum_{i=0}^{\infty} a_i \leq \sum_{i=0}^{k} |a_i|$$

Thus $|\sum_{i=0}^{\infty} a_i| \leq \sum_{i=0}^{\infty} |a_i|$ by sequence comparison. □

Problem 10: 7.3.3

Let $\sum_{i=0}^{\infty} a_i$ be a series.

i. Suppose the series converges absolutely. Show that for any subsequence a_{i_j}, $\sum_{i=0}^{\infty} a_{i_j}$ converges.

   Proof. Observe that:

   $$\sum_{j=0}^{N} |a_{i_j}| \leq \sum_{i=0}^{i_N} |a_i| \leq \sum_{i=0}^{\infty} |a_i|$$

   so the sequence of partial sums of $\sum_{j=0}^{\infty} |a_{i_j}|$ is non-decreasing and bounded above so that it converges by the completeness axiom. Hence $\sum_{j=0}^{\infty} a_{i_j}$ is absolutely convergent and thus is convergent. □

ii. Show that the statement fails when absolute convergence is removed from the hypotheses.

Proof. Consider the example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges to $\log(2)$, but:

$$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{2n} = \frac{1}{2} \sum_{1}^{\infty} \frac{1}{n}$$

which diverges. □