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Math 3110

HW5 Solutions

Problem 1: 7.2.1

Given that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

Solution: Let

$$a_n = \begin{cases} \frac{1}{n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad b_n = \begin{cases} 0 & n \text{ even} \\ \frac{1}{n^2} & n \text{ odd} \end{cases}.$$

Observe $\sum_{i=1}^{k} a_n \leq \sum_{n=1}^{k} \frac{1}{n^2} \leq \pi^2/6$, so the partial sums of $\sum_{i=1}^{\infty} a_{2n}$ are increasing and bounded above, so $\sum_{n=1}^{\infty} a_n$ converges. By a similar argument, $\sum_{n=1}^{\infty} b_n$ converges.

Since $a_n + b_n = \frac{1}{n^2}$, $\sum a_n$ converges and $\sum b_n$ converges, by linearity:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Moreover:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/24$$

Hence:

$$\pi^2/6 = \pi^2/24 + \sum_{n=1}^{\infty} b_n \Rightarrow \sum_{n=1}^{\infty} b_n = \pi^2/8$$

and the $2n+1$st partial sum of $\sum_{n=1}^{\infty} b_n$ is the nth partial sum of $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$, so the respective sequences of partial sums converge to the same limit by the subsequence theorem.

Problem 2: 7.2.3

Let a_n, b_n be non-negative series and let $\sum a_n$ and $\sum b_n$ converge. Prove $\sum a_n b_n$ converges using the methods:

i. Use an inequality relating the partial sums of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} a_n b_n$.

Proof. Let s_k, s'_k, s''_k denote the kth partial sums of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty} a_n b_n$ respectively.

We know that s_k, s'_k converge to some values s, s' respectively, so by the product theorem for sequences, the sequence $s_k s'_k \to ss'$. Furthermore, a_n, b_n are non-negative, so s_k, s'_k are increasing sequences. Consequently, $s_k \leq s$, $s'_k \leq s'$, so $s_k s'_k < ss'$ for all k. Since $a_i b_j \geq 0$ for all i, j:

$$s''_k = \sum_{n=1}^{k} a_n b_n \leq \left(\sum_{n=1}^{k} a_n \right) \left(\sum_{m=1}^{k} b_m \right) \leq s_k s'_k,$$

so s''_k is an increasing (because each $a_n b_n \geq 0$) sequence which is bounded above. Hence $\sum_{n=1}^{\infty} a_n b_n = \lim_{k\to\infty} s''_k$ converges by the completeness axiom. □

ii. By using the comparison test.

Proof. Since a_n is non-negative, $a_n = |a_n|$ and a_n is absolutely convergent. Since $\sum b_n$ converges, $b_n \to 0$ as $n \to \infty$ by the divergence test and in particular, b_n is bounded.[1] The result now follows by problem 7.3.2 (see below). □

Problem 3: 7.2.5
Let $\sum_{n=0}^{\infty} a_n$ be a convergent series with the sum S. Create a sequence b_k such that $b_k = a_{2k} + a_{2k+1}$.

Proof. Observe the $2k+1$th partial sum of $\sum_{n=0}^{\infty} a_n$ is the kth partial sum of $\sum_{n=0}^{\infty} b_n$, so the partial sums of the first series form a subsequence of the partial sums of the second. The result now follows by the subsequence theorem. □

Problem 4: 7.3.5
Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent and let b_n be a bounded sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof. Let $|b_n| \leq M$ for some $M \in \mathbb{R}$. Then $|a_n b_n| \leq M|b_n|$, so by the comparison test for series: $\sum |a_n b_n|$ converges. By the absolute convergence theorem, $\sum a_n b_n$ converges. □

Problem 5: 7.3.6
Let $\sum a_n$ be a conditionally convergent series. Prove there exist infinitely many positive and negative terms of a_n.

Proof. Suppose toward a contradiction there are only finitely many negative terms of a_n. Then there exists $N \in \mathbb{N}$ such that for $n \geq N$, $a_n \geq 0$. We know $\sum_{n=N}^{\infty} a_n$ converges by the tail convergence theorem and:

$$\sum_{n=N}^{\infty} |a_n| = \sum_{n=N}^{\infty} a_n$$

so by the tail convergence theorem $\sum_{n=1}^{\infty} |a_n|$ converges. Thus $\sum a_n$ is absolutely convergent, contradicting our hypothesis.

Observe that $\sum -a_n$ is also conditionally convergent, so the above argument shows that $-a_n$ has infinitely many positive terms. Hence a_n has infinitely many negative terms. □

[1] Given $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|b_n| < \epsilon$, then $M = \max(b_1, \ldots, b_N, 1)$ is an upper bound for b_n.