Solutions for HW 3

5.1.2 Prove directly that $a_n \to L,\ b_n \to M \Rightarrow a_n + b_n \to L+M$.

Proof: Given $\varepsilon > 0$. As $\{a_n\}, \{b_n\}$ converge, $\exists\ N_1$ and N_2 s.t. for all $n > N_1$ $|a_n - L| < \frac{\varepsilon}{2}$, for all $n > N_2$ $|b_n - M| < \frac{\varepsilon}{2}$. Then for all $n > \max\{N_1, N_2\}$ we get

$$|(a_n + b_n) - (L+M)| = |(a_n - L) + (b_n - M)| \le |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore $a_n + b_n \to L+M$.

Prove directly that $a_n \to L,\ b_n \to 0 \Rightarrow a_n b_n \to 0$.

Proof: Given $\varepsilon > 0$. As $\{a_n\}, \{b_n\}$ converge, $\exists\ N_1$ and N_2 s.t. for all $n > N_1$, $|a_n - L| < \varepsilon$, for all $n > N_2$, $|b_n| < \varepsilon$. Then for all $n > \max\{N_1, N_2\}$ we get

$$|a_n b_n| = |a_n b_n - Lb_n + Lb_n| \le |b_n||a_n - L| + |L||b_n| < \varepsilon^2 + \varepsilon|L|$$

As we only really care about what happens for small errors, ie small ε, we may assume $\varepsilon < 1$. Then $|a_n b_n| < \varepsilon(|L|+1)$ when $n > N$, and therefore by the K-ε principle $\{a_n b_n\}$ converges to 0.

5.1.4 Prove if $\frac{a_n}{b_n} \to L$, $b_n \neq 0$ for all n and $b_n \to 0$ then $a_n \to 0$.

Proof: Conserve energy. Instead of reproving, we cite 5.1.2 (b). Let $A_n = \frac{a_n}{b_n}$. This sequence converges to L, and b_n converges to 0, so their product $A_n b_n = \frac{a_n}{b_n} b_n = a_n$ converges to 0.

5.3.2 Prove $\lim a_n > M \Rightarrow a_n > M$ for $n \gg 1$

Proof: Let $L := \lim_{n\to\infty} a_n$. Then $L - M > 0$, by hypothesis, so for $\varepsilon = L - M$, as $\{a_n\}$ converges, there is $N_\varepsilon \in \mathbb{N}$ s.t. $\forall n > N_\varepsilon$, $|a_n - L| < \varepsilon = L - M$.

In particular, for $n > N_\varepsilon$, $-\varepsilon < a_n - L < \varepsilon$ so $M - L < a_n - L < L - M$ so $M < a_n < 2L - M$.

That $a_n > M$ for $n > N_\varepsilon$ means that $a_n > M$ for $n \gg 1$.

5.3.3 (a) Prove that if $a_n \to \infty$ and $b_n \to L > 0$, then $a_n b_n \to \infty$.

Proof: Given $M > 0$, we want to show that $\exists\ N$ s.t. $\forall\ n > N$, $a_n b_n > M$.

Let $M' = \frac{2M}{L}$. As $a_n \to \infty$, $\exists\ N_1$ s.t. for $n > N_1$, $a_n > M'$

As $L > 0$, so is $\frac{L}{2}$. Let $\varepsilon = \frac{L}{2}$. As $b_n \to L$, $\exists\ N_2$ s.t. for $n > N_2$, $b_n > L - \varepsilon = \frac{L}{2}$.

For all $n > N := \max\{N_1, N_2\}$, $a_n b_n > \frac{2M}{L}\left(\frac{L}{2}\right) = M$, as desired.

(b) If $L \geq 0$, we can no longer say anything.

Indeed 1. Consider $a_n = n$, $b_n = \frac{1}{n^2}$. Then $a_n \to \infty$, $b_n \to 0$ and $a_n b_n = \frac{1}{n} \to 0$, so the statement is false for these sequences.

2. Consider $a_n = n^2$, $b_n = \frac{1}{n}$. Then $a_n \to \infty$, $b_n \to 0$ and $a_n b_n = n \to \infty$ so the statement is true for these sequences.