c. Prove it for any k.

Following the same notation, let $\{x_n^i\}$ be the sequence of color #i for $i \in \{1,\dots,k\}$.

Given $\varepsilon > 0$. For each $i \in \{1,\dots,k\}$, as $\lim_{n\to\infty} x_n^i = L$, there exists N_i s.t. for $n > N_i$, $|x_n^i - L| < \varepsilon$. Let M_i be the index of $x_{N_i}^i$ in the sequence $\{a_n\}$. Define $M = \max\{M_1,\dots,M_k\}$.

Suppose $n > M$. By hypothesis, a_n corresponds to x_j^i for some $i \in \{1,\dots,k\}$ and some $j > \max\{N_1,\dots,N_k\} \geq N_i$. Therefore $|a_n - L| = |x_j^i - L| < \varepsilon$.

In the proof when we talked about corresponding indices what was meant?
You can think of subsequences of $\{a_n\}$ as coming from strictly increasing maps $\mathbb{N} \xrightarrow{f} \mathbb{N}$, which count out terms: $x_n^i = a_{f(n)}$, that is the n^{th} term of $\{x_n^i\}_{n\in\mathbb{N}}$ is the $f(n)^{th}$ term of $\{a_n\}_{n\in\mathbb{N}}$. Since f is strictly increasing, it is injective, which is why we know that $n > \max\{M_1,\dots,M_k\}$, and for $a_n = x_j^i$, that $j > \max\{N_1,\dots,N_k\}$.

6.4.1. Prove that every convergent sequence is a Cauchy sequence. Let $\{a_n\}$ be convergent, with limit L.

Given $\varepsilon > 0$. We want to find M s.t. if $n,m > M$ then $|a_n - a_m| < \varepsilon$.

As $\{a_n\}$ is convergent, there is N s.t. if $n > N$, $|a_n - L| < \frac{\varepsilon}{2}$.

Let $M = N$. Then if $n,m > M$,
$$|a_n - a_m| \underset{\substack{\uparrow\\ \text{adding in}\\ 0=-L-(-L)}}{=} |(a_n - L) - (a_m - L)| \underset{\substack{\uparrow\\ \Delta\text{-inequality}}}{\leq} |a_n - L| + |a_m - L| \underset{\substack{\uparrow\\ \text{since } n>N\\ m>N}}{<} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $\{a_n\}$ is Cauchy.

6.4.2. Suppose a sequence $\{a_n\}$ has this property: there exist constants C and K, with $0 < K < 1$, such that $|a_n - a_{n+1}| < CK^n$ for $n \gg 1$. Prove that $\{a_n\}$ is a Cauchy sequence.

Note that it IS NOT enough to show that $|a_n - a_{n+1}| \to 0$ to prove Cauchyness. Indeed, $a_n = \sqrt{n}$ is definitely not a Cauchy sequence, it satisfies $\lim_{n\to\infty} a_n = \infty$, but

* $|a_n - a_{n+1}| = |\sqrt{n} - \sqrt{n+1}| = \left|\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}(\sqrt{n}-\sqrt{n+1})\right| = \left|\frac{n-(n+1)}{\sqrt{n}+\sqrt{n+1}}\right| = \frac{1}{\sqrt{n}+\sqrt{n+1}} \leq \frac{1}{2}\left(\frac{1}{\sqrt{n}}\right)$ and so this has limit 0. *

Given $\varepsilon > 0$.

$|a_n - a_{n+1}| < CK^n$ for $n \gg 1$ means that $\exists\, N_1$ s.t. for $n > N_1$, $|a_n - a_{n+1}| < CK^n$. Let $m > n > N_1$. Then

$$|a_n - a_m| = |a_n + (-a_{n+1} + a_{n+1}) + \dots + (-a_{m-1} + a_{m-1}) - a_m| = |(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + \dots + (a_{m-1} - a_m)|$$
$$\underset{\substack{\uparrow\\ \Delta\text{-ineq.}}}{\leq} |a_n - a_{n+1}| + \dots + |a_{m-1} - a_m| < CK^n + CK^{n+1} + \dots + CK^{m-1} = CK^n(1 + K + K^2 + \dots + K^{m-1-n}) \leq CK^n\left(\frac{1}{1-K}\right)$$

(Indeed, if $S = \sum_{i=0}^{m-1-n} K^i$, then $KS = \sum_{i=1}^{m-n} K^i$, and $(1-K)S = 1 - K^{m-n} < 1$, so $S < \frac{1}{1-K}$

We are using here that $0 < K < 1$ to guarantee $1 - K > 0$ and $1 - K^{m-n} < 1$.)