As $|K|<1$, $\lim_{n\to\infty} K^n = 0$, and so $\exists\ N_2$ s.t. if $n>N_2$ $|K^n|<\varepsilon$

Taking $N=\max\{N_1,N_2\}$, if $m>n>N$, then

$|a_n - a_m| < CK^n\left(\frac{1}{1-K}\right) < \frac{C}{1-K}\,\varepsilon.$

By the K-ε principle, this implies that $\{a_n\}$ is a Cauchy sequence.

6.5.2. Let S be a non-empty bounded set of real numbers, and $\bar{m}=\sup(S)$. Prove that

$\inf\{\bar{m}-x : x\in S\} = 0.$

Proof:

From the definition of supremum, $\bar{m}\geq x$, for all $x\in S$, so in particular, $\bar{m}-x\geq 0$ for all $x\in S$. Therefore 0 is a lower bound to the set $\{\bar{m}-x : x\in S\}$, so the infimum exists. Call it b. As b is the greatest of the lower bounds, $b\geq 0$.

Further, $\bar{m}-x\geq b$ for all $x\in S$ implies that $\bar{m}-b\geq x$ for all $x\in S$, so that $\bar{m}-b$ is an upperbound for the set S. As $\bar{m}$ is the supremum of S, it is the least of the upperbounds, so $\bar{m}\leq\bar{m}-b$. Since $b\geq 0$, $\bar{m}-b\leq\bar{m}$, so $\bar{m}\leq\bar{m}-b\leq\bar{m}$ implies that $b=\inf\{\bar{m}-x : x\in S\}=0$, as desired.

6.5.4. Let S and T be non-empty subsets of $\mathbb{R}$, and suppose that for all $s\in S$ and $t\in T$, we have $s\leq t$. Prove that $\sup S\leq\inf T$.

Choose $t\in T$ arbitrary. We have that for all $s\in S$, $s\leq t$. This implies that t is an upperbound for S. Therefore $\sup(S)$ exists, and $\sup(S)\leq t$, as the supremum is the least of the upperbounds of S. Since t was arbitrary, for any $t\in T$, the inequality $\sup(S)\leq t$ holds and thus $\sup(S)\leq t$ for all $t\in T$, and $\sup(S)$ is a lowerbound for T. Therefore $\inf(T)$ exists, and $\sup(S)\leq\inf(T)$, as the infimum of T is the largest of T's lower bounds.