Taylor series Homework set solutions.

17.1.1 Let $f(x) = (1+x)^r$. Then, by the usual differentiation formulas, we have that

$$f^{(k)}(x) = r(r-1)\cdots(r-k+1)(1+x)^{r-k}.$$

Therefore,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{r(r-1)\cdots(r-k+1)}{k!} x^k.$$

Observation 1 *The formula*

$$\binom{r}{k} := \frac{r(r-1)\cdots(r-k+1)}{k!} \tag{1}$$

is sometimes used to represent the coefficient appearing on the right hand side of equation (1). Observe that when $r \geq k$ is an integer, this formula agrees with its usual definition.

17.2.1 We will prove Lemma 17.2 using induction.

Base Case. By the usual Rolle's theorem, if $f(a) = f(b) = 0$, then there exists $c \in (a,b)$ such that $f'(c) = f^{(0+1)}(c) = 0$.

Inductive Step. Now assume that the result holds up to a *fixed* $n \in \mathbb{N}$, that is, assume that if $f^{(n+1)}$ exists on $[a,b]$ and $f(a) = f'(a) = \ldots = f^{(n)}(a) = f(b) = 0$, then there exists c between a and b such that $f^{(n+1)}(c) = 0$.

We want to show that the result holds for $n+1$, that is, we want to show that if $f^{(n+2)}$ exists on $[a,b]$ and $f(a) = f'(a) = \ldots = f^{(n+1)}(a) = f(b) = 0$, then there exists c between a and b such that $f^{(n+2)}(c) = 0$. Let $g(x) = f'(x)$. By the usual Rolle's theorem, there exists $c_0 \in (a,b)$ such that

$$g(c_0) = f'(c_0) = 0.$$

Hence $g^{(n+1)} = f^{(n+2)}$ exists and $g(a) = g'(a) = \ldots = g^{(n)}(a) = g(c_0) = 0$. Therefore, by induction hypothesis, there exists c between a and c_0 such that $g^{(n+1)}(c) = 0$. Since $(a, c_0) \subset (a,b)$, we conclude that

$$f^{(n+2)}(c) = g^{(n+1)}(c) = 0 \qquad \text{for some } c \in (a,b),$$

as we wanted to show.

17.2.2 a) Assume that

$$P(x) = \sum_{k=0}^{n} b_k x^k = b_0 + b_1 x + \cdots + b_n x^n$$

is a polynomial of degree n. If $x = u + a$, then we can rewrite the above expression as

$$\begin{aligned} P(x) &= P(u+a) \\ &= \sum_{k=0}^{n} b_k (u+a)^k \\ &= \sum_{k=0}^{n} b_k \Big(\sum_{l=0}^{k} \binom{k}{l} a^{k-l} u^l \Big) \\ &= \sum_{k=0}^{n} \sum_{l=0}^{k} b_k \binom{k}{l} a^{k-l} u^l \\ &= \sum_{l=0}^{n} \left[\sum_{k=l}^{n} b_k \binom{k}{l} a^{k-l} \right] u^l, \qquad (2) \end{aligned}$$

where the last equality follows from the identities of iterated sums. Substituting $u = x - a$ in (2) we get

$$P(x) = \sum_{l=0}^{n} \left[\sum_{k=l}^{n} b_k \binom{k}{l} a^{k-l} \right] (x-a)^l. \qquad (3)$$

On the other hand, using the usual formulas for the derivatives, we obtain that

$$P^{(l)}(x) = \sum_{k=l}^{n} b_k\, k(k-1)\cdots(k-l+1) x^{k-l}.$$

Hence

$$T_n(x) = \sum_{l=0}^{n} \frac{P^{(l)}(a)}{l!}(x-a)^l = \sum_{l=0}^{n} \frac{\sum_{k=l}^{n} b_k\, k(k-1)\cdots(k-l+1) a^{k-l}}{l!}(x-a)^l. \qquad (4)$$

Comparing equations (3) and (4), we conclude that

$$P(x) = T_n(x).$$

b) Observe that, if we set $x = u - 1$,

$$\begin{aligned} P(x) &= (u-1)^3 - 2(u-1) + 2 \\ &= u^3 - 3u^2 + u + 3 \\ &= (x+1)^3 - 3(x+1)^2 + (x+1) + 3. \end{aligned}$$

On the other hand, we have that

$$\begin{array}{lll} P(x) = x^3 - 2x + 2, & \text{hence} & P(-1) = 3. \\ P'(x) = 3x^2 - 2, & \text{hence} & P'(-1) = 1. \\ P''(x) = 6x, & \text{hence} & P''(-1) = -6. \\ P'''(x) = 6, & \text{hence} & P''(-1) = 6. \end{array}$$

Therefore,

$$P(x) = 3 + (x+1) + \frac{(-6)}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3 = 3 + (x+1) - 3(x+1)^2 + (x+1)^3.$$

17.2.3 We will prove this result using induction on n.

Base Case. For $n = 0$, the problem is asking us to show that if $g(a) = g(b) = 0$, then there exist $c \in (a,b)$ such that $g^{(0+1)}(c) = g'(c) = 0$. But this is precisely the statement of Rolle's theorem.

Inductive Step. Now assume that the result holds up to a *fixed* $n \in \mathbb{N}$. If we set $a_0 = a$, and $a_{n+1} = b$, then this is equivalent to assuming that if g has $(n+1)$ derivatives, and

$$g(a_0) = g(a_1) = \cdots g(a_{n+1}) = 0,$$

then there exists $c \in (a,b)$ such that $g^{(n+1)}(c) = 0$.

We want to prove that this result holds for $n+1$, that is, we want to show that if g has $(n+2)$ derivatives, and if

$$g(a_0) = g(a_1) = \cdots g(a_{n+2}) = 0,$$

then there exists $c \in (a,b)$ such that $g^{n+2}(c) = 0$. Let's set $f(t) = g'(t)$. Then, by Rolle's theorem, there exists $c_k \in (a_k, a_{k+1})$ for $k = 0, 1, \ldots, n+1$ such that

$$f(c_k) = g'(c_k) = 0 \qquad \text{for } k = 0, \ldots, n+1.$$

Hence, by induction hypothesis, there exists $c \in (c_0, c_{n+1}) \subset (a,b)$ such that

$$f^{(n+1)}(c) = g^{(n+2)}(c) = 0,$$

as we wanted to show.

17.2.4 a) Since $f(a) = f(b) = 0$, then, by Rolle's theorem, there exists $c_0 \in (a,b)$ such that $f'(c_0) = 0$. Applying Rolle's theorem again on the intervals $[a, c_0]$ and $[c_0, b]$ for the function $g(x) = f'(x)$, we get that there exists $c_1 \in [a, c_0]$ and $c_2 \in [c_0, b]$ such that

$$f''(c_1) = f''(c_2) = 0.$$

Yet another application of Rolle's theorem gives us a $c \in (c_1, c_2)$ such that $f'''(c) = 0$.

b) Observe that

$$f'(x) = 2(x-a)(x-b)^2 + 2(x-a)^2(x-b).$$

Hence $f(a) = f(b) = f'(a) = f'(b) = 0$, that is, the hypothesis of part a) apply. Now, by the usual differentiation formulas, we obtain that

$$f'''(x) = 24x - 12a - 12b.$$

Hence, the equation $f'''(c) = 0$ is equivalent to the equation

$$\begin{aligned} 24c - 12a - 12b &= 0 \\ c &= \frac{a+b}{2} \end{aligned}$$

17.3.1 Let $f(x) = e^{-x}$. Then,

$$T_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} x^k = 1 - x + \frac{x^2}{2}.$$

We want to estimate the magnitude of the residue function

$$R_2(x) = f(x) - T_2(x) \qquad \text{for } x \in [0, 0.1].$$

By Taylor's theorem with Lagrange remainder, we know that for all such x, there exists a c between 0 and x such that

$$R_2(x) = \frac{f^{(3)}(c)}{3!} x^3.$$

So the magnitude of the error is given by

$$|R_2(x)| = |e^{-c}|\,|\frac{x^3}{3!}| \qquad \text{for some } 0 \le c \le x \le 0.1.$$

Therefore, we get the bound

$$|R_2(x)| \le e^0 \frac{(0.1)^3}{6} = \frac{(0.1)^3}{6}.$$

17.3.2 Let $f(x) = \cos x$. Observe that, since $f'''(0) = -\sin 0 = 0$, the expression $1 - x^2/2$ is not only a degree 2 approximation, but actually a degree 3 approximation! (In other words, $T_2(x) = T_3(x) = 1 - x^2/2$.) Just as in the last problem, we obtain an approximation of the remainder function of the form

$$|R_3(x)| = |\frac{f^{(4)}(c)}{4!} x^4| = |\cos c|\,\frac{|x|^4}{24} \qquad \text{for some } 0 \le c \le x.$$

Therefore, if we want to bound the error by .0001 it suffices to choose an interval of the form $[-b, b]$ with

$$\begin{aligned} \frac{b^4}{24} &\le .0001 \\ b &\le \sqrt[4]{24}/10 \approx 0.221336 \end{aligned}$$

17.3.3 Let $f(x) = \sin x$. We want to bound the remainder function $T_n(x)$ by .0001 for $|x| < .5$. Proceeding as in the last problem, we can quickly get a bound of the form

$$|R_n(x)| = |\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}| < \frac{1}{2^{n+1}(n+1)!}.$$

Now, observe that $3840 = 2^{4+1}(4+1)! < 10,000 < 2^{5+1}(5+1)! = 46,080$. Hence, it is enough to take T_5. However, since $T_4 = T_5$ (for similar reasons as in problem 17.3.2) it suffices to take $n = 4$.

17.3.4 Let $f(x) = \cos x$. Once again, we observe that since $T_4 = T_5$, we have an estimate of the form

$$|R_4(x)| = |R_5(x)| \leq \frac{x^6}{6!}.$$

Now, since

$$(.1)^6/6! < (.1)^8,$$

we conclude that $T_4(.1)$ is a good enough approximation to $\cos .1$. Calculating we get the approximation

$$\cos .1 \approx 1 - (.1)^2/2 + (.1)^4/24 \approx .9950041$$

17-1 a) Let $P(x)$ be a polynomial of degree n. Then,

$$P(x) = \sum_{l=0}^{n} c_l (x-a)^l,$$

where $c_l = P^{(l)}(a)/l!$. Therefore, if we assume that

$$P(a) = P'(a) = \ldots = P^{(k-1)}(a) = 0, \qquad P^{(k)}(a) \neq 0,$$

then

$$P(x) = (x-a)^k \left(\sum_{l=k}^{n} c_l (x-a)^{l-k} \right).$$

If we set

$$Q(x) = \sum_{l=k}^{n} c_l (x-a)^{l-k},$$

then it is clear that $P(x) = (x-a)^k Q(x)$ and $Q(a) = c_k = P^{(k)}(a)/k! \neq 0$, that is, a is k-fold zero of $P(x)$.

Now assume that a is a k-fold zero of $P(x)$, that is, assume that there exists a polynomial $Q(x)$ such that $P(x) = (x-a)^k Q(x)$ and $Q(a) \neq 0$. Expressing the polynomial $Q(x)$ as

$$Q(x) = \sum_{l=0}^{m} b_l (x-a)^l,$$

we obtain that

$$\begin{aligned} P(x) &= (x-a)^k \sum_{l=0}^{m} b_l (x-a)^l \\ &= \sum_{l=0}^{m} b_l (x-a)^{k+l} \\ &= \sum_{l=k}^{k+m} b_{l-k} (x-a)^l \end{aligned}$$

with $b_0 \neq 0$. From this expression, it is immediate that

$$P(a) = P'(a) = \ldots = P^{(k-1)}(a) = 0, \qquad \text{and } P^{(k)}(a) = b_0 k! \neq 0,$$

b) Assume that a is a double zero of the polynomial $P(x) = 2x^3 - bx^2 + 1$. Then, according to part a) we should have that

$$0 = P'(a) = 6a^2 - 2ba = a(6a - 2b).$$

Hence, the only options for a are $a = 0$ or $a = b/3$. Since $P(0) \neq 0$, we find ourselves looking for a value of b such that

$$\begin{aligned}
0 &= f(a) = f(b/3) = 2\left(\frac{b}{3}\right)^3 - b\left(\frac{b}{3}\right)^2 + 1 \\
-1 &= \frac{2}{3}\left(\frac{b^3}{9}\right) - \left(\frac{b^3}{9}\right) \\
-1 &= -\frac{1}{3}\left(\frac{b^3}{9}\right) \\
b^3 &= 27 \\
b &= 3.
\end{aligned}$$

Pluggin back into our equations, we can see that, effectively, for $b = 3$ the value $a = 1$ is a double zero of the polynomial $P(x)$

c) If in exercise 17.2.3 we take all the roots to be equal to a instead of being all different, then part a) of this problems says that the conclusion in exercise 17.2.3 still holds and is equivalent to the Extended Rolle's Theorem.