Hw 11 Soln

15.2.1 a) Prove if $f'(x)$ is bounded on a finite interval I then $f(x)$ is bounded on I.
If finite is eliminated, does the statement still hold?

Pf// Let $x_0 \in I$. For $a \in I$, $a \neq x_0$, by MVT, $\exists\, c \in I$ s.t.

$$\frac{f(x_0) - f(a)}{x_0 - a} = f'(c) \Rightarrow f(x_0) - f(a) = f'(c)(x_0 - a).$$

Since $\exists\, M > 0$ s.t. $|f'(x)| \leq M$,

$$|f(x_0) - f(a)| \leq |f'(c)|\,|x_0 - a| \leq M\,|I|$$

so $f(x)$ is bounded on I.

If we remove the hypothesis that I is finite,
consider $f(x) = x$ on $I = \mathbb{R}$. Then $f(x)$ is unbounded,
but $f'(x) = 1$ $\forall x \in \mathbb{R}$.

b) Show the converse is false.
For example: $f(x) = \sqrt{x}$ on $(0,1)$.
Then $0 \leq f(x) \leq 1$, but $\lim_{x \to 0^+} f(x) = 0$, so since
$f'(x) = \frac{1}{2\sqrt{x}}$, $\lim_{x \to 0^+} f'(x) = +\infty$, so $f'(x)$ is unbounded on $(0,1)$.
(Note $f'(x) \to +\infty$ as $x \to 0^+$ $\Rightarrow$ given $M \in \mathbb{R}$, $\exists\, \delta > 0$ s.t.
if $0 < x < \delta$, $f'(x) > M$)

15.2.3 Prove if g' is continuous and positive on (a,b) and g is continuous on $[a,b]$, then $g(x)$ is strictly increasing on $[a,b]$.

Pf// Given $x_1, x_2 \in [a,b]$ s.t. $x_2 > x_1$, by MVT, $\exists c \in (x_1, x_2)$ s.t. $\frac{g(x_2) - g(x_1)}{x_2 - x_1} = g'(c) > 0$. Since $x_2 - x_1 > 0$,

$g(x_2) - g(x_1) > 0 \Rightarrow g(x)$ is strictly increasing //

15.3.1 Computation - Straightforward.

15.4.2 Evaluate $\lim_{x \to 0} \frac{x - \tan x}{x - \sin x}$

By elementary results, $x - \tan x$, $x - \sin x$ are differentiable for $x \approx 0$, also $\lim_{x \to 0} x - \tan x = \lim_{x \to 0} x - \sin x = 0$.

Thus by L'Hopital's rule,

$$\lim_{x \to 0} \frac{x - \tan x}{x - \sin x} = \lim_{x \to 0} \frac{1 - \sec^2 x}{1 - \cos x} = \lim_{x \to 0} \frac{-\tan^2 x}{1 - \cos x} \quad *$$

Again $\tan^2 x$, $\cos x$ are diff'ble near 0 and $\lim_{x \to 0} -\tan^2 x$, $\lim_{x \to 0} 1 - \cos x$ are both 0

so by L'H rule:

$$* = \lim_{x \to 0} \frac{-2 \tan x \sec^2 x}{\sin x} = \lim_{x \to 0} \frac{-2}{\cos^3 x} = -2$$

15.4.3 Prove L'Hopital's rule ~~Thm 15.4A~~ for the $0/0$ form as $x \to \infty$, using the case $t \to 0^+$.

Pf// Let f, g be differentiable for $f, g >> 1$ and $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$

Let $x = \frac{1}{t}$ so $x \to \infty \Leftrightarrow t \to 0^+$. Hence

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{f(1/t)}{g(1/t)}$. Since f, g are differentiable and

$1/t$ is differentiable for $t > 0$, by the chain rule, $f(1/t), g(1/t)$

are differentiable with derivatives $\frac{f'(1/t)}{-t^2}$, $\frac{g'(1/t)}{-t^2}$

$$\text{Then } \lim_{x\to\infty} \frac{f(x)}{g(x)} \underset{\text{L'Hopital's rule}}{=} \lim_{t\to 0^+} \left(\frac{\left(\frac{f'(1/t)}{-t^2}\right)}{\left(\frac{g'(1/t)}{(-t^2)}\right)} \right) = \lim_{t\to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

//

15.4.4 a) Write out a statement of L'Hopital's rule in the ∞/∞ form for the case $x \to a^+$.

Soln: ~~Copy Thm 15.4C~~ Suppose f, g are differentiable functions for $x \approx a^+$. ~~If $\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists, Then~~

~~lim~~

Suppose $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty$ and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists,

Then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$. //

15.4.46. Evaluate $\lim_{x\to 0^+} x(\log x)$

Observe $1/x$, $\log x$ are differentiable for $x \simeq 0^+$ with derivatives $-1/x^2$, $1/x$ respectively.

Thus by part a), since $\underbrace{\lim_{x\to 0^+} 1/x}_{}$, $\lim_{x\to 0^+} \log x = \infty$,
$$\lim_{x\to 0^+} \log x = -\infty$$

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x$$
$$= 0. //$$

16.1.1. Using Thm 16.1B, prove these estimates on the given interval:

a) $e^x > 1+x+x^2/2$ for $x \in (0,1]$, find a quadratic upper estimate.

By 16.1B since $(e^x)' = (e^x)'' = e^x$,

for each $x \in (0,1]$, $\exists c \in (0,x)$ s.t.
$$e^x = 1 + (x) + \frac{e^c}{2} x^2$$

$\Rightarrow$ since on $(0,1]$ $1 < e^x \le e$,

and $x^2 \geq 0$
$$1 + x + \frac{x^2}{2} < e^x \leq 1 + x + \frac{e}{2} x^2$$

16.1.1b) $\log(1+x) > x - x^2/2$ find a quadratic upper estimate

By 16.1B ~~...~~ if $f(x) = \cancel{\ }\log(1+x)$, $f'(x) = \frac{1}{1+x}$ and

$$f''(x) = \frac{-1}{(1+x)^2}, \text{ Then } \exists c \in (0, x] \text{ s.t.}$$

for $x \in (0,1]$: $f(x) = \log(1) + \frac{1}{1+0} x + \frac{-1}{(1+c)^2} x^2$

$$= x - \frac{1}{(1+c)^2} x^2$$

observe that for $x \in (0,1]$

$$1 < 1+c \leq 2$$

$$\Rightarrow \quad 1 < (1+c)^2 \leq 4$$

$$\Rightarrow \quad -1 < \frac{-1}{(1+c)^2} \leq \frac{-1}{4}$$

$$\Rightarrow \quad x - x^2 < f(x) \leq x - \tfrac{1}{4} x^2 \;//$$

16.1.4 Suppose $f''(x)$ exists on I and $f(a) = f(b) = 0 = f(c)$ where $a, b, c \in I$ and $a < b < c$. Prove $f''(x) = 0$ for some $x \in I$.

Pf// By Rolle's Thm $\exists\, x_1 \in (a,b)$ and $x_2 \in (b,c)$ s.t.

$$f'(x_1) = f'(x_2) = 0, \text{ and } x_1 < x_2.$$

Hence by Rolle's Thm $\exists\, x_3$ s.t. $f''(x_3) = 0$.

16.2.2 Let $f(x)$ be differentiable on I, an interval.
Prove that if $f(x)$ is ~~a~~ geometrically convex then $f(x)$ is convex.

Pf// ~~Pf~~ Fix $a, b \in I$. By geometric convexity for all x s.t. ~~for~~ ~~a~~ $a < x < b$:

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a}$$

Taking limits on both sides with x as variable

$$f'(a) = \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a} \leq \lim_{x \to a^+} \frac{f(b)-f(a)}{b-a} = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow f(a) + f'(a)(b-a) \leq f(b)$$

$$\Rightarrow f \text{ is convex } //$$

16.2.4a) Let f be geometrically concave on I and $a, b \in I$

Show the value of $f(x)$ at the average of a, b is at least the average of $f(a)$ and $f(b)$.

Since $f(x)$ is geometrically concave:

For $a < \frac{a+b}{2} < b$:

$$\frac{f\left(\frac{a+b}{2}\right) - f(a)}{\left(\frac{b-a}{2}\right)} \geq \frac{f(b) - f(\cancel{a+b}\, a)}{b-a}$$

$$\Leftrightarrow \quad f\left(\frac{a+b}{2}\right) \geq \frac{f(b) - f(a)}{2} + f(a) = \frac{f(b) + f(a)}{2} \quad //$$

b) Use equivalence of geometric and regular concavity to prove $\sqrt{ab} \leq (a+b)/2$

Observe $f(x) = \log x$ is diff'ble on $(1, \infty)$
and $f''(x) = \frac{-1}{x^2} < 0 \;\; \forall x \in (1, \infty)$

Hence f is concave on $(1, \infty)$.

~~Observe if $a=0$, the~~

WLOG assume $a \leq b$. If $a = 0$, the result follows immediately, so assume $a > 0$.

By the preceding, then

$$\log(\sqrt{ab}) = \frac{\log(a) + \log(b)}{2} \leq \log\left(\frac{a+b}{2}\right).$$

Since e^x is strictly increasing and hence preserves order: $\sqrt{ab} \leq \frac{a+b}{2}$. //