HW12 Solutions:

18.1.1 If an n-partition P of $[a,b]$ has mesh $\frac{(b-a)}{n}$ is P the standard partition? Prove or give a counterexample.

True: observe if $a = x_0 < x_1 < \dots < x_n = b$ and $\max_{0 \le i \le n-1} (x_{i+1} - x_i) \le \frac{b-a}{n}$

Then
$$x_n - x_0 = \left(\sum_{i=0}^{n-1} (x_{i+1} - x_i)\right)$$
$$\le n \cdot \frac{b-a}{n} \le b-a$$

but if $x_{i+1} - x_i < \frac{b-a}{n}$ for any i,

Then $x_n - x_0 < b-a \Rightarrow \Leftarrow$. //

18.2.1 Prove using definitions only that x^2 is integrable on any $[a,b] \subset [0,\infty)$

Observe x^2 is continuous and $[a,b]$ is compact so x^2 is bded on $[a,b]$.

Let P be a partition of $[a,b]$ where $P = \{x_0 < x_1 < \dots < x_n\}$

observe x^2 is increasing, so the upper sum of this partition is $U \equiv \sum_{i=0}^{n-1} (x_{i+1})^2 (x_{i+1} - x_i)$ and the lower sum is

$L \equiv \sum_{i=0}^{n-1} (x_i)^2 (x_{i+1} - x_i)$ so $U - L = \sum_{i=0}^{n-1} (x_{i+1}^2 - x_i)^2 (x_{i+1} - x_i)$

$\Rightarrow |U-L| \leq \sum_{i=0}^{n-1} [x_{i+1}^2 - x_i^2] |P| = |P| \sum_{i=0}^{n-1} (x_{i+1}^2 - x_i^2)$ See next page

~~Hence to show integrability, it suffices to show~~

~~$\sum |x_{i+1}^2 - x_i^2|$ is bounded~~

~~observe $\left|\sum_{i=0}^{n-1} (x_{i+1}^2 - x_i^2)\right| = \sum_{i=0}^{n-1} |(x_{i+1} - x_i)(x_{i+1} + x_i)|$~~

~~$\leq |P| \sum |x_{i+1} + x_i|$~~

~~so as $|P| \to 0$,~~ (See next page)

~~$|U(P) - L(P)| \leq M |P|^2 (2b) \to 0$.~~

18.2.2

Let $f(x) = \begin{cases} 0 & x = m/2^n \text{ for some } m,n \in \mathbb{Z}^{>0} \\ 1 & \text{otherwise} \end{cases}$

Prove $f(x)$ is not Riemann integrable on $[0,1]$.

Claim: every non-trivial closed interval $[a,b] \subseteq [0,1]$ contains a dyadic (number of the form $m/2^n$ for $m,n \in \mathbb{Z}^{>0}$).

Observe that $\frac{m+1}{2^n} - \frac{m}{2^n} = \frac{1}{2^n}$.

Given a,b w/ $a \neq b$ and $a<b$:

Choose N s.t. $\frac{1}{2^N} < b-a$. ~~and~~ If $a=0$ or $b=1$, the proof is easy, so assume not.

Observe ~~if~~ that for $L_- \equiv \{ \frac{m}{2^n} \mid m,n \in \mathbb{Z}^+ \text{ and } 0 \leq m/2^N \leq a \}$

$L_+ \equiv \{ \frac{m}{2^n} \mid m,n \in \mathbb{Z}^+ \text{ and } 1 \geq m/2^N \geq b \}$

if $L_+ \cap [a,b] = L_- \cap [a,b] = \emptyset$, $(*)$

Then $\min L_+ > b$ and $\max L_- < a$

but then $\min L_+ - \min L_- > b-a > \frac{1}{2^N}$, but by $(*)$ $L_+ \cup L_- =$

$L \equiv \{ m/2^N : 0 \leq m/2^N \leq 1 \} \Rightarrow\Leftarrow$ b/c L_+ and L_- partition L, but ~~...~~

18.2.2 cont.

By the above claim: every partition $P = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$

has $p_1, p_0 \in [x_i, x_{i+1}]$ s.t. $f(p_0) = 0$ (b/c every non-trivial closed interval contains an irrational)

$f(p_1) = 1$ (by above claim)

Hence $U(P) = 1$ and $L(P) = 0$, so ~~for any par~~ even as $|P| \to 0$, $U(P) - L(P) = 1$.

18.7.1 (cont)

$$\sum_{i=0}^{n-1} (x_{i+1}^2 - x_i^2) = \sum_{i=0}^{n-1} (x_{i+1} - x_i)(x_{i+1} + x_i)$$

$$\leq \sum_{i=0}^{n-1} (x_{i+1} - x_i)(2b)$$

$$= 2x_n b - x_0 (2b) \text{ (Telescoping series)}$$

$$= 2b^2 - 2ab$$

Hence $|P| \sum_{i=0}^{n-1} (x_{i+1}^2 - x_i^2) \to 0$ as $|P| \to 0$.

Thus $|U(P) - L(P)| \to 0$ as $|P| \to 0 \Rightarrow x^2$ is integrable on $[0,1]$

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18.3.1 Fix $k \geq 0$, $k \in \mathbb{Z}$.
Use the definition of integrable to show

$$f_k(x) = \begin{cases} 0 & \text{if } x = 1/k, 2/k, \dots, \frac{k-1}{k} \\ 1 & \text{~~if not~~ otherwise} \end{cases}$$

is integrable on $[0,1]$.

Observe $f_k(x) = 0$ or 1 for all $x \in [0,1]$.
For P be a partition of $[0,1]$ let $n(P)$ be the number of intervals in the partition.
Observe that as $|P| \to 0$, $n(P) \to \infty$ because $n(P)\,|P| \geq 1$ since P partitions $[0,1]$.

At most $k-1$ intervals of the partition P may have a value where $f(x)=0$, so given an arbitrary partition P, $L(P) \geq 1-(k-1)|P| \to 1$ as $|P| \to 0$

$U(P) = 1$, so $U(P) - L(P) \to 0$ as $|P| \to 0$. //

19.2.1 Evaluate $\int_0^1 e^x dx$ directly, using (8) applied to lower sums taken over the standard n-partition.

Pf// e^x is continuous and hence integrable on $[0,1]$.
Hence (8) applies:

$$\int_0^1 e^x dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(e^{i/n} \cdot \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} (e^{1/n})^i = \lim_{n\to\infty} \frac{1}{n}(n)(e^{1/n}-1)$$

$$= e - 1$$

(by finite geometric sum formula) //

19.2.3 Evaluate $\int_1^a x^k dx$ for $k \in \mathbb{Z}^+$ using upper sums, (8)

and the partition $1 < r < r^2 < \_ < r^{n-1} < a$

Since x^k is increasing for $x \geqslant 1$,

$$\int_1^a x^k dx = \lim_{n\to\infty} \sum_{i=1}^n (r^i - r^{i-1}) r^{ik} = \lim_{n\to\infty} \left(\sum_{i=1}^n r^{i+ik} - r^{-1} \sum_{i=1}^n r^{i+ik} \right)$$

$$= \lim_{n\to\infty} \left((1-r^{-1}) \sum_{i=1}^n (r^{1+k})^i \right) \cancel{= (1-r^{-1}) \left(\frac{r^{(1+k)(n+1)} - 1}{r-1} - 1 \right)}$$

$$= \lim_{n\to\infty} (1-r^{-1}) r^{k+1} \sum_{i=1}^n (r^{(1+k)})^{i-1}$$

$$= \lim_{n\to\infty} (r-1) r^k \sum_{j=0}^{n-1} (r^{(1+k)})^j$$

$$= \lim_{n\to\infty} (r-1) r^k \left(\frac{r^{(1+k)n} - 1}{r^{1+k} - 1} \right)$$

$$= \lim_{n\to\infty} \frac{r^k (r^{(1+k)n} - 1)}{1 + r + r^2 + \_ + r^k} = \frac{a^{k+1} - 1}{k+1}$$

because $\lim_{n\to\infty} r^{\ell} = \lim_{n\to\infty} a^{\ell/n} = 1$.

for all $\ell \in \mathbb{N}$.

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19.3.1 Let $f(x)$ be integrable on $[a,b]$ and $f(x)=0$ for $x \in \mathbb{Q}$.

Prove $\int_a^b f(x)\,dx = 0$.

Pf// Let P_n be the standard n partition of $[a,b]$ with $P_n = \{a = x_0^n < x_1^n < \dots < x_n^n = b\}$

Note superscripts are indices, not powers

Then in $[x_i^n, x_{i+1}^n]$, ~~take the x~~ there exists $q_i^n \in \mathbb{Q}$ s.t. $q_i^n \in [x_i^n, x_{i+1}^n]$

Thus the Riemann sum $R_n = \sum_{i=0}^{n-1} f(q_i^n) | [x_i^n, x_{i+1}^n] | = 0$.

Hence by Thm 19.3, $\int_a^b f(x)\,dx = 0$. //

19.3.3 Is the trapezoidal sum a Riemann sum for

a) $f(x)$ monotone.

No. e.g. $f(x) = \begin{cases} \frac{1}{4}(x-n) + n & x \in [n, n+1) \end{cases}$

For $k \in \mathbb{Z}$, the standard k partition of $[0,k]$

$$\frac{f(x_{i+1}) + f(x_i)}{2} = k + \tfrac{1}{2} \neq f(x) \text{ for any } x \in [x_i, x_{i+1}]$$

b) $f(x)$ cont: yes, By IVT, $\exists\, x \in [x_i, x_{i+1}]$ s.t.

$$f(x) = \frac{f(x_i) + f(x_{i+1})}{2} \text{ b/c}$$

The average of $f(x_i)$ and $f(x_{i+1})$ lies between $f(x_i)$ and $f(x_{i+1})$. //

19.3.4

19.4.1 Prove: If $f(x)$ is integrable on $[a,b]$, then $f(x-c)$ is integrable on $[a+c, b+c]$ and $\int_{a+c}^{b+c} f(x-c)\,dx = \int_a^b f(x)\,dx$.

Pf// Let P be a partition of $[a+c, b+c]$.

Then if $P = \{x_0 < x_1 < \cdots < x_n\}$, let $P_c = \{x_0 - c < x_1 - c < \cdots < x_n - c\}$

Observe $U_{f(x)}(P_c) = U_{f(x-c)}(P)$

and $L_{f(x)}(P_c) = L_{f(x-c)}(P)$

The partitions P of $[a,b]$ are in one to one correspondence with the partitions of $[a+c, b+c]$ by the map $P \mapsto P_c$ and $|P| = |P_c|$

so then $\lim_{|P| \to 0} U_{f(x-c)}(P) - L_{f(x-c)}(P) =$

$\lim_{|P_c| \to 0} U_{f(x)}(P_c) - L_{f(x)}(P_c) \to 0.$

Hence $f(x-c)$ is integrable on $[a+c, b+c]$ and the upper sums of f over P_c equals the upper Riemann sum of $f(x-c)$ over P so the result follows by 19.2.