Homework 7 Solutions

11.1.1 Prove from the definition that $\frac{x}{1+x}$ is continuous at $x=1$.

Given $\varepsilon>$ for $\delta<\min\{2\varepsilon, 1\}$ if $|x-1|<\delta$ then

$$|f(x)-f(1)| = \left|\frac{x}{1+x}-\frac{1}{1+1}\right| = \left|\frac{2x}{2(1+x)}-\frac{(1+x)}{(1+x)2}\right| = \left|\frac{x-1}{2(1+x)}\right| < \frac{|x-1|}{2}$$ as $x>0$ by our choice of δ, so $1+x>1$

and $\frac{|x-1|}{2}<\frac{2\varepsilon}{2}=\varepsilon$. Thus f is continuous at $x=1$.

11.1.4. Show that if e^x is continuous at 0, it is continuous for all x.

For $x_0\in\mathbb{R}$, given $\varepsilon>0$, by continuity of e^x at 0, $\exists\ \delta$ s.t. if $|x-x_0|=|(x-x_0)-0|<\delta$, then

$|e^{x-x_0}-e^0|<\frac{\varepsilon}{e^{x_0}}$. Notice that we used here that $e^{x_0}>0$ so that $\frac{\varepsilon}{e^{x_0}}>0$ is an acceptable ε' for e^x at 0.

Now for $|x-x_0|<\delta$, $|e^x-e^{x_0}|=|e^{x-x_0+x_0}-e^{x_0}|=|e^{x_0}(e^{x-x_0}-e^0)|=|e^{x_0}||e^{x-x_0}-e^0|\le e^{x_0}\left(\frac{\varepsilon}{e^{x_0}}\right)=\varepsilon$, as desired.

11.1.5 f is continuous at x_0. Show there exists $\delta>0$, $B>0$ s.t. if $|y-x_0|<\delta$ then $|f(y)|<B$.
ie f is locally bounded at x_0.

Take $\varepsilon=1$. By the continuity of f at x_0, there is δ s.t. if $|y-x_0|<\delta$, then

$|f(y)-f(x_0)|<1$. In particular,

$$-1<f(y)-f(x_0)<1$$

$$f(x_0)-1<f(y)<f(x_0)+1$$

Set $B=\max\{|f(x_0)-1|, f(x_0)+1\}$. If $f(x_0)\geq 0$, then $B=f(x_0)+1>0$, and if $f(x_0)<0$, then $B=-f(x_0)+1>0.$, so $B>0$

and f is locally bounded at x_0.

11.2.2. Let $f(x)$ be even. Prove that $\lim_{x\to 0^+} f(x)=L \Rightarrow \lim_{x\to 0} f(x)=L$.

We want to show that $\lim_{x\to 0^-} f(x)=L$, as then by Theorem 11.2 $\lim_{x\to 0^+} f(x)=L$ and $\lim_{x\to 0^-} f(x)=L$ implies $\lim_{x\to 0} f(x)=L$

Given $\varepsilon>0$, as $\lim_{x\to 0^+} f(x)=L$, $\exists\ \delta$ s.t. if $0<x<\delta$ then $|f(x)-L|<\varepsilon$.

If $0<-x<\delta$ then $|f(x)-L|=|f(-x)-L|<\varepsilon$, by f being even. Thus $\lim_{x\to 0^-} f(x)=L$ as desired.
ie
$-\delta<x<0$

11.2.3 Prove if $f(x) \to \infty$ as $x \to a$, and $g(x) \geq b > 0$ for $x \approx a$ then $f(x)g(x) \to \infty$ as $x \to a$

Given $M > 0$, $\exists\ \delta_1$ st if $0 < |x-a| < \delta_1$ then $f(x) > \frac{M}{b}$, as $f(x) \to \infty$ as $x \to a$.

$\exists\ \delta_2$ s.t. if $0 < |x-a| < \delta_2$ then $g(x) \geq b$, by assumption.

For $\delta = \min(\delta_1, \delta_2)$, if $0 < |x-a| < \delta$, then $f(x)g(x) > \frac{M}{b}(b) = M$, so $f(x)g(x) \to \infty$ as $x \to a$.

11.3.4 Prove the Function Location Theorem 11.3D

Claim: If the limit exists, $\lim_{x \to x_0} f(x) < M \Rightarrow f(x) < M$ for $x \underset{\neq}{\approx} x_0$.

Let $L = \lim_{x \to x_0} f(x)$. As $L < M$, $0 < \frac{M-L}{2}$. Set $\varepsilon = \frac{M-L}{2}$. By definition of limit there exists $\delta > 0$ s.t. if $0 < |x - x_0| < \delta$ then $|f(x) - L| < \frac{M-L}{2}$ and

$$-\left(\frac{M-L}{2}\right) < f(x) - L < \left(\frac{M-L}{2}\right) \text{ so}$$

$$L - \left(\frac{M-L}{2}\right) < f(x) < \frac{M+L}{2} < \frac{M+M}{2} = M \text{ as } L < M \text{ implies } M+L < 2M.$$

Therefore for $0 < |x - x_0| < \delta$, $f(x) < M$.

11.4.2. If $f(x)$ is continuous on $[a,b]$ and strictly increasing on (a,b), prove it is strictly increasing on $[a,b]$.

We will show that $f(a) < f(x)$ for $x \in (a,b)$, and leave the other direction to you.
Suppose $f(a) \geq f(x)$ for some $x \in (a,b)$. If $f(a) = f(x)$, then by strict increasing of f on (a,b), we can consider instead $y = \frac{x+a}{2}$, and $f(a) > f(y)$. So without loss of generality, $f(a) > f(x)$.
Let $\varepsilon = \frac{f(a) - f(x)}{2}$. By continuity of f at a, $\exists\ \delta > 0$ s.t. if $z - a < \delta$, $|f(a) - f(z)| < \varepsilon$

We can also require $\delta < x - a$, so that $a < z < x$. Thus $-\varepsilon < f(a) - f(z) < \varepsilon$

so

$$f(z) - \varepsilon < f(a) < f(z) + \varepsilon. \quad \text{As} \quad f(a) - \varepsilon < f(z),$$

$$f(z) > f(a) - \frac{f(a) - f(x)}{2} = \frac{f(a) + f(x)}{2} > f(x)$$

This is a contradiction because f strictly increasing on (a,b) implies

$$f(z) < f(x) \quad \text{as} \quad z < x.$$

Therefore f is strictly increasing on $[a,b)$.

11.4.4 Prove that max(f,g) and min(f,g) are continuous if f and g are.

max (f,g) is defined pointwise by taking the maximum of $f(x)$ and $g(x)$ at each x.

$$\max(f,g)(x) = \frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2}$$

$$\min(f,g)(x) = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$$

The first term takes the average. Thesecond adds distance between the average and the original terms

Following the book, we want to show that absolute value function $\text{abs}(x)=|x|$, and then using composition and linear combinations of continuous functions, we end up w/ a continuous function.

Given $\varepsilon>0$ for $x_0 \in \mathbb{R}$, there is $\delta=\varepsilon$ s.t. if $|x-x_0|<\delta$,

$|\text{abs}(x)-\text{abs}(x_0)| = \big||x|-|x_0|\big| \leq |x-x_0| < \varepsilon$, by the reverse triangle inequality.

$\frac{f(x)+g(x)}{2}$ is continuous as f and g are continuous (linear comb. of continuous functions are cont.)

$\left|\frac{f(x)-g(x)}{2}\right|$ is continuous as a composition $\text{abs}\circ\left(\frac{f-g}{2}\right)(x)$, of which all pieces are continuous.

Hence max(f,g) and min(f,g) are continuous.

11.5.1 a) If f is continuous and $f(x)=0$ for all x rational, then $f(x)=0 \;\forall x$.

Here we make use of sequential continuity. If f is continuous at x_0, then any sequence $(x_n)_1^\infty$ s.t. $\lim_{n\to\infty} x_n = x_0$ also satisfies $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Every irrational x_0 has a sequence (x_n) of rationals s.t. $\lim_{n\to\infty} x_n = x_0$. In particular, you can use x_n = truncation of x_0 at the n^{th} decimal place.

By sequential continuity $f(x_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$, so $f(x)=0$ for all $x\in\mathbb{R}$.

b) Prove if $f(x)$ and $g(x)$ are continuous, and $f(x)\leq g(x)$ for all x rational, then $f(x)\leq g(x)$ for all x. Show that $\leq$ cannot be replaced with $<$ throughout.

Again, if x_0 is irrational, take (x_n) a sequence of rationals s.t. $\lim_{n\to\infty} x_n = x_0$.

Then $f(x_n)\leq g(x_n)$ for all n and so by limit location theorem

$f(x_0) = \lim_{n\to\infty} f(x_n) \leq \lim_{n\to\infty} g(x_n) = g(x_0)$, so $f(x)\leq g(x)$ for all x.

Consider $f(x)=0$, $g(x)=|x-\pi|$ for all x. Then $f(x)<g(x)$ for all $x\neq\pi$ (in particular for x rational)

but $f(x) \not< g(x)$ for all $x\in\mathbb{R}$.