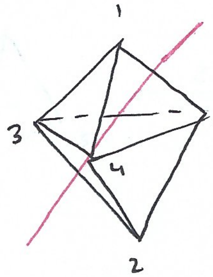
3560 HW 1

(1)



This rotation by π about the red axis corresponds to the permutation $(12)(35)$.

There are 2 more analagous rotations about an axis through 3 and edge $\{4,5\}$, and through 5 and edge $\{3,4\}$, which correspond respectively to $(12)(45)$ and $(12)(34)$.

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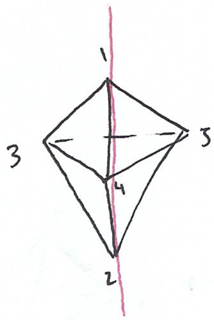
3
Symme



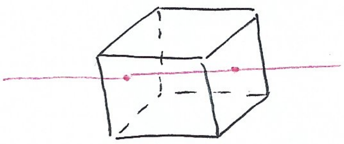
Two non-identity rotations by $2\pi/3$, $4\pi/3$ about red axis through vertices 1 and 2. These correspond to permutations (345) and (354).

2
Symm

There is also the identity symmetry } 1

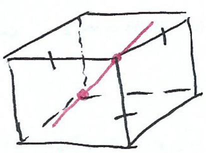
Total: 6 rotational symmetries

(2)

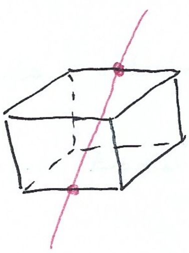


3 non-identity rotations about this axis $(\pi/2, \pi, 3\pi/2)$ and there are 2 more axes that go through opposite faces, so $3 \cdot 3 = 9$.

picture this symmetry as permuting the marked edges



2 non-identity rotations about this axis $(2\pi/3, 4\pi/3)$ and there are 3 more axes that go through opposite vertices, so $4 \cdot 2 = 8$.

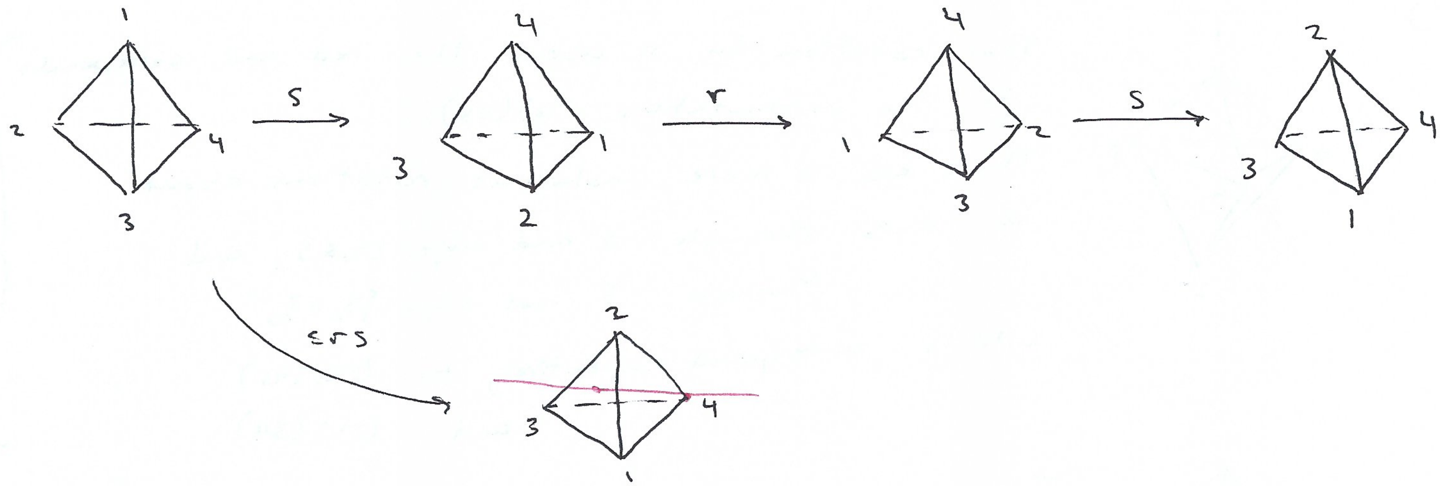


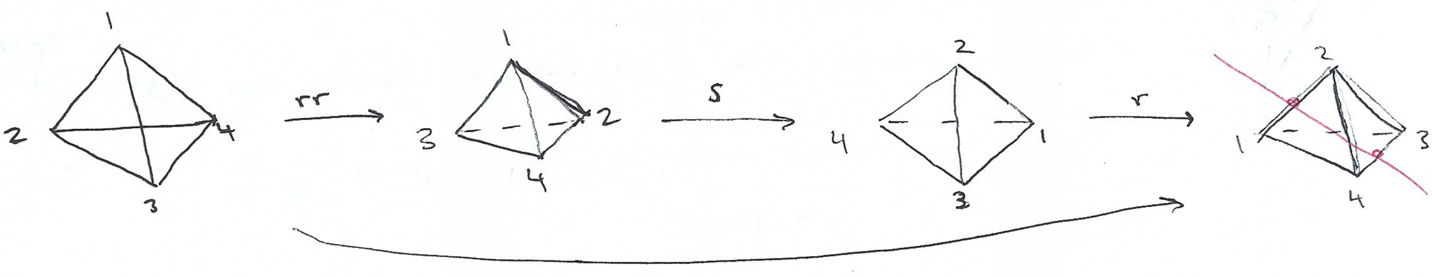
1 non-identity rotation about this axis (π) and there are 5 more axes that go through opposi edges, so $6 \cdot 1 = 6$.

1 identity rotation.

Total = 24. rotational symmetries.

3





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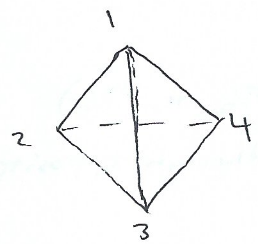


$\{ij\}$ denotes edge connecting vertex i and j.

We have e, r, r^2, s.

Previous exercise gave us $srs, (srs)^2$ for the two rotations ~~throug~~ about axis through 4.

Check that $sr, (sr)^2$ are the two rotations about axis through vertex 3.

Check that $srr, (srr)^2$ are the two for axis through vertex 2.

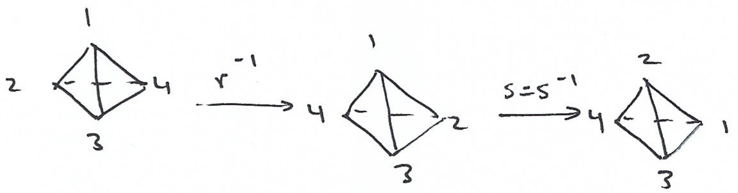
Previous exercise also gave us $rsrr$ as rotation about axis through $\{12\}$ and $\{34\}$.

Finally, check that $rrsr$ is the rotation about axis through $\{13\}$ and $\{24\}$.

(5) Since $r^3 = e$, $r^2 = r^{-1}$. Likewise, $s^2 = e \Rightarrow s^{-1} = s$.

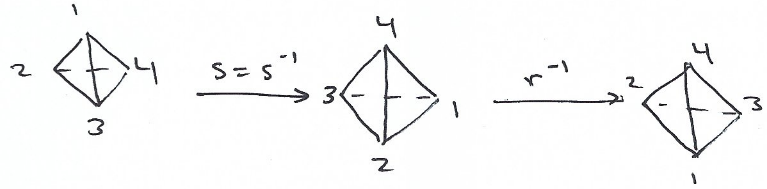
Note that (rs) means first do s then do r, so to undo (rs) first to undo r, then undo s. In group theory notation,

$$(rs)^{-1} = s^{-1}r^{-1}.$$



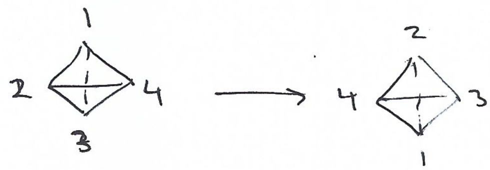
which we see is the same as srr from exercise 3.

Also,

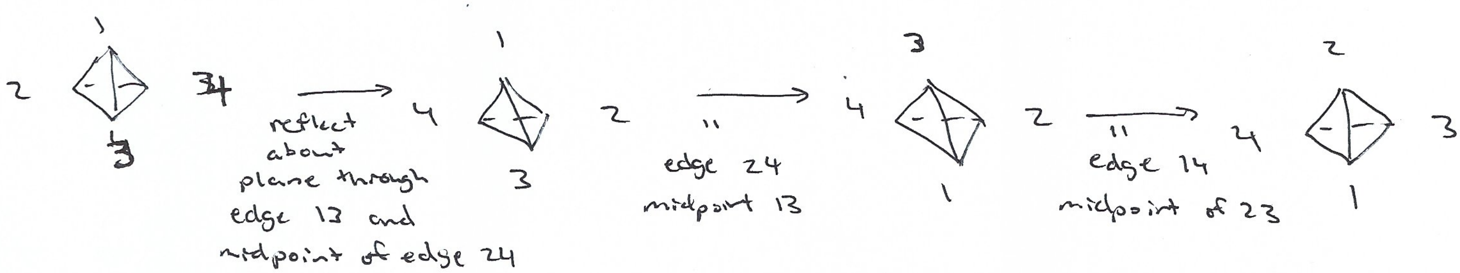


and check that this is rss.

(6) The symmetry



is a symmetry which is not a reflection (b/c there is no edge such that its vertices remain fixed) and is not a rotation (b/c no single vertex is fixed and no pair of edges have their end points flipped).



To see that there are 24 symmetries, consider how the symmetries permute the vertices $\{1,2,3,4\}$ of the tetrahedron. Check that using the reflections it is possible to get any permutation of the 4 vertices. Since there are $4! = 24$ permutations total, there must be exactly 24 symmetries of the tetrahedron.

//

(2.1) Check $\mathbb{R}^+$ closed under mult., 1 is identity, inverse of $x \in \mathbb{R}^+$ is $1/x$, and associativity holds b/c multiplication is associative.

(2.2) a) This is not a group for consider $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $ac = 0 \neq 1 = b^2$

and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $ac = 0 \neq 1 = b^2$,

two matrices in the specified set. Their product, however

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

Thus, the set is not even closed under multiplication.

b) This is not a group b/c $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ are in the set

but their product $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is not in the set.

c) Let $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ be two matrices w/ $a_1c_1 \neq 0$ $a_2c_2 \neq 0$.

Their product is $\begin{pmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$

and since $a_1c_1 \neq 0$, $a_2c_2 \neq 0$ we have that $a_1a_2c_1c_2 \neq 0$.

Likewise, the product $\begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ is also of the same form.

The identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is contained in the set and is the multiplicat identity under matrix multiplication.

The determinant of $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$ is $a_1c_1 - 0b_1 = a_1c_1$ and if this is nonzero then it has an inverse matrix. Check that $\begin{pmatrix} 1/a_1 & -b_1/(c_1a_1) \\ 0 & 1/c_1 \end{pmatrix}$ works and is of the desired form, i.e. $\left(\frac{1}{a_1}\right)\left(\frac{1}{c_1}\right) \neq 0$.

Multiplication of matrices is associative so the group law for this set is also associative.

d) Not a group because check that $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has det $\neq 0$ but its inverse is $\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$ which has entries not all integers.

Or considering a diagonal matrix is even easier. //

2.5 Suppose $f: \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry.

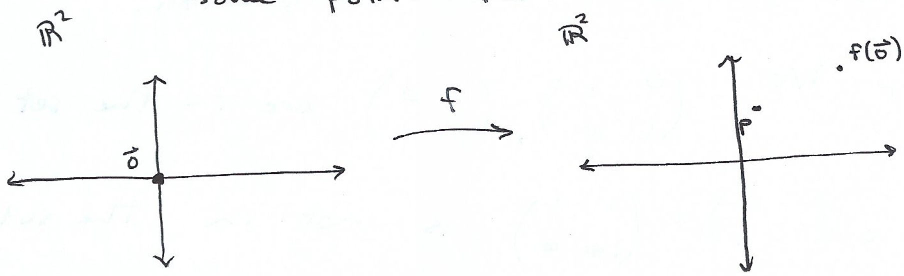
Or can think about triangles. for alternat[e] proof.

Injectivity: Suppose $a, b \in \mathbb{R}^2$ s.t. $f(a) = f(b)$. Then $d(f(a), f(b)) = 0$, implies $d(a,b) = 0$ because f is an isometry.

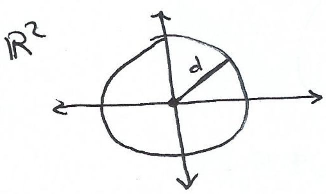
Then $a = b$.

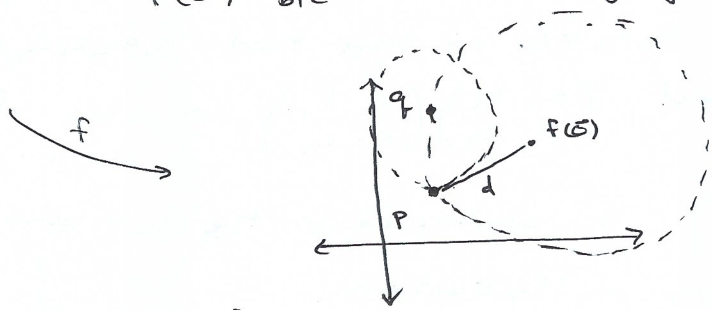
Surjectivity: Take a point $p \in \mathbb{R}^2$ and we will show $\exists\, a \in \mathbb{R}^2$ s.t. $f(a) = p$.

Consider the origin $\vec{o} = (0,0) \in \mathbb{R}^2$ and its image under f is some point $f(\vec{o}) \in \mathbb{R}^2$. If $f(\vec{o}) = p$ we are done, so suppose $f(\vec{o}) \neq p$.



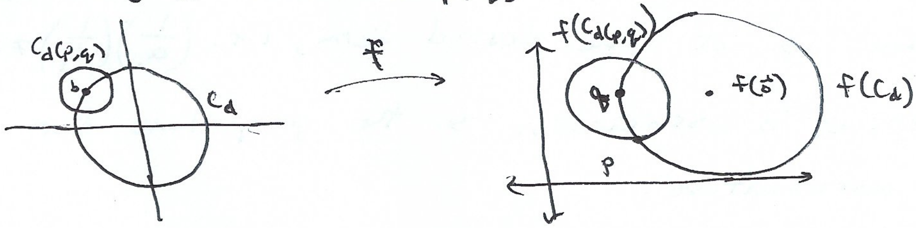
Let $d = d(p, f(\vec{o}))$ be the distance from the image of the origin to the point p. Consider the circle C_d of radius d in $\mathbb{R}^2$ centered at the origin. The image $f(C_d)$ of the circle is contained in the circle of radius d centered at $f(\vec{o})$ b/c f is an ~~isogeny~~ isometry.





So if p is in the image of f then it is in $f(C_d)$. There must be at least one point in the image $f(C_d)$, call i[t] $q \in f(C_d)$.

Then there is a $b \in C_d \subseteq \mathbb{R}^2$ s.t. $f(b) = q$. Then ~~the image~~ let $C_{d(p,q)}$ be the circle of radius $d(p,q)$ centered at b. Then $C_{d(p,q)}$ intersects C_d



in two points, so the[y] map to two distinct points under f because f is injective. Thus, one of these points must be p because $f(C_{d(p,q)} \cap C_d) = f(C_{d(p,q)}) \cap f(C_d)$.

Thus $\exists\, a \in \mathbb{R}^2$ s.t. $f(a) = p$, so f is surjective. //

Composition of two isometries is an isometry, function composition is assoc., identity function is an isometry, each isometry is a bijection so has an inverse, which

2.8 We would like to show that $x^{-1}y^{-1}$ is the inverse of (yx).

Since $(x^{-1}y^{-1})(yx) = x^{-1}(y^{-1}y)x = x^{-1}ex = x^{-1}x = e$

and $(yx)(x^{-1}y^{-1}) = y(xx^{-1})y^{-1} = yey^{-1} = yy^{-1} = e$,

we see that $x^{-1}y^{-1}$ satisfies the properties for the inverse to (yx).

Since inverses are unique, the inverse of yx is $x^{-1}y^{-1}$, or

in group theory notation, $(yx)^{-1} = x^{-1}y^{-1}$.

Proving that $(xy)^{-1} = y^{-1}x^{-1}$ is similar. //