Groups and Symmetry HW5 Solutions

October 2, 2014

Exercise 1. 6.1

The multiplication table for S_3 is the following table, where the entry in column α, row β corresponds to the permutation $\beta \circ \alpha$, which means do α first and then β.

Proof.

	e	(12)	(13)	(23)	(123)	(132)
e	e	(12)	(13)	(23)	(123)	(132)
(12)	(12)	e	(132)	(123)	(23)	(13)
(13)	(13)	(123)	e	(132)	(12)	(23)
(23)	(23)	(132)	(123)	e	(13)	(12)
(123)	(123)	(13)	(23)	(12)	(132)	e
(132)	(132)	(23)	(12)	(13)	e	(123)

□

Exercise 2. 6.2

Proof. (a) Start with 1 and apply the permutation recursively to the image until you get back to 1. Then do the same to the smallest number not yet visited, in this case this is 2. At this point, 5 is the smallest not yet visited. The result is (1734)(26)(58). We can rewrite this in terms of transpositions as (17)(73)(34)(26)(58), or also as (14)(13)(17)(26)(58).

(b) Note that the notation (4568)(1245) means first apply (1245) and then (4568). Start with 1 and apply the two permutations: 1 goes to 2 and 2 is not included in the second permutation, so in total 1 goes to 2. Now 2 goes to 4 in the first permutation, but 4 goes to 5 in the second permutation, so in total 2 goes to 5. Continuing on, 5 goes to 1 in the first permutation and 1 is not mentioned in the second permutation, so 5 goes to 1 in total. This gives our first cycle (125). The element 3 is now the smallest one that has not been visited, but it is not included in either permutation, so 3 maps to 3 (so we may omit writing it). Continue on with 4, which goes to 5 in the first permutation and 5 goes to 6 in the second permutation, so in total 4 goes to 6. The element 6 is not included in the first permutation, which means implicitly that the first permutation sends 6 to 6, but then 6 maps to 8 under the second permutation. Finally, 8 is mapped to 4, so the final answer is (125)(468) where 3 and 7 are fixed, so we don't include them in the notation. As a product of transpositions we may write this as (12)(25)(46)(68).

(c) (25687)(34)=(25)(56)(68)(87)(34)

□

Exercise 3.

Proof.

$$
\begin{aligned}
&(143)P(x_1, x_2, x_3, x_4) \\
=&(143)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \\
=&(x_4 - x_2)(x_4 - x_1)(x_4 - x_3)(x_2 - x_1)(x_2 - x_3)(x_1 - x_3) \\
=&P(x_1, x_2, x_3, x_4),
\end{aligned}
$$

since there are 4 inversions.

Rewriting gives $(23)(412) = (1324)$, so

$$
\begin{aligned}
&(1324)P(x_1, x_2, x_3, x_4) \\
=&(1324)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \\
=&(x_3 - x_4)(x_3 - x_2)(x_3 - x_1)(x_4 - x_2)(x_4 - x_1)(x_2 - x_1) \\
=&-P(x_1, x_2, x_3, x_4),
\end{aligned}
$$

since there are 5 inversions. □

Exercise 4. 6.6

Proof. Since H is not contained in A_5, there exists at least one odd permutation in H. Choose one and call it α. Consider the subset $E = A_n \cap H \subseteq H$ of even permutations in H. Translate this subset by α to get the subset $\alpha E = \{\alpha\beta \mid \beta \in E\}$. Note that every element of αE is an odd permutation because $\alpha\beta$ is odd when β is even. Therefore, E and αE are disjoint subsets of H, and E and αE must have the same cardinality because multiplication by α is a one-to-one map ($\alpha\beta_1 = \alpha\beta_2 \implies \beta_1 = \beta_2$). If we can show that the two subsets E and αE fill up all of H, then we are done.

Let γ be some element of H. If it is even, then $\gamma \in E$. If it is odd, then $\alpha^{-1}\gamma$ is even, so $\alpha^{-1}\gamma \in E$. But then $\gamma = \alpha(\alpha^{-1}\gamma)$ is in αH. Thus, the union of the two disjoint sets E and αE is in fact all of H. □

Exercise 5. 6.8

Proof. The sign of of α is the same as the sign of $-\alpha$, since if α is written as disjoint cycles, then α^{-1} can be written as the same number of disjoint cycles of the same length, so the number of transpositions has the same parity for both of them. Therefore, of the four elements $\alpha, \alpha^{-1}, \beta, \beta^{-1}$, either all four are even, two are even and two are odd, or all four are odd. In every case, the product $\alpha\beta\alpha^{-1}\beta^{-1}$ has even order, so is contained in A_n.

Suppose that β is an even permutation. Then if α is even then $\alpha\beta\alpha^{-1}$ is a product of even permutation, so is itself even. If α is odd, then $\alpha\beta\alpha^{-1}$ is a product of two odd permutations and one even permutation so is altogether an even permutation.

Write $(2134) = (4321)$ which has inverse (1234). Then $(4321)(423)(1234)^{-1} = (4321)(423)(1234) = (123)$. The inverse of (234) is (324), so

$$\begin{aligned}
&(4321)(423)(1234)^{-1}(234)^{-1} \\
=&\left[(4321)(423)(1234)^{-1}\right](234)^{-1} \\
=&(123)(324) \\
=&(124).
\end{aligned}$$

□