Groups and Symmetry HW8 Solutions

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Exercise 1. 15.1

Proof. Suppose that $HJ = JH$. We will show that HJ is a subgroup. The identity element is in HJ. If $h_1j_1, h_2j_2 \in HJ$, then there exists some $h' \in H, j' \in J$ such that $j_1h_2 = h'j'$, which implies $h_1j_1h_2j_2 = h_1h'j'j_2 \in HJ$, so HJ is closed under multiplication. If $hj \in HJ$, then $(hj)^{-1} = j^{-1}h^{-1} \in JH = HJ$, so HJ is a sungroup.

Conversely, suppose that HJ is a subgroup. Let $h \in H$ and $j \in J$. Then since $hj \in HJ$, the inverse $j^{-1}h^{-1} = (hj)^{-1}$ is in HJ. Thus, there exist $h' \in H, j' \in J$ such that $j^{-1}h^{-1} = h'j'$. This implies $jh = (j')^{-1}(h')^{-1} \in JH$. Since h and j were arbitrary, this shows that $HJ \subseteq JH$ and likewise $JH \subseteq HJ$. □

Exercise 2. 15.2

Proof. (a) The proper normal subgroups of $D_4 = \{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$ are $\{e, r, r^2, r^3\}$, $\{e, r^2, s, r^2s\}$, $\{e, r^2, rs, r^3s\}$, and $\{e, r^2\}$. To see this note that s is conjugate to r^2s (conjugate by r), so if a subgroup contains s for it to be normal it must contain r^2s. But if it contains s and r^2s then it must also contain r^2. Check that these four elements indeed form a subgroup, and since this subgroup has index 2 is normal.

Likewise, rs is conjugate to r^3s, so if a subgroup contains one of them it must contain both to be normal. But then the subgroup must also contain $r^3srs = r^2$. Check that $\{e, rs, r^3s, r^2\}$ also forms a subgroup. It is also normal because it has index 2.

If a subgroup contains r then it contains the subgroup generated by r which has index 2, so is normal.

Finally, r^2 commutes with every other element, so $\{e, r^2\}$ is a normal subgroup. These exhaust all of the possibilities for proper normal subgroups, as you can verify (note that once you have strictly more than 4 elements, the subgroup cannot be proper because it's order has to divide 8).

(b) The only proper normal subgroup of $D_5 = \{e, r, r^2, r^3, r^4, s, rs, r^2s, r^3s, r^4s\}$ is $< r >$. To see this let's analyze the general case of D_n. Conjugating s we see that $(r^as)s(r^as)^{-1} = r^asssr^{-a} = r^asr^{-a} = r^{2a}s$. Therefore, s is conjugate to $r^{2a}s$ for every integer a (i.e. all even powers of r times s). Conjugating by just a power of r gives $r^asr^{-a} = r^{2a}s$ as well. Therefore, the set

$$E = \{r^{2a}s \mid a \in \mathbb{Z}/n\mathbb{Z}\}$$

is a conjugacy class. Similarly, conjugating rs gives $(r^as)rs(r^as)^{-1} = r^asrssr^{-a} = r^asr^{-a+1} = r^{2a-1}s$ and $r^arsr^{-a} = r^{2a+1}s$ (i.e. all odd powers of r times s). Therefore,

$$O = \{r^{2a+1}s \mid a \in \mathbb{Z}/n\mathbb{Z}\}$$

is a conjugacy class as well. However, if n is odd, then these conjugacy classes coincide and $E = O = \{r^as \mid a \in \mathbb{Z}/n\mathbb{Z}\}$.

Any element of the form r^a is conjugate only to r^{n-a} (check!), so the sets $R_a = \{r^a, r^{n-a}\}$ are conjugacy classes. Therefore, the obvious proper normal subgroups are the subgroups generated by r^a where a is a divisor of n (and $a \neq 1$), since for any r^{ak} in $< r^a >$, r^{n-ak} is also in $< r^{ak} >$. This coincides with us finding that $< r^2 >$ is a normal subgroup of D_4 but not of D_5 and that $< r >$ is a normal subgroup in both D_4 and D_5.

If n is odd, then $E = O$ and this conjugacy class contains n elements. Together with the identity this make $n+1$ elements, so it cannot possible be a proper subgroup (because its order has to divide $2n$). Therefore, if n is odd, there are no other proper normal subgroups.

Now suppose n is even and that the conjugacy class E is contained in some potentially normal subgroups N. Then since s and r^2s are in this subgroup, so is r^2. Since n is even, r^2 does not generate all of $\{e, r, r^2, \dots, r^{n-1}\}$, but just the subgroup of $< r^2 >$. Check that $E\cup < r^2 >$ does indeed form a subgroup (i.e. the elements don't generate anything outside of this list of elements). The subgroup $< r^2 >$ is also a union of conjugacy classes, so $E\cup < r^2 >$ is a proper normal subgroup.

Now suppose that the conjugacy class O is contained in some potentially normal subgroup N. Then since rs and r^3s are in O, r^2 is again in the subgroup N. Again since n is even, check that $O\cup < r^2 >$ indeed forms a subgroup, which is again normal since it is a union of conjugacy classes.

To summarize, D_n always has the proper normal subgroups $< r^a >$ where a is a divisor of n. If n is odd then these are all of them. If n is even then D_n also has $E\cup < r^2 >$ and $O\cup < r^2 >$ as proper normal subgroups (check this against what we got for D_4 and D_5).

$\square$

Exercise 3. 15.5

Proof. Let $G = D_4$, $H = \{e, s, r^2, r^2 s\}$, which is normal in G because it has index 2. Let $J = \{e, s\}$, which is normal in H because it has index 2. However, J is not normal in G because $r^{-1}sr = r^{-2}s = r^2 s$, which is not in J. □

Exercise 4. 15.6

Proof. Let $x \in H$ and $y \in J$. Since J is normal, $xyx^{-1} \in xJx^{-1} = J$, which implies that $xyx^{-1}y^{-1}$ is in J. Likewise, since H is normal, $yx^{-1}y \in y^{-1}Hy = H$, which implies that $xyx^{-1}y^{-1} \in H$. Therefore $xyx^{-1}y^{-1}$ is in both H and J, but they only have the identity element in common, so $xyx^{-1}y^{-1} = e$ implies $xy = yx$. □

Exercise 5. 15.8

Proof. The commutator subgroup of A_4 is $\{e, (12)(34), (13)(24), (14)(23)\}$ as can be checked by direct computation.

Now consider A_n for $n \geq 5$. Recall that A_n is generated by all of the 3-cycles, so if we can show that $[A_n, A_n]$ contains every 3-cycle then we are done. Let (abc) be an arbitrary 3-cycle, so that a, b, c are distinct numbers between 1 and n. We would like to write it in the form $ghg^{-1}h^{-1}$ for two elements $g, h \in A_n$. Rewrite $(abc) = (acb)(acb) = (ab)(ac)(ab)(ac)$. Now choose two distinct numbers x, y that are different from a, b, and c. This is possible since $n \geq 5$, so note that this proof fails for A_4 as we know it should. Then

$$\begin{aligned}(abc) &= (acb)(acb)\\ &= (ab)(ac)(ab)(ac)\\ &= (xy)(xy)(xy)(xy)(ab)(ac)(ab)(ac)\\ &= (xy)(ab)(xy)(ac)(xy)(ab)(xy)(ac)\end{aligned}$$

where we can commute (xy) with both (ab) and (ac) because they are disjoint cycles. Setting $g = (xy)(ab) \in A_n$, $h = (xy)(ac) \in A_n$ we have written (abc) as $ghg^{-1}h^{-1}$. Therefore, every 3-cycle is contained in the commutator subgroup, and therefore all of A_n is contained in the commutator subgroup, so $[A_n, A_n] = A_n$ for $n \geq 5$. □

Exercise 6. 16.1

Proof. (a) Yes. Let $z = a + bi$ and $w = x + yi$. Then $zw = ax - by + (ay + bx)i$, so $\phi(zw) = ax - by - (ay + bx)i = (a - bi)(x - yi) = \phi(z)\phi(w)$, so ϕ is a homomorphism.

(b) Yes. We have $\phi(zw) = (zw)^2 = z^2 w^2 = \phi(z)\phi(w)$.

(c) No. We have $\phi(1 \cdot i) = -1$, but $\phi(1)\phi(i) = i \cdot -1 = -i$, and in fact the identity 1 does not map to the identity.

(d) Yes. Compute $\phi(zw) = \sqrt{(ax-by)^2+(ay+bx)^2} = \sqrt{a^2+b^2}\sqrt{x^2+y^2} = \phi(z)\phi(w)$.

□

Exercise 7. 16.2

Proof. (a) No. For matrices A, B, $\phi(AB) = (AB)^t = B^tA^t = \phi(B)\phi(A)$, and there certainly exist matrices in $GL_n(\mathbb{C})$ that do not commute so there are matrices such that $\phi(AB) = \phi(B)\phi(A) \neq \phi(A)\phi(B)$.

(b) Yes. We have $\phi(AB) = ((AB)^{-1})^t = (B^{-1}A^{-1})^t = (A^{-1})^t(B^{-1})^t = \phi(A)\phi(B)$.

(c) No. For matrices $A, B,$, $\phi(AB) = (AB)^2 = ABAB$, which is not always equal to A^2B^2.

(d) No. We have $\phi(AB) = (AB)^* = B^*A^*$ which is not equal in general to A^*B^* where recall A^* denotes the adjoint of A.

□

Exercise 8. 16.3

Proof. Define the map $\phi : G \times H \to H$ by $\phi((g,h)) = h$. This map is surjective onto H. If $\phi((g,h)) = e$, then $h = e$, which shows that the elements of $\ker\phi$ are of the form (g, e). Furthermore, for any $g \in G$, $\phi((g,e)) = e$ so in fact all such elements are in the kernel, i.e. $\ker\phi = G \times \{e\}$. Kernel's are always normal and by the isomorphism theorem in the text, $G \times H/(G \times \{e\}) \cong H$. □

Exercise 9. 16.4

Proof. Consider the homomorphism $\phi : G \times H \to (G/A) \times (H/B)$ defined by $\phi((g,h)) = (gA, hB)$. This is a homomorphism because $\phi((g_1,h_1),(g_2,h_2)) = \phi((g_1g_2, h_1h_2)) = (g_1g_2A, h_1h_2B) = (g_1A, h_1B)(g_2A, h_2B) = \phi((g_1,h_1))\phi((g_2,h_2))$. The kernel of ϕ is any element (g,h) such that $g \in A$ and $h \in B$, which is $A \times B$. Therefore, $A \times B$ is a normal subgroup of $G \times H$ and $G \times H/A \times B \cong G/A \times H/B$. □

Exercise 10. 16.8

Proof. Suppose that ϕ is a homomorphism. Let $H = \{(g, \phi(g)) \mid g \in G\} \subseteq G \times G'$ and let $(h, \phi(h)), (g, \phi(g)) \in H$. Then

$$\begin{aligned}(h,\phi(h))(g,\phi(g))^{-1} &= (h,\phi(h))(g^{-1},\phi(g)^{-1})\\ &= (hg^{-1}, \phi(h)\phi(g)^{-1}),\end{aligned}$$

and since ϕ is a homomorphism this is equal to $(hg^{-1}, \phi(hg^{-1})) \in H$. Thus, H is a subgroup.

Conversely, suppose that H is a subgroup. Then let $g, h \in G$. Since H is a subgroup, $(g,\phi(g)) = (h,\phi(h)) = (gh, \phi(g)\phi(h))$ must be some element $(x, \phi(x)) \in H$. This implies $x = gh$, so that $(gh, \phi(g)\phi(h)) = (gh, \phi(gh))$. Thus, $\phi(gh) = \phi(g)\phi(h)$ and ϕ is a homomorphism. □