(Very) basic introduction to special relativity

Math 3560 Groups and symmetry, Fall 2014

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Newton's reference frames

Figure : Isaac Newton

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3. Law of action-reaction For every action there is an equal and opposite reaction.

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Although this statement may seem obvious, it is not clear at all how to choose such a reference frame in the universe.

Example

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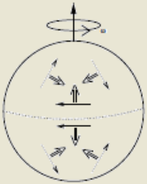
If we use a fixed point on earth to define a reference frame, then we would obtain an example of a non inertial frame. In this frame the centrifugal force and the Coriolis force (which is responsible for the formation of hurricanes) violate Newton's first law.



Systems of particles

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Decompose $\mathbf{F}_i$ as

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$$\begin{aligned} \sum_i F_i &= \sum_{i,j,j \neq i} \mathbf{F}_{ij} + \sum_i \mathbf{F}_i^{\text{ext}} \\ &= \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) + \sum_i \mathbf{F}_i^{\text{ext}}. \end{aligned}$$

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But according to Newton's third law, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, hence

$$\sum_i \mathbf{F}_i = \sum_i \mathbf{F}_i^{\text{ext}} =: \mathbf{F}^{\text{ext}},$$

where $\mathbf{F}^{\text{ext}}$ is the total exterior force acting on the system.

We define the total mass of the system to be

$$M = \sum_i m_i,$$

and the center of mass to be

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In a system where there are no external forces (like the universe) this equation says that fixing the origin at the center of mass gives you an inertial reference frame.

Example

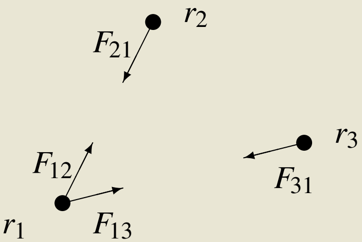
Let

$$\mathbf{r}_1 = (0,0) \quad \mathbf{r}_2 = (3,6) \quad \mathbf{r}_3 = (8,2),$$

and assume that

$$\mathbf{F}_{12} = (1,2) \quad \mathbf{F}_{13} = (2,1/2) \quad \mathbf{F}_{23} = (0,0).$$

The we obtain the following picture:



Absolute vs. relative point of view

Figure : Gottfried Leibniz

Classical Electromagnetism

Figure : James Clerk Maxwell

Maxwell equations (1862)

Let $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{J}$ be the electric field, magnetic field and current density, respectively, and let ρ be a charge distribution in $\mathbb{R}^3$.

1. Gauss law.

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0.$$

2. Gauss law for magnetism.

$$\nabla \cdot \mathbf{B} = 0.$$

3. Fadaray Law.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

4. Ampère circulation law (with Maxwell correction.)

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).$$

where ε_0 and μ_0 are the electric and magnetic constants.

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More precisely, if $\rho = 0$, then Maxwell equations are equivalent to the system of equations:

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

and

$$\frac{\partial^2 \mathbf{B}}{\partial x^2} + \frac{\partial^2 \mathbf{B}}{\partial y^2} + \frac{\partial^2 \mathbf{B}}{\partial z^2} - \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0,$$

plus a condition relating $\mathbf{E}$ and $\mathbf{B}$.

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Based on this observation, Maxwell predicted that light was just an example of an electromagnetic wave.

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This was supposed to be the medium through which electromagnetic waves propagated. The speed calculated by Maxwell was then the speed of light with respect to this medium.

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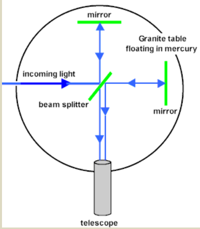

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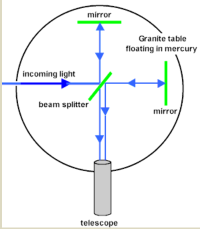

The result however, was totally unexpected. The speed of light was the same in every direction!

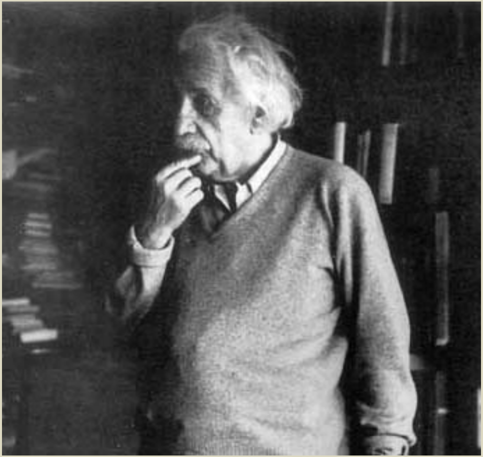

Figure : Albert Einstein

Space-time coordinates

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Observe that if $r(t)$ represents the position of a particle at time t, then we can describe its position as a point in $\mathbb{R}^2$ using the coordinates $(t, r(t))$.

The Minkowski metric

Figure : Hermann Minkowski

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But if we ask about the distance between this moment and the homecoming event, we only need two numbers: Schoellkopf Field is half a mile from here, and it's been four days since Friday.

Can we combine this 2 numbers to get a notion of "distance" between this two events?

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Unfortunately this is not very useful because the laws of physics are not invariant under the action of $SO(4, \mathbb{R})$.

For example, if we write Maxwell equations and then we transform everything using an element of $SO(4, \mathbb{R})$ we get an equation that looks very different.

This is clear if we look at the wave equation in one dimension:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0.$$

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Definition

Let $v_1 = (t_1, x_1)$ and $v_2 = (t_2, x_2)$ be two vectors in $\mathbb{R}^2$. We define the Minkowski product of v_1 and v_2 by

$$v_1 \circ v_2 = t_1 t_2 - x_1 x_2.$$

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However the Minkowski product of a vector in $\mathbb{R}^2$ with itself is well defined and is a very useful quantity as we will see soon.

Special relativity

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As we will see shortly, this is extremely useful.

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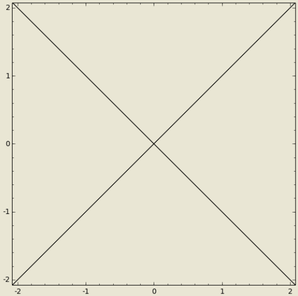

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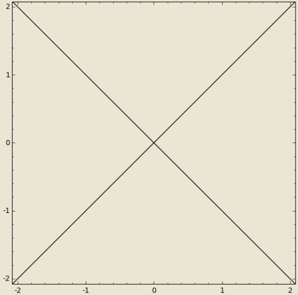

This are called the light rays, precisely because they describe the trajectory of light coming up from the origin.

The rest of the plane gets divided into two regions, the time-like directions:

$$\{v \in \mathbb{R}^2 \,|\, v \circ v > 0\},$$

and the space-like directions:

$$\{v \in \mathbb{R}^2 \,|\, v \circ v < 0\}.$$

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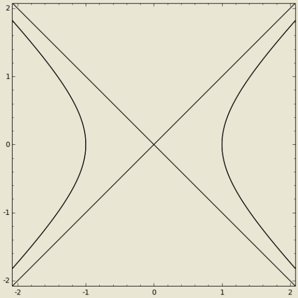
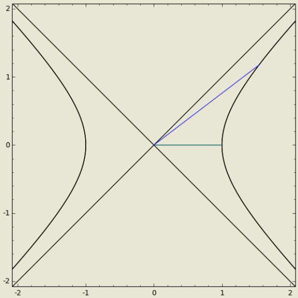


Now we will consider two particles, r_1 and r_2 traveling at constant speed from the origin and whose graph in $\mathbb{R}^2$ are given by the parametric equations: $t \mapsto (t,0)$, $t \mapsto (t\cosh\alpha, t\sinh\alpha)$.



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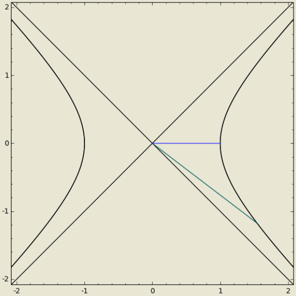

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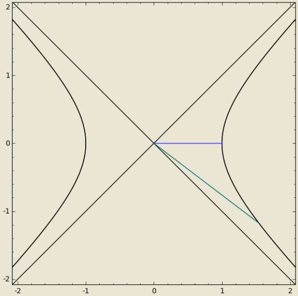

Observe that this means that for both observers their relative velocities are the same, and the speed of light is the same with respect to both observers!

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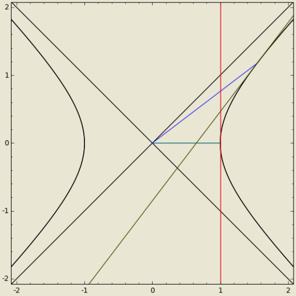

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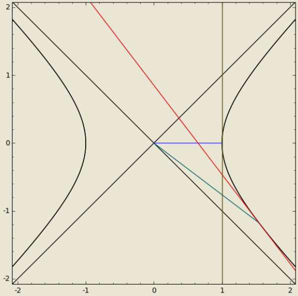

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However, the observer at the coast will see the light arriving to the different walls at different times.