MATH 6210 HOMEWORK 6

1. Let X be a set, and let $\mathscr{C} \subset \mathscr{P}(X)$ be a σ-algebra generated by a collection of sets $\mathscr{E}$. Then $\mathscr{C}$ is the union of the σ-algebras generated by $\mathscr{F}$ as $\mathscr{F}$ ranges over all countable subsets of $\mathscr{E}$. (Hint: Show that the latter object is a σ-algebra.)

2. If $(X, \mathscr{C}, \mu)$ is a measure space and E, $F \in \mathscr{C}$, then

$$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F).$$

3. Let $(X, \mathscr{C}, \mu)$ be a finite measure space. (That is, $\mu(X) < \infty$.) Given E, $F \in \mathscr{C}$ we define its *symmetric difference* to be the set $E \Delta F = (E \cap F^c) \cup (E^c \cap F)$.

a) If E, $F \in \mathscr{C}$ and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$.

b) We say that $E \sim F$ if $\mu(E \Delta F) = 0$; then $\sim$ is an equivalence relation on $\mathscr{C}$.

c) For E, $F \in \mathscr{C}$, define $\rho(E, F) = \mu(E \Delta F)$. Then, if E, F, $G \in \mathscr{C}$,

$$\rho(E, G) \leq \rho(E, F) + \rho(F, G),$$

and hence ρ defines a metric on the space $\mathscr{C}/\sim$ of equivalence classes.

4. Let $\mathscr{M}$ be the σ-algebra of Lebesgue measurable sets in $\mathbb{R}$. If $E \in \mathscr{M}$ and $\mu(E) > 0$ (where μ is the Lebesgue measure) then for any $\alpha < 1$ there is an open interval I such that

$$\mu(E \cap I) > \alpha\mu(I).$$

5. If $E \in \mathscr{M}$ and $\mu(E) > 0$, the set

$$E - E = \{x - y \, : \, x, \, y \in E\}$$

contains an interval centered at 0. (Hint: If I is as in exercise 4 with $\alpha > \frac{3}{4}$, then $E - E$ contains the open interval $(-\frac{1}{2}\mu(I), \frac{1}{2}\mu(I))$.)