MATH 713 SPRING 2012

LECTURE NOTES ON FUNCTIONAL ANALYSIS

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Contents

1. Topological Vector Spaces

We consider only real or complex vector spaces.

Definition 1.1. A *topological vector space* is a Hausdorff topological space X which is also a vector space such that the maps

a) $x, y \mapsto x - y$ is continuous from $X \times X$ to X and
b) $\alpha, x \mapsto \alpha x$ is continuous from $\{\text{scalars}\} \times X$ to X.

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Exercise 1.1. Prove that if X is a topological space and a vector space such that a) and b) hold and such that for every $x \neq 0$ $\exists$ a neighborhood U of 0 such that $x \notin U$ then X is a Hausdorff space (and is therefore a topological vector space).

Exercise 1.2. Consider $\mathbb{R}^2$ with the following topology: A set $V \subset \mathbb{R}^2$ is open iff for each point $(a,b) \in V$ there is an $\varepsilon > 0$ such that $\{(x,b) : |x-a| < \varepsilon\} \subset V$.

(1) Prove that $\mathbb{R}^2$ is a Hausdorff space in this topology.
(2) Determine whether or not $\mathbb{R}^2$ is a topological vector space in this topology (with the usual vector space operations).

Example 1.2. Any normed linear space is a topological vector space in the metric topology determined by the norm:

$$d(x,y) = \|x-y\|.$$

Definition 1.3. A *semi–norm* N on a vector space X is a function $N : X \to \mathbb{R}$ such that

(1) Positivity: $N(x) \geq 0 \ \forall x \in X$.
(2) Positive homogeneity: $N(\alpha x) = |\alpha| N(x)$
(3) Subadditivity: $N(x+y) \leq N(x) + N(y)$

(Recall that a norm is a semi–norm such that $N(x) = 0$ implies that $x = 0$.) If N is a semi–norm on X and $a \in X$ and $\rho > 0$, then

$$S_N(a,\rho) := \{x \in X : N(x-a) < \rho\}$$

is called the *open* N *ball* of radius ρ centered at a.

Example 1.4. Let L be any linear functional on X. Then $N(x) = |L(x)|$ is a semi–norm.

Definition 1.5. A collection $\mathcal{N}$ of semi–norms on a vector space X is called *separating* if $N(x) = 0 \ \forall N \in \mathcal{N}$ implies that $x = 0$.

If $\mathcal{N}$ is a family of semi–norms on X, a set S of the form

$$S = S_{N_1}(a,\rho_1) \cap S_{N_2}(a,\rho_2) \cap \ldots \cap S_{N_k}(a,\rho_k), \tag{1.1}$$

where $a \in X$, $N_1, N_2, \ldots, N_k \in \mathcal{N}$ and $\rho_1, \rho_2, \ldots \rho_k \in (0,\infty)$, is called an *open* $\mathcal{N}$*–ball centered at* a.

Definition 1.6. Given a family $\mathcal{N}$ of semi-norms on a vector space X, let $X^{\mathcal{N}}$ denote X equipped with the topology having the open $\mathcal{N}$ – balls as a basis. (You should check that the $\mathcal{N}$ balls form a basis for a topology.) Explicitly, a set $V \subset X$ is open iff for all $a \in V$ there exists an open $\mathcal{N}$ ball S centered at a such that $S \subset V$. (It is easy to verify that $X^{\mathcal{N}}$ is Hausdorff iff $\mathcal{N}$ is separating and that $X^{\mathcal{N}}$ is a topological vector space when $\mathcal{N}$ is separating.)

Examples 1.7. Some examples of topological vector spaces.

(1) $\mathcal{S}(\mathbb{R}) = C^\infty$ complex valued functions f on $\mathbb{R}$ such that $x^n f^{(k)}(x) \in L^2$ $\forall n \geq 0$ and $k \geq 0$. Let $\|f\|_{n,k} = \|x^n f^{(k)}\|_{L^2}$. If $\mathcal{N} = \{\| \cdot \|_{n,k}\}$ then $\mathcal{S}^{\mathcal{N}}$ is called the Schwartz space of rapidly decreasing functions. $\mathcal{N}$ is separating because $\|f\|_{0,0} = 0$ implies that $f = 0$.
(2) If Y is a set of linear functionals on X which separates points of X then the Y topology of X is the topology determined by the semi–norms $N_L(x) = |L(x)|$, $L \in Y$. Notation: $\sigma(X,Y)$.

(3) Special Case. $X = C([0,1])$, Y = point evaluations, i.e., Y = all finite linear combinations of L_t, $t \in [0,1]$ where $L_t(f) = f(t)$. Note that a sequence $f_n \in X$ converges to $f \in X$ in this topology iff $f_n(t) \to f(t)$ for each $t \in [0,1]$.

Remark 1.8. The Hahn–Banach theorem may be stated thus. If N is a semi–norm on a linear space X and V is a subspace of X and f is a linear functional on V such that $|f(x)| \leq aN(x)$, $x \in V$ then $\exists$ a linear functional g on X such that $|g(x)| \leq aN(x)$ $\forall x \in X$ and $f(x) = g(x)$, $x \in V$.

Remark 1.9. If $\mathcal{N}$ is a separating family of semi–norms on X then the space $(X^{\mathcal{N}})^*$ of continuous linear functionals on $X^{\mathcal{N}}$ separates points of X. For if $x_0 \neq 0$ let $f(\alpha x_0) = \alpha$ on span$(\{x_0\})$. There exist $N \in \mathcal{N}$ such that $N(x_0) \neq 0$. Then

$$|f(\alpha x_0)| = |\alpha| = \frac{N(\alpha x_0)}{N(x_0)}$$

and hence there exists a linear functional g on X such that

$$g(x_0) = 1 \text{ and } g(x) \leq \frac{N(x)}{N(x_0)} \ \forall x \in X.$$

Clearly $g \in (X^{\mathcal{N}})^*$.

Exercise 1.3. (0) For $f \in C^\infty(\mathbb{R})$ define $\|f\|'_{n,k} = \sup_{x\in\mathbb{R}} |x^n f^{(k)}(x)|$. Show that $\mathcal{S}(R) = \{f \in C^\infty(\mathbb{R}) : \|f\|'_{n,k} < \infty \ \forall n \geq 0 \text{ and } \forall k \geq 0\}$. If $\mathcal{N}' = \{\|\cdot\|'_{n,k} : n \geq 0, k \geq 0\}$ show that the topologies $\mathcal{S}^{\mathcal{N}}$ and $\mathcal{S}^{\mathcal{N}'}$ are the same.

A function $f : \mathbb{R} \to \mathbb{C}$ is said to be of <u>polynomial growth</u> if

$$|f(x)| \leq C(1 + |x|^k)$$

for some constant C, some integer k and for all x. A Borel measure μ on $\mathbb{R}$ is said to be of <u>polynomial growth</u> if

$$\int_{\mathbb{R}} (1 + |x|^k)^{-1} d\mu(x) < \infty$$

for some integer k. For example Lebesgue measure is of polynomial growth.

(1) Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a Borel measurable function of polynomial growth and μ is a measure of polynomial growth. For φ in $\mathcal{S}(\mathbb{R})$ [see Example 1.7 above] write

$$(f\mu)(\varphi) = \int_{-\infty}^{\infty} \varphi(x) f(x) d\mu(x). \tag{1.2}$$

Show that the integral in equation (1.2) exists and defines a continuous linear functional (which we denote by $f\mu$) on $\mathcal{S}(\mathbb{R})$.

(2) Show that the operator

$$\frac{d}{dx} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$$

is everywhere defined and continuous.

(3) If μ is Lebesgue measure we will write $f dx$ instead of $f\mu$. By virtue of b) the operator $(-\frac{d}{dx})$ has an adjoint $D : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ where $\mathcal{S}'(\mathbb{R})$ denotes the dual space of $\mathcal{S}(\mathbb{R})$ (i.e., the space of continuous linear functionals on $\mathcal{S}(\mathbb{R})$). Suppose that g is a continuously differentiable function of polynomial growth whose derivative g' also has polynomial growth. Describe explicitly the linear functional $D(gdx)$ by writing it in the form (1.2) for some wise choice of f and μ and compute it in case $g(x) = x^2 + 3x$.

(4) Suppose

$$g(x) = \chi_{[0,\infty)}(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

Describe $D(gdx)$ explicitly by writing it in the form (1.2) for some wise choice of f and μ.

(5) Since D is an everywhere defined linear operator on $\mathcal{S}'(\mathbb{R})$ so is D^n for $n = 1, 2, 3, \ldots$. In part d) above $D^2(gdx)$ is therefore an element of $\mathcal{S}'(\mathbb{R})$. Determine whether it can be written in the form (1.2). Prove your claim.

Definition 1.10. Let X be a linear space. A set $A \subset X$ is called *convex* if for any $x, y \in A$, $\alpha x + (1-\alpha)y \in A$ whenever $0 \leq \alpha \leq 1$.

Definition 1.11. A topological linear space (X, τ) is *locally convex* if the topology τ has a neighborhood base at 0 consisting of convex sets, i.e., for any open neighborhood U of 0 $\exists$ a convex open neighborhood V of 0 such that $V \subset U$.

Theorem 1.12. *Let (X, τ) be a topological linear space. Let $\mathcal{N}$ be the family of continuous semi–norms on (X, τ). Then $X^{\mathcal{N}} = (X, \tau)$ if and only if $(X.\tau)$ is locally convex. (Usually we will just write X for a topological linear space when the topology τ is clear.)*

The proof of this theorem will be given after the proof of Lemma 1.17. Before starting into the proof we state a corollary.

Corollary 1.13. *If X is a locally convex topological linear space then the topological dual space X^* separates points of X.*

Proof. It follows from Theorem 1.12 and Remark 1.9. ■

The proof of Theorem 1.12 depends on the following lemmas.

Definition 1.14. Let $S \subset X$ be a set. Then

(1) S is called *symmetric* if $x \in S$ implies that $\alpha x \in S$ whenever $|\alpha| = 1$.
(2) S is *absorbing* if for every $x \in X$, $\exists \alpha > 0$ such that $x \in \alpha S$.
(3) S is *linearly open* if for every $x_0 \neq 0$, $\{\alpha : \alpha x_0 \in S\}$ is open.

Remarks 1.15. Let $S \subset X$ then

(1) if S is absorbing then $0 \in S$.
(2) If S is nonempty, convex and symmetric then $0 \in S$.
(3) The intersection of convex sets is convex.

Lemma 1.16. *Let X be a linear space. A set $S \subset X$ is convex, symmetric, absorbing and linearly open iff $\exists$ a semi–norm $N \ni S = \{x \in X : N(x) < 1\}$.*

Proof. ($\Leftarrow$) Let $S = \{x : N(x) < 1\}$. S is convex since $N(\alpha x + (1-\alpha)y) \leq \alpha N(x) + (1-\alpha)N(y) < 1$ if $x, y \in S$. S is clearly symmetric.

S is absorbing: $N(x) = 0$ implies that $x \in S$. If $N(x) \neq 0$ let $\alpha = 2N(x)$. Then $x/\alpha \in S$.

S is linearly open: $\{\alpha : \alpha x_0 \in S\} = \{\alpha : N(\alpha x_0) < 1\} = \{\alpha : |\alpha| N(x_0) < 1\}$ which is open.

($\Rightarrow$) Assume S is a symmetric, absorbing, convex, linearly open set. Define $N(x) = \inf\{\alpha : \alpha > 0, x \in \alpha S\}$. Then $N(0) = 0$ since $0 \in S$ (because S is absorbing). If $\beta \neq 0$ then

$$\begin{aligned} N(\beta x) &= \inf\{\alpha : \alpha > 0, \beta x \in \alpha S\} = \inf\left\{\alpha : \alpha > 0, x \in \frac{\alpha}{\beta} S\right\} \\ &= \inf\left\{\alpha : \alpha > 0, x \in \frac{\alpha}{|\beta|} S\right\} \text{ by symmetry of } S \\ &= \inf\{|\beta|\gamma : \gamma > 0, x \in \gamma S\} = |\beta| N(x). \end{aligned}$$

This shows positive homogeneity. We must show *subadditivity.* Given $x, y \in X$. Take $\alpha > N(x)$, $\beta > N(y) \ni \exists\, u, v \in S$ with $x = \alpha u$, $y = \beta v$. Then

$$\frac{x+y}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta} u + \frac{\beta}{\alpha+\beta} v \in S$$

$$\begin{aligned} &\therefore x + y \in (\alpha+\beta) S \\ &\therefore N(x+y) \leq \alpha + \beta \text{ for all such } \alpha \text{ and } \beta \\ &\therefore N(x+y) \leq \inf_{\alpha} (\alpha + \beta) = N(x) + \beta \ \forall \text{ such } \beta \\ &\therefore N(x+y) \leq N(x) + \inf \beta = N(x) + N(y) \\ &\therefore N \text{ is a seminorm.} \end{aligned}$$

Now suppose $N(x) < 1$. $\exists \alpha < 1 \ni x \in \alpha S$, i.e., $x/\alpha \in S$. Then $x = \alpha(1/\alpha)x + (1-\alpha)0 \in S$. Conversely if $x \in S$ then $\alpha^{-1}x \in S$ for some $\alpha < 1$ because S is linearly open. So $N(x) < 1$.

$$\therefore S = \{x : N(x) < 1\}$$

■

Lemma 1.17. *A topological vector space is locally convex iff the convex symmetric neighborhoods of zero form a base at* 0*.*

Proof. $\Leftarrow$ trivial

$\Rightarrow$ Let V be a convex neighborhood of 0. Let $V_1 = \bigcap_{|p|=1} pV$. Then $V_1 \subset V$ and $0 \in V_1$. V_1 is an intersection of convex sets so is convex. If $|\beta| = 1$ then $\beta V_1 = \bigcap_{|p|=1} \beta p V = \bigcap_{|\gamma|=1} \gamma V = V_1$. Therefore V_1 is symmetric.

We show next that the interior of V_1 is not empty. Since $(\alpha, x) \mapsto \alpha x$ is continuous and $0 \cdot 0 = 0 \in V$ $\exists$ an $a > 0$ and a neighborhood V_2 of $0 \ni \alpha V_2 \subset V$ whenever $|\alpha| < a$. Put $V_3 = (a/2)V_2$. Then if $|p| = 1$, $pV_3 = \frac{pa}{2} V_2 \subset V$. $\therefore V_3 \subset pV$ if $|p| = 1$. $\therefore V_3 \subset V_1$. So V_1 has a non–empty interior W and $0 \in W \subset V$.

Claim: W is convex and symmetric.

Convexity: Let $x, y \in W$ and let $\alpha, \beta > 0$, with $\alpha + \beta = 1$. Since V_1 is convex and $W \subset V_1$ we may conclude that $\alpha x + \beta y \in V_1$. Since W is open x has a neighborhood $U \subset W$ and we may similarly conclude that the open set $\alpha U + \beta y$ is contained in V_1, and hence in its interior, W. In particular $\alpha x + \beta y \in W$. So W is convex.

Symmetry: If $x \in W$ and $|\beta| = 1$ then $\beta x \in \beta W \subset \beta V_1 = V_1$. Since βW is open $\beta x \in int V_1$. ■

We are now ready for the proof of Theorem 1.12. The proof given here will follow Rudin 1.33–1.39, p. 24–28] [1975].

Proof of Theorem 1.12. Assume $X = X^{\mathcal{N}}$. If V is a neighborhood of 0 then by definition there exists $N_1, \dots, N_k \in \mathcal{N}$ and $P_j > 0$, $j = 1, 2, \dots, k$ such that $V \supset \{x : N_j(x) < P_j,\ j = 1, 2, \dots, k\}$. This is an intersection of convex sets by Lemma 1.16 and is therefore a convex neighborhood of 0. Thus X is locally convex. Conversely, suppose X is locally convex. Clearly $X \supset X^{\mathcal{N}}$, i.e., $X^{\mathcal{N}}$ is a weaker topology than the original X topology. Suppose V is an open neighborhood of 0. By Lemma 1.17 $\exists$ a convex symmetric neighborhood W of $0 \ni W \subset V$. W is absorbing, for if $x \in X$ then $0 \cdot x = 0 \in W$. $\therefore \exists a > 0 \ni \alpha x \in W$ for $|\alpha| < a$. Take $\alpha = a/2$. W is linearly open since $\{\alpha : \alpha x_0 \in W\}$ is the inverse image of an open set W under the continuous map $\alpha \to \alpha x_0$. Hence by Lemma 1.16 there exists a semi–norm N on $X \ni W = \{N(x) < 1\}$. N is continuous at 0 since $\{N(x) < \varepsilon\} = \varepsilon W$ which is open. $\therefore N$ is continuous $\therefore X^{\mathcal{N}} \supset X$. ∎

Remark 1.18. If V is a neighborhood of zero in a locally convex space then $\exists$ a continuous semi–norm N with $\{x : N(x) < 1\} \subset V$, for we know $X = X^{\mathcal{N}}$ where $\mathcal{N}$ is the family of continuous semi–norms

$$\therefore \exists N_j, \rho_j \text{ such that } \bigcap_{j=1}^{m} \{x : N_j(x) < \rho_j\} \subset V$$

$$\therefore \text{ Take } N = \sum_{j=1}^{m} \rho_j^{-1} N_j$$

Definition 1.19. Let X be a normed linear space and X^* its dual space (all continuous linear functionals on X).

(1) The weak topology on X is the X^* topology of X, i.e., $\sigma(X, X^*)$.
(2) The weak* topology on X^* is the X topology of X^*, i.e., $\sigma(X^*, \widetilde{X})$ where $\widetilde{X}$ is the image of X in X^{**} .[Recall topology of product spaces and Tychonoff's Theorem.]

Theorem 1.20 (Banach–Alaoglu Theorem)**.** *Let X be a normed linear space. Then the unit ball of X^* is weak* compact.*

Proof. Let $A = \{\text{scalar valued functions } \xi \text{ on } X : |\xi(x)| \le \|x\| \forall x\}$. For $x \in X$ let $B_x = \{\lambda : |\lambda| \le \|x\|\}$. B_x is compact. Therefore $A = \prod_{x \in X} B_x$ is compact. A basic neighborhood of a point ξ_0 is

$$\{\xi : |\xi(x_j) - \xi_0(x_j)| < \varepsilon, j = 1, \dots, n\}$$

The projection map $\xi \mapsto \xi(x)$ from A to {scalars} is continuous for each x in this (product) topology. Now the unit ball of $X^* \subset A$. The induced topology is the weak* topology on X^*. It remains to show the unit ball is closed in A.

Let $x, y \in X$, α, β scalars

$$\xi(\alpha x + \beta y) - \alpha\xi(x) - \beta\xi(y) \text{ is a continuous function of } \xi \text{ on } A.$$

$$\therefore \{\xi : \xi(\alpha x + \beta y) - \alpha\xi(x) - \beta\xi(y) = 0\} \text{ is closed in } A.$$

Hence

$$\bigcap_{x,y,\alpha,\beta} \{\xi : \xi(\alpha x + \beta y) - \alpha\xi(x) - \beta\xi(y) = 0\} \text{ is closed in } A.$$

But this is the unit ball. ∎

Exercise 1.4. Let X be a separable Banach space. Show that the weak* topology on the closed unit ball B of X^* is metrizable. **Hint:** let $\{x_1, x_2, \ldots\}$ be a sequence of unit vectors in X which is dense in $\{x \in X : \|x\| = 1\}$. Consider

$$d(\xi, \eta) := \sum_{n=1}^{\infty} 2^{-n} |(\xi - \eta)(x_n)|.$$

Note: The Banach–Alaoglu theorem together with Exercise 1.4 shows that B is *sequentially* compact when X is separable.

1.1. **The Krein-Milman theorem.**

Definition 1.21. Let X be a topological vector space.

1. A set $K \subset X$ is called *compex* if it is compact and convex.
2. A *segment* is a set of the form $\{\alpha x + \beta y : \alpha + \beta = 1,\ 0 \leq \alpha \leq 1\}$ for some $x, y \in X$. . For $x \neq y$, the *interior of a segment* is $\{\alpha x + \beta y : \alpha, \beta > 0\ \alpha + \beta = 1\}$.
3. A point $x \in K$ is an *extreme point* of K if it is not contained in the interior of any segment of K.
4. If K is compex $F \subset K$ is a *face* of K if F is compex and every segment in K having a point of F in its interior is contained in F, i.e., $\alpha x + \beta y \in F$ for some α, β with $0 < \alpha, \beta < 1$ and $\alpha + \beta = 1$ implies that $\alpha x + \beta y \in F$ $\forall \alpha$, β such that $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$.

Notation 1.22. Let S be a subset of X. The *compex hull* of S is the smallest compex set, if any, containing S. If there is any compex set containing S then the intersection of all such compex sets is clearly the compex hull of S. Let K be a non-empty compex set. We will denote by $\hat{K}$ the compex hull of the set of extreme points of K, i.e.

$$\widehat{K} = \cap\{\text{compex sets containing all extreme points of } K\}.$$

It follows immediately from the definition that $\hat{K} \subset K$.

Theorem 1.23. *(Krein–Milman Theorem) Let X be a locally convex topological vector space. If K is compex, then $\widehat{K} = K$.*

We will need two lemmas before giving the proof of the theorem.

Lemma 1.24. *Let X be a real locally convex topological vector space and K compex $\subset X$. If $x_0 \notin K$ then $\exists$ a continuous linear functional $\xi_0 \ni \xi_0(x_0) \notin \xi_0(K)$.*

Proof. Step 1: Reduction to finite dimensions. Without loss of generality we may take $x_0 = 0$. Since the complement of K is open there is a convex, symmetric (absorbing, linearly open) neighborhood, S, of 0 which is disjoint from K. There exists a continuous semi–norm N such that $S = \{x : N(x) < 1\}$. By the Hahn-Banach theorem there exists, for each $x_0 \in X$ a continuous linear functional ξ_{x_0} such that

a. $|\xi_{x_0}(y)| \leq N(y)$ and

b. $\xi_{x_0}(x_0) = N(x_0)$. (Use the construction method of Remark 1.9, starting with $\xi_{x_0}(\alpha x_0) = \alpha N(x_0)$ on span x_0.)

If $x \in K$ then $N(x) \geq 1$. Therefore $\xi_x(x) = N(x) \geq 1$. So the collection of open sets $\{y : |\xi_x(y)| > 1/2\}_{x \in K}$ is an open cover of K. Since K is compact there exists $x_1, \ldots, x_n$ such that

$$K \subset \bigcup_{j=1}^{n} \{y : |\xi_{x_j}(y)| > \frac{1}{2}\}.$$

Define $A : X \to \mathbb{R}^n$ by

$$Ax = (\xi_{x_1}(x), \xi_{x_i}(x), \ldots, \xi_{x_n}(x)).$$

Then A is continuous and linear. Let $\widetilde{K} = A(K)$. Then $0 \notin \widetilde{K}$ because if $(\alpha_1, \ldots, \alpha_n) \in \widetilde{K}$ then $\exists j$ such that $|\alpha_j| = |\xi_{x_j}(x)| > \frac{1}{2}$. Thus we now need only construct a linear functional $\eta : \mathbb{R}^n \to \mathbb{R}$ such that $0 \notin \eta(\widetilde{K})$ and then put $\xi_0 = \eta \circ A$. For then $0 \notin \xi_0(K)$. We have thus reduced the problem to a finite dimensional one.

Step 2: The finite dimensional case. Construction of η. Denote by $(\ ,\ )$ the standard inner product on $\mathbb{R}^n$ and $\|\cdot\|$ the associated norm. Given a compact convex set $\tilde{K}$ not containing 0 there exists a vector $r \in \tilde{K}$ such that $\|r\| = \inf_{y \in \tilde{K}} \|y\|$. Then $r \neq 0$. Let $\eta(y) = (y, r)$. We assert that if $s \in \tilde{K}$ then $\eta(s) \neq 0$. Suppose that $s \in \tilde{K}$ and $\eta(s) = 0$. Then $r \perp s$. Let $t = (1-\alpha)r + \alpha s$. Then $t \in \tilde{K}$ for all $\alpha \in [0, 1]$. But $\|t\|^2 = (1-\alpha)^2\|r\|^2 + \alpha^2\|s\|^2 = \|r\|^2 + \alpha\{\alpha(\|r\|^2 + \|s\|^2) - 2\|r\|^2\}$. The expression in braces is clearly negative for small positive α and for such α we therefore have $\|t\| < \|r\|$, which contradicts the definition of r. ∎

Remark 1.25. The previous lemma holds for a complex vector space X also with a complex liner functional ξ_0. The proof is similar.

For the next lemma we will need the following machinery.

Remarks 1.26. The following properties are easily checked:

(1) $\cap$ faces = face
(2) A face of a face is a face.
(3) If $A : H \to \widetilde{H}$ is continuous and linear and K is compex then $A(K)$ compex
(4) If $\widetilde{F}$ is a face of $\widetilde{K} \equiv A(K)$ then $F \equiv K \cap A^{-1}(\widetilde{F})$ is a face of K.
(5) If $\widetilde{F} \subsetneqq \widetilde{K} \equiv A(K)$ then $F \equiv K \cap A^{-1}(\widetilde{F}) \neq K$

Definition 1.27. F is an extreme face of K if it is a face of K such that the faces of it (F) are precisely $\emptyset$ and F.

Lemma 1.28. *Let X be a topological vector space such that X^* separates points of X (e.g. X locally convex). Then a non–empty extreme face is an extreme point.*

Proof. View X as a vector space over $\mathbb{R}$, and let ξ be a continuous real linear functional on X. Let F be an extreme face of K. $\xi(F)$ is a compact convex subset of the reals $\therefore \xi(F) = [\alpha, \beta]$.

$\{\alpha\}$ is a face of $\xi(F)$.

$$\therefore F \cap \xi^{-1}\{\alpha\} \text{ is a non-empty face of } F$$

$$\therefore F \cap \xi^{-1}\{\alpha\} = F$$

$$\text{Hence } \xi(F) = \{\alpha\} \text{ by item 5 of Remark 1.26}$$

But if F has two distinct points, then $\exists$ a real linear functional which separates them. Therefore, F has only one point. ∎

Proof of Krein-Milman theorem. Clearly $\widehat{K} \subset K$. For any real linear continuous functional, ξ, on X, $\xi(\widehat{K}) \subset \xi(K)$. Let α be an endpoint of $\xi(K)$. $K \cap \xi^{-1}\{\alpha\}$ is a face of K. Let $T = \{$non–empty faces of $K \cap \xi^{-1}\{\alpha\}\}$. Order T by reverse inclusion. (T is partially ordered.) The intersection of a simply ordered collection (chain) in T is non–empty because of the finite intersection property of compact sets. By Zorn's Lemma, $\exists$ an extreme face in $K \cap \xi^{-1}\{\alpha\}$. This is an extreme point by Lemma 1.28 $\therefore$ $(K \cap \xi^{-1}\{\alpha\}) \cap \widehat{K} \neq \emptyset$. Hence $\alpha \in \xi(\widehat{K})$ $\therefore$ endpoints of $\xi(K) \in \xi(\widehat{K})$ so $\xi(K) \subset \xi(\widehat{K})$. Hence $\xi(K) = \xi(\widehat{K})$ $\forall$ real linear functional ξ. Theorem 1.23 now follows from Lemma 1.24. ■

Application 1. Let H be a Banach space and S be the unit sphere in H^*. In the w^* topology on H^*, S is compex. Hence $\widehat{S} = S$. Therefore, if a Banach space is the dual of a normed linear space, its unit sphere must satisfy the condition $\widehat{S} = S$. For example we may use this to prove that Real $C[0,1]$ is not a dual space under $\| \ \|_\infty$ of any Banach space.

Proof. Suppose that $f \in$ Real $C([0,1])$ and is an extreme point of the unit ball. Let $g(s) = f(s) - |f(s)| + 1$ and $h(s) = f(s) + |f(s)| - 1$. Then $\|g\|_\infty \leq 1$ and $\|h\|_\infty \leq 1$ since $\|f\|_\infty \leq 1$, as we see by considering for each s the cases $f(s) \geq 0$ and $f(s) < 0$. But $f = (1/2)g + (1/2)h$. Hence since f is an extreme point we must have $g = h$. That is, $|f(s)| = 1$ for all s. Hence $f \equiv 1$ or $f \equiv -1$. These are the only extreme points of the closed unit ball S. Hence $\widehat{S} \neq S$. So S is not compact in any locally convex topology on Real $C([0,1])$. Therefore Real $C([0,1])$ is not a dual space of any Banach space. ■

Exercise 1.5. Prove that the closed unit ball of real $L^1(0,1)$ has no extreme points and therefore $L^1(0,1)$ is not a dual space.

Definition 1.29. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent* if $\exists$ constants $m > 0$, $M > 0$ such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1, \quad \forall x \in X.$$

Exercise 1.6. Show that any two norms on a finite dimensional linear space are equivalent. [**Hint:** Show that any norm is equivalent to a Euclidean norm.]

Proposition 1.30. *A finite dimensional subspace F of a normed linear space X is closed in X.*

Proof. By Exercise 1.6 the norm on F is equivalent to any Euclidean norm on F and therefore F is complete in its own norm. Suppose then that $\{x_n\}_{n=1}^\infty$ is a sequence in F that converges to a point x in X. Then $\{x_n\}$ is a Cauchy sequence in F which therefore converges by completeness of F to a point y in F. By uniqueness of limits $x = y$. So x is in F and F is closed in X. ■

Fact 1. If a finite dimensional vector space is a topological vector space in two topologies T_1, T_2, then $T_1 = T_2$.

Lemma 1.31. *Let H be a normed linear space and H_0 a closed proper subspace. For any $\varepsilon > 0$, there exists $x_0 \in H$ such that $\|x_0\| = 1$ and $\|x - x_0\| \geq 1 - \varepsilon$ whenever $x \in H_0$.*

Proof. Can assume $\varepsilon < 1$. Take any $z_0 \notin H_0$. Let $d = \inf\limits_{x \in H_0} \|x - z_0\|$. For any $\delta > 0$, there exists $z \in H_0$, such that $\|z - z_0\| \leq d + \delta$. Take $\delta = \frac{\varepsilon d}{1-\varepsilon}$. Let

$x_0 = (z - z_0)/\|z - z_0\|$, where z is determined for this δ. Then $\|x_0\| = 1$, and if $x \in H_0$,

$$\|x - x_0\| = \frac{\|(\|z - z_0\|)x - z + z_0\|}{\|z - z_0\|} \geq \frac{d}{\|z - z_0\|} \geq \frac{d}{d + \delta} = 1 - \varepsilon.$$

■

Proposition 1.32. *A locally compact Banach space is finite dimensional.*

Proof. We prove that an infinite dimensional Banach space is not locally compact. We construct a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $\|x_n\| = 1$, $\|x_i - x_j\| \geq 1/2$, $i \neq j$. Take x_1 to be any unit vector. Suppose vectors $x_1, \ldots, x_n$ are constructed. Let $H_0 =$ linear manifold spanned by $x_1, \ldots, x_n$. By Proposition 1.30, H_0 is closed. By Lemma 1.31, $\exists\ x_{n+1} \ni \|x_i - x_{n+1}\| \geq 1/2$, $i = 1, \ldots, n$. Now the sequence just constructed has no Cauchy subsequence. Hence the closed unit ball is not compact. Similarly the closed ball of radius $r > 0$ is also not compact. ■

2. Banach Algebras

Definition 2.1. An associative algebra $\mathcal{A}$ over a field F is a vector space over F with a bilinear, associative multiplication: i.e.,

$$\begin{aligned}(ab)c &= a(bc)\\ a(b+c) &= ab+ac\\ (a+b)c &= ac+bc\\ a(\lambda c) &= (\lambda a)c = \lambda(ac)\end{aligned}$$

Definition 2.2. A Banach Algebra is a real or complex Banach space which is an associative algebra such that

$$\|ab\| \leq \|a\|\ \|b\|.$$

Examples 2.3.
(1) X = topological space, $C(X)$ = bounded, complex valued, continuous functions on X, with $\|f\| = \sup_{x\in X} |f(x)|$. $C(X)$ is a commutative Banach algebra under pointwise multiplication. The constant function 1 is an identity element.
(2) V = Banach space, $\mathcal{B}(V)$ = all bounded operators $V \to V$. $\mathcal{B}(V)$ is a Banach algebra in operator norm with identity. $\mathcal{B}(V)$ is not commutative if $\dim V > 1$.
(3) $\mathcal{A} = L^1(\mathbb{R}^1)$ Multiplication = convolution. $\mathcal{A}$ is a commutative Banach algebra without identity.

Proposition 2.4. *Let $\mathcal{A}$ be a (complex) Banach algebra without identity. Let*

$$\mathcal{B} = \{(a,\alpha) : a \in \mathcal{A}, \alpha \in \mathbb{C}\} = \mathbb{A} \oplus \mathbb{C}.$$

Define

$$(a,\alpha)(b,\beta) = (ab + \alpha b + \beta a, \alpha\beta)$$

and

$$\|(a,\alpha)\| = \|a\| + |\alpha|.$$

Then $\mathcal{B}$ is a Banach algebra with identity $e = (0,1)$, and the map $a \to (a,0)$ is an isometric isomorphism onto a closed two sided ideal in $\mathcal{B}$.

Proof. Straightforward. ∎

Definition 2.5. Let $\mathcal{A}$ be a Banach algebra with identity 1. If $a \in \mathcal{A}$, then a is *right* invertible if $\exists\ b \in \mathcal{A} \ni ab = 1$. b is called a *right inverse.* (Similarly for left inverse.) a is called *invertible* if it has a left and a right inverse.

Note: If $ab = 1$ and $ca = 1$ then $c = cab = b$. Therefore if a has left and right inverses they are equal, unique, and called *the* inverse of a.

Proposition 2.6. *$\mathcal{A}$ = Banach algebra with 1. If $\|a\| < 1$, then $1-a$ is invertible and $\|(1-a)^{-1}\| \leq \frac{1}{1-\|a\|}$.*

Proof. Let $b = \sum_{n=0}^{\infty} a^n$. Since $\|a^n\| \leq \|a\|^n$, the series converges ($\mathcal{A}$ is complete). Clearly, $(1-a)b = b(1-a) = 1$. Also $\|b\| \leq \sum_{n=0}^{\infty} \|a\|^n = \dfrac{1}{1-\|a\|}$. ∎

Corollary 2.7. *If $\mathcal{A}$ is a Banach algebra with 1, then the set $\mathcal{U}$ of invertible elements is an open set.*

Proof. Let $a \in \mathcal{U}$. Suppose $\|x - a\| < \|a^{-1}\|^{-1}$. Then: $\|a^{-1}x - 1\| = \|a^{-1}(x - a)\| \le \|a^{-1}\| \cdot \|x - a\| < 1$. $\therefore 1 - (1 - a^{-1}x)$ is invertible, i.e., $a^{-1}x$ has an inverse b. $\therefore (ba^{-1})x = 1$ and $a^{-1}xb = 1$. $\therefore xb = a$, $xba^{-1} = 1$. ∎

Exercise 2.1. Prove that the map $x \to x^{-1}$ from the set $\mathcal{U}$ of invertible elements in $\mathcal{A}$ (= Banach algebra with 1) is continuous.

Henceforth all Banach algebras $\mathcal{A}$ are complex and have an identity. $\mathcal{U}$ = set of all invertible elements.

Convention: We write λ instead of $\lambda 1$.

Definition 2.8. Let $x \in \mathcal{A}$. The *spectrum* of x is

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda \text{ is not invertible}\}.$$

The *resolvent set* of x is

$$\rho(x) = \{\lambda \in \mathbb{C} : x - \lambda \text{ is invertible}\}.$$

The *resolvent* of x is the function

$$R(\lambda) = (x - \lambda)^{-1} \quad \text{defined for} \quad \lambda \in \rho(x).$$

The *spectral radius*, $r(x)$, of x is

$$r(x) \equiv \sup\{|\lambda| : \lambda \in \sigma(x)\}.$$

(Note: We will show later that $\sigma(x) \neq \emptyset$.)

Exercise 2.2. Let $X = C([0,1])$ (complex functions with sup norm). Let $\mathcal{B}$ denote the Banach algebra of all bounded operators on X. Define $(Tf)(s) = \int_0^s f(t)dt$. Show that $T \in \mathcal{B}$ and find the spectrum of T in the algebra $\mathcal{B}$.

Theorem 2.9. *For all $a \in \mathcal{A}$,*
1. *$r(a) \le \|a\|$*
2. *$\sigma(a)$ is compact*
3. *$\sigma(a)$ is nonempty*

Proof of 1. and 2. Since $\lambda \in \mathbb{C} \to a - \lambda \in \mathcal{A}$ is continuous and $\rho(a) = \{\lambda : a - \lambda \in \mathcal{U}\}$, $\rho(a)$ is open and hence $\sigma(a) = \rho(a)^c$ is closed. If $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$ and

$$\begin{aligned}
&\therefore\ \lambda^{-1}a - 1 \in \mathcal{U} \\
&\therefore\ a - \lambda \in \mathcal{U} \text{ since } \lambda \neq 0 \\
&\therefore \lambda \in \rho(a) \text{ whenever } |\lambda| > \|a\| \\
&\therefore\ r(a) \le \|a\| \text{ and } \sigma(a) \text{ is compact.}
\end{aligned}$$

∎

In order to prove that $\sigma(a)$ is non-empty we will need to replace the finite dimensional proof of this, which you may recall is based on the fact that $det(a - \lambda)$ has at least one zero, by a different use of analytic function theory.

Definition 2.10. A function φ from an open set $V \subset \mathbb{C}$ to a complex Banach space is *weakly analytic* on V if $\xi \circ \varphi$ is analytic on V for every $\xi \in \mathcal{A}^*$.

Proposition 2.11. *Let $\mathcal{A}$ be a complex Banach algebra with 1 and let $a \in \mathcal{A}$. Then $R(\lambda) = (a-\lambda)^{-1}$ is weakly analytic on $\rho(a)$ and $\|R(\lambda)\| \to 0$ as $\lambda \to \infty$.*

Proof. Let $\lambda_0 \in \rho(a)$. Now

$$a - \lambda = (a-\lambda_0)(1-(a-\lambda_0)^{-1}(\lambda-\lambda_0)).$$

So $a-\lambda$ is invertible if $\|(a-\lambda_0)^{-1}(\lambda-\lambda_0)\| < 1$ and then:

$$(a-\lambda)^{-1} = \sum_{n=0}^{\infty}(a-\lambda_0)^{-n}(\lambda-\lambda_0)^n(a-\lambda_0)^{-1}$$

$$\therefore\ \xi((a-\lambda)^{-1}) = \sum_{n=0}^{\infty}\xi((a-\lambda_0)^{-n-1})(\lambda-\lambda_0)^n$$

$$\therefore\ \xi(R(\lambda)) \text{ is analytic.}$$

Finally, $(a-\lambda)^{-1} = [\lambda(\lambda^{-1}a-1)]^{-1} = \lambda^{-1}(\lambda^{-1}a-1)^{-1}$ and $\|(\lambda^{-1}a-1)^{-1}\| \leq \frac{1}{1-|\lambda|^{-1}\|a\|} \to 1$ as $\lambda \to \infty$ $\therefore$ $\|R(\lambda)\| \to 0$ as $\lambda \to \infty$. ∎

Proof that $\sigma(a)$ is not empty. Suppose $\sigma(a)$ is empty. Then for any $\xi \in \mathcal{A}^*$, $\lambda \to \xi((a-\lambda)^{-1})$ is an entire function and goes to 0 as $\lambda \to \infty$. Then, by Liouville's theorem,

$$\xi[(a-\lambda)^{-1}] \equiv 0$$

Therefore $(a-\lambda)^{-1} \equiv 0\ \forall\lambda$. This is impossible. This completes the proof of theorem 2.9. ∎

Theorem 2.12. *(Spectral Mapping Theorem) If p is a polynomial then $p(\sigma(a)) = \sigma(p(a))$.*

Proof. Let $\lambda_0 \in \sigma(a)$. We will show that $p(a) - p(\lambda_0) \notin \mathcal{U}$. Let q be such that $p(\lambda) - p(\lambda_0) = (\lambda-\lambda_0)q(\lambda)$.

Suppose there exists b so that $b(p(a)-p(\lambda_0)) = (p(a)-p(\lambda_0))b = 1$. Then

$$bq(a)(a-\lambda_0) = (a-\lambda_0)q(a)b = 1.$$

Thus $a-\lambda_0$ is invertible. *Contradiction.* Thus $p(\sigma(a)) \subset \sigma(p(a))$.

Suppose $\lambda_0 \in \sigma(p(a))$. Let $\lambda_1, \ldots, \lambda_n$ be the roots of $p(\lambda) = \lambda_0$. Thus $p(\lambda)-\lambda_0 = \alpha(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$. But $p(a)-\lambda_0 = \alpha(a-\lambda_1)\cdots(a-\lambda_n)$ is not invertible. Hence at least one of the factors, say $a-\lambda_j$ is not invertible. Thus $\lambda_j \in \sigma(a)$. Thus $\lambda_0 = p(\lambda_j) \in p(\sigma(a))$. ∎

Corollary 2.13. $r(a^n) = r(a)^n$.

Proof. Since $\sigma(a)$ is compact $\exists\ \lambda \in \sigma(a)$ so that $|\lambda| = r(a)$. Hence $\lambda^n \in \sigma(a^n)$ so $r(a^n) \geq |\lambda^n| = r(a)^n$.

Conversely, $\exists\lambda_0 \in \sigma(a^n)$ so that $r(a^n) = |\lambda_0|$. By Theorem 2.12, $\exists\lambda \in \sigma(a)$ such that $\lambda^n = \lambda_0$. Thus $r(a)^n \geq |\lambda|^n = |\lambda_0| = r(a^n)$. ∎

Corollary 2.14. $r(a) = \lim\limits_{n\to\infty}\|a^n\|^{1/n}$.

Proof. For λ sufficiently small:

$$(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} a^n\lambda^n \text{ and } \xi((1-\lambda a)^{-1}) = \sum_{n=0}^{\infty}\xi(a^n)\lambda^n.$$

By Theorem **??**, $\xi((1-\lambda a)^{-1})$ is analytic for $\frac{1}{\lambda} \notin \sigma(a)$. Hence $\sum_{n=0}^{\infty} \xi(a^n)\lambda^n$ converges when $\frac{1}{|\lambda|} > r(a)$, i.e., when $|\lambda| < 1/r(a)$.

Thus $\{|\lambda^n|\ |\xi(a^n)| : n = 0, 1, 2, \ldots\}$ is a bounded set for each $\xi \in \mathcal{A}^*$.

By the uniform boundedness principle $\{\lambda^n a^n\}$ is a bounded set: So

$$\|\lambda^n a^n\| \le K \quad (K > 0).$$

Hence $|\lambda| \|a^n\|^{1/n} \le K^{1/n}$ and $|\lambda| \limsup \|a^n\|^{1/n} \le 1$. Thus $\limsup \|a^n\|^{1/n} \le 1/|\lambda|$ whenever $r(a) < \frac{1}{|\lambda|}$. Hence $\limsup \|a^n\|^{1/n} \le r(a)$. But $r(a)^n = r(a^n) \le \|a^n\|$ or $r(a) \le \|a^n\|^{1/n}$. Therefore $r(a) \le \liminf \|a^n\|^{1/n}$. Consequently $\lim \|a^n\|^{1/n}$ exists and $r(a) = \lim \|a^n\|^{1/n}$. ∎

Theorem 2.15. *(Gelfand–Mazur) The only complex Banach algebra with unit which is a division algebra is $\mathbb{C}$.*

Proof. Let $a \in \mathcal{A}$ and $\lambda \in \sigma(a)$. Then $a - \lambda 1$ is not invertible. Thus $a - \lambda 1 = 0$ so $a = \lambda 1$. Hence $\mathcal{A} = \{\text{scalar multiples of } 1\}$. ∎

2.1. Commutative Banach algebras.

In this subsection $\mathcal{B}$ will denote a commutative Banach algebra with identity.

Example 2.16. Let X be a compact Hausdorff space. Then $C(X)$ is a commutative Banach algebra with identity in the sup norm under pointwise multiplication. We will refer to this space from time to time and always take these functions to be complex valued.

Example 2.17. Let $\mathcal{B}_2$ denote the set of 2 by 2 complex matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with a and b complex. This is a commutative Banach algebra in the operator norm and the usual matrix product.

Example 2.18. Let $\mathcal{B}_a$ be the set of continuous complex valued functions on the unit disk $\{z \in \mathbb{C} : |z| \le 1\}$ which are analytic in the interior of the disk. Then $\mathcal{B}_a$ is a Banach algebra in the sup norm and pointwise multiplication.

Our goal in this section and the next is to determine which commutative Banach algebras with identity are isomorphic to the simplest of these three examples, $C(X)$. It is clear that the second example, $\mathcal{B}_2$, cannot be isomorpic to $C(X)$ for any compact Hausdorff space X because it contains the nonzero element $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ whose square is zero. The third example is also not isomorphic to any $C(X)$, as we will see later.

The next two theorems are devoted to producing a compact Hausdorff space X from a given commutative Banach algebra $\mathcal{B}$ such that $\mathcal{B}$ *might* be isomorphic to $C(X)$.

Definition 2.19. A *character* of $\mathcal{B}$ is a nonzero multiplicative linear functional on $\mathcal{B}$, i.e., $\alpha(ab) = \alpha(a)\alpha(b)$. (We do not assume α is bounded.)

The *spectrum* of $\mathcal{B}$ is the set $\widetilde{\mathcal{B}}$ of all characters of $\mathcal{B}$.

Theorem 2.20. *In any commutative Banach algebra with identity $\widetilde{\mathcal{B}}$ is a weakly closed subset of the unit ball of $\mathcal{B}^*$ in the weak* topology. In particular $\widetilde{\mathcal{B}}$ is a compact Hausdorff space in this topology.*

Notation 2.21. (The *Gelfand map.*) For $a \in \mathcal{B}$ and $\alpha \in \widetilde{\mathcal{B}}$ define

$$\hat{a}(\alpha) = \alpha(a) \tag{2.1}$$

From the definition of the weak* topology its clear that $\hat{a}$ is a continuous function on the compact Hausdorff space $\widetilde{\mathcal{B}}$. The map $a \mapsto \hat{a}$ from $\mathcal{B}$ into $C(\widetilde{\mathcal{B}})$ is called the canonical map or the Gelfand map.

Theorem 2.22. *(Gelfand) The canonical map is a homomorphism from $\mathcal{B}$ into $C(\widetilde{\mathcal{B}})$ with norm at most one.*

The rest of this section will be devoted to proving these two theorems.

Definition 2.23. An ideal $I \subset \mathcal{B}$ is called a *maximal ideal* if $I \neq \mathcal{B}$ and I is not contained in any larger proper ideal.

Remarks 2.24.
(1) If $\{0\}$ is the only proper ideal in $\mathcal{B}$ then $\mathcal{B}$ is a field.
(2) If I is a maximal ideal in $\mathcal{B}$ then $\mathcal{B}/I$ is a field.
(3) Let $a \in \mathcal{B}$. a is invertible if and only if a belongs to no maximal ideal. [If a is not invertible then the proper ideal $\mathcal{B}a$ is contained in a maximal ideal.]
(4) $\cup$ (maximal ideals) $= \mathcal{S} =$ (singular elements).
(5) If $\alpha \in \widetilde{\mathcal{B}}$ then $\alpha(1) = 1$.
(6) If I is a proper ideal in $\mathcal{B}$ then $\overline{I}$ is a proper ideal. Here $\overline{I}$ denotes the closure of I. Proof. I is a subspace and if $b \in \mathcal{B}$, $a \in \overline{I}$ and $a_n \in I$, $a_n \to a$ then $ba = \lim ba_n \in \overline{I}$. Hence $\overline{I}$ is an ideal. Now, $I \subset \mathcal{S}$ and $\mathcal{S}$ is closed. Thus $\overline{I} \subset \mathcal{S}$ so $\overline{I}$ is proper.
(7) If I is a maximal ideal then $I = \overline{I}$.

Definition 2.25. The *radical* of $\mathcal{B} = \cap$ (maximal ideals) . It is clear from item (7) above that the radical of $\mathcal{B}$ is closed. $\mathcal{B}$ is called *semisimple* if its radical $= \{0\}$.

Exercise 2.3. Let B be a Banach space and K a closed subspace.
(1) On the quotient space B/K define $\|x + K\| = \inf\{\|y\| : y \in x + K\}$. Prove this is a norm on B/K and that B/K is a Banach space in this norm.
(2) Suppose further that B is a Banach algebra with identity and K is a closed proper two sided ideal in B. Show that B/K is a Banach algebra in the norm described in part (1).

Exercise 2.4. Prove that if B is a Banach space and ξ is a linear functional on B then ξ is continuous if and only if $\ker \xi$ is closed.

Lemma 2.26. *Any character is continuous.*

Proof. If α is a character of $\mathcal{B}$ then $I := \{a : \alpha(a) = 0\} = \ker(\alpha)$ is an ideal which is proper since $\alpha(1) = 1$. For any $a \in \mathcal{B}$ we have:

$$a = \Big(a - \alpha(a)1\Big) + \alpha(a)1 \in I \oplus \mathbb{C}1.$$

This shows that I has codimension 1 (i.e., $\dim(\mathcal{B}/I) = 1$)[1]. So I is maximal and thus I is closed. Hence α is continuous by Exercise 2.4. ■

Lemma 2.27. *There is a one to one correspondence between characters and maximal ideals given by $\alpha \to \ker \alpha$.*

[1] Alternatively, α descends to an algebra isomorphism of $\mathcal{B}/I \to \mathbb{C}$ showing that $\dim(\mathcal{B}/I) = 1$.

Proof. By the proof of Lemma 2.26 we see that $\ker \alpha$ is a maximal ideal. Now if I is any maximal ideal then it is closed by item (7) of Remark 2.24. Hence not only is $\mathcal{B}/I$ a field by item (2) of Remark 2.24 but also $\mathcal{B}/I$ is a complex Banach algebra by Exercise 2.3, part (2). Hence by Theorem 2.15, $\mathcal{B}/I$ is isomorphic to $\mathbb{C}$ (under the map $u \to u1_{\mathcal{B}/I}$ $u \in \mathbb{C}$). If $\beta : \mathcal{B} \to \mathcal{B}/I \equiv \mathbb{C}$ is the natural homomorphism then β is a character. Clearly $I = \ker \beta$. So any maximal ideal is the kernel of some character. Finally, if $\ker \alpha = \ker \beta = I$ then since I has codimension 1 (see Lemma 2.26) and 1 is not in I we may write any element as $a = c + u1$ with c in I and u in $\mathbb{C}$. Then $\alpha(a) = u = \beta(a)$. So $\ker \alpha$ uniquely determines α. ∎

Notation 2.28. Terminology: $\widetilde{\mathcal{B}}$ is sometimes called the *maximal ideal space* of $\mathcal{B}$.

Proposition 2.29. *If $\alpha \in \widetilde{\mathcal{B}}$ then $\|\alpha\| \le 1$.*

Proof. We must prove that $|\alpha(a)| \le \|a\|$ or, equivalently, if $\|a\| \le 1$ then $|\alpha(a)| \le 1$. Now $\|a\| \le 1$ implies that $\|a^n\| \le 1$. So $\{a^n\}$ is a bounded set. Suppose $|\alpha(a)| > 1$. Since $|\alpha(a^n)| = |\alpha(a)|^n$, α sends a bounded set onto an unbounded set. Thus α is not bounded. This contradicts Lemma 2.26. ∎

Corollary 2.30. *$\widetilde{\mathcal{B}} \subset$ unit ball of $\mathcal{B}^*$.*

Corollary 2.31. *$\widetilde{\mathcal{B}}$ is a w^*–closed subset of the unit ball in $\mathcal{B}^*$.*

Proof. $\{\xi \in \mathcal{B}^* : \xi(ab) = \xi(a)\xi(b)\}$, ($a, b$ fixed) is closed in the w^*–topology since both sides of the equation are w^*–continuous functions of ξ. Thus $\bigcap_{a,b} \{\xi \in \mathcal{B}^* : \xi(ab) = \xi(a)\xi(b)\}$ is w^*–closed. Also $\{\xi \in \mathcal{B}^* : \xi(1) = 1\}$ is w^*–closed. Thus $\widetilde{\mathcal{B}}$ is w^* closed. ∎

Corollary 2.32. *$\widetilde{\mathcal{B}}$ is a compact Hausdorff space in the w^* topology.*

This concludes the proof of Theorem 2.20.

Remark 2.33. If $\mathcal{B}$ is a commutative Banach algebra *without* identity and we define a character as a continuous nonzero homomorphism $\alpha : \mathcal{B} \to \mathbb{C}$. Then the preceding arguments shows that $\widetilde{\mathcal{B}} \subset$ (unit ball of $\mathcal{B}^*$) but may not be closed because 0 is a limit point of $\widetilde{\mathcal{B}}$. In this case $\widetilde{\mathcal{B}}$ is locally compact.

Proof of Theorem 2.22. $\widehat{ab}(\alpha) = \alpha(ab) = \alpha(a)\alpha(b) = \widehat{a}(\alpha)\widehat{b}(\alpha)$. Thus $\widehat{\ }$ is a homomorphism. Now $\forall \alpha \in \widetilde{\mathcal{B}}$ we have:

$$|\widehat{a}(\alpha)| = |\alpha(a)| \le \|a\| \quad \text{so} \quad \|\widehat{a}\|_\infty \le \|a\|.$$

Thus the norm of the canonical mapping is ≤ 1. ∎

Corollary 2.34. *The kernel of the canonical map is the radical of $\mathcal{B}$.*

Proof. If $\widehat{a} = 0$ then $\alpha(a) = 0$ $\forall \alpha$. Hence $a \in$ kernel of every α. Therefore $a \in$ every maximal ideal. So $a \in$ radical of $\mathcal{B}$. Conversely, to see that radical $\subset$ kernel note that each of the last four steps is reversible. ∎

Remark 2.35. We see now that the cononical map is one-to-one if and only if the radical of $\mathcal{B}$ is $\{0\}$. Surjectivity can still fail. (But wait till the next section.)

Remarks 2.36 (Continuation of Remark 2.24). a
(8) $\widehat{1}(\alpha) = \alpha(1) = 1 \; \forall \alpha \in \widetilde{\mathcal{B}}$
(9) $\lambda \in \sigma(a) \Leftrightarrow \lambda \in$ range of $\widehat{a}$, i.e., $\sigma(a) = \mathcal{R}(\widehat{a})$.
Proof:

$$\begin{aligned} \text{a inv.} &\Leftrightarrow a \text{ is not in any maximal ideal} \\ &\Leftrightarrow \widehat{a}(\alpha) \neq 0 \text{ for each } \alpha \\ \therefore \lambda \in \sigma(a) &\Leftrightarrow a - \lambda 1 \in \text{ some maximal ideal} \\ &\Leftrightarrow \exists \alpha \ni \alpha(a - \lambda 1) = 0 \end{aligned}$$

i.e., $\widehat{a}(\alpha) - \lambda\widehat{1}(\alpha) = 0$, i.e., $\widehat{a}(\alpha) = \lambda$.
(10) The spectral mapping theorem follows from (12). For

$$\sigma(P(a)) = \mathcal{R}(\widehat{P}(a)) = \mathcal{R}(P(\widehat{a})) = P(\mathcal{R}(\widehat{a})) = P(\sigma(a)).$$

(11) $r(a) = \|\widehat{a}\|_\infty \leq \|a\|$ from (12).

$$\therefore \; r(a+b) \leq r(a) + r(b) \text{ and } r(ab) \leq r(a)r(b).$$

(12)The following are equivalent:
$a \in$ radical, $\widehat{a} = 0$, $\|\widehat{a}\|_\infty = 0$, $r(a) = 0$.

(13) $\|\widehat{a}\|_\infty = \|a\| \Leftrightarrow \|a^2\| = \|a\|^2 \; \forall a \in \mathcal{B}$.
Proof: Recall $\|\widehat{a}\|_\infty = \|a\| \Leftrightarrow r(a) = \|a\|$.
$\Leftarrow$: $\|a^2\| = \|a\|^2$ implies that $\|a^{2^n}\| = \|a\|^{2^n}$ which implies that$\|a\| = \|a^{2^n}\|^{1/2^n}$
which implies that $\|a\| = \lim_n \|a^{2^n}\|^{1/2^n} = r(a)$
$\Rightarrow$: $r(a) = \|a\| \; \forall \; a \Rightarrow \|a^2\| = r(a^2) = r(a)^2 = \|a\|^2$.

Remark 2.37. If $\mathcal{B}$ does not have a unit then a similar theory can be developed in which $\widetilde{\mathcal{B}}$ is locally compact.

2.2. $*-$Algebras (over complexes).

Definition 2.38. An involution on a Banach algebra $\mathcal{B}$ is a map $\mathcal{B} \to \mathcal{B}$, $a \to a^*$ which is:

(1) involutory $a^{**} = a$
(2) additive $(a+b)^* = a^* + b^*$
(3) conjugate homogeneous $(\lambda a)^* = \overline{\lambda} a^*$
(4) anti–automorphic $(ab)^* = b^* a^*$

Notice that we automatically have $1^* = 1$ because applying $*$ to the equation $1 \cdot 1^* = 1^*$ gives $1^{**} \cdot 1^* = 1^{**}$. Thus $1 \cdot 1^* = 1$. So $1^* = 1$.

Definition 2.39. An element a is *Hermitian* if $a = a^*$, *strongly positive* if $a = b^* b$ for some b, positive if $\sigma(a) \subset [0, \infty)$ and real if $\sigma(a) \subset \mathbb{R}$ is real.

Definition 2.40. An involution $*$ in a Banach algebra $\mathcal{B}$ with unit is *symmetric* if $1 + a^* a$ is invertible for all $a \in \mathcal{B}$.

Proposition 2.41. *Let $\mathcal{B}$ be a symmetric Banach algebra, then (1) if a is Hermitian then a is real and (2) if a is strongly positive then a is positive.*

Proof. (1) Suppose a is hermitian ($a^* = a$) and $\lambda = \alpha + \beta i \in \mathbb{C}$ with $\beta \neq 0$. We must show $a - \lambda$ is invertible. Since

$$a - \lambda = (a - \alpha) - \beta i = \beta\big(\frac{a - \alpha}{\beta} - i\big),$$

we must show that $a = a^*$ implies $a - i$ is invertible. But

$$(a - i)(a + i)(1 + a^* a)^{-1} = 1 \text{ and } (1 + a^* a)^{-1}(a + i)(a - i) = 1$$

which shows $a - i$ is invertible.

(2) Suppose that a is strongly positive, $a = b^* b$. Then $a^* = b^* b = a$ showing that a is hermitian and hence by (1) that $\sigma(a) \subset \mathbb{R}$. Let $\alpha < 0$, then

$$b^* b - \alpha = -\alpha\big(\frac{b^* b}{-\alpha} + 1\big) = -\alpha\big(\big(\frac{b}{\sqrt{-\alpha}}\big)^*\big(\frac{b}{\sqrt{-\alpha}}\big) + 1\big)$$

which is invertible showing $\sigma(a) \subset [0, \infty)$. ■

Proposition 2.42. *Let $\mathcal{B}$ be a commutative $*$ algebra with unit. The following are equivalent:*

1. *$\mathcal{B}$ is symmetric*
2. *Hermitian implies real*
3. $\widehat{a}^*(\alpha) = \overline{\widehat{a}(\alpha)}$
4. *max. ideal implies $*$ ideal. That is, every maximal ideal is closed under $*$.*

Proof. 1) $\Rightarrow$ 2) This is Proposition 2.41.

2) $\Rightarrow$ 3) Let $a \in \mathcal{B}$, $b = a + a^*$ and $c = i(a - a^*)$. Then b and c are hermitian and hence $\sigma(b) \subset \mathbb{R}$ and $\sigma(c) \subset \mathbb{R}$. Therefore by Remark 2.36, if α is a character then $\alpha(b)$ and $\alpha(c)$ are real numbers. Hence

$$\overline{\alpha(a)} + \overline{\alpha(a^*)} = \alpha(a) + \alpha(a^*) \tag{2.2}$$

and

$$-i(\overline{\alpha(a)} - \overline{\alpha(a^*)}) = i(\alpha(a) - \alpha(a^*)),$$

or equivalently

$$\overline{\alpha(a)} - \overline{\alpha(a^*)} = -\alpha(a) + \alpha(a^*). \tag{2.3}$$

Adding Eqs. (2.2) and (2.3) shows, $\overline{\alpha(a)} = \alpha(a^*)$.

3) $\Rightarrow$ 4) Let $\mathcal{I}$ be a maximal ideal. Let $\alpha =$ char. with kernel $\mathcal{I}$.

If $a \in \mathcal{I}$ then $\alpha(a) = 0$ so that $\alpha(a^*) = \overline{\alpha(a)} = 0$. Hence $a^* \in \mathcal{I}$.

4) $\Rightarrow$ 1) Let $a \in \mathcal{B}$. We first prove that if α is a character then $\alpha(a^*) = \overline{\alpha(a)}$. Let $b = a - \alpha(a)$. Then $\alpha(b) = \alpha(a) - \alpha(a) = 0$. $\therefore b \in$ kernel α $\therefore b^* \in$ kernel α, i.e., $\alpha(b^*) = \alpha(a^*) - \overline{\alpha(a)} = 0$. $\alpha(a^*) = \overline{\alpha(a)}$.

Now $\alpha(a^* a) = \alpha(a^*)\alpha(a) = \overline{\alpha(a)}\alpha(a) = |\alpha(a)|^2$ for any character α $\therefore$ $\alpha(1 + a^* a) = 1 + |\alpha(a)|^2 \neq 0$. $\therefore 1 + a^* a \notin$ any maximal ideal. $\therefore 1 + a^* a$ is invertible. ■

Remark 2.43 (Stone–Weierstrass theorem). Recall if T is a compact Hausdorff space and B is a norm closed* subalgebra, $\subset C(T)$ such that given ξ_1, ξ_2, $t_1 \neq t_2$ $\exists\, x \in B \ni x(t_1) = \xi_1$, $x(t_2) = \xi_2$, then $B = C(T)$. ($* =$ conjugation)

Theorem 2.44. *If $\mathcal{B}$ is commutative, symmetric (with unit), the image of $\mathcal{B}$ under the canonical map is dense in $C(\widetilde{\mathcal{B}})$.*

Proof. Let $\alpha_1 \neq \alpha_2 \in \widetilde{\mathcal{B}}$. Let ξ_1, ξ_2 be complex. $\alpha_1(a) \neq \alpha_2(a)$ for some $a \in \mathcal{B}$. There exist $\lambda, \mu \ni \lambda\alpha_1(a) + \mu = \xi_1$, $\lambda\alpha_2(a) + \mu = \xi_2$. Let $b = \lambda a + \mu$. Then $\widehat{b}(\alpha_1) = \xi_1$, $\widehat{b}(\alpha_2) = \xi_2$. Therefore Theorem 2.44 follows from the Stone–Weierstrass theorem, since image of $\mathcal{B}$ is closed under conjugation by Proposition 2.42. ■

Definition 2.45. A Banach $*$ algebra $\mathcal{B}$ is

(1) $*$ multiplicative if $\|a^*a\| = \|a^*\|\ \|a\|$
(2) $*$ isometric if $\|a^*\| = \|a\|$
(3) $*$ quadratic if $\|a^*a\| = \|a\|^2$

Remark 2.46. Conditions 1) and 2) in Definition 2.45 are equivalent to condition 3), i.e. $*$ is multiplicative & isometric iff $*$ is quadratic.

Proof. $\Rightarrow$ clear.

$$\begin{aligned}&\Leftarrow \|a\|^2 = \|a^*a\| \leq \|a^*\|\ \|a\|\\ &\therefore \|a\| \leq \|a^*\|. \text{ This also holds for } a^*\\ &\therefore \|a\| = \|a^*\|.\end{aligned}$$

So

$$\|a^*a\| = \|a\|^2 = \|a^*\|\ \|a\|.$$

■

Definition 2.47. A B^* algebra is a quadratic $*$ algebra. [Nowadays, (2002), this is called a C^* algebra.]

Theorem 2.48. *If $\mathcal{B}$ is a commutative B^* algebra with identity, then the canonical map is an isometric isomorphism onto $C(\widetilde{\mathcal{B}})$.*

Lemma 2.49. *If $\mathcal{B}$ is a commutative $*$–multiplicative Banach algebra with identity then*

$$\|a\| = r(a) \quad \forall a \in \mathcal{B}.$$

Proof. If b is Hermitian, then $\|b^2\| = \|b\|^2$, $\|b^{2^n}\| = \|b\|^{2^n}$. Hence $r(b) = \|b\|$. Let a be arbitrary. Since a^*a is Hermitian we have

$$\begin{aligned}r(a^*a) &= \|a^*a\| = \|a^*\|\ \|a\|\\ \|a^*\|\ \|a\| &= r(a^*a) \leq r(a^*)r(a) \text{ by Remark 2.36}\end{aligned}$$

So

$$\|a^*\|\ \|a\| \leq \|a^*\| r(a).$$

Hence

$$\|a\| \leq r(a).$$

Since $r(a) \leq \|a\|$ by Remark 2.36 we have $\|a\| = r(a)$. ■

Lemma 2.50. *A commutative B^* algebra with identity is symmetric and semi–simple.*

Proof. We will show that if $a^* = a$ then $\sigma(a)$ real. As in Proposition 2.41 it suffices to prove $a - i$ is invertible, i.e., $1 + ia$ is invertible, i.e., $1 \notin \sigma(-ia)$. This is equivalent to $\lambda + 1 \notin \sigma(\lambda - ia)$ for some real λ. But if $\lambda + 1 \in \sigma(\lambda - ia)$, then

$$(\lambda+1)^2 \leq \|\lambda - ia\|^2 = \|(\lambda + ia)(\lambda - ia)\| = \|\lambda^2 + a^2\| \leq \lambda^2 + \|a^2\|.$$

Hence $2\lambda + 1 \le \|a^2\|$. But, this inequality fails for λ large enough. The semi–simplicity follows from Lemma 2.49 and Remark (15). ■

Proof of Theorem 2.48. By Lemma 2.50 and Theorem 2.44 the image of $\mathcal{B}$ is dense in $C(\mathcal{B})$ under the canonical map. By Lemma 2.49, the image is complete, hence closed, hence equal to $C(\widetilde{\mathcal{B}})$ and the canonical map is therefore an isometric isomorphism onto $C(\widetilde{\mathcal{B}})$. ■

Corollary 2.51. *A commutative B^* algebra with identity is isometrically isomorphic to the algebra of complex valued continuous functions on a compact Hausdorff space.*

2.3. **Problems on Banach algebras.**

In each of the following two problems a commutative $*$ algebra $\mathcal{A}$ with identity is given. In each case

1. (1) Find the spectrum of $\mathcal{A}$.
2. (2) Determine whether $\mathcal{A}$ is semi–simple or symmetric or a B^* algebra, or several of these.
3. (3) Determine whether the Gelfand map is one to one, or onto or both or neither or has dense range.

Exercise 2.5. $\mathcal{A} =$ all 2×2 complex matrices of the form $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Define $A^* = \begin{pmatrix} \overline{a} & \overline{b} \\ 0 & \overline{a} \end{pmatrix}$. Define $\|A\|$ to be the operator norm where $\mathbb{C}^2$ is given the norm $\left\|\binom{c}{d}\right\| = (|c|^2 + |d|^2)^{1/2}$.

Exercise 2.6. $\mathcal{A} = \ell^1(\mathbb{Z})$ where $\mathbb{Z}$ is the set of all integers. For f and g in $\mathcal{A}$ define

$$(fg)(x) = \sum_{n=-\infty}^{\infty} f(x-n)g(n)$$

and $f^*(x) = \overline{f(-x)}$. Show first that $\mathcal{A}$ is a commutative $*$ Banach algebra with identity. You may cite any results from Rudin's "Real and Complex Analysis".

Exercise 2.7. Let X be a compact Hausdorff space. Show that $C(X)$ in sup norm and pointwise multiplication is a B^* algebra with respect to the $*$ operation given by $f^*(x) = \overline{f(x)}$. For each $x \in X$ let

$$\alpha_x(f) = f(x), \quad f \in C(X).$$

Prove that the map $x \to \alpha_x$ is a homeomorphism of X onto the spectrum of $C(X)$.

Exercise 2.8. Using the previous problem show that if X and Y are compact Hausdorff spaces and $\varphi : C(X) \to C(Y)$ is an algebraic, $*$ preserving, isomorphism of these algebras then there exists a unique homeomorphism $T : Y \to X$ which induces φ. I.e., such that

$$(\varphi f)(y) = f(Ty), \ y \in Y, \ f \in C(X).$$

Exercise 2.9. If $\mathcal{A}$ is an n–dimensional commutative B^* algebra with identity show that the spectrum of $\mathcal{A}$ consists of exactly n points ($n < \infty$).

3. The Spectral Theorem

Let A be a bounded operator on a complex Hilbert space H. If y is in H, the map $x \to (Ax, y)$ is a continuous linear functional on H. Hence, by the Riesz representation theorem, there exists a unique element z in H such that $(Ax, y) = (x, z)$ for all x in H. Define A^* by $A^*y = z$. Thus A^* is defined for all y in H and satisfies

$$(Ax, y) = (x, A^*y) \quad x, y \in H. \tag{3.1}$$

If α, β are scalars then for all x

$$\begin{aligned}(x, A^*(\alpha y_1 + \beta y_2)) &= (Ax, \alpha y + \beta y_2) \quad \text{by (3.1)}\\ &= \overline{\alpha}(Ax, y_1) + \overline{\beta}(Ax, y_2)\\ &= (x, \alpha A^* y_1 + \beta A^* y_2) \quad \text{by (3.1) again.}\end{aligned}$$

Therefore A^* is linear.

Put $x = A^*y$ in (3.1) to get

$$\|A^*y\|^2 = (AA^*y, y) \le \|A\| \, \|A^*y\| \, \|y\|$$

Therefore

$$\|A^*y\| \le \|A\| \, \|y\|.$$

Hence A^* is bounded with $\|A^*\| \le \|A\|$. Now A^* is uniquely determined by equation (3.1) and taking the complex conjugate of (3.1), we see $A^{**} = A$. Hence $\|A\| \le \|A^*\|$. Thus we have the following properties.

Properties 3.1.
(1) A^* is linear and bounded and $\|A^*\| = \|A\|$
(2) $A^{**} = A$
(3) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$ (exercise).
(4) a) $\|AB\| \le \|A\| \, \|B\|$
 b) $(AB)^* = B^* A^*$
(5) $\|A^*A\| = \|A\|^2$. Proof: $\|A^*A\| \le \|A^*\| \, \|A\| = \|A\|^2$. Also $\|Ax\|^2 = (A^*Ax, x) \le \|A^*A\| \, \|x\|^2$. Therefore $\|A\|^2 \le \|A^*A\|$.
 Terminology: A^* is called the adjoint of A.
(6) Recall: The set $\mathcal{B}(X)$ of all bounded operators on X is a Banach algebra in operator norm whenever X is a Banach space.

Definition 3.2. A C^* *algebra* on a Hilbert space is a subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$ which is closed in norm and such that $A \in \mathcal{A} \Rightarrow A^* \in \mathcal{A}$. A subalgebra closed under taking adjoints is called a $*$ *subalgebra* of $\mathcal{B}(H)$.

Example 3.3. $\mathcal{B}(H)$ is a C^* algebra.

Definition 3.4. A *maximal abelian self–adjoint (m.a.s.a.) algebra* on H is a commutative algebra $\mathcal{A} \subset \mathcal{B}(H)$ which is not contained in any larger commutative subalgebra and such that $\mathcal{A}$ is a $*$ subalgebra.

Notation 3.5. If $S \subset \mathcal{B}(H)$ then $S' = \{A \in \mathcal{B}(H), AB = BA \ \forall B \in S\}$. S' is clearly a subalgebra of $\mathcal{B}(H)$ for any set S. S' is called the *commutor algebra* of S.

Proposition 3.6. *Let H be a Hilbert space.*
(1) *A subalgebra $\mathcal{A} \subset \mathcal{B}(H)$ is a maximal abelian algebra iff $\mathcal{A}' = \mathcal{A}$.*
(2) *A m.a.s.a. algebra $\mathcal{A}$ is a C^* algebra.*

Proof.

(1) Suppose $\mathcal{A}$ is m.a. Suppose $B \in \mathcal{A}'$. Then the set of all operators of the form $A_0 + A_1 B + A_2 B^2 + \cdots + A_n B^n$. $A_j \in \mathcal{A}$ is a commutative algebra containing $\mathcal{A}$. Therefore it is $\mathcal{A}$ $\therefore$ $B \in \mathcal{A}$ $\therefore$ $\mathcal{A}' \subset \mathcal{A}$. Clearly $\mathcal{A} \subset \mathcal{A}'$ since $\mathcal{A}$ is commutative. Therefore $\mathcal{A}' = \mathcal{A}$.

Conversely, if $\mathcal{A}' = \mathcal{A}$ then since any larger commutative algebra C containing $\mathcal{A}$ is contained in $\mathcal{A}'$, it follows that $C = \mathcal{A}$. Therefore $\mathcal{A}$ is m.a.

(2) Suppose that $\mathcal{A}$ is maximal abelian. If $A_n \in \mathcal{A}$ and $\|A_n - A\| \to 0$ then, for any operator $B \in \mathcal{A}$, $AB - BA = \lim(A_n B - BA_n) = 0$. Therefore $A \in \mathcal{A}'$, which $= \mathcal{A}$, by Part 1. So $\mathcal{A}$ is norm closed. Thus if $\mathcal{A}$ is also self-adjoint then it's a C^* algebra.

■

Example 3.7. Let (X, μ) be a measure space. Let $f \in L^\infty(\mu)$. Define $M_f : L^2(\mu) \to L^2(\mu)$ by $M_f g = fg$. Then clearly, since $fg \in L^2$ when g is in L^2, M_f is everywhere defined and

$$\|M_f g\|_2^2 = \int |fg|^2 d\mu \leq \|f\|_\infty^2 \|g\|_2^2$$

Therefore $\|M_f\| \leq \|f\|_\infty$. Note $M_{fg} = M_f M_g$, $M_{\alpha f + \beta g} = \alpha M_f + \beta M_g$, $M_f^* = M_{\overline{f}}$.

Assumption 1. *Assume* that every measurable set in X of positive measure contains a subset of finite strictly positive measure. (We say μ has no infinite atoms.)

Lemma 3.8. *Under Assumption 1, $\|M_f\| = \|f\|_\infty$.*

Proof. Can assume $\|f\|_\infty > 0$. Suppose $0 < a < \|f\|_\infty$. Then $\mu(\{x : |f(x)| > a\}) > 0$. Therefore there exists a measurable set S of finite positive measure $\ni |f(x)| > a$ on S. Then $\|M_f \chi_S\|_2^2 = \int |f(x)|^2 \chi_S^2 d\mu \geq a^2 \int \chi_S^2 d\mu = a^2 \|\chi_S\|_2^2$. Therefore $\|M_f\| \geq a$. Hence $\|M_f\| \geq \|f\|_\infty$. ■

Definition 3.9. Let (X, μ) be a measure space. The *multiplication algebra* (denoted by $\mathcal{M}(X, \mu)$) of (X, μ) is the algebra of operators on $L^2(X, \mu)$ consisting of all M_f, $f \in L^\infty$.

Proposition 3.10. *If (X, μ) is a σ–finite measure space, then $\mathcal{M}(X, \mu)$ is a m.a.s.a. algebra.*

Proof. Assume first $\mu(X) < \infty$. Write $\mathcal{M} = \mathcal{M}(X, \mu)$ and assume $T \in \mathcal{M}'$. Let $g = T(1)$. If $f \in L^\infty$ then $TM_f 1 = M_f T1$. Therefore $T(f) = fg$. Thus $Tf = M_g f$ for f in L^∞. The proof in the preceding example shows $\|g\|_\infty \leq \|T\|$. Since M_g is bounded the equation $T \mid L^\infty = M_g \mid L^\infty$, already established, extends by continuity to L^2. Hence $T \in \mathcal{M}$ and $\mathcal{M}$ is maximal abelian. Since $M_g^* = M_{\overline{g}}$, $\mathcal{M}$ is self–adjoint.

In the general case, write $X = \cup_{j=1}^\infty X_j$, where the X_j are disjoint subsets of finite measure. If T is in $\mathcal{M}'$ it commutes with $M\chi_{X_j}$ and therefore leaves invariant the subspace $L^2(X_j)$ which we identify with $\{f \in L^2(X) : f = 0 \text{ off } X_j\}$. Apply the finite measure case and piece together the result to get the general case. ■

Definition 3.11. Let $D(w, \varepsilon) = \{z \in \mathbb{C} : |z - w| < \varepsilon\}$, then if $f \in L^\infty(X, \mu)$ the essential range of f is

$$\{w \in \mathbb{C} : \mu(f^{-1}(D(w, \varepsilon))) > 0 \text{ for all } \varepsilon > 0\}.$$

Exercise 3.1. Prove that the spectrum of M_f = essential range of f when X has no infinite atoms.

Definition 3.12. If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(H)$ a vector x in H is called a *cyclic* vector for $\mathcal{A}$ if $\mathcal{A}x \equiv \{Ax : A \in \mathcal{A}\}$ is dense in H.

Remark 3.13. Let $\mathcal{A}$ be any $*$ subalgebra of $\mathcal{B}(H)$. Suppose K is a closed subspace of H and P is the projection on K. Then K is invariant under $\mathcal{A}$ iff $P \in \mathcal{A}'$.

Proof. ($\Leftarrow$) If $P \in \mathcal{A}'$, $x \in K$ then $Ax = APx = PAx \in K$.

($\Rightarrow$) If $AK \subset K$ then $APx \in K$. $\therefore$ $APx = PAPx$. Also, $A^* \in \mathcal{A}$. So $A^*P = PA^*P$. Therefore $PA = P^*A = (A^*P)^* = (PA^*P)^* = PAP = AP$. Hence $P \in \mathcal{A}'$. ■

Lemma 3.14. *If H is separable and $\mathcal{A}$ is a m.a.s.a. on H then $\mathcal{A}$ has a cyclic vector.*

Proof. For any $x \in H$, let $\overline{\mathcal{A}x}$ be the closed subspace containing $\mathcal{A}x$. Since $I \in \mathcal{A}$, $x \in \mathcal{A}x$. Since $\mathcal{A}x$ is invariant under $\mathcal{A}$, so is $\overline{\mathcal{A}x}$. Note that if $y \perp \mathcal{A}x$ then $\mathcal{A}y \perp \mathcal{A}x$ since $(Ay, Bx) = (y, A^*Bx) = 0$. Let $E = \{x_\alpha\}$ be an orthonormal set such that $\mathcal{A}x_\alpha \perp \mathcal{A}x_\beta$ if $\alpha \neq \beta$. Such sets exist (e.g. singletons). Zorn's lemma gives us a maximal such set. For this E, H = closed $\text{span}_\alpha\{\mathcal{A}x_\alpha\}$ for otherwise we could adjoin to E any unit vector in $(\text{span}\{\mathcal{A}x_\alpha\})^\perp$. Now, since H is separable, E is countable; $E = \{x_1, x_2, \ldots\}$ put $z = \sum_{n=1}^\infty 2^{-n}x_n$. *Claim:* z is a cyclic vector for $\mathcal{A}$. The projection P_n onto $\overline{\mathcal{A}x_n}$ is in $\mathcal{A}'$ by the above remark. Therefore $P_n \in \mathcal{A} = \mathcal{A}'$.

$$\therefore \mathcal{A}z \supset \mathcal{A}P_n z = \mathcal{A}2^{-n}x_n = \mathcal{A}x_n \ \forall n$$

$$\therefore \overline{\mathcal{A}z} \supset \text{closed span}_n\{\mathcal{A}x_n\} = H.$$

■

Example 3.15. Let

$$C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

The algebra $\mathcal{A}$ generated by C, i.e. the smallest algebra containing C and I, consists of all polynomial functions of C and is therefore abelian (and self-adjoint). But it's not maximal abelian (clear?). And it doesn't have a cyclic vector, does it?

Definition 3.16. A *unitary operator* U from Hilbert space H to Hilbert space K is a linear operator from H onto K such that $\|Ux\| = \|x\| \ \forall x \in H$. We may emphasize that $U : H \to K$ is surjective by writing $U : H \twoheadrightarrow K$.

Theorem 3.17. *Let $\mathcal{A}$ be a m.a.s.a. on separable Hilbert space H. Then there exists finite measure space (X, μ) and a unitary operator $U : H \twoheadrightarrow L^2(X, \mu)$ such that $U\mathcal{A}U^{-1} = \mathcal{M}(X, \mu)$.*

Proof. Let z be a unit cyclic vector for $\mathcal{A}$. Then z is also a *separating vector* for $\mathcal{A}$ (i.e., if $A \in \mathcal{A}$ and $Az = 0$ then $A = 0$) since if $Az = 0$ then $\forall\ B \in \mathcal{A}$, $ABz = BAz = 0$. Therefore $A\mathcal{A}z = 0$. But $\mathcal{A}z$ is dense. $\therefore A = 0$. We have seen that $\mathcal{A}$ is a C^* algebra. Let $X = \text{spectrum}(\mathcal{A})$. Then the Gelfand map $A \to \widehat{A}$ is an isometric isomorphism $\mathcal{A} \twoheadrightarrow C(X)$.

Define Λ on $C(X)$ by

$$\Lambda(\widehat{A}) = (Az, z)$$

Λ is clearly a bounded linear functional on $C(X)$. Indeed, $|\Lambda(\widehat{A})| \leq \|A\| = \|\widehat{A}\|$. Λ is positive since

$$\Lambda(\overline{\widehat{A}}\widehat{A}) = (A^*Az, z) = \|Az\|^2 \geq 0$$

Therefore there exists a unique regular Borel measure μ on X such that

$$\Lambda(\widehat{A}) = \int \widehat{A} d\mu$$

$\mu(X)$ is finite because $\mu(X) = \int 1 d\mu = \Lambda(1) = \|z\|^2 = 1$. Define U_0 on $\mathcal{A}z$ by

$$U_0 Az = \widehat{A}.$$

U_0 is well defined since $Az = 0 \Rightarrow A = 0$. U_0 is thus linear and densely defined. Moreover

$$\|U_0 Az\|^2 = \int \overline{\widehat{A}}\widehat{A} d\mu = \Lambda(\widehat{A^*A}) = (Az, Az) = \|Az\|^2.$$

Hence U_0 is isometric from $\mathcal{A}z$ into $L^2(X,\mu)$. Since U_0 is continuous it extends by continuity to an operator $U : H \to L^2(X,\mu)$ such that

$$\|Ux\| = \|x\| \quad \forall x \in H$$

Since range(U) is a complete (therefore closed) subspace of $L^2(\mu)$ which contains $C(X)$ it is all of $L^2(\mu)$.

Now, if $A, B \in \mathcal{A}$ then

$$UAU^{-1}\widehat{B} = UABz = \widehat{AB} = M_{\widehat{A}}\widehat{B}$$

Therefore

$$UAU^{-1} = M_{\widehat{A}} \tag{3.2}$$

on a dense set and thus, on all of $L^2(\mu)$.

Let $\mathcal{N} = U\mathcal{A}U^{-1}$ and let $\mathcal{M}$ be the multiplication algebra of (X,μ). Clearly $\mathcal{N} \subset \mathcal{M}$ by (3.2).

If $T \in \mathcal{M}$ then $T \in \mathcal{N}'$, therefore $U^{-1}TU \in \mathcal{A}'$. But $\mathcal{A}' = \mathcal{A}$

$$\therefore U^{-1}TU \in \mathcal{A}$$
$$\therefore T \in \mathcal{N}$$
$$\therefore \mathcal{M} = \mathcal{N}.$$

∎

Definition 3.18. A bounded operator $A : H \to H$ is

(1) *normal* if $A^*A = AA^*$
(2) *Hermitian* if $A = A^*$
(3) *unitary* if A is onto and $\|Ax\| = \|x\|$ $\forall x \in H$
(4) *orthogonal* if H is real and A is unitary.

Proposition 3.19. *Let H be a Hilbert space. Suppose $A : H \to H$ is linear and* $(Ax, x) = 0$ $\forall x \in H$ then

(a) *if H is complex then $A = 0$*
(b) *if H is real and $A^* = A$ then $A = 0$.*

Proof. *Polarization identity*:

$$(A(x+y), x+y) - (A(x-y), (x-y)) = 2(Ax, y) + 2(Ay, x)$$

Therefore

$$(Ax, y) + (Ay, x) = 0 \quad \forall x, y \tag{3.3}$$

If H is real and $A^* = A$ then

$$(Ay, x) = (y, Ax) = (Ax, y).$$

Therefore $(Ax, y) = 0\ \forall x, y$. $\therefore Ax = 0$, $\forall x$.

If H is complex, replace x by ix in (3.3) to get

$$i(Ax, y) - i(Ay, x) = 0$$

Divide by i and add to (3.3) to get:

$$(Ax, y) = 0 \quad \forall x, y \qquad \therefore Ax = 0 \quad \forall x.$$

■

Corollary 3.20. *An operator $U : H \to H$ is unitary iff U is bounded and*

$$U^*U = UU^* = I \tag{3.4}$$

Proof. *Assume U is bounded and that (3.4) holds.* Then $\|Ux\|^2 = (U^*Ux, x) = \|x\|^2$. Since $U(U^*x) = x$, U is onto. Therefore U is unitary.

Assume U is unitary. Then

$$((U^*U - I)x, x) = \|Ux\|^2 - \|x\|^2 = 0$$

Therefore $U^*U - I = 0$. Since U is onto, $(\forall x \in H)(\exists y \in H)(x = Uy)$. Therefore $UU^*x = UU^*Uy = Uy = x$. $\therefore UU^* = I$. ■

Proposition 3.21. *Let (X, μ) be a σ–finite measure space. Let $f \in L^\infty$. Then*

(1) *M_f is normal*
(2) *(M_f is Hermitian) $\leftrightarrow$ (f is real a.e.)*
(3) *(M_f is unitary) $\leftrightarrow$ ($|f| = 1$ a.e.)*

Proof. (1) $M_f^* M_f = M_{\overline{f}} M_f = M_{\overline{f}f} = M_f M_f^*$. Therefore M_f normal.
(2) $(M_f^* = M_f) \leftrightarrow (M_{\overline{f}} = M_f) \leftrightarrow (f = \overline{f}$ a.e.)
(3) $(M_f^* M_f = I) \leftrightarrow (M_{\overline{f}f} = M_1) \leftrightarrow (\overline{f}f = 1$ a.e.) ■

Theorem 3.22 (Spectral Theorem)**.** *Let $\{A_\alpha\}_{\alpha \in I}$ be a family of bounded normal operators on a complex separable Hilbert space. Assume that the family is a commuting set in the sense that:*

$$A_\alpha A_\beta = A_\beta A_\alpha \quad \forall \alpha, \beta$$

and

$$A_\alpha A_\beta^* = A_\beta^* A_\alpha \quad \forall \alpha, \beta$$

Then there exists a finite measure space (X, μ) and a unitary operator $U : H \to L^2(X, \mu)$ and for each α there exists a function $f_\alpha \in L^\infty$ such that

$$UA_\alpha U^{-1} = M_{f_\alpha}.$$

Proof. Let $\mathcal{A}_0$ be the algebra generated by the $\{A_\alpha, A_\alpha^*\}_{\alpha\in I}$. Then $\mathcal{A}_0$ is a commutative $*$ algebra. Order the set of all commutative self-adjoint algebras containing $\mathcal{A}_0$ by inclusion. By Zorn's lemma there exists a largest such algebra, $\mathcal{A}$. We assert that $\mathcal{A} = \mathcal{A}'$. Indeed if $B \in \mathcal{A}'$ then $B^* \in \mathcal{A}'$ also because $\mathcal{A}$ is self-adjoint. Hence $C := B + B^* \in \mathcal{A}'$. But the algebra generated by $\mathcal{A}$ and C is commutative and self-adjoint. Therefore $C \in \mathcal{A}$. Similarly $i(B - B^*) \in \mathcal{A}$. Hence $B \in \mathcal{A}$. So $\mathcal{A}' = \mathcal{A}$. Therefore $\mathcal{A}$ is maximal abelian and self-adjoint.

Now by the preceding theorem there exists (X, μ) with $\mu(X) = 1$ and a unitary $U : H \to L^2(X)$ such that $U\mathcal{A}U^{-1} = \mathcal{M}(X, \mu)$. Therefore $UA_\alpha U^{-1} = M_{f_\alpha}$ for some $f_\alpha \in L^\infty$. ∎

3.1. **Problems on the Spectral Theorem (Multiplication Operator Form).**

Exercise 3.2. If A is a Hermitian operator on an n–dimensional unitary space $(n < \infty)$ V prove that there is an orthonormal basis of V which diagonalizes A by applying the theorem that a m.a.s.a. algebra is unitarily equivalent to a multiplication algebra.

Exercise 3.3. Let H be a Hilbert space with O. N. basis $e_1, e_2, \dots$. Let θ_j be a sequence of real numbers in $(0, \pi/2)$. Let

$$x_j = (\cos\theta_j)e_{2j} + (\sin\theta_j)e_{2j-1} \quad j = 1, 2, \dots$$

and

$$y_j = -(\cos\theta_j)e_{2j} + (\sin\theta_j)e_{2j-1} \quad j = 1, 2 \dots .$$

Let

$$M_1 = \text{closedspan } \{x_j\}_{j=1}^\infty \text{ and}$$
$$M_2 = \text{closedspan } \{y_j\}_{j=1}^\infty.$$

(1) Show that the closed span of M_1 and M_2 (i.e., the closure of $M_1 + M_2$) is all of H.
(2) Show that if $\theta_j = 1/j$ then the vector

$$z = \sum_{j=1}^\infty j^{-1} e_{2j-1}$$

is not in $M_1 + M_2$, so that $M_1 + M_2 \neq H$.

Exercise 3.4. Let

$$H = \ell^2(Z) = \{\text{all square summable 2−sided complex sequences } a \text{ with } \|a\|^2 = \sum_{j=-\infty}^\infty |a_j|^2\}.$$

Define $U : H \to L^2(-\pi, \pi)$ by

$$(Ua)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^\infty a_n e^{in\theta}.$$

It is well known that U is unitary. For f in $\ell^1(Z)$ define

$$(C_f a)_n = \sum_{k=-\infty}^\infty f(n-k)a_k.$$

(1) Show that C_f is a bounded operator on H.
(2) Find C_f^* explicitly and show that C_f is normal for any f in $\ell^1(Z)$.

(3) Show that UC_fU^{-1} is a multiplication operator.
(4) Find the spectrum of C_f, where

$$f(j) = \begin{cases} 1 & \text{if} \quad |j| = 1 \\ 0 & \text{otherwise} \end{cases} .$$

Exercise 3.5. Define f on $[0, 1]$ by

$$f(x) = \begin{cases} 2 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases} .$$

Find the spectrum of M_f as an operator on $L^2(0, 1)$.

Exercise 3.6. Find a bounded Hermitian operator A with both of the following properties:

(1) A has no eigenvectors
(2) $\sigma(A)$ is set of Lebesgue measure zero in $\mathbb{R}$.

Hint 1: Such an operator is said to have singular continuous spectrum.
Hint 2: Consider the Cantor set. See Rudin, 3rd Edition, Section 7.16.

3.2. **Integration with respect to a Projection Valued Measure.**

We now study a second form of the spectral theorem.

Definition 3.23. A sequence A_n of bounded operators on a Banach space B *converges strongly* to a bounded operator A if $A_nx \to Ax$ for each $x \in B$. A_n *converges weakly* if $\langle A_nx, y\rangle \to \langle Ax, y\rangle$ $\forall x \in B$, $y \in B^*$. If B is a Hilbert space weak convergence is equivalently defined as $(A_nx, y) \to (Ax, y)$ $\forall x, y \in H$.

Definition 3.24. If P and Q are two projections in H, then P is called *orthogonal* to Q if $\mathcal{R}(P) \perp \mathcal{R}(Q)$.

Proposition 3.25. *A bounded operator P with range M is the orthogonal projection onto M iff $P^2 = P$ and $P^* = P$.*

Proof. We already know that the orthogonal projection onto a closed subspace M has these properties. Suppose then that $P^2 = P$ and $P^* = P$ and $M =$ range P. If $x \in M$ then $x = Py$ for some y. Hence: $Px = P^2y = Py = x$. So $P \mid M = I$ on M. M is closed, for if $x_n \in M$ and $x_n \to x$ then $Px = \lim Px_n = \lim x_n = x$. Hence $x \in M$. It remains to show that $\mathcal{N}(P) = M^\perp$.

If $x \in M$ and $Py = 0$ then $(x, y) = (Px, y) = (x, Py) = 0$. Therefore $\mathcal{N} \subset M^\perp$. If $y \in M^\perp$ then $\forall x \in H$, $(Px, y) = 0$. Therefore $(x, Py) = 0$ $\forall x \in H$. Therefore $Py = 0$. So $y \in \mathcal{N}$. ■

Note: Henceforth projection means "*orthogonal projection*".

Corollary 3.26. *If P_1, P_2 are two projections with ranges M_1, M_2, respectively, then*

a) $M_1 \perp M_2 \Rightarrow P_1P_2 = P_2P_1 = 0$
b) $P_1P_2 = 0 \Rightarrow M_1 \perp M_2$
c) In case of a) or b) $P_1 + P_2$ is the projection onto span $\{M_1, M_2\}$.

Proof. a) Assume $M_1 \perp M_2$. For any $x \in H$, $P_1x \in M_1 \subset M_2^\perp = \mathcal{N}(P_2)$. Therefore $P_2P_1x = 0$, etc.

b) Assume $P_1P_2 = 0$. If $x \in M_1$, $y \in M_2$ then $(x, y) = (P_1x, P_2y) = (x, P_1P_2y) = 0$. Therefore $M_1 \perp M_2$.

c) Assume $P_1P_2 = 0$. Then $(P_1+P_2)^2 = P_1^2 + P_1P_2 + P_2P_1 + P_2^2 = P_1 + P_2$. Clearly $(P_1+P_2)^* = P_1 + P_2$. Therefore $P = P_1 + P_2$ is the projection onto some closed subspace M. If $x \in M_1$, $y \in M_2$ then $P(x+y) = P_1x + P_2x + P_1y + P_2y = P_1x + P_2y = x + y$. Therefore $M \supseteq M_1 + M_2$. If $z \in M$, then $z = Pz = P_1z + P_2z \in M_1 + M_2$. ∎

Proposition 3.27. *If P_n is a sequence of mutually orthogonal projections, then strong $\lim_{n\to\infty} \sum_{k=1}^n P_k$ exists and is the projection onto the closure of span $\{\mathcal{R}(P_n)\}_{n=1}^\infty$.*

Proof. Let $Q_n = \sum_{k=1}^n P_k$.

Now Q_n is the projection on $M_1 + \cdots + M_n$ where $M_j = \mathcal{R}(P_j)$ by Corollary 3.26 and induction. Therefore $\|Q_nx\|^2 \leqq \|x\|^2 \ \forall x$, i.e.,

$$\|x\|^2 \geqq \Big\| \sum_{k=1}^n P_kx \Big\|^2 = \Big(\sum_{k=1}^n P_kx, \sum_{j=1}^n P_jx \Big) = \sum_{k=1}^n \|P_kx\|^2.$$

Hence the series $\sum_{k=1}^\infty \|P_kx\|^2$ converges and is $\leqq \|x\|^2$. But if $n > m$,

$$\|(Q_n - Q_m)x\|^2 = \sum_{k=m+1}^n \|P_kx\|^2.$$

Therefore $\|(Q_n - Q_m)x\| \to 0$ as $n, m \to \infty$. Hence Q_nx converges as $n \to \infty$. Call the limit Qx. Q is clearly a bounded linear operator and $\|Q\| \leqq 1$. Moreover $(Qx, y) = \lim(Q_nx, y) = \lim(x, Q_ny) = (x, Qy)$. Therefore $Q^* = Q$. Note that $Q_mQ_n = Q_m$ if $n \geqq m$.

$$\therefore\ (Q^2x, y) = \lim_m (Q_mQx, y) = \lim_m \lim_n (Q_mQ_nx, y) = \lim_m (Q_mx, y) = (Qx, y) \quad \forall x, y.$$

$$\therefore\ Q^2 = Q.$$

Thus Q is the projection on some closed subspace M. If $x \in M_k$, then $Q_nx = x$ for $n \geqq k$. Therefore $Qx = x$.

$$\therefore\ M_k \subset M \qquad \therefore\ M \supset \overline{\text{span}\{M_n\}} \equiv N.$$

If $x \perp N$, then $x \perp M_j \ \forall j$. Therefore $Q_nx = 0 \ \forall n$. Therefore $Qx = 0$. $\therefore\ x \perp M$, i.e., $N^\perp \subset M^\perp$, $\therefore\ N \supset M$. ∎

Definition 3.28. Let X be a set and let $\mathcal{S}$ be a σ–field in X. A projection valued measure on $\mathcal{S}$ is a function $E(\cdot)$ from $\mathcal{S}$ to projections on a Hilbert space H such that

(1) $E(\emptyset) = 0$
(2) $E(X) = I$
(3) $E(A \cap B) = E(A)E(B)$ where $A, B \in \mathcal{S}$
(4) If $A_1, A_2, \ldots$ is a disjoint sequence in $\mathcal{S}$ then

$$E(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty E(A_n) \text{(strong sum)}$$

Remarks 3.29. (1) and (3) in Definition 3.28 imply that if $A \cap B = \emptyset$ then $E(A)$ and $E(B)$ are mutually orthogonal. Hence the strong sum in (4) of Definition 3.28 converges to a projection by Proposition 3.27.

Example 3.30. Let (Y, μ) be a measure space. Let f be a complex valued measurable function on Y. For any Borel set $A \subseteq \mathbb{C}$, define $E(A) = M_{\chi_{f^{-1}(A)}}$ on $L^2(Y, \mu)$. It is straightforward to verify that $E(\cdot)$ is a projection valued measure on the Borel sets.

Let $(X, \mathcal{S})$ be a measurable space and $E(\cdot)$ a projection valued measure on $\mathcal{S}$ with values in $\mathcal{B}(H)$.

Note: If $x, y \in H$, then $B \to (E(B)x, y)$ is a complex measure on $\mathcal{S}$.

Definition 3.31. If $f = \sum_{j=1}^n a_j \chi_{B_j}$ is a simple complex valued measurable function on X, let

$$\int f dE = \sum_{j=1}^n a_j E(B_j).$$

Properties 3.32. Properties of $\int$: If f is simple, then

(1) $\int f dE$ is well defined because the corresponding bilinear form $(\int f dEx, y) = \int f d(Ex, y)$ is well defined.
(2) $\| \int f dE\| \le \sup_{s \in X} |f(s)|$.
(3) $\int \alpha f + \beta g dE = \alpha \int f dE + \beta \int g dE$.
(4) If g is also simple then

$$\int fg dE = \Big(\int f dE \Big)\Big(\int g dE \Big).$$

(5) $\int \overline{f} dE = (\int f dE)^*$.

Proof. 2) Write $f = \sum a_j \chi_{B_j}$ where $B_i \cap B_j = \emptyset$ if $i \neq j$. Then

$$\Big\| \Big(\int f dE \Big) x \Big\|^2 = \Big\| \sum_j a_j E(B_j) x \Big\|^2 = \sum_{ij} (a_i E(B_i)x, a_j E(B_j)x)$$

$$\begin{aligned} \sum_{ij} a_i \overline{a}_j (E(B_j)E(B_i)x, x) &= \sum_j |a_j|^2 (E(B_j)x, x) \\ &\le (\max |a_j|^2) \sum_j (E(B_j)x, x) = \Big(\sup_{s \in X} |f(s)|^2 \Big) (E(\cup B_j)x, x) \\ &\le \sup |f(s)|^2 \|x\|^2 \end{aligned}$$

3) The bilinear forms of both sides are the integrals with respect to a complex measure.

4) If $g = \sum_{j=1}^n b_j \chi_{C_j}$ and $f = \sum_{j=1}^m a_j \chi_{B_j}$, then by taking a common refinement of the $\{B_j\}$ and $\{C_j\}$, we may assume that $B_j = C_j$ and $B_i \cap B_j = \emptyset$ if $i \neq j$. Then

$$\int fg dE = \sum a_j b_j E(B_j) = \Big(\sum a_j E(B_j) \Big) \Big(\sum b_i E(B_i) \Big).$$

■

Definition 3.33. If f is a bounded measurable function, let f_n be a sequence of simple measurable functions converging to f uniformly. Then by (2) of Property 3.32

$$\Big\| \int f_n dE - \int f_m dE \Big\| \to 0 \text{ as } n, m \to \infty.$$

Define $\int f dE = \lim \int f_n dE$ (in operator norm).

Properties (1)–(4)of Property 3.32 hold for a bounded measurable f – the proofs are straightforward.

Remark 3.34. If H, K are Hilbert spaces and $A : H \to K$ is bounded linear then $A^* : K \to H$ can be defined by $(Ax, y) = (x, A^*y)$, $x \in H$, $y \in K$ just as if $K = H$. Usual properties hold:

$$\begin{aligned} (AB)^* &= B^*A^* \\ A^{**} &= A, \text{ etc.} \end{aligned}$$

Definition 3.35. A projection valued measure $E(\cdot)$ on the Borel sets in $\mathbb{C}$ *is supported in a closed set* K if $E(K^c) = 0$. The *support set* of E is the complement of $\cup\{V \mid E(V) = 0,\ V \text{ open}\}$.

Note: If E has compact support, define

$$\int_{\mathbb{C}} zdE = \int_K zdE.$$

(This is clearly well defined.)

Example 3.36. Resume notation from Example 3.30. Assume f is bounded. Then $E(\cdot)$ has compact support and

$$\int_{\mathbb{C}} zdE = M_f.$$

Exercise 3.7. Prove the assertions in Example 3.36. Verify first that $E(\cdot)$ is indeed a projection valued measure.

Theorem 3.37 (Spectral Theorem)**.** *Let A be a bounded normal operator on a separable Hilbert space H. There exists a unique projection valued Borel measure E on $\mathbb{C}$ with compact support such that*

$$A = \int_{\mathbb{C}} zdE.$$

Furthermore if D is any bounded operator on H then D commutes with A and A^* iff D commutes with $E(B)$ for all Borel sets B.

Proof. *Existence:* By the form of the spectral theorem given on page 24, there exists a measure space (X, μ) and a unitary operator $U : H \to L^2(X, \mu)$ such that

$$UAU^{-1} = M_f$$

where f is a bounded measurable function on X. If B is a measurable set in X, let $G(B) = M_{\chi_{f^{-1}(B)}}$.

In view of Examples 3.30 and 3.36, we see that $G(\cdot)$ is a projection valued measure with compact support in $\mathbb{C}$ and

$$\int_{\mathbb{C}} zdG = M_f.$$

Let $E(B) = U^{-1}G(B)U = U^*G(B)U$. Then one sees easily that $E(\cdot)$ is also a projection valued measure on $\mathbb{C}$ with compact support. If h is a simple function on $\mathbb{C}$ (say $h = \sum a_j\chi_{B_j}$) then

$$\int h(z)dE(z) = \sum a_jE(B_j) = U^{-1}\Big(\sum a_jG(B_j)\Big)U = U^{-1}\int h(z)dG(z)U.$$

Taking uniform limits, we see that

$$\int h(z)dE(z) = U^{-1}\int h(z)dG(z)U. \tag{3.5}$$

holds for all bounded measurable h. In particular, since E has compact support, we may take $h(z) = z$ and obtain

$$\int zdE(z) = U^{-1}M_f U = A.$$

Uniqueness: Suppose that $F(\cdot)$ is another projection valued Borel measure on $\mathbb{C}$ with compact support, such that $A = \int zdF(z)$. We wish to show $E = F$. Let K be a compact set in $\mathbb{C}$ containing the supports of E and F. By (1)– (5) of Property 3.32, we have

$$p(A, A^*) = \int p(z, \overline{z})dE(z) \tag{3.6}$$

for any polynomial $p(\cdot)$ in the commuting operators A and A^*. (Note that $(\int fdE)^* = \int \overline{f}dE$ must be used here.) Since these polynomials are dense in $C(K)$ in sup norm (by Stone–Weierstrass theorem) it follows that

$$\int_K f(x, y)dE(z) = \int_K f(x, y)dF(z) \tag{3.7}$$

for any f in $C(K)$ because (3.6) implies its validity for polynomials. Now let u and v be in H. Then

$$\int_K fd(E(z)u, v) = \int_K fd(F(z)u, v) \tag{3.8}$$

for all $f \in C(K)$. Hence the two complex measures $B \to (E(B)u, v)$ and $B \to (F(B)u, v)$ are equal since the dual space of $C(K)$ is the space of complex measures on K. Thus for any Borel set B, $((E(B) - F(B))u, v) = 0$ $\forall u, v \in H$. Thus $E(B) = F(B)$, proving uniqueness.

For the final assertion of the theorem, suppose that $DE(B) = E(B)D$ $\forall$ Borel sets B. Then D commutes with all operators of the form $\Sigma a_j E(B_j)$ and with their uniform limits. In particular, D commutes with $A = \int zdE$ and $A^* = \int \overline{z}dE$.

Conversely, suppose that D commutes with both A and A^*. Then D commutes with all polynomials $p(A, A^*)$, and hence with their uniform limits. Thus if support $E = K$ then D commutes with $\int_K fdE$ for any f in $C(K)$. If u and v are in H, then

$$\Big(\Big(\int_K fdE\Big)Du, v\Big) = \Big(\Big(\int_K fdE\Big)u, D^*v\Big).$$

That is,

$$\int_K f(z)d(E(z)Du, v) = \int f(z)d(E(z)u, D^*v).$$

Hence, just as in the uniqueness proof, it follows that for any Borel set B,

$$(E(B)Du, v) = (E(B)u, D^*v) \text{ for all } u, v \in H.$$

That is

$$((E(B)D - DE(B))u, v) = 0 \text{ for all } u, v \in H$$

and therefore $E(B)D = DE(B)$. ■

Definition 3.38. The projection valued measure $E(\cdot)$ appearing in the preceding theorem is called the *spectral resolution* of A.

Corollary 3.39. *If A is a bounded normal operator on H with spectral resolution $E(\cdot)$ then support $E = \sigma(A)$.*

Proof. From the construction of E, we see that the support of E is the essential range of f. But essential range of $f = \sigma(M_f) = \sigma(A)$ since unitary equivalences preserve spectrum. ■

Corollary 3.40. *A point $\lambda \in \mathbb{C}$ is an eigenvalue for A iff λ is an atom for E. Moreover if λ is an eigenvalue for A then $\mathcal{R}E(\{\lambda\})$ is the corresponding eigenspace.*

Proof. Assume $E(\{\lambda\}) \neq 0$. Let $x \in \mathcal{R}E(\{\lambda\})$, $x \neq 0$. Then $E(F)x = 0$ if $\lambda \notin F$, and so

$$\begin{aligned} \|(A-\lambda)x\|^2 &= ((A-\lambda)^*(A-\lambda)x, x) \\ &= \int_{\mathbb{C}} |z-\lambda|^2 d(\mathbb{E}(\cdot)x, x) = \int_{\{\lambda\}} |z-\lambda|^2 d(\mathbb{E}(\cdot)x, x) = 0 \end{aligned} \tag{3.9}$$

Hence x is an eigenvector for A with eigenvalue λ. Conversely, if there exists $x \neq 0$ such that $Ax = \lambda x$ then (3.9) shows that

$$\int_{\mathbb{C}} |z-\lambda|^2 d(\mathbb{E}(\cdot)x, x) = 0.$$

This implies that $(E(F)x, x) = 0$ if $\lambda \notin F$. But $(E(\mathbb{C})x, x) = \|x\|^2 \neq 0$, and hence $(E(\{\lambda\}x, x) = \|x\|^2$. Therefore $E(\{\lambda\}) \neq 0$ and $E(\{\lambda\})x = x$. ■

Lemma 3.41. *For any bounded operator A, $\sigma(A^*) = \overline{\sigma(A)}$; further, if A is invertible, then $\sigma(A^{-1}) = \sigma(A)^{-1}$.*

The proof follows easily from the definitions of $\sigma(A)$, A^* and A^{-1}.

Proposition 3.42. *a) If A is Hermitian then $\sigma(A)$ is real.*
b) If U is unitary then $\sigma(U) \subset \{z : |z| = 1\}$.

Proof. (a) We have already proved this for a $*$ quadratic normed $*$ algebra. But it also follows immediately from Lemma 3.41.

(b) $\|U\| = 1$ and so $\sigma(U) \subseteq \{z : |z| \leq 1\}$. If $0 < |z| < 1$ and $z \in \sigma(U)$ then

$$z^{-1} \in \sigma(U^{-1}) = \sigma(U^*) \subset \{z : |z| \leq 1\},$$

a contradiction. Finally, it is clear that $o \notin \sigma(U)$. ■

Corollary 3.43 (Spectral theorem for a bounded Hermitian operator)**.** *If A is a bounded Hermitian operator on a separable Hilbert space H, then there exists a unique projection–valued Borel measure $E(\cdot)$ on the line with compact support such that*

$$A = \int_{-\infty}^{\infty} \lambda dE(\lambda).$$

For all real Borel sets B, $E(B) \subset \{A\}''$.

Proof. $\sigma(A) \subset (-\infty, \infty)$ by the proposition. Apply Corollary 3.39. ■

Corollary 3.44 (Spectral theorem for a unitary operator)**.** *If U is a unitary operator on a separable Hilbert space, then there exists a unique projection–valued Borel measure $E(\cdot)$ on $[0, 2\pi)$ such that*

$$U = \int_0^{2\pi} e^{i\theta} dE(\theta),$$

and $E(B) \subset \{U\}''$ for all Borel sets B.

Proof. The same as for Corollary 3.43 if we map $[0, 2\pi)$ onto $\{z : |z| = 1\}$ with $\theta \to e^{i\theta}$. ∎

Example 3.45. $H = \ell^2$. If $x = \{a_n\}_{n=1}^{\infty} \in \ell^2$, put

$$Ax = \left\{\frac{1}{n} a_n\right\}_{n=1}^{\infty}.$$

Then A is a bounded multiplication operator by a real function, and is hence Hermitian.

$$\sigma(A) = \{1, 1/2, 1/3, \ldots, 0\}.$$

Each point is an eigenvalue, except 0. The eigenvector corresponding to $1/n$ is

$$x_n = (0, 0, \ldots, 1, 0, \ldots).$$

Definition 3.46. Let A be any bounded operator. The set $\sigma_p(A)$ of all eigenvalues of A is called the *point spectrum* of A. Let H_p be the closed subspace of H spanned by the eigenvectors of A. If $H_p = H$ then A is said to have *pure point spectrum*.

In Example 3.45 $H_p = H$ but $\sigma_p(A) \neq \sigma(A)$ since $0 \notin \sigma_p(A)$.

Definition 3.47. If $H_p = \{0\}$ then A is said to have *purely continuous spectrum*.

Example 3.48. $H = L^2(0, 1)$, $A = M_{x+2}$. Then A has no eigenvalues, as we have seen before. Hence $\sigma_p(A) = \emptyset$. Thus A has purely continuous spectrum. Note that $\sigma(A) = [2, 3]$.

Example 3.49. Let Q = rationals in $[0, 1]$ with the counting measure. Let $A = M_{x+2}$. Then

$$\sigma(A) = \text{ess. range of } x + 2 = [2, 3].$$

But every rational number in $[2, 3]$ is an eigenvalue of A because the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x = r \\ 0 & \text{if} \quad x \neq r,\ x \in [0, 1] \end{cases}$$

is an eigenfunction associated to the eigenvalue $2 + r$ if r is a rational in $[0, 1]$. Since these functions form an Orthonormal . basis of H we have $H_p = H$. Thus A has pure point spectrum in spite of the fact that $\sigma(A) = [2, 3]$, which is the same spectrum as in Example 3.48.

Definition 3.50. 1) If A is a bounded Hermitian operator we write $A \geq 0$ if $(Ax, x) \geq 0\ \forall x \in H$.

2) If A and B are bounded Hermitian operators, we write $A \leq B$ if $B - A \geq 0$.

Remark 3.51. The bounded Hermitian operators form a partially ordered set in this ordering.

Exercise 3.8. Prove Remark 3.51.

Exercise 3.9 (Decomposition by spectral type)**.** Let A be a bounded Hermitian operator on a complex Hilbert space H. Suppose that $A = \int_{-\infty}^{\infty} \lambda dE(\lambda)$ is its spectral resolution. Denote by H_{ac} the set of all vectors x in H such that the measure $B \to \|E(B)x\|^2$ is absolutely continuous with respect to Lebesgue measure.

(1) Show that H_{ac} is a closed subspace of H.
(2) Show that $H_p \perp H_{ac}$.

(3) Define $H_{sc} = (H_p + H_{ac})^\perp$. (So we have the decomposition $H = H_p \oplus H_{ac} \oplus H_{sc}$.) Show that if $x \in H_{sc}$ and $x \neq 0$ then the measure $B \to (E(B)x, x)$ has no atoms and yet there exists a Borel set B of Lebesgue measure zero such that $E(B)x \neq 0$.

(4) Show that the decomposition of part c) reduces A. That is, $AH_i \subset H_i$, for $i = p$, ac, or sc.

Exercise 3.10 (Behavior of the resolvent near an isolated eigenvalue)**.** We saw in the proof of Theorem 1.12 in Chapter 2 that if A is a bounded operator on a complex Banach space and λ_0 is not in $\sigma(A)$ then $(A-\lambda)^{-1}$ has a power series expansion: $(A-\lambda)^{-1} = \sum_{n=0}^\infty (\lambda-\lambda_0)^n B_n$ valid in some disk $|\lambda - \lambda_0| < \varepsilon$, where each B_n is a bounded operator.

(1) Suppose that A is the operator on the two dimensional Hilbert space $\mathbb{C}^2$ given by the two by two matrix
$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$
As you (had better) know, $\sigma(A) = \{3\}$. Show that the resolvent $(A-\lambda)^{-1}$ has a Laurent expansion near $\lambda = 3$ with a pole of order two. That is
$$(A-\lambda)^{-1} = (\lambda-3)^{-2}B_{-2} + (\lambda-3)^{-1}B_{-1} + \sum_{n=0}^\infty (\lambda-\lambda_0)^n B_n$$
which is valid in some punctured disk $0 < |\lambda - 3| < a$. Find B_{-2} and B_{-1} and show that neither operator is zero.

(2) Suppose now that A is a bounded Hermitian operator on a complex, separable, Hilbert space H. Suppose that λ_0 is an isolated eigenvalue of A, by which we mean that, for some $\varepsilon > 0$
$$\sigma(A) \cap \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\} = \{\lambda_0\}.$$
Prove that $(A-\lambda)^{-1}$ has a pole of order one around λ_0, in the sense that, for some $\delta > 0$,
$$(A-\lambda)^{-1} = (\lambda-\lambda_0)^{-1}B_{-1} + \sum_{n=0}^\infty (\lambda-\lambda_0)^n B_n, \ 0 < |\lambda-\lambda_0| < \delta,$$
where the operators B_j, $j = -1, 0, 1, \ldots$ are bounded operators on H. Express B_{-1} in terms of the spectral resolution of A.

3.3. **The Functional Calculus.**

Let A be a bounded normal operator on a separable complex Hilbert space H. Let
$$A = \int_{\mathbb{C}} z\,dE(z)$$
be its spectral resolution. For any bounded complex Borel measurable function f defined on $\sigma(A)$, we define
$$f(A) = \int_{\sigma(A)} f(z)\,dE(z).$$

Definition 3.52. If $f : \sigma(A) \to \mathbb{C}$ is a bounded Borel measurable function, then the *essential range* of f with respect to the spectral resolution $E(\cdot)$ of A consists of those $\lambda \in \mathbb{C}$ such that $E(f^{-1}(B)) \neq 0$ for all open sets B containing λ.

Clearly the essential range of f is closed. Define $F(B) = E(f^{-1}(B))$ for all Borel sets E in $\mathbb{C}$. Clearly $F(\cdot)$ is another projection valued measure in $\mathbb{C}$ with support equal to the essential range of f (because for any open set B, $F(B) = 0$ iff $B\cap$ ess. range $f = \emptyset$).

Properties 3.53.

(1) If g is another such function and α and β are scalars, then
 a) $(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A)$
 b) $(fg)(A) = f(A)g(A)$.
 Proof: These are properties 3 and 4 on page 27.

(2) If $f(z) = z^n$, then $f(A) = A^n$. Proof: b) above and induction.

(3) If $f : \sigma(A) \to \mathbb{C}$ is a bounded Borel measurable function, and $F(B) = E(f^{-1}(B))$, then

$$f(A) = \int_{\mathbb{C}} z d\mathbb{F}(z). \tag{3.10}$$

Hence, $F(\cdot)$ is the spectral resolution of $f(A)$.
 Proof.: If $h(z) = \sum_{j=1}^{n} a_j \chi_{B_j}$ is simple, then $h(f(z)) = \sum_{j=1}^{n} a_j \chi_{f^{-1}(B_j)}(z)$. Hence

$$\begin{aligned} \int h(z) dF(z) &= \sum a_j E(f^{-1}(B_j)) \\ &= \int h(f(z)) dE(z). \end{aligned}$$

Since any bounded Borel function g on ess. range f is a uniform limit of simple functions, we may take the norm limit of both sides of the above equation to obtain

$$\int g(z) dF(z) = \int g(f(z)) dE(z) \tag{3.11}$$

for any bounded measurable complex–valued function g on ess. range f. In particular, putting $g(z) = z$ yields Eq. (3.10).

(4) $\sigma(f(A)) =$ ess. range f if $f : \sigma(A) \to \mathbb{C}$ is a bounded complex–valued Borel function.
 Proof: $\sigma(f(A)) =$ supp F by (3.10) and Corollary 3.39. But supp $F =$ ess. range of f.

(5) If g : ess. range $f \to \mathbb{C}$ is a bounded complex–valued Borel function, then $(g \circ f)(A) = g(f(A))$.
 Proof:

$$\begin{aligned} (g \circ f)(A) &= \int (g \circ f)(z) dE(z) \text{ by definition} \\ &= \int g(z) dF(z) \text{ by (3.11)} \end{aligned}$$

Corollary 3.54. *If A is a bounded Hermitian operator and $A \geq 0$ then*

(a) $\sigma(A) \subset [0, \infty)$
(b) *the support of the spectral resolution of A is contained in* $[0, \infty)$
(c) *A has a unique positive square root C (that is, there is a unique positive (and a fortiori Hermitian) operator C such that $C^2 = A$).*

Proof. a) and b) are equivalent by Corollary 3.39. We prove b). Suppose $E((-\infty,0)) \neq 0$. Let $x \in$ range $E((-\infty,0))$. Then the Borel measure that takes B to $(E(B)x,x)$ is supported in $(-\infty,0)$ in the sense that $(E(B)x,x)=0$ if $B \subset [0,\infty)$ and $(E(B)(x),x)>0$ for some $B \subset (-\infty,0)$. Hence

$$(Ax,x) = \int_{-\infty}^{\infty} \lambda d(E(\lambda)x,x) = \int_{-\infty}^{0} \lambda d(E(\lambda)x,x) < 0,$$

contradicting $A \geq 0$.

To prove c), let $f(\lambda) = \lambda^{1/2}$ for $\lambda \geq 0$. Since $\sigma(A) \subset [0,\infty)$, $C = f(A)$ is well–defined. Since range $f \subset [0,\infty)$, $\sigma(C) \subset [0,\infty)$. The spectral theorem implies $C \geq 0$. Suppose D is another positive (hence Hermitian) square root of A. Let $g(\lambda) = \lambda^2$. Then $(f \circ g)(\lambda) = \lambda$. Hence

$$\begin{aligned} D &= (f \circ g)(D) = f(g(D)) \quad \text{by (5) in Property 3.53} \\ &= f(A) \qquad \text{by (2) in Property 3.53} \\ &= C. \end{aligned}$$

■

Definition 3.55. A one parameter unitary group is a function $U : \mathbb{R} \to$ unitary operators on a Hilbert space H such that

$$U(t+s) = U(t)U(s) \text{ for all real } t \text{ and } s.$$

Exercise 3.11. Let A be a bounded Hermitian operator on a separable Hilbert space H. Denote by $E(\cdot)$ its spectral resolution. Assume that $A \geq 0$ and write $P = E(\{0\})$ (which may or may not be the zero projection). Prove that for any vector u in H

$$\lim_{t \to +\infty} e^{-tA} u = Pu.$$

Exercise 3.12. Let V be a unitary operator on a separable complex Hilbert space H. Prove that there exists a one parameter group $U(t)$ on H such that

(a) $U(1) = V$
(b) $U(\cdot)$ is continuous in the operator norm.

4. Unbounded Operators

4.1. **Closed, symmetric and self-adjoint operators.**

Definition 4.1. If X and Y are Banach spaces and $\mathcal{D}$ is a subspace of X, then a linear transformation T from $\mathcal{D}$ into Y is called a linear transformation (or operator) from X to Y with domain $\mathcal{D}$. If $\mathcal{D}$ is dense in X, T is said to be *densely defined*.

Notation 4.2. If S and T are operators from X to Y with domains $\mathcal{D}_S$ and $\mathcal{D}_T$ and if $\mathcal{D}_S \subset \mathcal{D}_T$ and $Sx = Tx$ for $x \in \mathcal{D}_S$, then we say T is an *extension* of S and write $S \subset T$.

We note that $X \times Y$ is a Banach space in the norm

$$\|\langle x, y\rangle\| = \sqrt{\|x\|^2 + \|y\|^2}.$$

If X and Y are Hilbert spaces then $X \times Y$ is a Hilbert space in this norm with inner product

$$(\langle x, y\rangle, \langle x', y'\rangle) = (x, x') + (y, y').$$

Definition 4.3. If T is an operator from X to Y with domain $\mathcal{D}$, the *graph* of T is

$$G_T = \{\langle x, Tx\rangle : x \in \mathcal{D}\}.$$

Note that G_T is a subspace of $X \times Y$.

Definition 4.4. T is *closed* if G_T is closed in $X \times Y$.

It is easy to see that T is closed iff

$$\left.\begin{aligned} x_n &\to x \\ x_n &\in \mathcal{D} \\ Tx_n &\to y \end{aligned}\right\} \Rightarrow x \in \mathcal{D} \text{ and } Tx = y$$

Recall:

Theorem 4.5 (Closed Graph Theorem)**.** *If $T : X^{Ban} \to Y^{Ban}$ is closed and everywhere defined and linear, then T is bounded. (See Rudin, Chapter 5, Problem 16. The solution to this problem depends on Theorem 5.10 of Rudin.)*

Moral: Unbounded closed operators cannot be everywhere defined.

Exercise 4.1. Suppose that (X, μ) is a measure space and that $\mu(X) < \infty$. Let $T : L^2(\mu) \to L^2(\mu)$ be a bounded operator. Suppose that range T is contained in $L^5(\mu)$. Show that T is bounded as an operator from $L^2(\mu)$ into $L^5(\mu)$. **Hint:** Use the closed graph theorem.

Definition 4.6. Let H be a Hilbert space. Let $T : H \to H$ be linear and densely defined with domain $\mathcal{D}$. Define $\mathcal{D}_{T^*}$ as follows: $y \in \mathcal{D}_{T^*} \Leftrightarrow$ the map $x \to (Tx, y)$ is continuous from $\mathcal{D}$ to $\mathbb{C}$. For such y there exists a unique $y^* \in H$ such that $(Tx, y) = (x, y^*)$. We *define* $T^*y = y^*$. Thus

$$(Tx, y) = (x, T^*y) \quad \forall x \in \mathcal{D}_T,\ y \in \mathcal{D}_{T^*}.$$

Properties 4.7. $\mathcal{D}_{T^*}$ is a linear subspace. T^* is linear. (Same proof as for bounded case.) Even though $\mathcal{D}_T$ is dense, $\mathcal{D}_{T^*}$ need not be dense.

Exercise 4.2. Let $H = L^2(0, 1)$, $\mathcal{D} = C([0, 1])$. Let $(Tf)(x) \equiv f(0) =$ constant function. Then T is densely defined. $T : \mathcal{D} \to H$. Prove that $\mathcal{D}_{T^*}$ is not dense.

Definition 4.8. If A and B are operators on H define $A+B$ on $\mathcal{D}_A \cap \mathcal{D}_B$ by $(A+B)x = Ax + Bx$ and AB on $\{x \in \mathcal{D}_B : Bx \in \mathcal{D}_A\}$ by $(AB)x = A(Bx)$.

Properties of $*$:
(0) $(cA)^* = \bar{c}A^*$ if $c \neq 0$
(1) $(A^* + B^*) \subset (A+B)^*$ if $A+B$ is densely defined
(2) $(AB)^* \supset B^*A^*$ if AB is densely defined.
(3) $A \subset B \Rightarrow B^* \subset A^*$

Exercise 4.3. Prove that (1) and (2) are equalities if A is bounded and everywhere defined.

Definition 4.9. We write $H \oplus H$ instead of $H \times H$. Define $V : H \oplus H \to H \oplus H$ by $V\langle x, y\rangle = \langle y, -x\rangle$. V is unitary.

Lemma 4.10. *If $T : H \to H$ is linear and densely defined then $G_{T^*} = (VG_T)^\perp$.*

Proof. $(Tx, y) = (x, y^*) \Leftrightarrow (\langle Tx, -x\rangle, \langle y, y^*\rangle) = 0 \Leftrightarrow (V\langle x, Tx\rangle, \langle y, y^*\rangle) = 0$. Therefore $\langle y, y^*\rangle \in G_{T^*}$ iff $(V\langle x, Tx\rangle, \langle y, y^*\rangle) = 0$ $\forall x \in \mathcal{D}_T$, i.e., iff $\langle y, y^*\rangle \perp VG_T$. ■

Corollary 4.11. *T^* is always closed when T is densely defined.*

Theorem 4.12. *If T is densely defined in H and closed, then T^* is densely defined and $T^{**} = T$.*

Proof. Suppose $\mathcal{D}_{T^*}$ is not dense. Then $\exists z \neq 0 \ni z \perp \mathcal{D}_{T^*}$. So $\langle 0, z\rangle \perp \langle T^*y, -y\rangle$ $\forall y \in \mathcal{D}_{T^*}$. I.e., $\langle 0, z\rangle \perp VG_{T^*} = V(VG_T)^\perp$. Therefore $V\langle 0, z\rangle \perp (VG_T)^\perp$ since $V^2 = -I$. Hence $V\langle 0, z\rangle \in VG_T$ since G_T, and therefore VG_T, is closed. Consequently, $\langle 0, z\rangle \in G_T$. I.e., $z = T(0)$ - a contradiction. Therefore $\mathcal{D}_{T^*}$ is dense and so T^{**} exists. Now for any unitary V and any closed subspace M we have $V(M^\perp) = (VM)^\perp$. Hence $G_{T^{**}} = (VG_{T^*})^\perp = (V(VG_T)^\perp)^\perp = V^2G_T = G_T$. Therefore $T^{**} = T$. ■

Exercise 4.4. Prove that if T is densely defined in H then T^* is densely defined in H iff T has a closed linear extension. Show that in this case T^{**} is the smallest closed linear extension, i.e., it is contained in any other closed linear extension. Moreover $G_{T^{**}} = \overline{G}_T$.

Definition 4.13. If T is a densely defined linear operator in H and if T has a closed linear extension then *the closure of* T is the smallest closed linear extension.

In view of the preceding exercise, the closure of T, if it exists, is equal to T^{**} if it exists — which it does if T^* is densely defined.

Definition 4.14. A *core* for a closed operator T is a subspace $L \subset \mathcal{D}_T$ such that T is the closure of the restriction of T to L, i.e., $T = (T \mid L)$ closure.

Definition 4.15. Let A be densely defined in H A is *symmetric* if $A \subset A^*$. A is *self–adjoint* if $A = A^*$.

Notes: 1) To say that A is symmetric simply means that $(Ax, y) = (x, Ay)$ $\forall x, y \in \mathcal{D}_A$.
2) Since A^* is always closed, a self–adjoint operator is always closed. A symmetric operator need not be closed. But since a symmetric operator always has

a closed extension (namely A^*) it always has a closure. However, as we shall see later, a closed symmetric operator need not be self–adjoint.

3) It tends to be easy to show that an operator is symmetric but hard to show that it is self–adjoint (even if it is self–adjoint). The spectral theorem applies to self–adjoint operators but not to symmetric operators in general. We build now some machinery that is useful in proving self–adjointness.

Definition 4.16. Let H_n be a sequence of Hilbert spaces. Define $H = \sum_{n=1}^\infty H_n$ to be the set of all sequences $(x_1, x_2, \ldots)$ $x_j \in H_j \ni \sum_{j=1}^\infty \|x_j\|^2 < \infty$. For $x, y \in H$ define $(x, y) = \sum_{j=1}^\infty (x_j, y_j)$. The sum converges absolutely since

$$\sum |(x_j, y_j)| \le \sum \|x_j\| \, \|y_j\| \le \sum_{j=1}^\infty \frac{\|x_j\|^2 + \|y_j\|^2}{2} < \infty.$$

Then H is a Hilbert space in this inner product. (Proof — straightforward — completeness same as for ℓ^2.) H is called the *(exterior) direct sum* of the H_n. Each H_n may be naturally identified with the subspace of H consisting of all $(0, 0, \ldots, x, 0, \ldots)$, $x \in H_n$, with x in the nth place.

If H is a given Hilbert space and $\{H_n\}$ is a sequence of mutually orthogonal subspaces such that the set of all $\sum_{j=1}^n x_j$, $x_j \in H_j$, are dense in H, then every x in H can be uniquely written in the form $\sum_{j=1}^\infty x_j$, $x_j \in H_j$, with $\|x\|^2 = \sum \|x_j\|^2$, and every such sum determines a vector in H (when $\sum \|x_j\|^2 < \infty$). In this case H is called the *(interior) direct sum* of the subspaces H_n and we write $H = \sum_{n=1}^\infty \oplus H_n$.

Proposition 4.17. *Let H_n be a sequence of Hilbert spaces. Let $H = \sum_{n=1}^\infty H_n$. Let $A_n : H_n \to H_n$ be a bounded operator for each n. Define $A : H \to H$ by $A(x_1, x_2, \ldots) = (A_1 x_1, A_2 x_2, \ldots)$ with*

$$\mathcal{D}_A = \{x \in H : \sum_{j=1}^\infty \|A_j x_j\|^2 < \infty\}.$$

Note that $\mathcal{D}_A$ is a dense subspace of H because it contains all finitely non–zero sequences. Define $B(x_1, x_2, \ldots) = (A_1^* x_1, A_2^* x_2, \ldots)$ with

$$\mathcal{D}_B = \{x \in H : \sum_{j=1}^\infty \|A_j^* x_j\|^2 < \infty\}.$$

Then $A^* = B$. Moreover, the set of finitely nonzero sequences in H is a core for A and A is closed.

Proof. Clearly A takes $\mathcal{D}_A$ into H. If $x \in \mathcal{D}_A$ and $y \in \mathcal{D}_B$ then

$$(Ax, y) = \sum_{j=1}^\infty (A_j x_j, y_j) = \sum_{j=1}^\infty (x_j, A_j^* y_j) = (x, By).$$

Hence the map $x \to (Ax, y)$ is continuous on $\mathcal{D}_A$ and $y \in \mathcal{D}_{A^*}$, and $A^* y = By$. Thus $A^* \supset B$. Conversely, suppose that $z \in \mathcal{D}_{A^*}$. Then there exists $z^* \in H$ such that $(Ax, z) = (x, z^*)$, for all $x \in \mathcal{D}_A$. Then for all $x \in H_n$ we have:

$$(x, (A_n)^* z_n) = (A_n x, z_n) = (Ax, z_n) = (Ax, z) = (x, z^*) = (x, (z^*)_n).$$

Hence $(A_n)^* z_n = (z^*)_n$. Thus

$$\sum_{n=1}^\infty \|(A_n)^* z_n\|^2 = \sum_{n=1}^\infty \|(z^*)_n\|^2 = \|z^*\|^2 < \infty.$$

This shows that $z \in \mathcal{D}_B$ and $z^* = Bz$. Hence $A^* \subset B$. Thus $A^* = B$. Finally, if $x \in \mathcal{D}_A$, $x = (x_1, x_2, \ldots)$, then the element $z^j = (x_1, \ldots, x_j, 0, 0, \ldots)$ is a finitely nonzero sequence and $z^j \to x$ as $j \to \infty$, while $Az^j \to Ax$, in view of the definition of $\mathcal{D}_A$. Hence the finitely nonzero sequences form a core for A. Since adjoints are closed B is closed. A similar argument with A_n and A_n^* interchanged shows that $B^* = A$. So A is closed. ∎

Definition 4.18. The operator A in the proposition is called the *direct sum of the operators* A_n. Notation: $A = \sum_{n=1}^{\infty} \oplus A_n$

Corollary 4.19. *If, in the preceding proposition, each A_n is Hermitian then A is self–adjoint. Moreover it is the only self–adjoint operator whose domain contains each H_n and agrees with A_n there.*

Proof. $B = A$. Hence $A^* = A$. If C is another self–adjoint operator with $\mathcal{D}_C \supset H_n$ $\forall n$ and $C = A_n$ on H_n, then $C = A$ on the finitely nonzero sequences. Since this set is a core for A, it follows that $C \supset A$. E.g., if $x \in \mathcal{D}_A$ and z^j is as in the proof of the proposition, then $Cz^j = Az^j$ converges, as does z^j. Hence, since C is closed $x \in \mathcal{D}_C$ and $Cx = \lim Cz^j = \lim Az^j = Ax$. But a self–adjoint operator can never have a proper self–adjoint extension. For $C \supset A \Rightarrow C^* \subset A^*$, i.e., $C \subset A$, therefore $C = A$. ∎

Examples 4.20. Let (X, μ) be a measure space. Let f be a complex valued measurable function on X. Let $\mathcal{D} = \{g \in L^2(\mu) : fg \in L^2(\mu)\}$. Define $M_f g = fg$ for $g \in \mathcal{D}$. Then M_f is densely defined. $\mathcal{D}$ is its "natural domain." We shall always understand the domain of a multiplication operator to be its natural domain. Of course if f is bounded then its domain obviously is all of $L^2(\mu)$.

Corollary 4.21. $(M_f)^* = M_{\overline{f}}$. *If f is real then M_f is self–adjoint.*

Proof. This is of interest primarily if f is unbounded because we have already proved it if f is bounded. Let $E_n = \{x \in X : n - 1 \le |f(x)| < n\}$ for $n = 1, 2, \ldots$. Let H_n be the set of functions in $L^2(\mu)$ which are zero off E_n. Then clearly $L^2(\mu) = \sum_{n=1}^{\infty} \oplus H_n$. Moreover the restriction of M_f to H_n is bounded (with norm at most n) and the domains of M_f and $M_{\overline{f}}$ are precisely those described in the proposition for A and B. Hence $M_f^* = M_{\overline{f}}$. ∎

Exercise 4.5. In the notation of the proposition, prove that $\|A\| = \sup_n \|A_n\|$ if this is finite, and otherwise A is unbounded. That is, a direct sum of bounded operators is bounded iff they are uniformly bounded.

Exercise 4.6. Let H be a Hilbert space and A a closed symmetric operator on H. Suppose that there exists an increasing sequence of closed subspaces K_n of H, each of which is contained in $\mathcal{D}_A$ and is invariant under A, i.e., $AK_n \subset K_n$. Suppose moreover that $\bigcup_{n=1}^{\infty} K_n$ is a core for A. Prove that A is self–adjoint.

4.2. **Differential operators.**

One of the most important applications of the spectral theorem for self-adjoint operators is to the analysis of differential operators. In this section we are going to study examples of symmetric differential operators, some of which are actually self-adjoint and (for good reason) some are not. We are going to focus on techniques that are used daily in the study of differential operators.

Smoothing by convolution

Definition 4.22. For two real or complex valued measurable functions f and g on $\mathbb{R}^d$ their convolution is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy \tag{4.1}$$

when the integral exists.

In particular, if $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ then the integral exists for almost all x and

$$\|f * g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^p(\mathbb{R}^d)}, \quad 1 \leq p \leq \infty. \tag{4.2}$$

[Ref. 611. E.g. Rudin #2 Ch. 8, Problem 4.]

But the convolution may also exist in other circumstances. We say that a function g is in $L^p_{loc}(\mathbb{R}^d)$ if

$$\int_K |g(x)|^p dx < \infty \quad \text{for every compact set } K \subset \mathbb{R}^d. \tag{4.3}$$

For example the function $e^{|x|^2}$ is not in any L^p space but is in L^p_{loc} for all $p \in [1, \infty]$. Now suppose that $f \in C_c(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$. Then $(f * g)(x)$ exists for every $x \in \mathbb{R}^d$ because the integral in (4.1) is really just an integral over the compact set $\{y \in \mathbb{R}^d : x - y \in \text{supt } f\}$. The integrand is a bounded function, $(y \mapsto f(x-y))$, times a function which is integrable over this set.

Exercise 4.7. Prove the following lemma.

Lemma 4.23. *Suppose that $f \in C_c(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$. Then $f*g$ is continuous on $\mathbb{R}^d$.*

Hint: Choose a compact set $K \subset \mathbb{R}^d$ which contains all points within one inch of supt f. Let $x - K = \{x - z : z \in K\}$ (which equals $\{y : x - y \in K\}$). Then

$$(f * g)(x) = \int_{x_0 - K} f(x-y)g(y)dy \tag{4.4}$$

if $|x - x_0| \leq 1$.

Now that you've mastered this technique of proof, here is another lemma with almost the same proof.

Lemma 4.24. *Suppose that $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$. Then $f * g \in C^\infty(\mathbb{R}^d)$.*

Proof. Write $\partial_j = \partial/\partial x_j$ and $f_j = \partial_j f$. To prove the lemma all we need to do is to justify the identity

$$\partial_j(f * g)(x) = \int_{\mathbb{R}^d} (\partial_j f)(x-y)g(y)dy \tag{4.5}$$

because $\partial_j f$ is again in C_c^∞. Choose K as in the previous hint and write the difference quotient as

$$\begin{aligned}&\frac{(f*g)(x_0+he_j)-(f*g)(x_0)}{h}-(f_j*g)(x_0)\\&=\int_{x_0-K}\Big\{\frac{f(x_0+he_j-y)-f(x_0-y)}{h}-f_j(x_0-y)\Big\}g(y)dy\end{aligned}$$

which is valid if $|h|<1$. The integrand clearly goes to zero pointwise as $h\to 0$. Thus to maintain a clear conscience you need only verify for yourself that the factor in braces remains uniformly bounded as h goes to zero. You might try using the fundamental theorem of calculus or (if f is real valued) the mean value theorem, according to taste. ∎

Lemma 4.25. *Choose any function $\phi\in C_c^\infty(\mathbb{R}^d)$ such that*
a. $\phi\ge 0$ on $\mathbb{R}^d$.
b. $\int_{\mathbb{R}^d}\phi(x)dx=1$
c. $\phi(x)=0$ if $|x|\ge 1$.
Let $\phi_k(x)=k^n\phi(kx)$.
d. Then $\int_{\mathbb{R}^d}\phi_k(x)dx=1$ and $\phi_k(x)=0$ if $|x|\ge(1/k)$.
Let $g\in L^p(\mathbb{R}^d)$ for some $p\in[1,\infty)$. Then $\|\phi_k*g-g\|_{L^p(\mathbb{R}^d)}\to 0$ as $k\to\infty$.

Proof. Given $g\in L^p(\mathbb{R}^d)$ and $\epsilon>0$ there exists a function $h\in C_c(\mathbb{R}^d)$ such that $\|g-h\|_{L^p(R^d)}<\epsilon$. Then

$$\begin{aligned}\|\phi_k*g-g\|_{L^p}&\le\|\phi_k*(g-h)\|_{L^p}+\|\phi_k*h-h\|_{L^p}+\|h-g\|_{L^p}\\&\le\epsilon+\|\phi_k*h-h\|+\epsilon\end{aligned}$$

But

$$\begin{aligned}|(\phi_k*h)(x)-h(x)|&=|\int_{\mathbb{R}^d}\phi_k(x-y)(h(y)-h(x)dy|\\&\le\sup_{\{y:|x-y|\le(1/k)\}}|h(y)-h(x)|\end{aligned}$$

by d. Since h is continuous and has compact support the right side of the last line goes to zero uniformly in x as $k\to\infty$ and, for all $k\ge 1$, is zero on the compact set of points which are more than one inch from the support of h. Hence the right side of the last inequality goes to zero in L^p. ∎

The following corollary now follows from Lemmas 4.24 and 4.25.

Corollary 4.26. *$C^\infty\cap L^p(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ if $1\le p<\infty$.*

This corollary was based on the use of convolution. In the next theorem we will go a step further. The proof will be based on using truncation in addition to this corollary.

Theorem 4.27. *C_c^∞ is dense in $L^p(\mathbb{R}^d)$ for $1\le p<\infty$.*

Proof. Choose a function $\psi\in C_c^\infty(\mathbb{R}^d)$ such that
e. $\psi(x)=1$ for $|x|\le 1$.
Let $\psi_k(x)=\psi(x/k)$. Then the functions ψ_k are uniformly bounded and $\psi_k(x)\to 1$ for all x. Hence if $f\in C^\infty\cap L^p$ then $\psi_k f$ is in $C_c^\infty(\mathbb{R}^d)$ for all k and converges to f in the L^p sense (by dominated convergence.) Thus $C_c^\infty(\mathbb{R}^d)$ is dense in $C^\infty\cap L^p(\mathbb{R}^d)$ (in L^p norm) which, by Corollary 4.26, is dense in L^p. ∎

4.3. **The Laplacian over $\mathbb{R}^d$.**

Definition 4.28. The *Laplacian* is the second order differential operator defined by

$$\Delta f = \sum_{j=1}^{d} \partial^2 f/\partial x_j^2 \quad \text{for real or complex valued functions } f \text{ on } \mathbb{R}^d \tag{4.6}$$

As to what sense of differentiability we need to impose on f we will leave up in the air for the moment. But we will use the symbol Δf only when f is twice continuously differentiable. The issue for us will be how to extend or restrict the domain of this operator so as to get a self-adjoint operator in $L^2(\mathbb{R}^d)$.

> The Laplace operator in its various manifestations is the most beautiful and central object in all of mathematics. Probability theory, mathematical physics, Fourier analysis, partial differential equations, the theory of Lie groups, and differential geometry all revolve around this sun, and its light even penetrates such obscure regions as number theory and algebraic geomery.
>
> Edward Nelson, Tensor Analysis

Theorem 4.29. *Define*

$$Tf = \Delta f \quad \textit{for} \quad f \in C_c^\infty \tag{4.7}$$

Then

a. T is symmetric in $L^2(\mathbb{R}^d)$ and
b. Its closure, $\bar{T}$, is self-adjoint.

Proof. Two integrations by parts shows that $(Tf, g) = (f, Tg)$ for f and g in $C_c^\infty(\mathbb{R}^d)$. So T is symmetric. Its closure, $\bar{T}$, is therefore also symmetric. Since $(\bar{T})^* = T^*$ we need to show that $\bar{T} = T^*$ in order to show that $\bar{T}$ is self-adjoint. Since $T \subset T^*$ we already know that $\bar{T} \subset T^*$. So we only need to show the reverse containment. In words: we need to show that the closure of T is T^*. More explicitly: we need to show that for any function $f \in D(T^*)$ there is a sequence $f_n \in C_c^\infty$ such that f_n converges to f in L^2 norm while at the same time Tf_n converges to T^*f in L^2 norm. We are going to do this in two steps, analogous to the proof of Theorem 4.27. ∎

Remark 4.30. The definition (4.6) makes perfectly good classical sense if $f \in C^2(\mathbb{R}^d)$. If $f \in C^2(\mathbb{R}^d) \cap D(T^*)$ then for any function $h \in C_c^\infty(\mathbb{R}^d)$, the definitions of T^* and T give $(T^*f, h) = (f, Th) = (f, \Delta h) = (\Delta f, h)$, wherein we just did two integrations by parts in the last step. Since this holds for all $h \in C_c^\infty(\mathbb{R}^d)$, it follows that

$$T^*f = \Delta f \quad \text{for} \quad f \in C^2(\mathbb{R}^d) \cap D(T^*). \tag{4.8}$$

So T^* and Δ are equal wherever both are defined. In fact, T^* includes in its domain some less regular functions than just C^2 functions, as we will come to understand better in one dimension. The technically complicated issues that we are going to face below are aimed at getting this domain just right, so that $\bar{T}$ will be self-adjoint.

Lemma 4.31. *$C^\infty(\mathbb{R}^d) \cap D(T^*)$ is a core for T^*.*

Proof. Choose a sequence ϕ_k as in the proof of Lemma 4.25. If $f \in D(T^*)$ and $f_k = \phi_k * f$ then we already know, by Lemma 4.25, that $f_k \in C^\infty \cap L^2$ and converges to f in L^2 norm. So we must show that $f_k \in D(T^*)$ and $T^* f_k$ converges to $T^* f$ in L^2 norm. Suppose that $h \in C_c^\infty(\mathbb{R}^d)$. Then

$$\begin{aligned}
(f_k, Th) &= (\phi_k * f, Th) \\
&= (f, \phi_k * Th) \quad \text{by Fubini's theorem} \\
&= (f, T(\phi_k * h)) \quad \text{by (4.5)} \\
&= (T^* f, \phi_k * h) \quad \text{because} \quad \phi_k * h \in C_c^\infty \\
&= (\phi_k * (T^* f), h)
\end{aligned}$$

Since $\phi_k * (T^* f) \in L^2$ the right side of the last equality is a continuous function of h in L^2 norm. Hence (f_k, Th) is a continuous function of h in L^2 norm. Therefore $f_k \in D(T^*)$ and we may now write the last identity as $(T^* f_k, h) = (\phi_k * (T^* f), h)$. Since h is an arbitrary function in $C_c^\infty(\mathbb{R}^n)$ it follows that

$$T^* f_k = \phi_k * (T^* f). \tag{4.9}$$

In view of Lemma 4.25 we now see that $T^* f_k$ converges to $T^* f$ in L^2 norm. This proves the lemma. It's pleasing to note that (4.9) simply says that convolution by ϕ_k commutes with T^*. This is pleasing because we already know that convolution commutes with differentiation, and now we see that this holds even when all these delicate domain considerations are thrown in. ■

Next, we are going to carry out the analog of the truncation step. But first we will have to do a lot of work to justify a seemingly obvious integration by parts when we are dealing with functions that don't have compact support.

Lemma 4.32. *Suppose that $f \in C^\infty(\mathbb{R}^d) \cap D(T^*)$. Then*

$$(-\Delta f, f) = \sum_{j=1}^{d} \int_{R^d} (\partial_j f)^2 dx \tag{4.10}$$

Proof. Of course the identity (4.10) would follow immediately from an integration by parts if we could ignore the boundary terms. The fact that we will have to do a lot of work to justify ignoring the boundary terms should be a lesson to you!

Choose $\psi \in C_c^\infty(\mathbb{R}^d)$ as in the proof of Theorem 4.27, but this time we will need to choose ψ a little more carefully. Choose it so that it is decreasing as one moves radially outward from the origin. For example one could construct ψ in the form $\psi(x) = u(|x|)$ where $u : [0, \infty) \to [0, \infty)$ is ≥ 0 everywhere, has compact support on $[0, \infty)$, is equal to one on $[0, 1]$ and is decreasing thereafter. (Draw a picture of u.) Let $\psi_k(x) = \psi(x/k)$. Then $0 \leq \psi_k \in C_c^\infty(\mathbb{R}^d)$ and $\psi_k(x)$ *increases* to one for all x as $k \to \infty$. Integration by parts in the following identities is valid because each integrand has compact support.

$$\begin{aligned}\sum_{j=1}^{d}\int_{\mathbb{R}^d}\psi_k(x)((\partial_j f)(x))^2dx &= -\sum_{j=1}^{d}\int_{\mathbb{R}^d}\{(\partial_j\psi_k)\partial_j f+\psi_k(\partial_j^2 f)\}f dx\\ &= \sum_{j=1}^{d}\int_{\mathbb{R}^d}(\partial_j\psi_k)\partial_j f^2dx-(\psi_k\Delta f,f)\\ &= \int_{\mathbb{R}^d}(\Delta\psi_k)f^2dx-(\psi_k\Delta f,f)\end{aligned}$$

Now since $0\le\psi_k(x)\uparrow 1$ we may apply the monotone convergence theorem to the left side to find the limit $\int_{\mathbb{R}^d}\sum(\partial_j f(x)))^2dx$ (which may be infinite as far as we know right now.) Since $(\Delta f)f\in L^1(\mathbb{R}^d)$ we may apply the dominated convergence theorem to the last term on the right to find $\lim_{k\to\infty}(\psi_k\Delta f,f)=(\Delta f,f)$. Finally, we may compute, by the chain rule, $\Delta\psi_k(x)=(1/k^2)(\Delta\psi)(x/k)$, which goes to zero uniformly in x as $k\to\infty$. Hence $\int_{R^d}(\Delta\psi_k)f^2dx\to 0$ by DCT again. This proves (4.10) and in particular shows that the right side of (4.10) is finite after all.
∎

Question Why did we need to choose ψ in this lemma so that ψ_k is increasing, but not in the proof of Theorem 4.27?

Proof of Theorem 4.29. In view of Lemma 4.32 it suffices to show that if $f\in C^\infty(\mathbb{R}^d)\cap D(T^*)$ then there exists a sequence $f_n\in C_c^\infty(\mathbb{R}^d)$ such that f_n converges to f in T^* graph norm. To this end define ψ_n as in the proof of Theorem 4.27 and define $f_n(x)=\psi_n(x)f(x)$. Clearly $f_n\in C_c^\infty$ and f_n converges to f in L^2 norm. So it remains only to prove that Δf_n converges to Δf in L^2 norm also. (Be mindful of (4.8).) But

$$\Delta f_n=(\Delta\psi_n)f+\psi_n\Delta f+2\sum_{j=1}^{d}(\partial_j\psi_n)(\partial_j f)$$

Since $\Delta\psi_n\to 0$ uniformly and $f\in L^2$ the first term goes to zero in L^2 norm by DCT. Since $\psi_n\to 1$ boundedly the second term converges to Δf in L^2 norm by DCT also. Finally, since $\partial_j\psi_n(x)=(1/n)(\partial_j\psi)(x/n)$, which goes to zero uniformly on $\mathbb{R}^d$, while each term $\partial_j f\in L^2(\mathbb{R}^d)$ by Lemma 4.32, the last term goes to zero in L^2 norm. ∎

Remark 4.33. There is an important alternative way to prove Theorem 4.29, using the Fourier transform. We will sketch this later in Exercise 4.10.

4.4. **The Laplacian in one dimension.**

In one dimension the Laplacian of a function f is just f'', if you pay attention to Definition 4.28. But we are now dealing with the Laplacian as a closed operator in a Hilbert space and the domain is not entirely up to us to choose, if we wish the operator to be closed, not to mention self-adjoint. If f is a function in $D(T^*)$ then f doesn't change as an element of $L^2(\mathbb{R})$ if we change it on a set of measure zero. And yet such a change can make a smooth function into a nowhere differentiable function. We are going to show that by changing f on a set of measure zero (if necessary) we can make any element f in $D(T^*)$ into a C^1 function such that f' is absolutely continuous on $\mathbb{R}$ while f'', which exists a.e., is in $L^2(R)$. This gives an explicit characterization of the domain of d^2/dx^2 as a self-adjoint operator in $L^2(\mathbb{R})$.

Theorem 4.34. *Suppose that $d = 1$ and $h \in D(T^*)$. Then there exists a unique continuous function u such that*

a. $u = h$ a.e.

Moreover

b. $u \in C^1(\mathbb{R})$,

c. u' is absolutely continuous,

d. $u'' \in L^2(\mathbb{R})$ and $T^*h = u''$.

Proof. Choose $\phi \in C_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi(x)dx = 1$. For any function $f \in C^\infty(\mathbb{R})$ observe the identity

$$f'(x) = f'(t) + \int_t^x f''(s)ds. \tag{4.11}$$

Multiply this identity by $\phi(t)$ and integrate w.r.t. t. Do an integration by parts on the first term to find

$$f'(x) = -\int_{\mathbb{R}} \phi'(t)f(t)dt + \int_{\mathbb{R}} \phi(t) \int_t^x f''(s)dsdt. \tag{4.12}$$

Let $w_x(s) = \int_{\mathbb{R}} \phi(t)\chi_{[t,x]}(s)dt$ where $\chi_{[t,x]}$ means $-\chi_{[x,t]}$ if $t > x$. For fixed x, $w_x(s)$ is a bounded function of s with compact support. As a function of x the map $x \mapsto w_x$ is a continuous function into $L^2(\mathbb{R})$. (Check these statements now!) Then (4.12) may be written

$$f'(x) = -(\phi', f) + (w_x, f''). \tag{4.13}$$

This equation expresses information about f' *at a point* in terms of inner products. Suppose now that $h \in D(T^*)$. By Lemma 4.31 there exists a sequence $f_n \in C^\infty \cap L^2$ such that f_n converges to h in L^2 norm while f_n'' converges to $g \equiv T^*h$ in L^2 norm. Replace f by f_n in (4.13) and let $n \to \infty$ to find

$$\lim_{n\to\infty} f_n'(x) = -(\phi', h) + (w_x, g). \tag{4.14}$$

Denote by $v(x)$ the function on the right. Then v is continuous and locally bounded. Now repeat this argument starting with the identity

$$f_n(x) = f_n(t) + \int_t^x f_n'(s)ds \tag{4.15}$$

instead of (4.11). We find

$$f_n(x) = \int_{\mathbb{R}} \phi(t)f_n(t)dt + (w_x, f_n'). \tag{4.16}$$

Thus

$$\lim f_n(x) = (\phi, h) + (w_x, v) \tag{4.17}$$

for each x. But some subsequence f_{n_k} converges a.e. to h. Hence the continuous function

$$u(x) \equiv (\phi, h) + (w_x, v)$$

is equal to $h(x)$ a.e. We have now shown that $f_n(x)$ converges to $u(x)$ while f_n' converges to v boundedly on every finite interval. Taking the limit now in (4.15) we deduce that

$$u(x) = u(t) + \int_t^x v(s)ds \tag{4.18}$$

for all x and t. Since v is continuous the fundamental theorem of calculus now shows that $u' = v$ and therefore $u \in C^1(\mathbb{R})$. Going back a step further, to (4.11), with f replaced by f_n we find, upon letting $n \to \infty$

$$v(x) = v(t) + \int_t^x g(s)ds \tag{4.19}$$

Since $L^2_{loc} \subset L^1_{loc}$ we see that v is absolutely continuous and $v'(x) = g(x)$ a.e. This completes the proof of the theorem. ∎

4.5. **A non-self-adjoint version of the Laplacian.**

So far we have been proving only self-adjointness of the Laplacian on the entire real line. But when a boundary is present anything can happen. Here is an example of the Laplacian on a half line wherein self-adjointness fails.

Example 4.35. A closed symmetric operator which is not self–adjoint. Let $H = L^2(0,\infty)$, $D = C_c^\infty((0,\infty))$. Define $Tf = f''$ for f in D. Then T is densely defined. If $f, g \in D$, then

$$(Tf, g) = \int_0^\infty f''\overline{g}dx = -\int_0^\infty f'\overline{g'}dx = \int_0^\infty f\overline{g''}dx = (f, Tg).$$

Therefore $T \subset T^*$. Hence T^* is densely defined and T closure exists. Let $\bar{T}$ be the closure of T. Then $\bar{T}^* = (T^{**})^* = T^*$. But $T^* \supset T$. Thus $\bar{T}^* \supset \bar{T}$, i.e., $\bar{T}$ is symmetric.

Claim: $\bar{T}$ is not self–adjoint.

Proof. Let $f \in D(\bar{T})$ and let $g = \bar{T}f$. Then there exists $f_n \in D(T) \ni f_n \to f$ in L^2 and $Tf_n \to g$. Now,

$$\begin{aligned} f_n(x) &= \int_0^x f_n'(t)dt = \int_0^x \Big(\int_0^t f_n''(s)ds\Big)dt = \int_0^x \int_0^x \chi_{[0,t]}(s) f_n''(s)dsdt \\ &= \int_0^x \Big(\int_0^x \chi_{[0,t]}(s)f_n''(s)dt\Big)ds = \int_0^x (x-s)f_n''(s)ds \to \int_0^x (x-s)g(s)ds. \end{aligned}$$

Therefore

$$f(x) = \int_0^x (x-s)g(s)ds \quad \text{a.e.} \tag{4.20}$$

Since f is determined only up to a set of measure zero, we may assume (4.20) holds for all x by modifying f on a set of measure zero. Thus f is absolutely continuous on $[0,\infty)$ since the integrand in(4.20) is in L^1 locally. Clearly $f(0) = \lim_{x\to 0} f(x) = 0$. Moreover

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\Big(\int_0^{x+h} (x+h-s)g(s)ds - \int_0^x (x-s)g(s)ds\Big) \\ &= \frac{1}{h}\Big(\int_x^{x+h} (x+h-s)g(s)ds + h\int_0^x g(s)ds\Big) \end{aligned}$$

but

$$\Big|\frac{1}{h}\int_x^{x+h} (x+h-s)g(s)ds\Big| \le \int_x^{x+h} |g(s)|ds \to 0 \quad \text{as} \quad h \to 0.$$

Therefore $f'(x) = \int_0^x g(s)ds$ everywhere. Thus f' is absolutely continuous on $[0,\infty)$ and $f'(0) = 0$. Hence if $f \in D(\bar{T})$ then f and f' are absolutely continuous, $f'' = g \in L^2$, and $Tf = f''$. (This is a slightly different derivation of these statements from the full line case.) But also $f(0) = f'(0) = 0$. So suppose φ is an arbitrary function in $C^2([0,\infty))$ with compact support in $[0,\infty)$. Then $\forall f \in D(\bar{T})$; $(\bar{T}f,\varphi) = (f,\varphi'')$, by two integrtions by parts, using $f(0) = f'(0) = 0$. Therefore $\varphi \in D(\bar{T}^*)$. But we may take $\varphi(0) \neq 0$. Then $\varphi \notin D(\bar{T})$. Hence $\bar{T} \subsetneqq \bar{T}^*$. That is, $\bar{T}$ is not self-adjoint. ∎

Moral. In this example the domain of T and hence of $\bar{T}$ was too small, giving us a symmetric but not self-adjoint version of d^2/dx^2. Why is there such a difference between this half-line case and the full-line case? Answer: The domain of $\bar{T}$ is so small because the two conditions $f(0) = 0$ and $f'(0) = 0$ were forced by our choice of domain of T, namely, $C_c^\infty((0,\infty))$. These functions are zero in an entire neighborhood of zero. And when we take the closure of the operator there is still a remnant of this character in the form $f(0) = f'(0) = 0$, as we saw. But if we enlarge the domain of T by dropping one of these two conditions will its closure be self-adjoint? Answer: Yes. The next theorem contains a precise statement. It will be important to understand that "too small" refers to too many boundary conditions rather than too much regularity. To this end we will assume the validity of the following exercise.

Exercise 4.8. Define $Tf = f''$ with domain $D(T) = C_c^\infty((0,\infty))$, as in the previous example. So T is the "minimal" operator. Prove that $g \in D(T^*)$ if and only if

a. $g \in L^2((0,\infty))$,
b. $g \in C^1([0,\infty))$ after modification on a set of measure zero, and
c. g' is absolutely continuous and $g'' \in L^2((0,\infty))$.

Hint: Most of the proof will be the same as the proof of Theorem 4.34. However when convoluting near the left endpoint it would be wise to use a function $\phi(x)$ which is supported in $(-\infty,0]$.

Theorem 4.36. *Define $Sf = f''$ with $D(S) = \{f \in C_c^\infty([0,\infty) : f(0) = 0\}$. Then S is symmetric and $\bar{S}$ is self-adjoint.*

Proof. Suppose that $f \in C_c^\infty([0,\infty))$ and that g and g'' are in $L^2([0,\infty))$. The second condition should be interpreted, as usual, to mean that $g \in C^1([0,\infty))$ while g' is absolutely continuous and $g'' \in L^2$. We need the following integration by parts identity, which is so important that I'm going to derive it for you.

$$(f'',g) = (f,g'') + f(0)g'(0) - f'(0)g(0) \tag{4.21}$$

Proof of (4.21):

$$\begin{aligned}\int_0^\infty f''(x)g(x)dx &= -\int_0^\infty f'(x)g'(x)dx - f'(0)g(0)\\ &= \int_0^\infty f(x)g''(x)dx + f(0)g'(0) - f'(0)g(0)\end{aligned}$$

We have made no assumption so far about f or g at $x = 0$. Now let us assume that $f(0) = 0$. This puts f into $D(S)$ and we may then rewrite (4.21) as

$$(Sf,g) = (f,g'') - f'(0)g(0) \tag{4.22}$$

If g were in $D(S)$ then we would have $g(0) = 0$ and this identity therefore shows that S is symmetric, as claimed. Returning to the supposition at the beginning of this proof observe that according to the definition 4.6 the function g will be in $D(S^*)$ only if (Sf,g) is a continuous function of f in the L^2 norm. But the first term on the right of (4.22) is a continuous linear functional of f in L^2 norm because $g'' \in L^2([0,\infty))$. Therefore g will be in the domain of S^* only if the map $f \mapsto f'(0)g(0)$ is continuous in L^2 norm. But this is impossible unless the coefficient $g(0)$ is zero. (See exercise below.) Hence, in order for g to be in $D(S^*)$ it is necessary that $g(0) = 0$. That is, if $g \in D(S^*)$ then $g \in D(\bar{S})$. Since we already know that $\bar{S} \subset S^*$ we have now shown that $\bar{S} = S^*$. ∎

Exercise 4.9. Show that the linear maps $C_c^\infty([0,\infty)) \ni f \mapsto f(0)$ and $C_c^\infty([0,\infty)) \ni f \mapsto f'(0)$ are both discontinuous when the L^2 norm is used on the domain and therefore have no continuous extensions to L^2.

4.6. **von Neumann's criteria for self-adjointness.**

As in the case of bounded Hermitian operators we shall prove two forms of the spectral theorem for self–adjoint operators. Our proof depends on the following *basic criterion for self–adjointness.* (The organization of the next theorem and its corollary is taken from Reed and Simon, vol. 1.)

Theorem 4.37. *Let T be a symmetric operator on a Hilbert space H. Then the following three statements are equivalent:*

(a) *T is self–adjoint*
(b) *T is closed and* $\ker(T^* + i) = \ker(T^* - i) = \{0\}$.
(c) *Range*$(T + i) =$ *Range*$(T - i) = H$.

Proof. We prove (a) ⇒ (b) ⇒ (c) ⇒ (a).

Assume (a). Since $T = T^*$, T is closed. If $\varphi \in \ker(T^* - i)$ then $i(\varphi,\varphi) = (i\varphi,\varphi) = (T^*\varphi,\varphi) = (T\varphi,\varphi) = (\varphi, T^*\varphi) = (\varphi, i\varphi) = -i(\varphi,\varphi)$. Therefore $\|\varphi\|^2 = 0$ and $\varphi = 0$. A similar proof shows that $\ker(T^* + i) = \{0\}$.

Assume (b). If ψ were orthogonal to Range$(T + i)$ then $((T + i)\varphi, \psi) = 0$, $\forall \varphi \in \mathcal{D}_T$. Hence $\psi \in D_{T^*}$ and $(\varphi, (T^* - i)\psi) = 0$ $\forall \varphi \in \mathcal{D}_T$. Since $\mathcal{D}_T$ is dense $(T^* - i)\psi = 0$. But by (b) $\psi = 0$. Hence Range$(T + i)$ is dense in H. But for any $\varphi \in \mathcal{D}_T$,

$$\|(T + i)\varphi\|^2 = ((T + i)\varphi, (T + i)\varphi) = \|T\varphi\|^2 + \|\varphi\|^2 \geq \|\varphi\|^2. \tag{4.23}$$

Hence if $\psi \in H$ is arbitrary $\exists\ \varphi_n \in \mathcal{D}_T \ni (T + i)\varphi_n \to \psi$ and the last inequality shows that then φ_n is Cauchy. Hence $\exists \varphi \in H$ such that $\varphi_n \to \varphi$. But $T + i$ is closed. Hence $\varphi \in \mathcal{D}_T$ and $(T + i)\varphi = \psi$. Therefore Range$(T + i) = H$. The proof that Range$(T - i) = H$ is similar.

Assume (c). To show $T^* = T$ it suffices to show $\mathcal{D}_{T^*} \subset \mathcal{D}_T$ since T is symmetric. Let $\varphi \in \mathcal{D}_{T^*}$. Since Range$(T - i) = H$, $\exists\ y \in \mathcal{D}_T$ such that $(T - i)y = (T^* - i)\varphi$.

Since $\mathcal{D}_T \subset \mathcal{D}_{T^*}$, $\varphi - y \in \mathcal{D}_{T^*}$ and $(T^* - i)(\varphi - y) = 0$. But Range$(T + i) = H$ implies $\ker(T^* - i) = \{0\}$. Hence $\varphi - y = 0$. That is, $\varphi \in \mathcal{D}_T$. ∎

Definition 4.38. An operator T is called essentially self–adjoint if its closure, $\overline{T}$, exists and is self–adjoint.

Corollary 4.39. *Let T be a symmetric operator on a Hilbert space. The following are equivalent:*

(a) *T is essentially self–adjoint.*
(b) $\ker(T^* + i) = \ker(T^* - i) = \{0\}$.
(c) *Range*$(T + i)$ *and Range*$(T - i)$ *are dense in H.*

Proof. Since $T^* = \overline{T}^*$ (a) and (b) are equivalent by the theorem. For any symmetric operator T, the inequality (4.23) in the proof of the theorem, together with the definition of closure of an operator, imply that $\overline{\text{Range}(T+i)} = \text{Range}(\overline{T}+i)$. Similarly $\overline{\text{Range}(T-i)} = \text{Range}(\overline{T}-i)$. Hence condition (c) of the corollary is equivalent to condition (c) of the theorem for $\overline{T}$. ∎

Remark 4.40. If T is a closed symmetric operator then the inequality (4.23) in the proof of the theorem holds and the argument following it shows that the subspaces $K_\pm = \text{Range}(T \pm i)$ are closed. Of course T is self–adjoint if and only if $K_+ = K_- = H$. But if this fails then the numbers $m_\pm = \dim K_\pm^\perp$ are measures of the deviation of T from self–adjointness. The cardinal numbers $m_\pm$ are called the deficiency indices of T. It is a theorem that if $m_+ \neq m_-$ then T has no self–adjoint extensions. If $m_+ = m_- \neq 0$ then T has many self–adjoint extensions. Reference: Riesz and Nagy "Functional Analysis," Section 123.

Exercise 4.10. Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ as in (1) of Example 1.7. Define $L : \mathcal{S} \to \mathcal{S}$ by

$$Lf = -\frac{d^2 f}{dx^2}.$$

$\mathcal{S}$ is dense in $L^2(\mathbb{R})$. Regard L as a densely defined linear transformation in $L^2(\mathbb{R})$.

a) Show that L has a closed linear extension.

b) Show that the closure T of L is self–adjoint by using Corollary 4.39 and the Fourier transform. This gives a different proof from that of Theorem 4.29.

Definition 4.41. For any (possibly unbounded) linear operator $T : H \to H$, a complex number λ is said to be in the *resolvent set* of T if $T - \lambda I$ is one to one and onto and $(T - \lambda I)^{-1}$ is bounded. Otherwise λ is said to be in the *spectrum* of T.

Exercise 4.11. a) Restrict the operator L of the previous problem to the set $\mathcal{D}$ consisting of those f which vanish in a neighborhood of 0. (The neighborhood depends on f.) Call the restriction A. Prove that A is densely defined and symmetric but is not essentially self–adjoint. **Hint:** Let $\varphi =$ Fourier transform of $\frac{1}{t^2-i}$ and show that $A^*\varphi = i\varphi$. b) Find the spectrum of A^*.

Exercise 4.12. Suppose that A is a linear transformation in H and that A is densely defined, closed, one to one, and has dense range. Then clearly A^{-1} exists and is densely defined, and A^* and $(A^{-1})^*$ both exist. Prove

i) $\ker(A^*) = 0$

ii) Range A^* is dense in H

iii) $(A^{-1})^* = (A^*)^{-1}$ (which exists by i) and ii))

iv) A^{-1} is closed.

Exercise 4.13. Suppose that A is a bounded (everywhere defined) Hermitian operator on H which is one-to-one.

i) Show that its range $\mathcal{D}$ is dense in H and that

ii) $A^{-1} : \mathcal{D} \to H$ is self-adjoint.

4.7. The spectral theorem for unbounded self-adjoint operators.

Let $E(\cdot)$ be a projection valued measure on a Hilbert space H over a σ–field $\mathcal{S}$ of subsets of a set X. (cf. Definition n 3.28) For any vector u in H, the measure

$$B \to m_u(B) \equiv (E(B)u, u)$$

is a positive finite measure. If f is a bounded measurable function on X, then

$$\begin{aligned} \Big\|\Big(\int f dE\Big)u\Big\|^2 &= \Big(\int f dEu, \int f dEu\Big) \\ &= \Big(\Big(\int f dE\Big)^* \Big(\int f dE\Big)u, u\Big) \\ &= \Big(\Big(\int |f|^2 dE\Big)u, u\Big) = \int |f|^2 d(E(\cdot)u, u). \end{aligned} \tag{4.24}$$

thus

$$\Big\|\Big(\int f dE\Big)u\Big\|^2 = \int |f|^2 dm_u. \tag{4.25}$$

Hence the map $f \to (\int f dE)u$ defined for bounded measurable f on X extends uniquely to an isometry from $L^2(x, m_u)$ into H. We denote by $\int f dEu$ the value of this isometry for each f in $L^2(x, m_u)$.

Now let g be a fixed complex valued measurable function on X. We define an operator A on H as follows: $D_A = \{u \in H : g \in L^2(X, m_u)\}$ and on this domain, define $Au = \int g dEu$ and write $A = \int g dE$.

Proposition 4.42. *A is a closed operator and $A^* = \int \overline{g} dE$. If g is real, then A is self–adjoint.*

Proof. Let $B_n = \{x \in X : n-1 \leq |g(x)| < n\}$ for $n = 1, 2, \ldots$. Let $H_n =$ range $E(B_n)$. Since the B_n are disjoint and $\bigcup_1^\infty B_n = X$, we have $H = \sum_{n=1}^\infty \oplus H_n$. Let $A_n = \int g\chi_{B_n} dE$. Since $g\chi_{B_n}$ is a bounded function, A_n is a bounded operator and $E(B_n)A_n = A_n E(B_n) = \int g\chi_{B_n}\chi_{B_n} dE = \int g\chi_{B_n} dE = A_n$. Thus A_n leaves H_n invariant and is zero on the orthogonal complement of H_n. Moreover, $A_n^* = \int \overline{g}\chi_{B_n} dE$ also leaves H_n invariant and annihilates $H_n^\perp$. We may regard A_n as an operator defined in H_n and we consider the direct sum operator $\sum_{n=1}^\infty \oplus A_n$ defined in Proposition 4.17. If u is in H, then

$$\sum_{n=1}^\infty \|A_n u\|^2 = \sum_{n=1}^\infty \int |g\chi_n|^2 d(E(\cdot)u, u) = \sum_{n=1}^\infty \int |g|^2 \chi_n dm_u = \int |g|^2 dm_u.$$

Hence u is in the domain of $\sum_{n=1}^\infty \oplus A_n$ if and only if $u \in D_A$. Moreover if u is in D_A, then

$$Au = \int g dEu = \lim_{n\to\infty} \int g\chi_{\cup_{k=1}^n B_k} dEu$$

by the definition of A since

$$g\chi_{\cup_{k=1}^n B_n} \to g$$

in $L^2(m_u)$. Hence

$$Au = \lim_{n\to\infty} \sum_{k=1}^n \int g\chi_{B_k} dEu = \lim_{n\to\infty} \sum_{k=1}^n A_k E(B_k)u = \Big(\sum_{k=1}^\infty \oplus A_k\Big)u.$$

Thus A is a direct sum of bounded operators and is therefore closed by Proposition 4.17. Moreover $A^* = \sum_{k=1}^{\infty} \oplus A_k^*$ which, by what we have just shown, is equal to $\int \overline{g} dE$. Finally if g is real, then $A^* = \int \overline{g} dE = \int g dE = A$, so A is self–adjoint. ■

Theorem 4.43 (Spectral Theorem: multiplication operator form)**.** *Let T be a self–adjoint operator on a separable Hilbert space H. Then there exists a finite measure space (X, μ), a unitary operator $U : H \to L^2(X, \mu)$ and a real valued measurable function f on X such that*

$$UTU^{-1} = M_f. \tag{4.26}$$

Proof. By Theorem 4.37, $T + i$ is one to one and onto from $\mathcal{D}_T$ to H. Hence its inverse $(T + i)^{-1}$ exists. Moreover by (4.23) (page 41) $(T + i)^{-1}$ is bounded. Similarly $(T - i)^{-1}$ is also a bounded, everywhere defined operator. Moreover, by the preceding exercise $((T + i)^{-1})^* = (T - i)^{-1}$. Since $(T + i)$ commutes with $T - i$ (be careful with domains when verifying this) their inverses also commute. It follows that $(T + i)^{-1}$ is a normal (bounded) operator. Hence, by the spectral Theorem 3.22 there exists a finite measure space (X, μ) and a bounded measurable function g on X and a unitary operator $U : H \to L^2(X)$ such that

$$U(T + i)^{-1}U^{-1} = Mg.$$

Now $(T + i)^{-1}$ is one to one. Hence, so is Mg. Therefore g can be zero only on a set of measure zero. Thus the function $f(x) = 1/g(x) - i$ is well defined a.e. $[\mu]$ and we may define it to be zero where $g = 0$. Thus $g(x) = (f(x) + i)^{-1}$ a.e. If h is in $L^2(\mu)$ then each of the following assertions is clearly equivalent to the next:

1) $h \in \mathcal{D}_{M_f}$
2) $h \in \mathcal{D}_{M_{f+i}}$
3) $h \in \text{Range } M_{(f+i)^{-1}} = \text{Range } Mg$
4) $U^{-1}h \in \text{Range } (T + i)^{-1} = \text{Domain } (T + i) = \mathcal{D}_T$.

Thus $U\mathcal{D}_T = \mathcal{D}_{M_f}$. Moreover if $h \in \mathcal{D}_{M_f}$ and $\varphi = (f + i)h$ then $h = g\varphi$ so that $U^{-1}h = U^{-1}MgUU^{-1}\varphi = (T + i)^{-1}U^{-1}\varphi$. Thus $(T + i)U^{-1}h = U^{-1}\varphi$ or $U(T+i)U^{-1}h = M_{f+i}h = M_f h + ih$, so $UTU^{-1}h = M_f h$ and (4.26) holds. But for any unitary operator U and closed operator A, $(UAU^{-1})^* = UA^*U^{-1}$ (exercise). Hence (4.26) shows that M_f is self–adjoint. Thus by Corollary 4.21 f is real almost everywhere. ■

Theorem 4.44 (Spectral theorem: projection valued measure form)**.** *Let T be a self–adjoint operator on a separable complex Hilbert space H. Then there exists a projection valued measure $E(\cdot)$ on the Borel sets of the line such that*

$$T = \int_{-\infty}^{\infty} \lambda dE(\lambda). \tag{4.27}$$

Proof. Just as in the existence proof of the spectral Theorem 3.37 for bounded normal operators, it suffices, in view of the preceding theorem, to prove the theorem in case T is a multiplication operator. Thus if $T = M_f$ where f is a real–valued measurable function on (X, μ), we define $E(B) = M_{\chi_{f^{-1}(B)}}$ for any Borel set $B \subset \mathbb{R}$. Then if $g(\lambda) = \sum_{j=1}^{n} a_j \chi_{B_j}(\lambda)$ is a simple function on the line with $\{B_j\}$ disjoint we have

$$\int g(\lambda) dE(\lambda) = \sum a_j E(B_j) = \sum a_j M_{\chi_{f^{-1}(B_j)}} = M_{g \circ f}.$$

Thus

$$\int_{-\infty}^{\infty} g(\lambda)dE(\lambda) = M_{g\circ f} \tag{4.28}$$

when g is a simple function. By taking uniform limits of a sequence of simple functions we see that this continues to hold for any bounded measurable function g on $\mathbb{R}$. Thus, from (4.28) and the equations (4.24) and (4.25) we have

$$\int_X |(g\circ f)(x)u(x)|^2 d\mu(x) = \Big\|\Big(\int g dE\Big)u\Big\|^2 = \int_{-\infty}^{\infty} |g(\lambda)|^2 dm_\mu(\lambda).$$

Now, if g is an arbitrary measurable function on $\mathbb{R}$, we put $g_n(\lambda) = g(\lambda)$ if $|g(\lambda)| \le n$ and 0 otherwise. Then by monotone convergence we get

$$\int_X |(g\circ f)(x)u(x)|^2 d\mu(x) = \int_{-\infty}^{\infty} |g(\lambda)|^2 dm_\mu(\lambda) \tag{4.29}$$

for all g. In particular, putting $g(\lambda) = \lambda$ we see that

$$\int_X |f(x)u(x)|^2 d\mu(x) = \int_{-\infty}^{\infty} \lambda^2 dm_u(\lambda).$$

Hence $u \in \mathcal{D}_{M_f}$ iff $u \in \mathcal{D}_S$ where $S = \int_{-\infty}^{\infty} \lambda dE(\lambda)$. Thus if $u \in \mathcal{D}_{M_f}$, then

$$\begin{aligned} Su &= \lim_{m\to\infty} \int_{-m}^{m} \lambda dE(\lambda)u = \lim_{m\to\infty} \int_{-\infty}^{\infty} g_m(\lambda)dE(\lambda)u \\ &= \lim_{m\to\infty} M_{g_m\circ f}u = M_f u \end{aligned}$$

as one sees easily using the dominated convergence theorem twice. Thus

$$\int_{-\infty}^{\infty} \lambda dE(\lambda) = M_f$$

which proves the theorem. ∎

Remark 4.45. The uniqueness portion of the spectral theorem as in the proof of Theorem 3.37 holds in the unbounded case also. But we omit the proof.

Remark 4.46. Questions of commutativity for unbounded operators are delicate and dangerous. If T is unbounded and D is a bounded operator the useful definition is as follows: T commutes with D if

$$DT \subset TD.$$

Note that with this definition 0 commutes with all unbounded operators (but would not if we insisted on $DT = TD$). With this definition it is true that if $T = \int \lambda dE(\lambda)$ is self–adjoint, then a bounded operator D commutes with T iff it commutes with all $E(B)$ ($B =$ Borel set).

Remark 4.47. (Functional Calculus.) Just as for bounded operators, there is a functional calculus for unbounded self-adjoint operators also. For any Borel measurable function f on $\mathbb{R}$ and self-adjoint operator T define

$$f(T) = \int_{-\infty}^{\infty} f(\lambda)dE(\lambda) \tag{4.30}$$

where $E(\cdot)$ is the spectral resolution of T. We will omit here further discussion of the consistency of this definition for unbounded functions. For bounded functions some of the discussion in Subsection 3.3 applies.

Exercise 4.14. Suppose that A is a self-adjoint operator on a separable complex Hilbert space H. suppose further that A is positive in the sense that $(Ax, x) \geq 0$ for all $x \in \mathcal{D}(A)$. Show that A has a positive square root $A^{1/2}$ and that $\mathcal{D}(A) \subset \mathcal{D}(A^{1/2})$. Fact: The positive square root is unique. For a proof in the bounded case see Corollary 3.54.

Exercise 4.15. Let A be a self–adjoint operator on a complex separable Hilbert space H and define e^{itA} by using $f(\lambda) = e^{it\lambda}$. The usual power series for the unitary operator e^{itA} cannot, of course, converge in norm if A is unbounded because each term of the series is an unbounded operator. But it could happen that for some vectors x the series

$$\sum_{n=0}^{\infty} (n!)^{-1} (itA)^n x \tag{4.31}$$

converges and actually equals $e^{itA}x$. A vector x is called an <u>analytic vector</u> for A if

a) $x \in \mathcal{D}(A^n)$ for $n = 1, 2, \ldots$ and
b) the series (4.31) converges absolutely for all t in some interval $(-\varepsilon, \varepsilon)$ depending on x.

Prove that any self–adjoint operator A has a dense set of analytic vectors. [One says $\sum_{n=0}^{\infty} y_n$ converges absolutely if $\sum_{n=0}^{\infty} \|y_n\| < \infty$.]

4.8. **Quadratic forms and the Friedrichs extension theorem.**

Definition 4.48. Let H be a complex Hilbert space and $\mathcal{D}$ a subspace. A quadratic form with domain $\mathcal{D}$ is a map $Q : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ such that

(1) $\mathcal{D} \ni x \mapsto Q(x, y)$ is linear for each $y \in \mathcal{D}$ and $\mathcal{D} \ni y \mapsto Q(x, y)$ is conjugate linear for each $x \in \mathcal{D}$.

(2) $Q(x, y) = \overline{Q(y, x)}$ for all $x, y \in \mathcal{D}$.

A quadratic form is *bounded below* if there exists $c \in \mathbb{R}$ such that

$$Q(x, x) \geq -c\|x\|^2 \text{ for all } \ x \in \mathcal{D} \tag{4.32}$$

A quadratic form Q is called *positive* if

$$Q(x, x) \geq 0 \text{ for all } \ x \in \mathcal{D} \tag{4.33}$$

As we will see, it does not alter a quadratic form in a serious way if one adds on a multiple of the inner product. If a quadratic form Q is bounded below then, taking c as defined in (4.32), and defining $\hat{Q}(x, y) = Q(x, y) + a(x, y)$, we have

$$\hat{Q}(x, x) \geq (a - c)\|x\|^2 \text{ for all } \ x \in \mathcal{D}. \tag{4.34}$$

Therefore $\hat{Q}$ is positive if $a \geq c$. The most useful quadratic forms are those that are positive.

Definition 4.49. A quadratic form Q with domain $\mathcal{D}$ is called *closed* if for any sequence $x_n \in \mathcal{D}$ such that

a)$\|x_n - x\| \to 0$ and b) $Q(x_n - x_k, x_n - x_k) \to 0$,

it follows that

a') $x \in \mathcal{D}$ and b') $Q(x_n - x, x_n - x) \to 0$.

Observe how similar this definition is to that of closed operator.

Exercise 4.16. Suppose that

$$Q(x,x) \geq \|x\|^2 \text{ for all } x \in \mathcal{D}. \tag{4.35}$$

(Of course this can be arranged for a quadratic form that is bounded below by taking $a = c+1$ in (4.34)). Show that Q is closed if and only if $\mathcal{D}$ is a Hilbert space in the inner product $Q(x,y)$.

Beware of the following mistake. To say that Q is closed is not the same as saying that its domain $\mathcal{D}$ is closed in H. In fact in almost all interesting cases $\mathcal{D}$ is dense and never closed.

Example 4.50. Let A be a self-adjoint operator on H satisfying $A \geq 0$ (which means $(Ax,x) \geq 0$ for all $x \in \mathcal{D}(A)$. As we learned in Exercise 4.14, A has a unique positive square root and, $\mathcal{D}(A^{1/2}) \supset \mathcal{D}(A)$. Define

$$Q(x,y) = (A^{1/2}x, A^{1/2}y) \text{ with } \mathcal{D}_Q = \mathcal{D}(A^{1/2}) \tag{4.36}$$

Then Q is a closed quadratic form (as you will have the opportunity to show in the next exercise). If in addition $A \geq 1$, which means $(Ax,x) \geq \|x\|^2$ for all $x \in \mathcal{D}(A)$, then (4.35) holds. Notice that, for any $A \geq 0$, if we restrict x to be in the slightly smaller set $\mathcal{D}(A)$, then we can bring a factor $A^{1/2}$ over to the left in (4.36) and we have then

$$Q(x,y) = (Ax,y) \text{ if } x \in \mathcal{D}(A) \text{ and } y \in \mathcal{D}_Q. \tag{4.37}$$

Exercise 4.17. Prove that the quadratic form in Example 4.50 is closed.

4.8.1. *Closability of quadratic forms.*

If a quadratic form Q with domain $\mathcal{D}$ is not closed one can try to enlarge its domain so as to produce a closed form. Suppose again that (4.35) holds. Since $\mathcal{D}$ is now assumed to be incomplete one can complete it in the Q norm and get a Hilbert space $\overline{\mathcal{D}}$. But not so fast. What does this abstract completion have to do with H? An element $u \in \overline{\mathcal{D}}$ is an equivalence class of Cauchy sequences in H. Suppose that u is the class containing the Cauchy sequence $\{x_n\}$. Because of (4.35) the sequence is Cauchy in H norm also, not just in Q norm. Since H is complete the sequence converges in H to an element $v \in H$. We get this way a linear map $S : \overline{\mathcal{D}} \to H$ given by $Su = v$. It's easy to see that this map is well defined and has norm at most one. If S were actually one-to-one we could identify $\overline{\mathcal{D}}$ with its image in H via the map S. This would clearly be the natural thing to do, and would produce a closed form $\overline{Q}$ with domain $S(\overline{\mathcal{D}})$. Sadly, S need not be one-to-one in general.

Example 4.51. Let $H = L^2([0,1])$. Let $\mathcal{D} = C([0,1])$. Then $\mathcal{D}$ is dense in H. Define

$$Q(f,g) = (f,g) + f(0)\overline{g}(0) \text{ for } f,g \in \mathcal{D}, \tag{4.38}$$

where (f,g) denotes the usual inner product: $(f,g) = (f,g)_{L^2([0,1])}$. Then $Q(f,f) = \|f\|^2 + |f(0)|^2 \geq \|f\|^2$. So (4.35) is satisfied. Define a sequence $f_n \in \mathcal{D}$ by: $f_n(t) = 0$ for $(1/n) \leq t \leq 1$ and take f_n to be linear on $[0,1/n]$ with $f_n(0) = 1$ and $f_n(1/n) = 0$. Then $f_n \in \mathcal{D}$ and $Q(f_n - f_k, f_n - f_k) = \|f_n - f_k\|^2 \to 0$ as n and $k \to \infty$. So this sequence is Cauchy in Q norm (and of course also in H norm) and $\|f_n\| \to 0$. So $f_n \to 0$ in H. But $Q(f_n,f_n) = 1$ for all n and therefore the limit in $\overline{D}$ is not zero. So S is not 1-1 in this case.

Definition 4.52. A quadratic form Q is *closable* if it has a closed extension.

The quadratic form Q of Example 4.51 is not closable because, for any closed extension $\overline{Q}$ with domain $\overline{\mathcal{D}}$, say, the sequence f_n, constructed in the example, is Cauchy in $\overline{Q}$ inner product, but must converge to zero in $\overline{Q}$ norm by Definition 4.49, which it doesn't.

If a quadratic form has a closed extension it may have more than one. We will see this in Exercises 4.19 and 4.20.

Definition 4.53. The *closure* of a closable quadratic form is the smallest closed extension. This is the extension constructed in the first paragraph of this subsection. Namely, it is $\overline{\mathcal{D}}$ identified with a subspace of H via the injection S. The closure of a quadratic form is clearly unique if it exists. If Q is a closed quadratic form with domain $\mathcal{D}$ then a subspace $\mathcal{D}_0 \subset \mathcal{D}$ is called a *core* for Q if the closure of $Q|\mathcal{D}_0 \times \mathcal{D}_0$ is Q.

Exercise 4.18. Prove that $\mathcal{D}(A)$ is a core for the quadratic form given in Example 4.50.

The next theorem asserts that Example 4.50 is the only example of a closed quadratic form satisfying (4.35).

Theorem 4.54. *Suppose that Q is a densely defined closed quadratic form satisfying* (4.35)*. Then there exists a self-adjoint operator A such that*

1) $A \geq 1$

2) $\mathcal{D}(A) \subset \mathcal{D}(Q)$

3) $Q(x, y) = (x, Ay)$ if $x \in \mathcal{D}(Q)$ and $y \in \mathcal{D}(A)$.

A is unique under these three conditions. Moreover $\mathcal{D}_Q = \mathcal{D}(A^{1/2})$ and Q is given by (4.36)*.*

Proof. Let $y \in H$ and consider the linear functional $\mathcal{D}_Q \ni x \mapsto (x, y)$. This is a continuous linear functional on the Hilbert space $\mathcal{D}_Q$ because $|(x, y)| \leq \|x\|\|y\| \leq Q(x, x)^{1/2}\|y\|$ by (4.35). Therefore there is a unique vector $z \in \mathcal{D}_Q$ such that $(x, y) = Q(x, z)$ for all $x \in \mathcal{D}_Q$. We have constructed thereby a map $T : H \to \mathcal{D}_Q$, given by $y \mapsto z$, such that

$$(x, y) = Q(x, Ty) \text{ for all } x \in \mathcal{D}_Q \text{ and } y \in H. \tag{4.39}$$

T is clearly linear. Now

$$\|Ty\|^2 \leq Q(Ty, Ty) = (Ty, y) \leq \|Ty\|\|y\| \tag{4.40}$$

Therefore

$$\|Ty\| \leq \|y\|. \tag{4.41}$$

So T is a bounded operator on H with norm at most one. If it should happen that $Ty = 0$ for some $y \in H$ then we have

$$(x, y) = Q(x, Ty) = 0 \text{ for all } x \in \mathcal{D}_Q.$$

But since $\mathcal{D}_Q$ is dense in H it follows that $y = 0$. Hence T is one-to-one. Furthermore, $(Tx, x) = Q(Tx, Tx) \geq 0$. So $T \geq 0$ and is therefore Hermitian (because H is complex). One more easy piece of information: Range T is dense in H because if $y \perp Tx$ for all $x \in H$ then, taking $x = Ty$ in (4.39), we find $0 = (Ty, y) = Q(Ty, Ty) \geq \|Ty\|^2$. So $Ty = 0$ and therefore $y = 0$. Wait: you are about to guess that Range $T = \mathcal{D}_Q$. That's wrong. Of course Range $T \subset \mathcal{D}_Q$ and is dense in H. Moreover Exercise 4.13 shows that T^{-1} is self-adjoint. Let

$A = T^{-1}$. Then $\mathcal{D}(A)$ =Range $T \subset \mathcal{D}_Q$. Moreover $A \geq 1$ because $\|T\| \leq 1$ and A is self-adjoint. Finally, if $z \in \mathcal{D}(A)$ then $TAz = z$. Put $y = Az$ in (4.39). This gives $(x, Az) = Q(x, z)$ for all $x \in \mathcal{D}_Q$ and $z \in \mathcal{D}(A)$. This proves 1), 2) and 3).

Uniqueness: Suppose B is another self-adjoint operator satisfying 1), 2), 3) (with B instead of A). By 1), B^{-1} exists and is bounded from H into H. Put $y = B^{-1}z$ in 3) to find $Q(x, B^{-1}z) = (x, z)$ for all $x \in \mathcal{D}(Q)$ and $z \in H$. Of course the same equation holds for A. So $Q(x, B^{-1}z) = Q(x, A^{-1}z)$ for all $x \in \mathcal{D}(Q)$ and all $z \in H$. Therefore $B^{-1}z = A^{-1}z$ for all $z \in H$. Hence $\mathcal{D}(B) = \mathcal{D}(A)$ because these domains are the ranges of their inverses. Let $w \in \mathcal{D}(A)$ and put $z = Aw$. Then $B^{-1}Aw = w$. Multiply by B (allowed because $w \in \mathcal{D}(B)$) to find $Aw = Bw$ for all w in their common domain. This proves uniqueness.

It remains to prove that Q is exactly that given by (4.36). Define $\hat{Q}$ to be the closed quadratic form given by (4.36). If x and y are both in $\mathcal{D}(A)$ then 2) and 3) show that $Q = \hat{Q}$ on $\mathcal{D}(A)$. But $\mathcal{D}(A)$ is a core for $\hat{Q}$ by Exercise 4.18. Since Q is closed it follows that $\mathcal{D}(Q) \supset \mathcal{D}(\hat{Q})$ and $Q = \hat{Q}$ on $\mathcal{D}(\hat{Q})$ (take sequences to see this.) We only need to show that $\mathcal{D}(Q) = \mathcal{D}(\hat{Q})$. Now $\mathcal{D}(\hat{Q})$ is a closed subspace of $\mathcal{D}(Q)$ in the Q inner product because $\mathcal{D}(\hat{Q})$ is complete in this inner product. If there were a vector $x \in \mathcal{D}(Q)$ which is orthogonal to $\mathcal{D}(\hat{Q})$ then $Q(x, z) = 0$ for all $z \in \mathcal{D}(\hat{Q})$, and in particular for all $z \in \mathcal{D}(A)$. For example if $z = Tw$ then $z \in \mathcal{D}(A)$ for all $w \in H$, as we saw in the first part of this proof. Hence $0 = Q(x, Tw) = (x, w)$ for all $w \in H$. Therefore $x = 0$. So $\mathcal{D}(Q) = \mathcal{D}(\hat{Q})$. ∎

Lemma 4.55. *Suppose that B is a densely defined symmetric operator. Assume that $B \geq 1$. Define*

$$Q(x, y) = (x, By) \text{ for } x \text{ and } y \in \mathcal{D} \equiv \mathcal{D}(B) \tag{4.42}$$

Then Q has a closed extension.

Proof. Since $B \subset B^*$, Q is a Hermitian quadratic form. Since $B \geq 1$, $Q(x, y)$ is an inner product on $\mathcal{D}(B)$. It suffices to show that if x_n is a Q-Cauchy sequence for which $x_n \to 0$ in H then $x_n \to 0$ in $\overline{\mathcal{D}}$. Let $u = \lim x_n$ in the sense of the $\overline{\mathcal{D}}$ topology. Then, for any $y \in \mathcal{D}$, we have $Q(u, y) = \lim Q(x_n, y) = \lim(x_n, By) = 0$. So u is orthogonal to the dense set $\mathcal{D} \subset \overline{\mathcal{D}}$. Therefore $u = 0$. Notice that we have used $B \subset B^*$ just to assure that Q is Hermitian. ∎

Theorem 4.56. (*Friedrichs extension theorem.*) *Suppose that B is a symmetric densely defined operator AND that $B \geq 0$. Then there exists a unique self-adjoint operator A such that*

1) $A \supset B$ and

2) $A \geq 0$

3) The closure of the quadratic form of B, given by (4.42)*, is the quadratic form of A, given by* (4.36).

Terminology 4.57. We say that A is the *form closure* of B.

Proof. By adding I onto B we can assume that $B \geq I$. We will construct a suitable self-adjoint operator A which is also ≥ 1 and then subtract I from it to get 2).

Define a quadratic form Q from B, with $\mathcal{D}(Q) = \mathcal{D}(B)$, using (4.42). By Lemma 4.55 Q has a closure, $\overline{Q}$. By Theorem 4.54 there exists a unique self-adjoint operator

A in H such that

$$\overline{Q}(x, y) = (x, Ay) \text{ for } x \in \mathcal{D}(\overline{Q}) \text{ and } y \in \mathcal{D}(A) \tag{4.43}$$

Moreover $A \geq 1$. Now $(x, By) = Q(x, y) = \overline{Q}(x, y)$ for all x and y in $\mathcal{D}(B)$. Since $\mathcal{D}(B)$ is a core for $\overline{Q}$ the last equation holds also for all $x \in \mathcal{D}(\overline{Q})$ and therefore for all $x \in \mathcal{D}(A)$ and therefore $(x, By) = (Ax, y)$ for all $x \in \mathcal{D}(A)$ and all $y \in \mathcal{D}(B)$. (Changing sides of A is OK as long as you change the location of x and y also. But why did we bother to extend the region of validity of the identity $(x, By) = \overline{Q}(x, y)$ to all $x \in \mathcal{D}(\overline{Q})$ before proceeding to introduce A into the identity?) Now the left side of the last equality is continuous in x in H norm. Therefore so is the right side. It follows that $y \in \mathcal{D}(A^*)$ and $(Ax, y) = (x, Ay)$ for all $x \in \mathcal{D}(A)$ and all $y \in \mathcal{D}(B)$ and therefore $(x, By) = (x, Ay)$ for all these x and y. Hence $By = Ay$ for all $y \in \mathcal{D}(B)$. Thus $B \subset A$. The uniqueness of A follows from the uniqueness portion of Theorem 4.54. ■

Exercise 4.19. Let $H = L^2([0, 1])$. Define

$$Q_0(f, g) = \int_0^1 f'(s)\overline{g'(s)}ds \quad \text{with domain } \mathcal{D}_0 = C_c^\infty((0, 1)) \tag{4.44}$$

Define also

$$T_0 f(s) = -f''(s) \quad \text{with domain} \quad \mathcal{D}(T_0) = C_c^\infty((0, 1)) \tag{4.45}$$

a) Show that Q_0 is a positive quadratic form.
b) Show that Q_0 is closable. (Hint: Explain relation to T_0.)

Culture: We already know from Example 4.35 that the closure of T_0 is not self-adjoint. But this exercise shows that the *form closure* of T_0 is self-adjoint. Fact: This self-adjoint extension is exactly the self-adjoint version of $-d^2/ds^2$ corresponding to Dirichlet boundary conditions (which mean $f = 0$ on the boundary.)

Exercise 4.20. Take $H = L^2([0, 1])$ again. Define

$$Q_1(f, g) = \int_0^1 f'(s)\overline{g'(s)}ds \quad \text{with domain } \mathcal{D}_1 = C^\infty([0, 1]) \tag{4.46}$$

Show that Q_1 is a closable quadratic form and is positive. What is the operator A associated to its closure?

5. Compact Operators

Definition 5.1. A linear map $A : H^{Ban} \to K^{Ban}$ is compact if the image of every bounded set has compact closure.

Remarks 5.2. Let $A : H^{Ban} \to K^{Ban}$ be a linear map. Then

(1) If A is compact then A is bounded.
(2) Let A be compact and define $S_n = \{x : \|x\| \leq n\}$. Then $\overline{AS_n}$ is compact and therefore separable. Since $ran(A) = \bigcup_{n=1}^{\infty} \overline{AS_n}$, the range of A is separable.
(3) Let $\{A_n\}$ be a sequence of compact operators such that $\|A_n - A\| \to 0$. Then A is compact.[2]

Proof. Given $\{x_n\}$ with $\|x_n\| \leq 1$ there is a subsequence $n_{j,1}$ such that $A_1 x_{n_{j,1}}$ converges as $j \to \infty$. This subsequence has in turn a subsequence $n_{j,2}$ such that $A_2 x_{n_{j,2}}$ converges, etc. Let $y_k = x_{n_{k,k}}$. Then $\{y_k\}$ is a subsequence of $\{x_n\}$. Moreover $A_n y_k$ converges in k for each n.

$$\begin{aligned}\|Ay_k - Ay_\ell\| &\leq \|Ay_k - A_i y_k\| + \|A_i y_k - A_i y_\ell\| + \|A_i y_\ell - Ay_\ell\| \\ &\leq 2\|A - A_i\| + \|A_i y_k - A_i y_\ell\|.\end{aligned}$$

Hence $\overline{\lim}_{k,\ell\to\infty}\|Ay_k - Ay_\ell\| \leq 2\|A - A_i\|$ which can be made arbitrarily small. ∎

Definition 5.3. A has finite rank if $\mathcal{R}(A)$ is finite dimensional, where $\mathcal{R}(A) =$ range A.

Example 5.4. Let ξ_j be in H^* and y_j be in K for $j = 1, \ldots, n$. Let $Ax = \sum_{j=1}^{n} \xi_j(x) y_j$. Then A is bounded and of finite rank.

Remarks 5.5. A bounded and finite rank $\Rightarrow$ A is compact. Every norm limit of bounded finite rank operators is compact.

Partial converse: If $A : H^{Ban} \to K^{Ban}$ is compact and has closed range then A is of finite rank.

Proof. By the open mapping theorem $A : H \to \mathcal{R}(A)$ is open. Therefore $\mathcal{R}(A)$ is locally compact and hence finite dimensional. ∎

Corollary 5.6. *If $A : H^{Ban} \to H^{Ban}$ is compact and H is infinite dimensional then $0 \in \sigma(A)$.*

Proof. If 0 is not in $\sigma(A)$ then $\mathcal{R}(A) = H$, which is closed but not finite dimensional. ∎

Example 5.7. Let $K(s,t)$ be continuous on $[0,1] \times [0,1]$. Define an operator $A : L^1(0,1) \to C([0,1])$ by

$$(Af)(s) = \int_0^1 K(s,t) f(t)\,dt \text{ for all } f \in L^1(0,1).$$

Af is continuous by the dominated convergence theorem. We will show that A is a compact operator.

---

[2]One can prove this by showing that AS_1 is totally bounded. Indeed if $\epsilon > 0$ is given and $\|A - A_m\| < \epsilon$, since $A_m S_1$ is totally bounded, there exists a finte set $\Lambda \subset K$ such that $A_m S_1 \subset \cup_{y\in\Lambda} B(y,\epsilon)$. It is now easily seen that $AS_1 \subset \cup_{y\in\Lambda} B(y, 2\epsilon)$.

Let $M = \sup\{|K(s,t)| : 0 \le s,t \le 1\}$. Then $|(Af)(s)| \le M$ for all s in $[0,1]$ if $\|f\|_1 \le 1$. Since K is uniformly continuous there is, for given $\varepsilon > 0$ a $\delta > 0$ such that $|K(s,t) - K(s_0,t)| \le \varepsilon$ whenever $|s - s_0| < \delta$. Hence if $\|f\|_1 \le 1$ then

$$|(Af)(s) - (Af)(s_0)| \le \int |K(s,t) - K(s_0,t)|\ |f(t)|dt \le \varepsilon$$

whenever $|s - s_0| < \delta$. Thus A (unit ball of L^1) is a pointwise bounded equicontinuous family of functions on $[0,1]$ and therefore has compact closure by Ascoli's theorem. So A is a compact operator.

Further examples can be obtained from the preceding example by changing the domain and/or range. Thus, since $\|f\|_p \ge \|f\|_1$ for $1 \le p \le \infty$, the restriction of A to L^p or L^∞ or $C([0,1])$ defines a compact operator into $C([0,1])$. Moreover, since a totally bounded subset of $C([0,1])$ is also totally bounded in L^p, for $1 \le p \le \infty$, one can simultaneously change the range also to any one of these spaces.

Theorem 5.8 (Schauder)**.** *An operator in $B(X,Y)$ is compact iff its adjoint is compact.*

Proof. Let S, S^* be the closed unit balls in X, Y^* respectively.

Let $T : X \to Y$ be compact and let $\{y_n^*\}$ be an arbitrary sequence in S^*. Let $B = \{y \in Y : \|y\| \le \|T\|\}$. The restriction of y_n^* to B gives a sequence of uniformly bounded functions on B which are equicontinuous since $|y_n^*(y) - y_n^*(z)| \le \|y - z\|$, $n = 1, 2, \ldots$. Since $\overline{TS}$ is a compact subset of B Ascoli's theorem shows that there is a subsequence $y_{n_j}^*$ which is Cauchy in sup norm on $\overline{TS}$. Thus $(T^* y_{n_j}^*)(x) = y_{n_j}^*(Tx)$ is Cauchy in sup norm on S. Thus $T^* y_{n_j}^*$ converges in norm to some continuous linear functional on X and so T^* is compact. Conversely, let T^* be compact. Then, by the point just proved, T^{**} is compact, hence if S^{**} is the closed unit ball in X^{**}, $T^{**}S^{**}$ is totally bounded. Thus if $\chi : Y \to Y^{**}$, is the natural imbedding, we have $\chi TS \subseteq T^{**}S^{**}$, χTS is totally bounded hence TS is totally bounded. Therefore $\overline{TS}$ is compact and T is compact. ■

Exercise 5.1. Let H be a separable complex Hilbert space. A bounded operator A is said to be of Hilbert–Schmidt type (or simply a Hilbert–Schmidt operator) if

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty \tag{5.1}$$

for some Orthonormal˙ basis $\{e_1, e_2, \ldots\}$ of H.

a) Prove that if A is of H.S˙ type then the sum in (5.1) is independent of the choice of Orthonormal˙ basis.

Denote the sum in (5.1) by $\|A\|_2^2$.

b) Prove that

$$\|A\| \le \|A\|_2. \tag{5.2}$$

c) Prove that the set of H.S˙ operators on H is itself a Hilbert space in the usual operations of addition and scalar multiplication if one defines

$$(A,B) = \sum_{n=1}^{\infty} (Ae_n, Be_n). \tag{5.3}$$

d) Prove that a Hilbert–Schmidt operator is compact.

e) Show that the set of Hilbert–Schmidt operators is a two sided ideal in $B(H)$.

5.1. **Riesz Theory of Compact Operators.**

Let H be a Banach space. Let $C : H \to H$ be compact. Let $B = I - C$ then $B^* = I - C^*$.

Lemma 5.9. *If $\{y_n\}$ is a bounded sequence and By_n converges then $\{y_n\}$ has a convergent subsequence.*

Proof. Since C is compact there is a subsequence y_{n_j} such that Cy_{n_j} converges, to z say. Since $y_{n_j} - Cy_{n_j}$ converges, to w say, it follows that y_{n_j} converges to $w + z$. ■

Definition 5.10. An operator $B : X^{normed} \to Y^{normed}$ is said to be *bounded below* if there is a constant $m > 0$ such that $\|Bx\| \geq m\|x\|$ for all $x \in X$.

Lemma 5.11. *If $B = I - C$ is one to one then B is bounded below.*

Proof. Suppose that B is not bounded from below. Then there exists a sequence y_n such that $\|y_n\| = 1$ while $By_n \to 0$. By Lemma 1 there is a convergent subsequence y_{n_j}. But if $x = \lim y_{n_j}$ then $\|x\| = 1$ and $Bx = 0$, so B is not one to one. ■

Proposition 5.12. $\ker(B)$ *is finite dimensional.*

Proof. Let $H_0 = \ker(B)$. x is in H_0 if and only if $Cx = x$. But C sends the unit ball of H_0 into a totally bounded set. Therefore H_0 is finite dimensional by Proposition 1.32. ■

Lemma 5.13. *Let F be a finite dimensional subspace of a Banach space H. Then there is a closed subspace $M \subset H$ such that*

$$H = F \oplus M$$

in the sense that every vector z, in H is uniquely of the form $z = x + y$ with x in F and y in M.

Proof. Let $x_1, \ldots, x_n$ be a basis of F. For each j the linear functional $\xi_j : F \to$ scalars, defined by $\xi_j(\sum_{k=1}^n a_k x_k) = a_j$, is a well defined linear functional on the finite dimensional space F, hence is continuous. It therefore has a continuous linear extension to all of H by the Hahn–Banach theorem. Denote the extension by ξ_j also. Define an operator $P : H \to H$ by

$$Px = \sum_{j=1}^n \xi_j(x) x_j.$$

If $x = \sum_{k=1}^n a_k x_k$ then clearly $Px = x$. Therefore, since range $P \subset F$ we have $P^2 = P$. P is a finite sum of continuous operators, hence is continuous. Let $M = \ker P$. Then M is closed and $M \cap F = \{0\}$. If $z \in H$ then $P(z - Pz) = Pz - P^2 z = 0$. So $y := z - Pz \in M$. I.e., $z = x + y$ with $x = Pz$. (Note that P is a projection onto F.) ■

Proposition 5.14. *Range B is closed.*

Proof. Since $\ker(B)$ is finite dimensional it has a closed complement M as in Lemma 5.13. Then $Range(B) = B(M)$. Suppose w is a limit point of $Range(B)$. Then there exists a sequence y_n in M such that $By_n \to w$. Now the proof of Lemma 5.11 shows that, as an operator from M into H, B is bounded below. That is, the restriction $B \mid M$ is bounded below. Hence the y_n form a bounded sequence. We may apply Lemma 5.9 to conclude that this sequence has a convergent subsequence $\{y_{n_k}\}$. Its limit v satisfies $Bv = \lim_{k\to\infty} By_{n_k} = w$. ■

Proposition 5.15. *If B is onto then B is one to one.*

Proof. Let $H_0 = \{0\}$. For $n \geq 1$ let $H_n = \ker(B^n)$ then $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$. If B is not 1:1 there exists $x \neq 0$ such that $Bx = 0$. Let $x_1 = x$, define x_n inductively such that $Bx_n = x_{n-1}$. Then $B^n x_n = 0$, $B^{n-1}x_n \neq 0$. Therefore $x_n \in H_n$ but $x_n \notin H_{n-1}$, hence H_{n-1} is properly contained in H_n. There exists $y_n \in H_n$ with $\|y_n\| = 1$ and $\|y_n - x\| \geq 1/2$ for all $x \in H_{n-1}$ (see Lemma 1.31). If $n > m$

$$Cy_n - Cy_m = y_n - [y_m + By_n - By_m] = y_n - x \text{ forsome } x \in H_{n-1}.$$

So $\|Cy_n - Cy_m\| \geq 1/2$. Therefore $C\{y_n\}$ is not totally bounded, contradiction. ■

Proposition 5.16. *If B is one to one then B is onto.*

Proof. Suppose B is one to one, by Lemma 5.11, B is bounded below. It follows that if $M \subset H$ is a closed subspace then $B(M)$ is closed also.

Suppose that Range $B \neq H$. Let $H_0 = H$, $H_1 = BH$, $H_2 = BH_1, \ldots$. Then H_{m+1} is a closed and proper (because B is 1:1) subspace of H_m. By Lemma 1.31 there exists a vector x_n in H_n such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, H_{n+1}) \geq 1/2$. But then if $n > m$

$$Cx_m - Cx_n = x_m - Bx_m - x_n + Bx_n = x_m - x \text{ where } x \in H_{m+1}.$$

Hence $\|Cx_m - Cx_n\| \geq d(x_m, H_{m+1}) \geq 1/2$. Thus the sequence Cx_n contains no Cauchy subsequence, contradicting the compactness of C. ■

Proposition 5.17. $\dim\ker B = \dim\ker B^*$.

Proof. Let $x_1, \ldots, x_n$ be a basis for $\ker B$ and let $\eta_1, \ldots, \eta_\nu$ be a basis for $\ker B^*$. By the Hahn–Banach theorem $\exists\ \xi_j \in H^*$ such that

$$\xi_j(x_i) = \delta_{i,j}, i, j = 1, \ldots, n. \tag{5.4}$$

Moreover, if $K = \ker B^*$ and $x \to \widehat{x}$ is the natural injection of H into H^{**} then the map $x \to \widehat{x} \mid K$ must map *onto* K^*, for if not then there is a non–zero vector u in the finite dimensional subspace K annihilated by all such $\widehat{x}$. That is $u(x) = \widehat{x}(u) = 0$ for all $x \in H$ — which means $u = 0$ after all. Thus there are vectors $y_1, \ldots, y_\nu$ in H such that

$$\eta_j(y_i) = \delta_{i,j}, i, j = 1, \ldots, \nu. \tag{5.5}$$

Now suppose $n < \nu$. Define

$$C'x = Cx + \sum_{j=1}^{n} \xi_j(x) y_j.$$

Then C' is a compact operator. If $B' = 1 - C'$ then we assert B' is one to one. For suppose $B'x_0 = 0$. Then

$$Bx_0 = \sum_{j=1}^{n} \xi_j(x_0) y_j. \tag{5.6}$$

But

$$\begin{aligned} 0 &= B^* \eta_i(x_0) \text{ because } \eta_i \in \ker B^* \\ &= \eta_i(Bx_0) \text{ by definition of } B^* \\ &= \sum_{j=1}^{n} \xi_j(x_0)\eta_i(y_j) \text{ by (5.6)} \\ &= \xi_i(x_0) \text{ by (5.5).} \end{aligned} \tag{5.7}$$

Hence $Bx_0 = 0$ by (5.6) and the last equality. Hence $x_0 = \sum_{j=1}^{n} \alpha_j x_j$ for some scalars α_j, because $x_1, \ldots, x_n$ spans $\ker B$. But by (5.7) and (5.4) $\alpha_i = \xi_i(x_0) = 0$. Hence $x_0 = 0$. This shows $\ker B' = 0$.

Thus by Proposition 5.17, B' is onto. Hence $\exists\ x \in H$ such that $y_{n+1} = B'x$. But then

$$\begin{aligned} 1 &= \eta_{n+1}(y_{n+1}) = \eta_{n+1}(B'x) \\ &= \eta_{n+1}(Bx) - \eta_{n+1}\Big(\sum_{j=1}^{n} \xi_j(x) y_j\Big) \\ &= (B^*\eta_{n+1})(x) - \sum_{j=1}^{n} \xi_j(x)\eta_{n+1}(y_j) = 0 - 0. \end{aligned}$$

Contradiction.

Thus we have shown $n \geq \nu$. I.e.,

$$\dim\ker B \geq \dim\ker B^*. \tag{5.8}$$

Since C^* is also compact we have

$$\dim\ker B^* \geq \dim\ker B^{**}. \tag{5.9}$$

But B^{**} "agrees" with B on the canonical image of H in H^{**}. Hence

$$\dim\ker B^{**} \geq \dim\ker B. \tag{5.10}$$

Combining (5.8), (5.9) and (5.10) shows that these are all equalities — which proves the proposition. ■

Theorem 5.18. *Let C be a compact operator on a Banach space H. Every non–zero point λ of the spectrum of C is an eigenvalue of finite multiplicity. (That is, $\dim\ker(\lambda - C)$ is finite.) Moreover the multiplicity of λ for C is the same as for C^*. The only possible cluster point of the spectrum of C is zero.*

Proof. If $\lambda \neq 0$ is in $\sigma(C)$ then $1 - \lambda^{-1}C$ is not invertible. Since $\lambda^{-1}C$ is compact $1 - \lambda^{-1}C$ can fail to be invertible either because it is not one to one — in which case λ is an eigenvalue (of finite multiplicity by Proposition 5.12 — or because it is not onto — in which case it is also not one to one by Proposition 5.16. If it is both one to one and onto it is of course invertible by the open mapping theorem (or Lemma 5.11). By Proposition 5.17 $\dim\ker(\lambda - C) = \dim\ker(\lambda - C^*)$ if $\lambda \neq 0$ since $\ker(\lambda - C) = \ker(1 - \lambda^{-1}C)$.

Finally, to prove that zero is the only possible cluster point of $\sigma(C)$ assume that $\exists\ \lambda_n \in \sigma(C)$ and that λ_n converges to some $\lambda \neq 0$. We may assume $\lambda_n \neq \lambda_m$ for $m \neq n$ and that $\exists\ \gamma > 0$ such that $|\lambda_n| \geq \gamma$ for all n. Then $\exists\ x_n \neq 0$ such that $Cx_n = \lambda_n x_n$. We assert that the set $\{x_n\}$ is linearly independent. If not let n be the first integer such that $x_n = \sum_{j=1}^{n-1} \alpha_j x_j$ with some $\alpha_j \neq 0$. Then

$$\lambda_n x_n = Cx_n = \sum_{j=1}^{n-1} \alpha_j \lambda_j x_j.$$

Thus

$$\lambda_n \sum_{j=1}^{n-1} \alpha_j x_j = \sum_{j=1}^{n-1} \alpha_j \lambda_j x_j$$

and

$$\sum_{j=1}^{n-1} \alpha_j (\lambda_n - \lambda_j) x_j = 0.$$

Therefore $\alpha_j = 0$, $j = 1, \ldots, n-1$ — contradiction.

Let $H_n = \operatorname{span}(x_1, \ldots, x_n)$. Then H_n is a properly increasing sequence of subspaces. $\exists\ y_n$ such that $\|y_n\| = 1$, $y_n \in H_n$ and $\|y_n - x\| \geq \frac{1}{2}\ \forall x \in H_{n-1}$ by Lemma 1.31.

Let $y \in H_n$. Then $y = \sum_{j=1}^{n} \alpha_j x_j$, and

$$Cy - \lambda_n y = \sum_{j=1}^{n} \alpha_j (\lambda_j - \lambda_n) x_j \in H_{n-1}.$$

If $n > m$ then

$$\begin{aligned}
\|Cy_n - Cy_m\| &= \|(Cy_n - \lambda_n y_n) + \lambda_n y_n - \lambda_m y_m + (\lambda_m y_m - Cy_m)\| \\
&= \|\lambda_n y_n - z\| \text{ where } z \in H_{n-1} \\
&\geq \frac{|\lambda_n|}{2} \geq \frac{\gamma}{2}.
\end{aligned}$$

Hence $\{Cy_m\}$ contains no Cauchy subsequence — contradiction. ∎

Corollary 5.19. *If C has an infinite number of eigenvalues then 0 is a cluster point of eigenvalues. Thus the eigenvalues can be arranged in a sequence converging to zero.*

Proof. If $Cx = \lambda x$, $x \neq 0$, then

$$|\lambda|\|x\| = \|\lambda x\| = \|Cx\| \leq \|C\|\ \|x\|$$

Therefore $|\lambda| \leq \|C\|$.

The set of proper values has at least one cluster point if there are an infinite number of them. This must be 0 by the theorem. Since only finitely many can lie outside the disc $|z| \leq 1/n$ they may be arranged in a sequence converging to zero. ∎

Corollary 5.20. *If C is a compact normal operator on a separable complex Hilbert space H there is a finite or infinite sequence P_n of mutually orthogonal finite dimensional projections such that*

$$C = \sum_{n=1}^{\infty} \lambda_n P_n \quad (\text{or } C = \sum_{n=1}^{k} \lambda_n P_n) \tag{5.11}$$

where $\{\lambda_n\}$ are the non–zero eigenvalues of C and the series converges in the operator norm. Moreover H has an orthonormal basis consisting of eigenvectors of C (i.e., C can be "diagonalized").

Proof. Let

$$C = \int_{\sigma(C)} \lambda dE(\lambda) \tag{5.12}$$

be the spectral representation of C and let $\lambda_1, \lambda_2, \dots$ be the non–zero eigenvalues of C. Then this set is finite or else $\lambda_n \to 0$ as $n \to \infty$ by Corollary 5.19. Let $P_n = E(\{\lambda_n\})$. Then $\dim P_n =$ multiplicity of $\lambda_n < \infty$. Then equation (5.12) reduces to equation (5.11) because the functions

$$f_n(\lambda) = \lambda \cdot \chi_{\{\lambda_1, \lambda_2, \dots \lambda_n\}}(\lambda) = \begin{cases} \lambda & \text{if } \lambda \in \{\lambda_1, \lambda_2, \dots \lambda_n\} \\ 0 & \text{otherwise} \end{cases}$$

form a sequence of simple functions on $\sigma(C)$ which converge uniformly on $\sigma(C)$ to λ, while

$$\int_{\sigma(C)} f_k(\lambda) dE(\lambda) = \sum_{n=1}^{k} \lambda_n P_n.$$

We may now choose an Orthonormal . basis $x_1, x_2, \dots$ of H such that each vector is in Range P_n for some n or is in Range $E(\{0\}) =$ null space C. C is diagonal on this basis. ∎

Exercise 5.2. Let $\mathcal{D} = \{f \in L^2(0,1) : f$ is absolutely continuous on $[0,1]$, f' is absolutely. continuous on $[0,1]$, f'' is in $L^2(0,1)$, and $f(0) = f(1) = 0\}$. Define $Tf = f''$ for f in $\mathcal{D}$.

a) Prove that T has a compact inverse.
b) Prove that T is self–adjoint.
c) Find the spectrum of T.

6. Semigroups of Operators

Definition 6.1. A semigroup of operators on a Banach space B is a function $s \hookrightarrow T_s$ from $[0, \infty)$ to bounded operators on B such that

a) $T_0 = I$

b) $T_{t+s} = T_t T_s$ for $s, t \geq 0$

The semigroup is called *strongly continuous* if for each $x \in B$, the function $t \to T_t x$ is continuous from $[0, \infty)$ into B.

Example 6.2. Let A be a bounded operator on B. Define $e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$. This series converges in norm because $\|\frac{(tA)^n}{n!}\| \leq \frac{\|tA\|^n}{n!}$. Thus $\|e^{tA}\| \leq e^{\|tA\|}$. Any elementary combinatorial (power series) proof that $e^{x+y} = e^x e^y$ shows, without change in proof, that $e^{(t+s)A} = e^{tA} e^{sA}$. Hence the function $T_t = e^{tA}$ defines a semigroup. This semigroup is not only strongly continuous but also *norm* continuous, i.e., $\|e^{tA} - e^{sA}\| \to 0$ as $t \to s$. To see this, note that

$$\|e^{tA} - e^{sA}\| = \|(e^{(t-s)A} - 1)e^{sA}\| \leq \|e^{(t-s)A} - 1\| \, \|e^{sA}\|.$$

But the power series representation of $e^{(t-s)A} - 1$ shows $\|e^{(t-s)A} - 1\| = O(t-s)$. Thus $\|e^{tA} - e^{sA}\| \to 0$ as $t \to s$.

Of course, norm continuity of any semigroup T_t is stronger than strong continuity — as follows from $\|T_t x - T_s x\| \leq \|T_t - T_s\| \, \|x\| \to 0$ as $t \to s$. It is a fact that the above example yields all the norm–continuous semigroups. The most important semigroups, however, are not norm continuous. They correspond — in a sense to be described below to an A which is unbounded.

Example 6.3. Let $E(\cdot)$ be a projection valued measure on the complex plane with values which are projections on a complex Hilbert space H. Assume that the support set of E is contained in the left half–plane $\mathbb{C}_- = \{z : Re\, z \leq 0\}$. Define

$$T_t = \int_{\mathbb{C}_-} e^{zt} dE(z), \quad t \geq 0.$$

Since $|e^{zt}| \leq 1$ for $z \in \mathbb{C}_-$ and $t \geq 0$, it follows that $\|T_t\| \leq 1$ for $t \geq 0$. Moreover, the functional calculus shows that $T_{t+s} = T_t T_s$ and of course $T_0 = 1$. Hence $T(\cdot)$ is a semi–group. It is strongly continuous, for if $x \in H$ and $m_x(B) = (E(B)x, x)$, then we have

$$\|T_t x - T_s x\|^2 = \int_{\mathbb{C}_-} |e^{zt} - e^{zs}|^2 dm_x(z).$$

Since the integrand is at most 4 on $\mathbb{C}_-$ and goes to zero pointwise as $t \to s$, the dominated convergence theorem shows that $\|T_t x - T_s x\| \to 0$ as $t \to s$. Thus $T(\cdot)$ is a strongly continuous semigroup.

Symbolically, if we write $A = \int_{\mathbb{C}_-} z dE(z)$, then in view of the functional calculus definition of e^{tA}, we have $T_t = e^{tA}$.

As we have seen, T_t is a contraction operator (i.e., $\|T_t\| \leq 1$) because the spectrum of A lies in the left half plane.

Informally, the function $u(t) = e^{tA}x$ solves the equation

$$\frac{du}{dt} = Au(t), \quad u(0) = x. \tag{6.1}$$

Two important special cases are:

1) $A = i(-\Delta + V)$ acting in $L^2(\mathbb{R}^n)$, where V is multiplication by a suitable real function. In this case, iA is self–adjoint and consequently $\sigma(A)$ lies on the imaginary axis. The equation (6.1) is the Schrödinger equation.

2) $A = \Delta$ acting in $L^2(\mathbb{R}^n)$. In this case, A is self–adjoint and negative, so $\sigma(A)$ lies along the negative real axis. The equation (6.1) is then the heat equation.

We remark also that the wave equation $\frac{\partial^2 u}{\partial t^2} = \Delta u$ can be reformulated so as to reduce to (6.1), with $\sigma(A)$ lying along the imaginary axis.

Definition 6.4. A semigroup of operators T_t is called a *contraction* semi–group if $\|T_t\| \leq 1$ for all $t \geq 0$. This is the most important class of semigroups.

Definition 6.5. Let T_t be a semigroup of linear operators in a Banach space B. Define $Af = \text{limit}_{h \downarrow 0} \frac{T_h - 1}{h} f$ with domain $\mathcal{D}_A = \{f \in B : \text{for which } Af \text{ exists}\}$. A is called the *infinitesimal generator* of the semigroup T_t.

The basic facts about semigroups of operators are the following theorems.

Theorem 6.6. *If T_t is a strongly continuous semigroup of bounded linear operators, then its infinitesimal generator A is a closed densely defined linear operator. T_t is uniquely determined by A in the sense that distinct semigroups have distinct infinitesimal generators. Moreover, if f is in $\mathcal{D}_A$, then $u(t) = T_t f$ solves the differential equation*

$$\frac{du}{dt}(t) = Au(t), \quad t \geq 0 \textit{ with } u(0) = f.$$

Theorem 6.7 (Hille Yosida theorem)**.** *A densely defined, closed, linear operator A is the infinitesimal generator of a strongly continuous contraction semigroup $\Leftrightarrow$ the positive half line $(0, \infty)$ is contained in the resolvent set of A and*

$$\|(\lambda - A)^{-1}\| \leq \frac{1}{\lambda} \quad \forall \lambda > 0.$$

For the proofs of these theorems and other exciting facts about semigroups, we refer the reader to E. B. Dynkin, Markov Process I, pages 22–33 which present a rather direct and self–contained account of the theory.

The earliest theorem of this type is the Stone–Von Neumann theorem.

Theorem 6.8 (Stone–Von Neumann)**.** *Every strongly continuous one parameter unitary group $U(t)$ on a complex Hilbert space H is of the form $U(t) = e^{itB}$ where B is a self–adjoint operator. The infinitesimal generator of $U(\cdot)$, in the sense of the previous definition, is precisely $A = iB$.*

Proof. See Reed and Simon, Functional Analysis, Vol. 1, 266–267. ■

Exercise 6.1. A vector x in a Banach space B is called a <u>C^∞ vector</u> for a densely defined operator A on B if $x \in \mathcal{D}(A^n)$ for $n = 1, 2, 3, \ldots$. Notation: $C^\infty(A) = \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$. Prove that if A is the infinitesimal generator of a contraction semigroup, T_s, on B then $C^\infty(A)$ is dense in B. **Hint:** Consider vectors $\int_0^\infty g(s) T_s x ds$ for wise choices of g.