Defining Equations for Bifurcations and Singularities

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1 Introduction

Locating bifurcations is one of the objectives of numerical methods for the analysis of dynamical systems. The presence of bifurcations is often deduced from trajectories; i.e., solutions of initial value problems, with different parameter values. However, methods that directly locate bifurcations are usually more efficient and more precise. Robust direct methods must take into account that the computations may seek objects that are inherently singular. Newton's method is a fundamental example of an algorithm that relies upon regularity of the equations being solved. Without regularity of the defining system, Newton's method and other equation solving algorithms suffer significant degradation of their convergence properties or fail altogether. Since the effectiveness of algorithms for numerical solution of systems of equations often depend upon the regularity of the equations being solved, one strategy for the formulation of the direct methods to compute singular objects is to decompose them into subsets that are manifolds. This paper discusses the implementation of algorithms that produce regular systems of defining equations for two problems: saddle-node bifurcation of periodic orbits of vector fields and the Thom-Boardman stratification of a smooth mapping. We describe these problems in more detail.

Let γ be a periodic orbit of period T for an n dimensional vector field $\dot{x} = f(x)$. If $p \in \gamma$, then the Jacobian $D\phi_T$ of the time T flow map ϕ_T at p is the **monodromy map** of γ based at p. Monodromy maps based at different points of γ are related by similarity transformations, so the spectrum of the monodromy map is independent of the base point p. The vector $f(p)$ is an eigenvector for monodromy map based at p with eigenvalue 1. The periodic orbit is isolated and varies smoothly with parameters if 1 is a simple eigenvalue of the monodromy map. Saddle-node bifurcation of a periodic orbit occurs when the return map of f has 1 as an eigenvalue and additional non-degeneracy conditions are satisfied. If 1 is an eigenvalue of the

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return map, then it is not a simple eigenvalue of the monodromy map. Defining equations for the periodic orbit must include an equation that determines whether 1 is an eigenvalue of the return map or has (algebraic) multiplicity two for the monodromy map. Multiple shooting boundary value solvers that we have formulated give a block representation of the monodromy map [7], so it is natural in this context to seek an explicit equation on the space of $n \times n$ matrices whose solutions give the matrices for which 1 is an eigenvalue of multiplicity larger than 1. Vanishing of the characteristic polynomial and its first derivative at 1 gives a pair of equations with the desired properties, but calculation of the characteristic polynomial is a poorly conditioned problem. Therefore, we discuss alternative sets of equations in the next section.

The second problem we discuss is about singularities of smooth mappings. If $\dot{x} = h(x, \lambda)$ is a generic, smooth family of vector fields, then its equilibrium points, the solutions of $h(x, \lambda) = 0$, form a smooth manifold E. However, bifurcations of h occur at the singularities of the projection of E onto the parameter space of the family. The resulting bifurcation locus in the parameter space may have singularities, so we want to pursue its decomposition into submanifolds. Here we consider a more general problem, the Thom-Boardman stratification of an arbitrary smooth mapping. Let $f : R^m \to R^n$ be a smooth mapping. The **Thom-Boardman** stratification of f is defined inductively, beginning with $\Sigma^i = \{x \in R^m \,|\, corank(Df_x) = i$. If $\Sigma^{i_1, i_2, \ldots, i_{k-1}}$ is defined and is a manifold, then $\Sigma^{i_1, i_2, \ldots, i_{k-1}, i_k}$ is defined to be the set of points for which the corank of Df restricted to the tangent space of $\Sigma^{i_1, i_2, \ldots, i_{k-1}}$ has corank i_k. Boardman [2] proved that for generic maps (with respect to the Whitney topology of $C^\infty(R^m, R^n)$, the sets $\Sigma^{i_1, i_2, \ldots, i_{k-1}} \subset R^m$ are manifolds. He gave a formula for the codimension of these manifolds.

The definition of the Thom-Boardman stratification given above does not lend itself readily to numerical computation. The operation of restriction of f to a submanifold is difficult to express directly if the submanifold is known only through the solution of equations at a discrete mesh of points. Moreover, the defining equations for the strata depend implicitly on derivatives of f of increasing order, suggesting that the lack of smooth approximations for each stratum is likely to lead to increasing errors in the computation of the next.

The strategy of creating regular systems of defining equations for bifurcations has become common in numerical algorithms for analysis of dynamical systems [3]. Many implementations of this strategy introduce extra variables and formulate systems whose dimension is larger than the minimum possible [12]. For example, in computing Hopf bifurcations of an n dimensional vector field f with a single parameter, one strategy to define a regular system is to introduce variables for vectors in a two dimensional invariant subspace and for the another variable for the magnitude of the pure imaginary eigenvalue. This yields $3n + 2$ variables, requiring a regular system of $3n + 2$ equations. Guckenheimer, Myers and Sturmfels [9, 8] explored this issue for Hopf bifurcation of equilibria [6] and found advantages to using systems of defining equations of minimal dimension. They advocated using singularity of the "bialternate product" of the Jacobian as a defining equation for Hopf bifurcation of an equilibrium. There is a trade-off in the definition of regular systems between the dimension of the system and the complexity of the defining equations reflected in this choice: singularity of the bialternate product is determined by application of methods from linear algebra whose complexity is reflected in the size of the bialternate product ($n(n-1)/2 \times n(n-1)/2$ for an n-dimensional vector field). This paper works with minimally augmented systems of defining equations that do not introduce extra variables. As in the Hopf example, the defining equations sometimes employ "standard" numerical algorithms to evaluate a function.

2 Defining equations for nilpotent Jordan blocks

This section presents an augmenting equation that can be used to determine when 1 is not a simple eigenvalue of the monodromy matrix of a periodic orbit. We shall denote the $n \times n$ monodromy matrix by A and assume that 1 is an eigenvalue of geometric multiplicity one. This assumption is valid generically at saddle-node bifurcations of periodic orbits. (The only 2×2 matrices with an eigenvalue of geometric multiplicity two are the diagonal matrices, so the set of matrices with an eigenvalue of geometric multiplicity two has codimension three.) Let w^t and v be the left and right null vectors of A. The generalized eigenspace of 0 has dimension larger than 1 if the equation $Ax = v$ has a non-zero solution. We then have $Ax \neq 0$ but $A^2x = Av = 0$. Moreover, since A has corank (rank deficiency) 1, the equation $Ax = b$ is solvable if and only if $w^tb = 0$. In particular, $w^tv = 0$ if and only if A has a two dimensional nilpotent subspace. To use this equation as part of a regular system of defining equations for the presence of a nilpotent subspace, we need an algorithm that produces v and w in a manner that depends smoothly upon A. Note that the eigenvalues of A do not vary smoothly near matrices that have multiple eigenvalues. Therefore, we want to avoid computing the eigenvalues of A. Instead, singular value decomposition can be used to both compute the smallest singular value of A and to compute its singular vectors. In the case of a singular matrix with rank deficiency 1, we can define a "signed" singular value s near 0 so that s and its singular vectors v, w^t vary smoothly with A. Then s together with w^tv give a pair of regular defining equations for determining the matrices A that have a two dimensional nilpotent subspace.

Guassian elimination with a bordered matrix can be used to produce a function whose the zero set includes the singular matrices of rank deficiency 1. Form an $(n+1) \times (n+1)$ bordered matrix

$$\bar{A} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

If A has rank deficiency 1, $AC^t = B^tA = 0$ and $CC^t = B^tB = 1$, then $\bar{A}$ is regular and

$$\bar{A} \begin{pmatrix} C^t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If A is perturbed, with B and C remaining constant, then the solution of the equation

$$\bar{A} \begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

varies smoothly with A. Observe that t is the $(n+1, n+1)$ component of $\bar{A}^{-1}$. Cramer's rule states that $t = \det(A)/\det(\bar{A})$, so $t = 0$ if and only if A is singular. Indeed, t is comparable to the smallest singular value of A and can be used as the defining function for detecting singularity of A.

3 Thom-Boardman Singularities

The Thom-Boardman stratification of a smooth mapping $f : R^m \to R^n$ is defined by a recursive procedure. From the definition, it is hardly apparent how to produce a regular set of defining equations for each stratum. Computing the corank of Df restricted to a submanifold P is problematic unless there is a representation of P, either explicitly as an embedding in R^m or implicitly as the solutions of a set of defining equations. Numerical implementation of the definition of the Thom-Boardman stratification does not give such a representation. Boardman [2] proved that there are submanifolds of jet spaces, the Thom-Boardman singularities, that can be used to give implicit equations for the stratification of a generic map. Specifically, the stratification is obtained by pulling back the Thom-Boardman singularities by the jet extensions of the map. Since the singularities are submanifolds of the jet space, systems of regular defining equations exist. Composing these equations with the jet extension of f gives a system of regular defining equations for the Thom-Boardman stratification of f. Boardman's construction of the singularity manifolds is complicated, but with some effort can be used to produce regular sets of defining equations that can be implemented numerically. This section describes two algorithms for producing the Thom-Boardman stratification of a map. More details about these algorithms and their implementation can be found in Xiang [13].

The defining equations for Σ^i are relatively easy to obtain. The rank of a matrix is the size of the largest non-vanishing minor. Let $x \in R^m$ be a point at which Df has corank i. Then there is an $(m-i) \times (m-i)$ minor of $Df(x)$ that is non-zero, but all $(m-i+1) \times (m-i+1)$ minors vanish. It is well known that the minors of the matrix are dependent polynomial equations, so we want to select an independent subset that generates the ideal of all the $(m-i+1) \times (m-i+1)$ minors. By reordering coordinates, if necessary, we assume that the upper left $(m-i) \times (m-i)$ minor of $Df(x)$ is not zero. This property then holds for $Df(u)$, u lying in a neighborhood of x. To obtain defining equations for Σ^i consider the block decomposition and factorization of $Df(u)$

$$Df(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix} \qquad (1)$$

with A an invertible $(m-i) \times (m-i)$ matrix. From the factorization it is evident that $Df(u)$ has corank i if and only if $D - CA^{-1}B = 0$. Thus $D - CA^{-1}B = 0$ constitutes a set of $i(n-m+i)$ defining equations for Σ^i. They have the form $g_1 \circ j^1(f)$ where $j^1(f)$ is the 1-jet extension of f and g_1 is a function defined on the space of 1-jets. Note that g_1 is a regular system of equations since its derivative with respect to the variables in D gives a set of spanning columns for Dg_1.

To find the higher strata of the Thom-Boardman stratification, we seek similar equations g_k defined on the k-jets of maps from R^m to R^n. There are two obstacles that need to be overcome. First, the definition of each stratum needs to be formulated so that we obtain defining equations that are defined on open sets of R^m rather than just on the restriction of f to the previous stratum. Second, the higher derivatives of f satisfy symmetry conditions that need to be taken into account when selecting a minimal set of defining equations. For general points, expressions for the defining equations become complicated, so it is convenient to apply coordinate transformations that implement a factorization of Df analogous to the one described above.

The formulation of the definition of the strata is based upon an elementary observation from linear algebra about the computation of the null space of one linear map restricted to the null space of a second. If V is the kernel of a linear map A and W is the kernel of a linear map B, then $V \cap W$ is the kernel of the "stacked" matrix

$$\begin{pmatrix} A \\ B \end{pmatrix} \tag{2}$$

This is a symmetric relationship in V and W, so we can regard $V \cap W$ as either the kernel of A restricted to W or the kernel of B restricted to V. This observation will be used in the construction of the stratification. If $P^p \subset R^m$ is a p-dimensional stratum that is locally the zero set of $g : R^m \to R^{m-p}$, then the tangent space $T_x P$ is the null space of $Dg(x)$. The next stratum with index j consists of points for which $Df|_{TP}$ has corank j. Now the dimension of the null space of $Df|_{TP}$ is the same as the dimension of $Dg|_Q$, Q being the kernel of Df. Thom proved that the sets Σ^i on which Df has corank i are submanifolds for generic f, and the procedure described above gives a regular systems of defining equations for Q. As we proceed through the stratification level by level, we obtain defining equations $g_1, g_2, \ldots, g_s$ that together constitute defining equations for stratum s. The map g_1 is the rational function $D - CA^{-1}B$ composed with Df expressed as above. Each successive g_i will be defined in terms of derivatives of g_{i-1}. Note that the submanifolds P and the functions g change as s increases, but the subspace Q does not. The equation $g_s = 0$ that identifies stratum s inside stratum $s-1$ will be given by the composition of a function defined on the space of k-jets with the kth jet extension $j^k(f)$ of f. In Boardman's theory [2] and the alternate approach of Mather [10], the ranks and coranks of the jets are determined initially by forming ideals that consist of minors of matrices constructed from partial derivatives of f. These constructions yield much larger systems of defining equations than the regular sets we seek. The codimensions of the singularities in the jet space determine the sizes of regular subsets of defining equations.

Let us now turn to the construction of defining equations for $\Sigma^{i,j}$ to illustrate how coordinate changes are used to simplify the calculations. Since we have ordered coordinates so that the upper left $(m-i) \times (m-i)$ minor is non-singular, we set $(f_1, \ldots, f_{m-i}) = u$, $(x_{m-i+1}, \ldots, x_m) = w$ and $(f_1, \ldots, f_{m-i}, x_{m-i+1}, \ldots, x_m) = (u, w) = \phi(x)$. Then ϕ is a local diffeomorphism, and the map $\bar{f} = f \circ \phi^{-1}$ represents f in the (u, w) coordinates. These coordinates implement the matrix factorization of Df described above in the sense that $D\bar{f}$ has the form

$$D\bar{f} = \begin{pmatrix} I & 0 \\ \bar{C} & \bar{D} \end{pmatrix}$$

with $\bar{D} = D - CA^{-1}B$ when

$$Df(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3}$$

Thus the defining equations for Σ^i of $\bar{f}$ are $g(u, w) = \bar{D} = 0$. Moreover, the null space of $D\bar{f}$ is the w-coordinate subspace. Therefore, $\Sigma^{i,j}$ will be given by points for which the matrix $D_w g$ has corank j.

Selecting a set of defining equations to express that $D_w g$ has corank j is more subtle than was the case for obtaining the defining equations of Σ^i. Begin the same way, however, by reordering coordinates and the $i(n-m+i)$ components of g so that the derivatives of the first $(i-j)$ components of g with respect to the first $(i-j)$ components of v is a nonsingular matrix. As before, we can perform a coordinate change in which the first $(i-j)$ coordinates from w are replaced by the first $(i-j)$ components of g. In this new coordinate system, $D_w g$ has corank j when the derivatives of the last $i(n-m+i)-(i-j)$ components of $D_w g$ with respect to the final j components of the new coordinates vanish. However, these functions are not independent: they are second derivatives of components of $\bar{f}$ and mixed partial derivatives are symmetric with respect to reordering their indices. Using the symmetry relations, we pick a maximal independent set of these to obtain defining equations for $\Sigma^{i,j}$.

The general case proceeds in a similar fashion. Assume that we have computed a regular system of defining equations $g_1, \ldots, g_t$ for the singularity submanifolds $\Sigma^{i_1,\cdots,i_t}$ for $t < s-1$. We then want to determine equations g_s so that $g = (g_1, \ldots, g_s)$ are a regular system of defining equations for $\Sigma^{i_1,\cdots,i_s}$. The inductive process we use for defining the g_t introduces a coordinate transformation at each step that makes the defining equations at the next step into the vanishing of a set of partial derivatives. Specifically, at stage t of the construction, coordinates are split into three groups: coordinates u that are complementary to the null space of Df and satisfy $f_l(x) = u_l$ in the current coordinate system for $l \le m-i$, $i_{t-1} - i_t$ coordinates $v_1, \ldots, v_t$ that have been determined inductively so that v_t is a subset of the defining functions g_t and the remaining coordinates w_t that are a subset of the original coordinates x. By reordering, we assume that at each stage the components of v_t are the first $i_{t-1} - i_t$ of w_{t-1}.

Now consider the construction of g_s. The first step in constructing g_s is to form the collection of functions $h(s)$ obtained by differentiating g_{s-1} with respect to the variables w_s. The criterion that $\mathrm{corank}(Df|\Sigma^{i_1,\cdots,i_s}) = \mathrm{corank}(Dg|\mathrm{Kernel}(Df)) = i_s$ is that the Jacobian of g_{s-1} with respect to the variables w_s have corank i_s. The components of this Jacobian J_s are the functions h_s. We assume that the first $i_{s-1} - i_s$ columns of the Jacobian are independent. Next we apply another coordinate transformation that replaces the coordinates corresponding to these columns with functions chosen from the set h_s so as to form a regular

$(i_{s-1} - i_s) \times (i_{s-1} - i_s)$ submatrix of the Jacobian J_s. In this new coordinate system, the Jacobian of g_{s-1} has a block triangular structure, so its corank is i_s if and only if its lower right block vanishes. Thus the component functions in the lower right block of the Jacobian in the new coordinate system are a set of defining equations for $\Sigma^{i_1,\cdots,i_s}$. Writing these functions as partial derivatives of f of order s, we can eliminate equations that are redundant by the equality of mixed partial derivatives by selecting those whose indices are non-decreasing. If the jet extension of f is transverse to the appropriate Thom-Boardman singularity, this subset of functions is a regular subset of defining functions. This finishes the inductive step of the set of defining functions. This construction is essentially an implementation of Porteous intrinsic derivatives [11]. Boardman gives combinatorial formulas for the codimensions of the singularity submanifolds.

An alternative method for computing the ranks of Df and higher derivatives relies upon the technique of "bordering matrices" from linear algebra [4]. The basic result is the following. Let A be an $n \times m$ matrix of rank i. We want to "embed" A as a submatrix of a square invertible matrix. Choose an $n \times (n-r)$ matrix B and an $(m-r) \times m$ matrix C with the properties that (A, B) has full rank m and

$$\begin{pmatrix} A \\ C \end{pmatrix}$$

has full rank m. Generic matrices B and C have these properties. Routine linear algebra calculations imply that

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is regular and the lower right hand block of size $m - r \times n - r$ in its inverse vanishes. We denote this block by $H(A)$. Further, if we perturb A to a matrix $\bar{A}$, then the rank of $\bar{A}$ is at least r and

$$\begin{pmatrix} \bar{A} & B \\ C & 0 \end{pmatrix}$$

remains invertible. The equation $H(\bar{A}) = 0$ is satisfied if and only if the rank of $\bar{A}$ is r. Since the inverse of a matrix is a rational function of its components, H is a smooth function.

Carrying out the construction above for $A = Df$, the equation $H(Df) = 0$ is a defining equation for Σ^i with $i = m - r$. In solving the equations

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

note that the columns of F give a basis for the kernel of A when $H = 0$. Moreover, we can differentiate this equation with respect to the components of A to obtain

$$\frac{\partial H_{p,q}}{\partial A_{i,j}} = -G(p, i)F(j, q)$$

This formula and the chain rule give the derivatives of the defining equations $H(Df)$ for Σ^i. A second bordering construction with the matrix of derivatives produces a set of (overdetermined) defining equations for $\Sigma^{i,j}$.

4 Computing the Thom-Boardman Stratification: an Example

For maps of $R^n \to R^n$, the singularity submanifold $\Sigma^{2,1}$ has codimension 7. Perhaps the simplest example of a map $f : R^7 \to R^7$ which has a transversal intersection with $\Sigma^{2,1}$ is given by

$$\begin{aligned} y_i &= x_i \qquad \text{for } 1 \le i \le 5 \\ y_6 &= x_6^2 + x_1 x_7 + x_7^3 \\ y_7 &= x_2 x_6 + x_3 x_7 + x_4 x_7^2 + x_5 x_6 x_7 \end{aligned} \tag{4}$$

We compute the Jacobian of f:

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ x_7 & 0 & 0 & 0 & 0 & 2x_6 & x_1 + 3x_7^2 \\ 0 & x_6 & x_7 & x_7^2 & x_6 x_7 & x_2 + x_5 x_7 & x_3 + 2x_4 x_7 + x_5 x_6 \end{pmatrix}$$

The 2 by 2 submatrix at the lower right corner being zero gives a set of 4 functions

$$g = \begin{pmatrix} 2x_6 \\ x_1 + 3x_7^2 \\ x_2 + x_5 x_7 \\ x_3 + 2x_4 x_7 + x_5 x_6 \end{pmatrix} \tag{5}$$

that define Σ^2. Differentiation of g yields

$$Dg = \begin{pmatrix} & & & & & 2dx_6 & \\ dx_1 & & & & & & +6x_7 dx_7 \\ & dx_2 & & & +x_7 dx_5 & & +x_5 dx_7 \\ & & dx_3 & +2x_7 dx_4 & & & +2x_4 dx_7 \end{pmatrix} \tag{6}$$

from which we compute the tangent space T of Σ^2, as well as the restriction of the original Jacobian J on the tangent space:

$$T = \begin{pmatrix} 0 & 0 & -6x_7 \\ 0 & -x_7 & -x_5 \\ -2x_7 & 0 & -2x_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad JT = \begin{pmatrix} 0 & 0 & -6x_7 \\ 0 & -x_7 & -x_5 \\ -2x_7 & 0 & -2x_4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_1 - 3x_7^2 \\ -x_7^2 & 0 & x_3 \end{pmatrix}.$$

Thus the conditions $\dim(\ker(J)) = 2$ and $\dim(\ker(JT)) = 1$ give 9 equations that define $\Sigma^{2,1}(f)$, namely the vanishing of the lower 2 by 2 block of J and the five entries in the last column of JT that are not identically zero. It is clear that 0 is the only solution of this over-determined system of equations.

Note that the Jacobian of g with respect to the variables x_6, x_7 that span the tangent space of Df is the matrix

$$Dg|_{\mathrm{Ker}(Df)} = \begin{pmatrix} 2 & 0 \\ 0 & 6x_7 \\ 0 & x_5 \\ 0 & 2x_4 \end{pmatrix} \tag{7}$$

which clearly has rank 1 if and only if $x_4 = x_5 = x_7 = 0$. Thus, g together with the last three components of the second column of this matrix give a regular set of defining equations for $\Sigma^{2,1}$ in this example.

We implemented both algorithms described in §3 and tested them with maps similar to the one above having an isolated point in $\Sigma^{2,1}$. Derivatives of the map were computed using ADOL-C [5], an automatic differentiation code. This avoids the truncation errors associated with computing derivatives with finite difference approximations. We began our tests with a map f that is a modification of the example described above

$$\begin{aligned} f_i(x) &= x_i \qquad \text{for } 1 \le i \le 5 \\ f_6(x) &= x_6^2 + x_1x_7 + x_7^3 \\ f_7(x) &= x_2x_6 + x_3x_7 + x_4x_7^2 + x_5x_6x_7 + x_6x_7^3 + x_7^4 \end{aligned}$$

The 2-jet of this map at the origin still has an isolated point of transversal intersection with the singularity submanifold $\Sigma^{2,1}$. We added the terms of degree 4 in f_7 to make the initial example a bit more complicated analytically than the original example. To further complicate the numerical tests, we added perturbations to f. The map f is stable [1], so perturbations also have an isolated point in $\Sigma^{2,1}$. The perturbations we studied have the form $(f + \epsilon g)(h(x))$. Here h is obtained by first using a random number generator to obtain a linear map $z = x + Rx$ with R a random matrix whose entries are normally distributed in $[0, 0.1]$. We then defined h by

$$\begin{aligned} h_i(x) &= z_i \qquad \text{for } i = 1,2,3,5,6 \\ h_4(x) &= z_4 + (e^{x_2} - 1)\sin(x_3) + \sin(x_5x_7) \\ h_7(x) &= z_7 + \tan(x_4x_6) \end{aligned}$$

The function g was defined by

$$\begin{aligned} g_i(y) &= 0 \qquad \text{for } i = 1,3,4,5 \\ g_2(y) &= \sin(z_4z_6) + \tan(z_6z_7) \\ g_4(y) &= e^{z_3z_5} - 1 \\ g_7(y) &= \cos(z_7) - 1 \end{aligned}$$

Table 1: Convergence results for $\epsilon = 0$ and two different starting values of x. Successive lines of the table are data from successive iterates of Newton's method. κ is the condition number of the Jacobian of the defining equations.

	test 1	test 2
$\|x\|$	$1.164210189976914e-01$	$1.386971169486891e-01$
	$1.856627930090916e-02$	$2.604948009547404e-02$
	$1.017859600357246e-04$	$1.121651418701914e-03$
	$4.136391532437480e-09$	$2.150917535176937e-07$
	$2.716823710082874e-17$	$7.143642136856968e-14$
κ	7.906258556895635e+00	7.906258556895608e+00

Table 1 gives data from successive iterates of Newton's method applied to solve the defining equations when $\epsilon = 0$. We observed convergence of the iteration at all randomly chosen points in the ball of radius 0.1 centered at the origin. The origin is the point in $\Sigma^{2,1}$. It is evident from the moderate value of the condition number that the equations are regular. It is also apparent that the convergence is quadratic. Table 2 shows analogous data when $\epsilon = 0.001$ together with the final values of x and the residual norm of the defining equations at the final values of x. The final test we did with this algorithm was to track the points of $\Sigma^{2,1}$ with increase ϵ. This continuation succeeded until $\epsilon = 1.66$, at which point it began to diverge.

The bordered matrix algorithm that we implemented did not converge as reliably as the direct calculation of solutions to the defining equations for $\Sigma^{2,1}$. From random initial conditions in the ball of radius 0.1, Newton's method applied to the first example above produced convergence in only 35 of 100 trials. For the second example, we obtained convergence in only 32 of 100 trials. When the bordered matrix algorithm converged, its run time was $2-3$ times faster than the direct solution of the defining equations. These tests, especially the first pair, demonstrate the feasibility of computing the Thom-Boardman stratification of a generic map by producing numerically and then solving systems of defining equations.

Table 2: Convergence results for $\epsilon = 0.001$ and two different starting values of x. Value is the norm of the defining function values at the final x and κ is the condition number.

	test 1	test 2
$\|x\|$	$1.505178047930191e-01$ $8.595870501003761e-02$ $1.315355832210626e-02$ $4.766145141188760e-04$ $4.854165624711747e-04$ $4.854166485805114e-04$	$1.969930058772691e-01$ $2.321363786625558e-02$ $8.129054141866592e-04$ $4.854171586870105e-04$ $4.854166485806945e-04$
final x	$1.0e-03*$ -0.03791640257908 -0.00698442166166 -0.00116737559545 0.48319254076764 -0.02524469932930 0.00123626681390 -0.00526018721355	$1.0e-03*$ -0.03791640257888 -0.00698442166186 -0.00116737559544 0.48319254076784 -0.02524469932924 0.00123626681389 -0.00526018721355
value	2.181602477619463e-15	6.029869709471259e-16
κ	7.390959824634281e+00	7.390959824634288e+00

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