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Structurally stable heteroclinic cycles

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This paper describes a previously undocumented phenomenon in dynamical systems theory; namely, the occurrence of heteroclinic cycles that are structurally stable within the space of C^r vector fields equivariant with respect to a symmetry group. In the space $X(M)$ of C^r vector fields on a manifold M, there is a residual set of vector fields having no trajectories joining saddle points with stable manifolds of the same dimension. Such heteroclinic connections are a structurally unstable phenomenon [**4**]. However, in the space $X_G(M) \subset X(M)$ of vector fields equivariant with respect to a symmetry group G, the situation can be quite different. We give an example of an open set U of topologically equivalent vector fields in the space of vector fields on $\mathbb{R}^3$ equivariant with respect to a particular finite subgroup $G \subset O(3)$ such that each $X \in U$ has a heteroclinic cycle that is an attractor. The heteroclinic cycles consist of three equilibrium points and three trajectories joining them.

The system we describe was first discussed by Busse and Heikes [**3**] in the context of Rayleigh–Benard convection. A similar, but more complicated, phenomenon [**1**] has been observed recently in models of flow in turbulent boundary layers [**2**] and by Kevrekides and Nicolaenko [**5**] in studies of the 'Kolmogorov–Sivashinsky' equation $\phi_t + \phi_{xxxx} + |\phi_x|^2 + \partial_x((2-\delta|\phi_x|^2)\phi_x) + \beta\phi = 0$. Here the group in question is $O(2)$ acting on $\mathbb{R}^4$ and families of heteroclinic cycles connecting pairs of equilibria are found.

The group $G \subset O(3)$ that forms the symmetry group of our examples has 24 elements and is generated by

$$r_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

G consists of cyclic permutations of the coordinate axes and reflections in the coordinate planes.

THEOREM A. *Consider the space $\chi_G(\mathbb{R}^3)$ of C^r vector fields on $\mathbb{R}^3$ that are equivariant with respect to the group G ($r \geq 1$). There is an open set $U \subset \chi_G$ of vector fields in χ_G such that*

(1) *all vector fields in U are topologically equivalent, and*

(2) *vector fields in U have heteroclinic cycles consisting of* 3 *saddle points and trajectories joining these. All trajectories that do not lie on the coordinate planes, or the lines $x = \pm y = \pm z$ are asymptotic to the heteroclinic cycles.*

The theorem is proved by displaying an element of U, analysing its dynamics, and proving that it is structurally stable. The example has cubic polynomial coefficients. We note that the set of cubic, equivariant vector fields forms a four-dimensional family.

The following lemmas are easily verified:

LEMMA 1. $X = f(x,y,z)\,\partial_x + g(x,y,z)\,\partial_y + h(x,y,z)\,\partial_z$ *is equivariant with respect to* G *if and only if*

(*a*) $f(x,y,z) = g(y,z,x) = h(z,x,y)$ *and*

(*b*) $f(x,y,z) = -f(-x,y,z) = f(x,-y,z) = f(x,y,-z)$.

LEMMA 2. *The lines defined by* $x = \pm y = \pm z$, *the coordinate axes, and the coordinate planes are invariant under the flow of any vector field* $X \in \chi_G(\mathbb{R}^3)$.

The only polynomials of degree 3 which satisfy (*b*) of Lemma 1 have the form $x(l + ax^2 + by^2 + cz^2)$. Thus, the system of equations defining a vector field in $\chi_G(\mathbb{R}^3)$ has a Taylor expansion at the origin of the form

$$\left.\begin{aligned} \dot{x} &= x(l + ax^2 + by^2 + cz^2), \\ \dot{y} &= y(l + ay^2 + bz^2 + cx^2), \\ \dot{z} &= z(l + az^2 + bx^2 + cy^2). \end{aligned}\right\} \tag{1}$$

The orbit structure of the group G forces certain subspaces of $\mathbb{R}^3$ to be invariant under the flow of an $X \in \chi_G$.

Without loss of generality, we can assume $|l| = 1$ and $|a+b+c| = 1$ in (1) by rescaling (x,y,z) and time.

LEMMA 3. *Consider the vector field* X *given by system* (1) *under the following conditions:*

(*a*) $l = 1$; (*b*) $a+b+c = -1$; (*c*) $-\frac{1}{3} < a < 0$; (*d*) $c < a < b < 0$.

Then all trajectories off the lines $x = \pm y = \pm z$ *and the coordinate planes themselves approach a cycle formed from three equilibrium points on the coordinate axes and trajectories in the coordinate planes joining these points.*

Proof. Consider the sphere S defined by $x^2 + y^2 + z^2 - 3 = G(x,y,z) = 0$ and the function $F(x,y,z) = xyz$. Calculate that

$$(\operatorname{grad} F).X = xyz(3 - (x^2 + y^2 + z^2)) = -F.G$$

and

$$\begin{aligned} \tfrac{1}{2}(\operatorname{grad} G).X &= (x^2+y^2+z^2) + a(x^4+y^4+z^4) + (b+c)(x^2y^2 + y^2z^2 + x^2z^2) \\ &= (x^2+y^2+z^2) + a(x^2+y^2+z^2)^2 - (1+3a)(x^2y^2+y^2z^2+x^2z^2) \\ &\geqslant (G+3)(1 + a(G+3)) - (\tfrac{1}{3} + a)(G+3)^2 = -\frac{G}{3}(G+3), \end{aligned}$$

since

$$x^2y^2 + y^2z^2 + x^2z^2 \leqslant \tfrac{1}{3}(x^2+y^2+z^2)^2.$$

It follows that G increases along non-zero trajectories while they lie inside S. Trajectories starting on S leave S transversely with the exception of the equilibria $(\pm 1, \pm 1, \pm 1)$. Note also that when $G \geqslant -3 - 1/a$, G decreases along trajectories. It follows that the closed spherical shell A with boundaries at radii $\sqrt{3}$ and $\sqrt{(-1/a)}$

is forward invariant under the flow and attracts all non-zero trajectories. Next observe that $(\operatorname{sgn} F)((\operatorname{grad} F).X) \leqslant 0$ in A, with equality only if $F = 0$ or $G = 0$. Therefore, all trajectories off the invariant lines $x = \pm y = \pm z$ approach the surface $F = 0$. This surface is just the union of the coordinate planes.

Consider next the flow in the coordinate planes. Requirements (*b*) and (*c*) imply that there are no equilibria of the flow off the coordinate axes in the coordinate planes. Moreover, the Poincaré–Bendixson theorem implies that all trajectories in a coordinate plane approach an equilibrium or periodic orbit. No periodic orbits exist, however, because there are no equilibria in the coordinate planes off the coordinate axes and the coordinate axes are invariant. In the (x, y) plane, the equilibria at $x = \pm\sqrt{(-1/a)}$ are stable. Thus, by cyclic permutation, there will be a heteroclinic cycle formed from trajectories in the coordinate planes that make a circuit from $(0, \sqrt{(-1/a)}, 0)$ to $(\sqrt{(-1/a)}, 0, 0)$ to $(0, 0, \sqrt{(-1/a)})$ to $(0, \sqrt{(-1/a)}, 0)$. This cycle and its reflections form the ω-limit set of all trajectories lying off the invariant lines $x = \pm y = \pm z$ and the coordinate planes.

All of the features we have described explicitly for this example persist for perturbations of X. There will still be a spherical annulus A which traps non-zero trajectories and the trajectories in this region approach an equilibrium on one of the lines $x = \pm y = \pm z$, or they approach the coordinate planes. Since the coordinate planes are invariant, and the flows within the coordinate planes are structurally stable, perturbations of X will have stable heteroclinic cycles. This completes the proof of the theorem.

Remark. The parameter choices of Lemma 3 yield an attracting heteroclinic cycle. Repelling homoclinic cycles occur with different choices and more degenerate phenomena can also appear. For example, if $a = -\frac{1}{3}$, $b + c = -\frac{2}{3}$ and $b \neq -\frac{1}{3}$, the sphere S is invariant and each octant of S is filled with closed orbits.

The example described above also yields a result about bifurcations of G-equivariant vector fields of arbitrary dimension. The vector field X considered above can be regarded as a scaled version of a truncated normal form [**4**] obtained from a one-parameter family of G-equivariant vector fields undergoing bifurcation. This yields the following result:

THEOREM B. *In the space of one-parameter families of G-equivariant vector fields, there is an open set V with the following property: If $X_\lambda \in V$, then there is a bifurcation of the origin at $\lambda = \lambda_0$ such that*

(*a*) *the origin loses stability as λ increases through λ_0*

(*b*) *for $\lambda > \lambda_0$, there is a stable heteroclinic cycle near the origin.*

Of course, other phenomena can occur for G-equivariant bifurcations corresponding to the values of (a, b, c) which appear in the normal form. In particular, if $b + c = 2a$ and $l > 0$ in the normal form, then fifth-order terms in the normal form can be expected to perturb the non-hyperbolic closed orbits of the cubic truncation to yield a non-degenerate Hopf bifurcation in the family of vector fields.

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