To fill in the missing step in this argument we will need a technical lemma about deforming maps to create some linearity. Define a **polyhedron** in $\mathbb{R}^n$ to be a subspace that is the union of finitely many convex polyhedra, each of which is a compact set obtained by intersecting finitely many half-spaces defined by linear inequalities of the form $\sum_i a_i x_i \le b$. By a **PL (piecewise linear) map** from a polyhedron to $\mathbb{R}^k$ we shall mean a map which is linear when restricted to each convex polyhedron in some decomposition of the polyhedron into convex polyhedra.

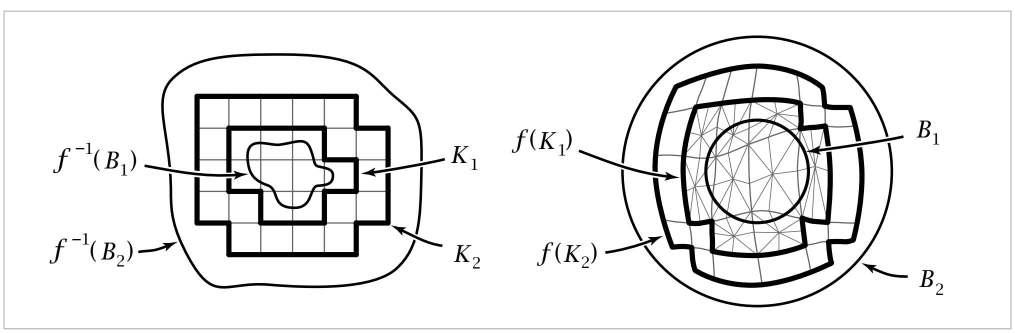
Lemma 4.10. *Let $f: I^n \to Z$ be a map, where Z is obtained from a subspace W by attaching a cell e^k. Then f is homotopic* rel $f^{-1}(W)$ *to a map f_1 for which there is a polyhedron $K \subset I^n$ such that*:

(a) *$f_1(K) \subset e^k$ and $f_1 | K$ is PL with respect to some identification of e^k with $\mathbb{R}^k$.*

(b) *$K \supset f_1^{-1}(U)$ for some nonempty open set U in e^k.*

Before proving the lemma, let us see how it finishes the proof of the cellular approximation theorem. Composing the given map $f: X^{n-1} \cup e^n \to Y^k$ with a characteristic map $I^n \to X$ for e^n, we obtain a map f as in the lemma, with $Z = Y^k$ and $W = Y^k - e^k$. The homotopy given by the lemma is fixed on ∂I^n, hence induces a homotopy f_t of $f | X^{n-1} \cup e^n$ fixed on X^{n-1}. The image of the resulting map f_1 intersects the open set U in e^k in a set contained in the union of finitely many hyperplanes of dimension at most n, so if $n < k$ there will be points $p \in U$ not in the image of f_1. □

Proof of 4.10: Identifying e^k with $\mathbb{R}^k$, let $B_1, B_2 \subset e^k$ be the closed balls of radius 1 and 2 centered at the origin. Since $f^{-1}(B_2)$ is closed and therefore compact in I^n, it follows that f is uniformly continuous on $f^{-1}(B_2)$. Thus there exists $\varepsilon > 0$ such that $|x - y| < \varepsilon$ implies $|f(x) - f(y)| < 1/2$ for all $x, y \in f^{-1}(B_2)$. Subdivide the interval I so that the induced subdivision of I^n into cubes has each cube lying in a ball of diameter less than ε. Let K_1 be the union of all the cubes meeting $f^{-1}(B_1)$, and let K_2 be the union of all the cubes meeting K_1. We may assume ε is chosen smaller than half the distance between the compact sets $f^{-1}(B_1)$ and $I^n - f^{-1}(\mathrm{int}(B_2))$, and then we will have $K_2 \subset f^{-1}(B_2)$.



Now we subdivide all the cubes of K_2 into simplices. This can be done inductively. The boundary of each cube is a union of cubes of one lower dimension, so assuming these lower-dimensional cubes have already been subdivided into simplices, we obtain a subdivision of the cube itself by taking its center point as a new vertex and joining this by a cone to each simplex in the boundary of the cube.

Let $g: K_2 \to e^k = \mathbb{R}^k$ be the map that equals f on all vertices of simplices of the subdivision and is linear on each simplex. Choose a map $\varphi: K_2 \to [0,1]$ with $\varphi(\partial K_2) = 0$ and $\varphi(K_1) = 1$, for example a map which is linear on simplices and has the value 1 on vertices of K_1 and 0 on all other vertices. Define a homotopy $f_t: K_2 \to e^k$ by the formula $(1 - t\varphi)f + (t\varphi)g$. Thus $f_0 = f$ and $f_1 | K_1 = g | K_1$. Since f_t is the constant homotopy on ∂K_2, we may extend f_t to be the constant homotopy of f on the rest of I^n.

The map f_1 takes the closure of $I^n - K_1$ to a compact set C which, we claim, is disjoint from the centerpoint 0 of B_1 and hence from a neighborhood U of 0. This will prove the lemma, with $K = K_1$, since we will then have $f_1^{-1}(U) \subset K_1$ with f_1 PL on K_1 where it is equal to g.

The verification of the claim has two steps:

(1) On $I^n - K_2$ we have $f_1 = f$, and f takes $I^n - K_2$ outside B_1 since $f^{-1}(B_1) \subset K_2$ by construction.

(2) For a simplex σ of K_2 not in K_1 we have $f(\sigma)$ contained in some ball B_σ of radius $1/2$ by the choice of ε and the fact that $K_2 \subset f^{-1}(B_2)$. Since $f(\sigma) \subset B_\sigma$ and B_σ is convex, we must have $g(\sigma) \subset B_\sigma$, hence also $f_t(\sigma) \subset B_\sigma$ for all t, and in particular $f_1(\sigma) \subset B_\sigma$. We know that B_σ is not contained in B_1 since σ contains points outside K_1 hence outside $f^{-1}(B_1)$. The radius of B_σ is half that of B_1, so it follows that 0 is not in B_σ, and hence 0 is not in $f_1(\sigma)$. □

This revised version of Lemma 4.10 requires a small adjustment in the proof of Theorem 4.23 on page 361. The sentence beginning "By repeated applications" about two-thirds of the way down the page should be modified to:

$\cdots\cdots$ By repeated applications of Lemma 4.10 we may homotope f, through maps $(I^i, \partial I^i, J^{i-1}) \to (X, B, x_0)$, so that there are simplices $\Delta_\alpha^{m+1} \subset e_\alpha^{m+1}$ and $\Delta^{n+1} \subset e^{n+1}$ for which the preimages $f^{-1}(\Delta_\alpha^{m+1})$ and $f^{-1}(\Delta^{n+1})$ are finite unions of convex polyhedra, on each of which f is the restriction of a linear map from $\mathbb{R}^i$ to $\mathbb{R}^{m+1}$ or $\mathbb{R}^{n+1}$. We may assume these linear maps are surjections by choosing the simplices Δ_α^{m+1} and Δ^{n+1} to lie in the complement of the images of the nonsurjective linear maps.