The Duality Theorem

The form of Poincaré duality we will prove asserts that for an R-orientable closed n-manifold, a certain naturally defined map $H^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism. The definition of this map will be in terms of a more general construction called *cap product*, which has close connections with cup product.

Cap Product

Let us see how cup product leads naturally to cap product. For an arbitrary space X and coefficient ring R let us for simplicity write $C_k(X;R)$ as C_k and its dual $C^k(X;R) = \mathrm{Hom}_R(C_k, R)$ as C^k. For cochains $\varphi \in C^k$ and $\psi \in C^\ell$ we have their cup product $\varphi \smile \psi \in C^{k+\ell}$. When we regard $\varphi \smile \psi$ as a function of ψ for fixed φ, we get a map $C^\ell \to C^{k+\ell}$, $\psi \mapsto \varphi \smile \psi$. One might ask whether this map $C^\ell \to C^{k+\ell}$ is the dual of a map $C_{k+\ell} \to C_\ell$. The answer turns out to be yes, and this map on chains will be the cap product map $\alpha \mapsto \alpha \frown \varphi$.

In order to define $\alpha \frown \varphi$ it suffices by linearity to take α to be a singular simplex $\sigma : \Delta^{k+\ell} \to X$, and then the formula for $\sigma \frown \varphi$ is rather like the formula defining cup product:

$$\sigma \frown \varphi = \varphi(\sigma \,|\, [v_0, \cdots, v_k])\, \sigma \,|\, [v_k, \cdots, v_{k+\ell}]$$

To say that for fixed φ the map $\alpha \mapsto \alpha \frown \varphi$ has as its dual the map $\psi \mapsto \varphi \smile \psi$ is to say that the composition $C_{k+\ell} \xrightarrow{\frown\varphi} C_\ell \xrightarrow{\psi} R$ is the map $\alpha \mapsto (\varphi \smile \psi)(\alpha)$, or in other words,

$$(*) \qquad \psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$$

This holds since for $\sigma : \Delta^{k+\ell} \to X$ we have

$$\begin{aligned}\psi(\sigma \frown \varphi) &= \psi(\varphi(\sigma|[v_0, \cdots, v_k])\sigma|[v_k, \cdots, v_{k+\ell}]) \\ &= \varphi(\sigma|[v_0, \cdots, v_k])\psi(\sigma|[v_k, \cdots, v_{k+\ell}]) = (\varphi \smile \psi)(\sigma)\end{aligned}$$

Another way to write the formula $(*)$ that may be more suggestive is in the form

$$\langle \alpha \frown \varphi, \psi \rangle = \langle \alpha, \varphi \smile \psi \rangle$$

where $\langle -, - \rangle$ is the map $C_i \times C^i \to R$ evaluating a cochain on a chain.

In order to show that the cap product $C_{k+\ell} \times C^k \to C_\ell$, $(\alpha, \varphi) \mapsto \alpha \frown \varphi$, induces a corresponding cap product $H_{k+\ell}(X;R) \times H^k(X;R) \to H_\ell(X;R)$ we need a formula for $\partial(\alpha \frown \varphi)$. This could be derived by a direct calculation similar to the one for cup products, but it is easier to use the duality relation $\langle \alpha \frown \varphi, \psi \rangle = \langle \alpha, \varphi \smile \psi \rangle$ to reduce to the cup product formula. We have

$$\begin{aligned}
\langle \partial(\alpha \frown \varphi), \psi \rangle &= \langle \alpha \frown \varphi, \delta\psi \rangle = \langle \alpha, \varphi \smile \delta\psi \rangle \\
&= (-1)^k \big[\langle \alpha, \delta(\varphi \smile \psi)\rangle - \langle \alpha, \delta\varphi \smile \psi \rangle\big] \\
&= (-1)^k \big[\langle \partial\alpha, \varphi \smile \psi \rangle - \langle \alpha, \delta\varphi \smile \psi \rangle\big] \\
&= (-1)^k \big[\langle \partial\alpha \frown \varphi, \psi \rangle - \langle \alpha \frown \delta\varphi, \psi \rangle\big]
\end{aligned}$$

Since this holds for all $\psi \in C^\ell$ and C_ℓ is a free R-module, we must have

$$\partial(\alpha \frown \varphi) = (-1)^k(\partial\alpha \frown \varphi - \alpha \frown \delta\varphi)$$

From this it follows that the cap product of a cycle α and a cocycle φ is a cycle. Further, if $\partial\alpha = 0$ then $\partial(\alpha \frown \varphi) = \pm(\alpha \frown \delta\varphi)$, so the cap product of a cycle and a coboundary is a boundary. And if $\delta\varphi = 0$ then $\partial(\alpha \frown \varphi) = \pm(\partial\alpha \frown \varphi)$, so the cap product of a boundary and a cocycle is a boundary. These facts imply that there is an induced cap product

$$H_{k+\ell}(X;R) \times H^k(X;R) \xrightarrow{\ \frown\ } H_\ell(X;R)$$

This is R-linear in each variable.

We have seen that the map $\varphi\smile : C^\ell \to C^{k+\ell}$ is dual to $\frown\varphi : C_{k+\ell} \to C_\ell$, but when we pass to homology and cohomology this may no longer be true since cohomology is not always just the dual of homology. What we have instead is a slightly weaker relation between cap and cup product in the form of the commutative diagram at the right. When the maps h are isomorphisms, for example when R is a field or when $R = \mathbb{Z}$ and the homology groups of X are free, then the map $\varphi \smile$ on cohomology is the dual of $\frown \varphi$ on homology. In these cases cup and cap product determine each other, at least if one assumes finite generation so that cohomology determines homology as well as vice versa. However, there are examples where cap and cup products are not equivalent when $R = \mathbb{Z}$ and there is torsion in homology.

$$\begin{array}{ccc}
H^\ell(X;R) & \xrightarrow{\ h\ } & \mathrm{Hom}_R(H_\ell(X;R),R) \\
\downarrow{\scriptstyle \varphi\smile} & & \downarrow{\scriptstyle (\frown\varphi)^*} \\
H^{k+\ell}(X;R) & \xrightarrow{\ h\ } & \mathrm{Hom}_R(H_{k+\ell}(X;R),R)
\end{array}$$

We shall also need relative forms of the cap product. Using the same formulas as before, one checks that there are relative cap product maps

$$H_{k+\ell}(X,A;R) \times H^k(X;R) \xrightarrow{\ \frown\ } H_\ell(X,A;R)$$
$$H_{k+\ell}(X,A;R) \times H^k(X,A;R) \xrightarrow{\ \frown\ } H_\ell(X;R)$$

For example, in the second case the cap product $C_{k+\ell}(X;R) \times C^k(X;R) \to C_\ell(X;R)$ restricts to zero on the submodule $C_{k+\ell}(A;R) \times C^k(X,A;R)$, so there is an induced cap product $C_{k+\ell}(X,A;R) \times C^k(X,A;R) \to C_\ell(X;R)$. The formula for $\partial(\sigma \frown \varphi)$ still holds, so we can pass to homology and cohomology groups. There is also a more general relative cap product

$$H_{k+\ell}(X, A \cup B;R) \times H^k(X,A;R) \xrightarrow{\ \frown\ } H_\ell(X,B;R),$$

defined when A and B are open sets in X, using the fact that $H_{k+\ell}(X, A\cup B;R)$ can be computed using the chain groups $C_n(X, A+B;R) = C_n(X;R)/C_n(A+B;R)$, as in the derivation of relative Mayer–Vietoris sequences in §2.2.

Cap product satisfies a naturality property that is a little more awkward to state than the corresponding result for cup product since both covariant and contravariant functors are involved. Given a map $f\colon X \to Y$, the relevant induced maps on homology and cohomology fit into the diagram shown below. It does not quite make sense to say this diagram commutes, but the spirit of commutativity is contained in the formula

$$\begin{array}{ccccc} H_k(X) & \times & H^\ell(X) & \xrightarrow{\frown} & H_{k-\ell}(X) \\ \big\downarrow f_* & & \big\uparrow f^* & & \big\downarrow f_* \\ H_k(Y) & \times & H^\ell(Y) & \xrightarrow{\frown} & H_{k-\ell}(Y) \end{array}$$

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$$

which is obtained by substituting $f\sigma$ for σ in the definition of cap product: $f\sigma \frown \varphi = \varphi(f\sigma\,|\,[v_0,\cdots,v_\ell])\,f\sigma\,|\,[v_\ell,\cdots,v_k]$. There are evident relative versions as well.

Now we can state Poincaré duality for closed manifolds:

Theorem 3.30 (Poincaré Duality). *If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D\colon H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.*

Recall that a fundamental class for M is an element of $H_n(M;R)$ whose image in $H_n(M\,|\,x;R)$ is a generator for each $x \in M$. The existence of such a class was shown in Theorem 3.26.

Example 3.31: Surfaces. Let M be the closed orientable surface of genus g. In Example 3.7 we computed the cup product structure in $H^*(M;\mathbb{Z})$ directly from the definitions using simplicial cohomology, and the same procedure could be used to compute cap products. However, let us instead deduce the cap product structure from the cup product structure. This is possible since we are in the favorable situation where the cohomology groups are exactly the Hom-duals of the homology groups. Using the notation from Example 3.7, the homomorphism $\alpha_i\smile$ sends β_i to γ and all other α_j's and β_j's to 0, so the dual homomorphism $\frown\alpha_i$ sends $[M]$ to the homology class $[b_i]$ Hom-dual to β_i. Thus $[M]\frown\alpha_i = [b_i]$ and so the Poincaré dual of α_i is $D(\alpha_i) = [b_i]$. Similarly we have $D(\beta_i) = -[a_i]$ where the minus sign comes from the relation $\beta_i \smile \alpha_i = -\gamma$.

It is tempting to try to interpret Poincaré duality geometrically, replacing cohomology by homology which is more geometric, and taking explicit cycles representing homology classes. Thus the geometric interpretation of the relation $D(\alpha_i) = [b_i]$ might be the statement that the cycle b_i is the Poincaré dual of the cycle a_i. The special feature of these two cycles is that they intersect transversely in a single point, while all other a_j's and b_j's are disjoint from a_i and b_i, or can be made disjoint after a homotopy. There are some difficulties with this geometric interpretation of Poincaré duality in terms of geometric intersections of cycles, however. For example, take the genus 1 case when the surface is a torus. Here there are other homology classes besides $[b_1]$ that are represented by cycles that are embedded loops intersecting a_1 transversely in one point, namely each homology class $[b_1]+m[a_1]$ is represented by such a loop, a loop going once around the torus in the b_1 direction and m times in the a_1 direction. Thus there is not a unique homology class Poincaré dual to $[a_1]$, in contrast with the situation for cohomology where the inverse of the Poincaré duality isomorphism $[M]\frown$ gives a unique cohomology class β_1 dual to $[a_1]$. The reason for this difference between homology and cohomology is that the isomorphism between H_1 and its Hom-dual H^1 is not canonical, but depends on a choice of basis, just as in linear algebra there is no canonical isomorphism between a vector space and its dual vector space.

Thus it is best to keep Poincaré duality as a statement involving both homology and cohomology. Based on the example of orientable surfaces, one might then guess that a reasonable geometric interpretation of Poincaré duality is to say that the Poincaré dual of a k-cycle in an n-manifold is the $(n-k)$-cocycle that assigns to each $(n-k)$-cycle the number of points in which it intersects the given k-cycle. Making this idea rigorous takes quite a bit of work, however, and this is not a book about manifolds so we will not attempt to do this here.

Poincaré duality for closed nonorientable surfaces using $\mathbb{Z}_2$ coefficients can be made explicit just as in the orientable case. With the notation of Example 3.8 we have $\alpha_i \smile \alpha_j$ nonzero only for $i=j$, so $D(\alpha_i)=[a_i]$. This corresponds to the geometric fact that the loops a_i can be homotoped to be all disjoint, and each a_i can be homotoped to a nearby loop a_i' that intersects a_i in one point transversely. Thus, loosely speaking, the loops a_i are their own Poincaré duals.

A nice high-dimensional example to think about is the n-dimensional torus T^n, the product of n circles. With $\mathbb{Z}$ coefficients the cup product ring is the exterior algebra $\Lambda_{\mathbb{Z}}[\alpha_1,\cdots,\alpha_n]$ where α_i is the Hom-dual of the homology class of the i^{th} circle factor S_i^1. The group $H_k(T^n;\mathbb{Z})$ is free of rank $\binom{n}{k}$ with basis the fundamental classes of the k-dimensional subtori $S_{i_1}^1\times\cdots\times S_{i_k}^1$ for $1\le i_1<\cdots<i_k\le n$. The Hom-dual of such a fundamental class is the product $\alpha_{i_1}\cdots\alpha_{i_k}$. When this product is multiplied by an $(n-k)$-dimensional class $\alpha_{j_1}\cdots\alpha_{j_{n-k}}$ the result is 0 except for the complementary product $\alpha_1\cdots\widehat{\alpha}_{i_1}\cdots\widehat{\alpha}_{i_k}\cdots\alpha_n$ where the product of these two monomials is a generator of $H^n(T^n;\mathbb{Z})$. As in the examples of surfaces it follows that the Poincaré dual of $\alpha_{i_1}\cdots\alpha_{i_k}$ is the fundamental class of the $(n-k)$-dimensional subtorus $S_1^1\times\cdots\times\widehat{S}_{i_1}^1\times\cdots\times\widehat{S}_{i_k}^1\times\cdots\times S_n^1$, up to a sign depending on orientations. This means that we again have the nice geometric interpretation of Poincaré duality in terms of intersections of subtori with subtori of complementary dimension.

Cohomology with Compact Supports

Our proof of Poincaré duality, like the construction of fundamental classes, will be by an inductive argument using Mayer–Vietoris sequences. The induction step requires a version of Poincaré duality for open subsets of M, which are noncompact and can satisfy Poincaré duality only when a different kind of cohomology called *cohomology with compact supports* is used. Before defining what this is, let us look at the conceptually simpler notion of simplicial cohomology with compact supports. Here one starts with a Δ-complex X which is locally compact. This is equivalent to saying that every point has a neighborhood that meets only finitely many simplices. Consider the subgroup $\Delta_c^i(X;G)$ of the simplicial cochain group $\Delta^i(X;G)$ consisting of cochains that are compactly supported in the sense that they take nonzero values on only finitely many simplices. The coboundary of such a cochain φ can have a nonzero value only on those $(i+1)$-simplices having a face on which φ is nonzero, and there are only finitely many such simplices by the local compactness assumption, so $\delta\varphi$ lies in $\Delta_c^{i+1}(X;G)$. Thus we have a subcomplex of the simplicial cochain complex. The cohomology groups for this subcomplex will be denoted temporarily by $H_c^i(X;G)$.

Example 3.32. Let us compute these cohomology groups when $X = \mathbb{R}$ with the Δ-complex structure having as 1-simplices the segments $[n, n+1]$ for $n \in \mathbb{Z}$. For a simplicial 0-cochain to be a cocycle it must take the same value on all vertices, but then if the cochain lies in $\Delta_c^0(X)$ it must be identically zero. Thus $H_c^0(\mathbb{R};G) = 0$. However, $H_c^1(\mathbb{R};G)$ is nonzero. Namely, consider the map $\Sigma : \Delta_c^1(\mathbb{R};G) \to G$ sending each cochain to the sum of its values on all the 1-simplices. Note that Σ is not defined on all of $\Delta^1(X)$, just on $\Delta_c^1(X)$. The map Σ vanishes on coboundaries, so it induces a map $H_c^1(\mathbb{R};G) \to G$. This is surjective since every element of $\Delta_c^1(X)$ is a cocycle. It is an easy exercise to verify that it is also injective, so $H_c^1(\mathbb{R};G) \approx G$.

Compactly supported cellular cohomology for a locally compact CW complex could be defined in a similar fashion, using cellular cochains that are nonzero on only finitely many cells. However, what we really need is singular cohomology with compact supports for spaces without any simplicial or cellular structure. The quickest definition of this is the following. Let $C_c^i(X;G)$ be the subgroup of $C^i(X;G)$ consisting of cochains $\varphi : C_i(X) \to G$ for which there exists a compact set $K = K_\varphi \subset X$ such that φ is zero on all chains in $X - K$. Note that $\delta\varphi$ is then also zero on chains in $X - K$, so $\delta\varphi$ lies in $C_c^{i+1}(X;G)$ and the $C_c^i(X;G)$'s for varying i form a subcomplex of the singular cochain complex of X. The cohomology groups $H_c^i(X;G)$ of this subcomplex are the **cohomology groups with compact supports**.

Cochains in $C_c^i(X;G)$ have compact support in only a rather weak sense. A stronger and perhaps more natural condition would have been to require cochains to be nonzero only on singular simplices contained in some compact set, depending on the cochain. However, cochains satisfying this condition do not in general form a subcomplex of the singular cochain complex. For example, if $X = \mathbb{R}$ and φ is a 0-cochain assigning a nonzero value to one point of $\mathbb{R}$ and zero to all other points, then $\delta\varphi$ assigns a nonzero value to arbitrarily large 1-simplices.

It will be quite useful to have an alternative definition of $H_c^i(X;G)$ in terms of algebraic limits, which enter the picture in the following way. The cochain group $C_c^i(X;G)$ is the union of its subgroups $C^i(X,X-K;G)$ as K ranges over compact subsets of X. Each inclusion $K \hookrightarrow L$ induces inclusions $C^i(X,X-K;G) \hookrightarrow C^i(X,X-L;G)$ for all i, so there are induced maps $H^i(X,X-K;G) \to H^i(X,X-L;G)$. These need not be injective, but one might still hope that $H_c^i(X;G)$ is somehow describable in terms of the system of groups $H^i(X,X-K;G)$ for varying K. This is indeed the case, and it is algebraic limits that provide the description.

Suppose one has abelian groups G_α indexed by some partially ordered index set I having the property that for each pair $\alpha,\beta \in I$ there exists $\gamma \in I$ with $\alpha \le \gamma$ and $\beta \le \gamma$. Such an I is called a **directed set**. Suppose also that for each pair $\alpha \le \beta$ one has a homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$, such that $f_{\alpha\alpha} = \mathbb{1}$ for each α, and if $\alpha \le \beta \le \gamma$ then $f_{\alpha\gamma}$ is the composition of $f_{\alpha\beta}$ and $f_{\beta\gamma}$. Given this data, which is called a **directed system** of groups, there are two equivalent ways of defining the **direct limit** group $\varinjlim G_\alpha$. The shorter definition is that $\varinjlim G_\alpha$ is the quotient of the direct sum $\bigoplus_\alpha G_\alpha$ by the subgroup generated by all elements of the form $a - f_{\alpha\beta}(a)$ for $a \in G_\alpha$, where we are viewing each G_α as a subgroup of $\bigoplus_\alpha G_\alpha$. The other definition, which is often more convenient to work with, runs as follows. Define an equivalence relation on the set $\coprod_\alpha G_\alpha$ by $a \sim b$ if $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$ for some γ, where $a \in G_\alpha$ and $b \in G_\beta$. This is clearly reflexive and symmetric, and transitivity follows from the directed set property. It could also be described as the equivalence relation generated by setting $a \sim f_{\alpha\beta}(a)$. Any two equivalence classes $[a]$ and $[b]$ have representatives a' and b' lying in the same G_γ, so define $[a]+[b] = [a'+b']$. One checks this is well-defined and gives an abelian group structure to the set of equivalence classes. It is easy to check further that the map sending an equivalence class $[a]$ to the coset of a in $\varinjlim G_\alpha$ is a homomorphism, with an inverse induced by the map $\sum_i a_i \mapsto \sum_i [a_i]$ for $a_i \in G_{\alpha_i}$. Thus we can identify $\varinjlim G_\alpha$ with the group of equivalence classes $[a]$.

A useful consequence of this is that if we have a subset $J \subset I$ with the property that for each $\alpha \in I$ there exists a $\beta \in J$ with $\alpha \le \beta$, then $\varinjlim G_\alpha$ is the same whether we compute it with α varying over I or just over J. In particular, if I has a maximal element γ, we can take $J = \{\gamma\}$ and then $\varinjlim G_\alpha = G_\gamma$.

Suppose now that we have a space X expressed as the union of a collection of subspaces X_α forming a directed set with respect to the inclusion relation. Then the groups $H_i(X_\alpha;G)$ for fixed i and G form a directed system, using the homomorphisms induced by inclusions. The natural maps $H_i(X_\alpha;G) \to H_i(X;G)$ induce a homomorphism $\varinjlim H_i(X_\alpha;G) \to H_i(X;G)$.

Proposition 3.33. *If a space X is the union of a directed set of subspaces X_α with the property that each compact set in X is contained in some X_α, then the natural map $\varinjlim H_i(X_\alpha;G) \to H_i(X;G)$ is an isomorphism for all i and G.*

Proof: For surjectivity, represent a cycle in X by a finite sum of singular simplices. The union of the images of these singular simplices is compact in X, hence lies in some X_α, so the map $\varinjlim H_i(X_\alpha;G)\to H_i(X;G)$ is surjective. Injectivity is similar: If a cycle in some X_α is a boundary in X, compactness implies it is a boundary in some $X_\beta\supset X_\alpha$, hence represents zero in $\varinjlim H_i(X_\alpha;G)$. □

Now we can give the alternative definition of cohomology with compact supports in terms of direct limits. For a space X, the compact subsets $K\subset X$ form a directed set under inclusion since the union of two compact sets is compact. To each compact $K\subset X$ we associate the group $H^i(X,X-K;G)$, with a fixed i and coefficient group G, and to each inclusion $K\subset L$ of compact sets we associate the natural homomorphism $H^i(X,X-K;G)\to H^i(X,X-L;G)$. The resulting limit group $\varinjlim H^i(X,X-K;G)$ is then equal to $H^i_c(X;G)$ since each element of this limit group is represented by a cocycle in $C^i(X,X-K;G)$ for some compact K, and such a cocycle is zero in $\varinjlim H^i(X,X-K;G)$ iff it is the coboundary of a cochain in $C^{i-1}(X,X-L;G)$ for some compact $L\supset K$.

Note that if X is compact, then $H^i_c(X;G)=H^i(X;G)$ since there is a unique maximal compact set $K\subset X$, namely X itself. This is also immediate from the original definition since $C^i_c(X;G)=C^i(X;G)$ if X is compact.

Example 3.34: $H^*_c(\mathbb{R}^n;G)$. To compute $\varinjlim H^i(\mathbb{R}^n,\mathbb{R}^n-K;G)$ it suffices to let K range over balls B_k of integer radius k centered at the origin since every compact set is contained in such a ball. Since $H^i(\mathbb{R}^n,\mathbb{R}^n-B_k;G)$ is nonzero only for $i=n$, when it is G, and the maps $H^n(\mathbb{R}^n,\mathbb{R}^n-B_k;G)\to H^n(\mathbb{R}^n,\mathbb{R}^n-B_{k+1};G)$ are isomorphisms, we deduce that $H^i_c(\mathbb{R}^n;G)=0$ for $i\neq n$ and $H^n_c(\mathbb{R}^n;G)\approx G$.

This example shows that cohomology with compact supports is not an invariant of homotopy type. This can be traced to difficulties with induced maps. For example, the constant map from $\mathbb{R}^n$ to a point does not induce a map on cohomology with compact supports. The maps which do induce maps on H^*_c are the *proper* maps, those for which the inverse image of each compact set is compact. In the proof of Poincaré duality, however, we will need induced maps of a different sort going in the opposite direction from what is usual for cohomology, maps $H^i_c(U;G)\to H^i_c(V;G)$ associated to inclusions $U\hookrightarrow V$ of open sets in the fixed manifold M.

The group $H^i(X,X-K;G)$ for K compact depends only on a neighborhood of K in X by excision, assuming X is Hausdorff so that K is closed. As convenient shorthand notation we will write this group as $H^i(X\,|\,K;G)$, in analogy with the similar notation used earlier for local homology. One can think of cohomology with compact supports as the limit of these 'local cohomology groups at compact subsets.'

Duality for Noncompact Manifolds

For M an R-orientable n-manifold, possibly noncompact, we can define a duality map $D_M:H^k_c(M;R)\to H_{n-k}(M;R)$ by a limiting process in the following way. For compact sets $K\subset L\subset M$ we have a diagram

$$\begin{array}{ccccc} H_n(M|L;R) & \times & H^k(M|L;R) & \xrightarrow{\frown} & \\ \downarrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i^*} & & H_{n-k}(M;R) \\ H_n(M|K;R) & \times & H^k(M|K;R) & \xrightarrow{\frown} & \end{array}$$

where $H_n(M\,|\,A;R) = H_n(M, M - A;R)$ and $H^k(M\,|\,A;R) = H^k(M, M - A;R)$. By Lemma 3.27 there are unique elements $\mu_K \in H_n(M\,|\,K;R)$ and $\mu_L \in H_n(M\,|\,L;R)$ restricting to a given orientation of M at each point of K and L, respectively. From the uniqueness we have $i_*(\mu_L) = \mu_K$. The naturality of cap product implies that $i_*(\mu_L)\frown x = \mu_L \frown i^*(x)$ for all $x \in H^k(M\,|\,K;R)$, so $\mu_K \frown x = \mu_L \frown i^*(x)$. Therefore, letting K vary over compact sets in M, the homomorphisms $H^k(M\,|\,K;R) \to H_{n-k}(M;R)$, $x \mapsto \mu_K \frown x$, induce in the limit a duality homomorphism $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$.

Since $H_c^*(M;R) = H^*(M;R)$ if M is compact, the following theorem generalizes Poincaré duality for closed manifolds:

Theorem 3.35. *The duality map $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k whenever M is an R-oriented n-manifold.*

The proof will not be difficult once we establish a technical result stated in the next lemma, concerning the commutativity of a certain diagram. Commutativity statements of this sort are usually routine to prove, but this one seems to be an exception. The reader who consults other books for alternative expositions will find somewhat uneven treatments of this technical point, and the proof we give is also not as simple as one would like.

The coefficient ring R will be fixed throughout the proof, and for simplicity we will omit it from the notation for homology and cohomology.

Lemma 3.36. *If M is the union of two open sets U and V, then there is a diagram of Mayer–Vietoris sequences, commutative up to sign*:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_c^k(U\cap V) & \longrightarrow & H_c^k(U)\oplus H_c^k(V) & \longrightarrow & H_c^k(M) & \longrightarrow & H_c^{k+1}(U\cap V) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle D_{U\cap V}} & & \downarrow{\scriptstyle D_U \oplus -D_V} & & \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{U\cap V}} & & \\ \cdots & \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U\cap V) & \longrightarrow & \cdots \end{array}$$

Proof: Compact sets $K \subset U$ and $L \subset V$ give rise to the Mayer–Vietoris sequence in the upper row of the following diagram, whose lower row is also a Mayer–Vietoris sequence.

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^k(M|K\cap L) & \longrightarrow & H^k(M|K)\oplus H^k(M|L) & \longrightarrow & H^k(M|K\cup L) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \approx} & & \downarrow{\scriptstyle \approx} & & & & \\ & & H^k(U\cap V|K\cap L) & & H^k(U|K)\oplus H^k(V|L) & & \downarrow{\scriptstyle \mu_{K\cup L}\frown} & & \\ & & \downarrow{\scriptstyle \mu_{K\cap L}\frown} & & \downarrow{\scriptstyle \mu_K\frown \,\oplus\, -\mu_L\frown} & & & & \\ \cdots & \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & \cdots \end{array}$$

The two maps labeled isomorphisms come from excision. Assuming this diagram commutes, consider passing to the limit over compact sets $K \subset U$ and $L \subset V$. Since each compact set in $U \cap V$ is contained in an intersection $K \cap L$ of compact sets $K \subset U$ and $L \subset V$, and similarly for $U \cup V$, the diagram induces a limit diagram having the form stated in the lemma. The first row of this limit diagram is exact since a direct limit of exact sequences is exact; this is an exercise at the end of the section, and follows easily from the definition of direct limits.

It remains to consider the commutativity of the preceding diagram involving K and L. In the two squares shown, not involving boundary or coboundary maps, it is a triviality to check commutativity at the level of cycles and cocycles. Less trivial is the third square, which we rewrite in the following way:

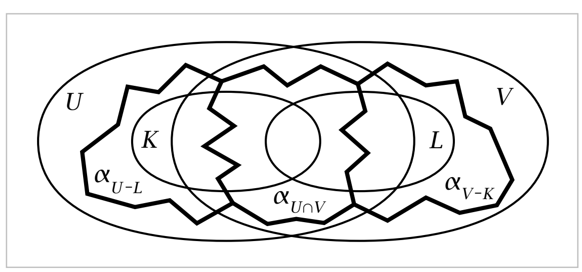
$$(*)\qquad \begin{array}{ccccc} H^k(M\,|\,K\cup L) & \xrightarrow{\ \delta\ } & H^{k+1}(M\,|\,K\cap L) & \xrightarrow{\ \approx\ } & H^{k+1}(U\cap V\,|\,K\cap L) \\ \big\downarrow{\scriptstyle \mu_{K\cup L}\frown} & & & & \big\downarrow{\scriptstyle \mu_{K\cap L}\frown} \\ H_{n-k}(M) & & \xrightarrow{\qquad\qquad \partial \qquad\qquad} & & H_{n-k-1}(U\cap V) \end{array}$$

Letting $A = M - K$ and $B = M - L$, the map δ is the coboundary map in the Mayer–Vietoris sequence obtained from the short exact sequence of cochain complexes

$$0 \longrightarrow C^*(M, A+B) \longrightarrow C^*(M,A) \oplus C^*(M,B) \longrightarrow C^*(M, A\cap B) \longrightarrow 0$$

where $C^*(M, A+B)$ consists of cochains on M vanishing on chains in A and chains in B. To evaluate the Mayer–Vietoris coboundary map δ on a cohomology class represented by a cocycle $\varphi \in C^*(M, A\cap B)$, the first step is to write $\varphi = \varphi_A - \varphi_B$ for $\varphi_A \in C^*(M,A)$ and $\varphi_B \in C^*(M,B)$. Then $\delta[\varphi]$ is represented by the cocycle $\delta\varphi_A = \delta\varphi_B \in C^*(M, A+B)$, where the equality $\delta\varphi_A = \delta\varphi_B$ comes from the fact that φ is a cocycle, so $\delta\varphi = \delta\varphi_A - \delta\varphi_B = 0$. Similarly, the boundary map ∂ in the homology Mayer–Vietoris sequence is obtained by representing an element of $H_i(M)$ by a cycle z that is a sum of chains $z_U \in C_i(U)$ and $z_V \in C_i(V)$, and then $\partial[z] = [\partial z_U]$.

Via barycentric subdivision, the class $\mu_{K\cup L}$ can be represented by a chain α that is a sum $\alpha_{U-L} + \alpha_{U\cap V} + \alpha_{V-K}$ of chains in $U - L$, $U \cap V$, and $V - K$, respectively, since these three open sets cover M. The chain $\alpha_{U\cap V}$ represents $\mu_{K\cap L}$ since the other two chains α_{U-L} and α_{V-K} lie in the complement of $K \cap L$, hence vanish in $H_n(M\,|\,K\cap L) \approx H_n(U\cap V\,|\,K\cap L)$. Similarly, $\alpha_{U-L} + \alpha_{U\cap V}$ represents μ_K.



In the square $(*)$ let φ be a cocycle representing an element of $H^k(M\,|\,K\cup L)$. Under δ this maps to the cohomology class of $\delta\varphi_A$. Continuing on to $H_{n-k-1}(U\cap V)$

we obtain $\alpha_{U\cap V} \frown \delta\varphi_A$, which is in the same homology class as $\partial\alpha_{U\cap V} \frown \varphi_A$ since

$$\partial(\alpha_{U\cap V} \frown \varphi_A) = (-1)^k(\partial\alpha_{U\cap V} \frown \varphi_A \;-\; \alpha_{U\cap V} \frown \delta\varphi_A)$$

and $\alpha_{U\cap V} \frown \varphi_A$ is a chain in $U \cap V$.

Going around the square $(*)$ the other way, φ maps first to $\alpha \frown \varphi$. To apply the Mayer–Vietoris boundary map ∂ to this, we first write $\alpha \frown \varphi$ as a sum of a chain in U and a chain in V:

$$\alpha \frown \varphi = (\alpha_{U-L} \frown \varphi) + (\alpha_{U\cap V} \frown \varphi + \alpha_{V-K} \frown \varphi)$$

Then we take the boundary of the first of these two chains, obtaining the homology class $[\partial(\alpha_{U-L} \frown \varphi)] \in H_{n-k-1}(U \cap V)$. To compare this with $[\partial\alpha_{U\cap V} \frown \varphi_A]$, we have

$$\begin{aligned}
\partial(\alpha_{U-L} \frown \varphi) &= (-1)^k \partial\alpha_{U-L} \frown \varphi && \text{since } \delta\varphi = 0 \\
&= (-1)^k \partial\alpha_{U-L} \frown \varphi_A && \text{since } \partial\alpha_{U-L} \frown \varphi_B = 0, \quad \varphi_B \text{ being} \\
& && \text{zero on chains in } B = M - L \\
&= (-1)^{k+1} \partial\alpha_{U\cap V} \frown \varphi_A
\end{aligned}$$

where this last equality comes from the fact that $\partial(\alpha_{U-L} + \alpha_{U\cap V}) \frown \varphi_A = 0$ since $\partial(\alpha_{U-L} + \alpha_{U\cap V})$ is a chain in $U - K$ by the earlier observation that $\alpha_{U-L} + \alpha_{U\cap V}$ represents μ_K, and φ_A vanishes on chains in $A = M - K$.

Thus the square $(*)$ commutes up to a sign depending only on k. ◻

Proof of Poincaré Duality: There are two inductive steps, finite and infinite:

(A) If M is the union of open sets U and V and if D_U, D_V, and $D_{U\cap V}$ are isomorphisms, then so is D_M. Via the five-lemma, this is immediate from the preceding lemma.

(B) If M is the union of a sequence of open sets $U_1 \subset U_2 \subset \cdots$ and each duality map $D_{U_i}: H_c^k(U_i) \to H_{n-k}(U_i)$ is an isomorphism, then so is D_M. To show this we notice first that by excision, $H_c^k(U_i)$ can be regarded as the limit of the groups $H^k(M\,|\,K)$ as K ranges over compact subsets of U_i. Then there are natural maps $H_c^k(U_i) \to H_c^k(U_{i+1})$ since the second of these groups is a limit over a larger collection of K's. Thus we can form $\varinjlim H_c^k(U_i)$ which is obviously isomorphic to $H_c^k(M)$ since the compact sets in M are just the compact sets in all the U_i's. By Proposition 3.33, $H_{n-k}(M) \approx \varinjlim H_{n-k}(U_i)$. The map D_M is thus the limit of the isomorphisms D_{U_i}, hence is an isomorphism.

Now after all these preliminaries we can prove the theorem in three easy steps:

(1) The case $M = \mathbb{R}^n$ begins the inductive process. For a ball $B \subset \mathbb{R}^n$ the maps $H^k(\mathbb{R}^n, \mathbb{R}^n - B) \to H_c^k(\mathbb{R}^n)$ are isomorphisms for all k, as we saw in Example 3.34, so we can consider the cap product map $H_n(\mathbb{R}^n, \mathbb{R}^n - B) \times H^k(\mathbb{R}^n, \mathbb{R}^n - B) \to H_{n-k}(\mathbb{R}^n)$. The only nontrivial case is $k = n$, where each group is isomorphic to the coefficient ring R and we need to show that the cap product of a generator with a generator is a generator. Using the formula $\langle \alpha \frown \varphi, \psi \rangle = \langle \alpha, \varphi \smile \psi \rangle$ with $\psi = 1 \in H^0(\mathbb{R}^n)$, we see

that if α and φ are generators of their respective homology and cohomology groups then $\langle \alpha \frown \varphi, 1\rangle = \langle \alpha, \varphi\rangle$ generates R so $\alpha \frown \varphi$ must generate $H_0(\mathbb{R}^n)$.

(2) More generally, D_M is an isomorphism for M an arbitrary open set in $\mathbb{R}^n$. To see this, first write M as a countable union of nonempty bounded convex open sets U_i, for example open balls, and let $V_i = \bigcup_{j<i} U_j$. Both V_i and $U_i \cap V_i$ are unions of $i-1$ bounded convex open sets, so by induction on the number of such sets in a cover we may assume that D_{V_i} and $D_{U_i \cap V_i}$ are isomorphisms. By (1), D_{U_i} is an isomorphism since U_i is homeomorphic to $\mathbb{R}^n$. Hence $D_{U_i \cup V_i}$ is an isomorphism by (A). Since M is the increasing union of the V_i's and each D_{V_i} is an isomorphism, so is D_M by (B).

(3) If M is a finite or countably infinite union of open sets U_i homeomorphic to $\mathbb{R}^n$, the theorem now follows by the argument in (2), with each appearance of the words 'bounded convex open set' replaced by 'open set in $\mathbb{R}^n$.' Thus the proof is finished for closed manifolds, as well as for all the noncompact manifolds one ever encounters in actual practice.

To handle a completely general noncompact manifold M we use a Zorn's Lemma argument. Consider the collection of open sets $U \subset M$ for which the duality maps D_U are isomorphisms. This collection is partially ordered by inclusion, and the union of every totally ordered subcollection is again in the collection by the argument in (B), which did not really use the hypothesis that the collection $\{U_i\}$ was indexed by the positive integers. Zorn's Lemma then implies that there exists a maximal open set U for which the theorem holds. If $U \neq M$, choose a point $x \in M - U$ and an open neighborhood V of x homeomorphic to $\mathbb{R}^n$. The theorem holds for V and $U \cap V$ by (1) and (2), and it holds for U by assumption, so by (A) it holds for $U \cup V$, contradicting the maximality of U. ◻

Corollary 3.37. *A closed manifold of odd dimension has Euler characteristic zero.*

Proof: Let M be a closed n-manifold. If M is orientable, we have $\operatorname{rank} H_i(M;\mathbb{Z}) = \operatorname{rank} H^{n-i}(M;\mathbb{Z})$, which equals $\operatorname{rank} H_{n-i}(M;\mathbb{Z})$ by the universal coefficient theorem. Thus if n is odd, all the terms of $\sum_i (-1)^i \operatorname{rank} H_i(M;\mathbb{Z})$ cancel in pairs.

If M is not orientable we apply the same argument using $\mathbb{Z}_2$ coefficients, with $\operatorname{rank} H_i(M;\mathbb{Z})$ replaced by $\dim H_i(M;\mathbb{Z}_2)$, the dimension as a vector space over $\mathbb{Z}_2$, to conclude that $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}_2) = 0$. It remains to check that this alternating sum equals the Euler characteristic $\sum_i (-1)^i \operatorname{rank} H_i(M;\mathbb{Z})$. We can do this by using the isomorphisms $H_i(M;\mathbb{Z}_2) \approx H^i(M;\mathbb{Z}_2)$ and applying the universal coefficient theorem for cohomology. Each $\mathbb{Z}$ summand of $H_i(M;\mathbb{Z})$ gives a $\mathbb{Z}_2$ summand of $H^i(M;\mathbb{Z}_2)$. Each $\mathbb{Z}_m$ summand of $H_i(M;\mathbb{Z})$ with m even gives $\mathbb{Z}_2$ summands of $H^i(M;\mathbb{Z}_2)$ and $H^{i+1}(M,\mathbb{Z}_2)$, whose contributions to $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}_2)$ cancel. And $\mathbb{Z}_m$ summands of $H_i(M;\mathbb{Z})$ with m odd contribute nothing to $H^*(M;\mathbb{Z}_2)$. ◻

Connection with Cup Product

Because of the close connection between cap and cup products, expressed in the formula $\langle \alpha \frown \varphi, \psi \rangle = \langle \alpha, \varphi \smile \psi \rangle$, Poincaré duality has nontrivial implications for the cup product structure of manifolds. For a closed R-orientable n-manifold M, consider the cup product pairing

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R, \qquad (\varphi, \psi) \mapsto \langle [M], \varphi \smile \psi \rangle = (\varphi \smile \psi)[M]$$

Such a bilinear pairing $A \times B \to R$ is said to be **nonsingular** if the maps $A \to \mathrm{Hom}(B,R)$ and $B \to \mathrm{Hom}(A,R)$, obtained by viewing the pairing as a function of each variable separately, are both isomorphisms.

Proposition 3.38. *The cup product pairing is nonsingular for closed R-orientable manifolds when R is a field, or when $R = \mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.*

Proof: Consider the composition

$$H^{n-k}(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_{n-k}(M;R),R) \xrightarrow{D^*} \mathrm{Hom}_R(H^k(M;R),R)$$

where h is the map appearing in the universal coefficient theorem, induced by evaluation of cochains on chains, and D^* is the Hom-dual of the Poincaré duality map $D : H^k \to H_{n-k}$. The composition D^*h sends $\psi \in H^{n-k}(M;R)$ to the homomorphism $\varphi \mapsto \langle [M] \frown \varphi, \rangle = \langle [M], \varphi \smile \psi \rangle$. For field coefficients or for integer coefficients with torsion factored out, h is an isomorphism. Nonsingularity of the pairing in one of its variables is then equivalent to D being an isomorphism. Nonsingularity in the other variable follows by commutativity of cup product. $\square$

Corollary 3.39. *If M is a closed connected orientable n-manifold, then for each element $\alpha \in H^k(M;\mathbb{Z})$ of infinite order that is not a proper multiple of another element, there exists an element $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $\alpha \smile \beta$ is a generator of $H^n(M;\mathbb{Z}) \approx \mathbb{Z}$. With coefficients in a field the same conclusion holds for any $\alpha \neq 0$.*

Proof: The hypotheses on α mean that it generates a $\mathbb{Z}$ summand of $H^k(M;\mathbb{Z})$. There is then a homomorphism $\varphi : H^k(M;\mathbb{Z}) \to \mathbb{Z}$ with $\varphi(\alpha) = 1$. By the nonsingularity of the cup product pairing, φ is realized by taking cup product with an element $\beta \in H^{n-k}(M;\mathbb{Z})$ and evaluating on $[M]$, so $\alpha \smile \beta$ generates $H^n(M;\mathbb{Z})$. The case of field coefficients is similar. $\square$

Example 3.40: Projective Spaces. The cup product structure of $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ as a truncated polynomial ring $\mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 2$ can easily be deduced from this as follows. The inclusion $\mathbb{C}\mathrm{P}^{n-1} \hookrightarrow \mathbb{C}\mathrm{P}^n$ induces an isomorphism on H^i for $i \leq 2n-2$, so by induction on n, $H^{2i}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is generated by α^i for $i < n$. By the corollary, there is an integer m such that the product $\alpha \smile m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. This can only happen if $m = \pm 1$, and therefore $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$. The same argument shows $H^*(\mathbb{H}\mathrm{P}^n;\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 4$. For $\mathbb{R}\mathrm{P}^n$ one can use the same argument with $\mathbb{Z}_2$ coefficients to deduce that $H^*(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ with $|\alpha| = 1$. The cup product structure in infinite-dimensional projective spaces follows from the finite-dimensional case, as we saw in the proof of Theorem 3.12.

Could there be a closed manifold whose cohomology is additively isomorphic to that of $\mathbb{C}\mathrm{P}^n$ but with a different cup product structure? For $n = 2$ the answer is no since duality implies that the square of a generator of H^2 must be a generator of H^4. For $n = 3$, duality says that the product of generators of H^2 and H^4 must be a generator of H^6, but nothing is said about the square of a generator of H^2. Indeed, for $S^2 \times S^4$, whose cohomology has the same additive structure as $\mathbb{C}\mathrm{P}^3$, the square of the generator of $H^2(S^2 \times S^4;\mathbb{Z})$ is zero since it is the pullback of a generator of $H^2(S^2;\mathbb{Z})$ under the projection $S^2 \times S^4 \to S^2$, and in $H^*(S^2;\mathbb{Z})$ the square of the generator of H^2 is zero. More generally, an exercise for §4.D describes closed 6-manifolds having the same cohomology groups as $\mathbb{C}\mathrm{P}^3$ but where the square of the generator of H^2 is an arbitrary multiple of a generator of H^4.

Example 3.41: Lens Spaces. Cup products in lens spaces can be computed in the same way as in projective spaces. For a lens space L^{2n+1} of dimension $2n+1$ with fundamental group $\mathbb{Z}_m$, we computed $H_i(L^{2n+1};\mathbb{Z})$ in Example 2.43 to be $\mathbb{Z}$ for $i = 0$ and $2n+1$, $\mathbb{Z}_m$ for odd $i < 2n+1$, and 0 otherwise. In particular, this implies that L^{2n+1} is orientable, which can also be deduced from the fact that L^{2n+1} is the orbit space of an action of $\mathbb{Z}_m$ on S^{2n+1} by orientation-preserving homeomorphisms, using an exercise at the end of this section. By the universal coefficient theorem, $H^i(L^{2n+1};\mathbb{Z}_m)$ is $\mathbb{Z}_m$ for each $i \le 2n+1$. Let $\alpha \in H^1(L^{2n+1};\mathbb{Z}_m)$ and $\beta \in H^2(L^{2n+1};\mathbb{Z}_m)$ be generators. The statement we wish to prove is:

$$H^j(L^{2n+1};\mathbb{Z}_m) \text{ is generated by } \begin{cases} \beta^i & \text{for } j = 2i \\ \alpha\beta^i & \text{for } j = 2i+1 \end{cases}$$

By induction on n we may assume this holds for $j \le 2n-1$ since we have a lens space $L^{2n-1} \subset L^{2n+1}$ with this inclusion inducing an isomorphism on H^j for $j \le 2n-1$, as one sees by comparing the cellular chain complexes for L^{2n-1} and L^{2n+1}. The preceding corollary does not apply directly for $\mathbb{Z}_m$ coefficients with arbitrary m, but its proof does since the maps $h: H^i(L^{2n+1};\mathbb{Z}_m) \to \mathrm{Hom}(H_i(L^{2n+1};\mathbb{Z}_m),\mathbb{Z}_m)$ are isomorphisms. We conclude that $\beta \smile k\alpha\beta^{n-1}$ generates $H^{2n+1}(L^{2n+1};\mathbb{Z}_m)$ for some integer k. We must have k relatively prime to m, otherwise the product $\beta \smile k\alpha\beta^{n-1} = k\alpha\beta^n$ would have order less than m and so could not generate $H^{2n+1}(L^{2n+1};\mathbb{Z}_m)$. Then since k is relatively prime to m, $\alpha\beta^n$ is also a generator of $H^{2n+1}(L^{2n+1};\mathbb{Z}_m)$. From this it follows that β^n must generate $H^{2n}(L^{2n+1};\mathbb{Z}_m)$, otherwise it would have order less than m and so therefore would $\alpha\beta^n$.

The rest of the cup product structure on $H^*(L^{2n+1};\mathbb{Z}_m)$ is determined once α^2 is expressed as a multiple of β. When m is odd, the commutativity formula for cup product implies $\alpha^2 = 0$. When m is even, commutativity implies only that α^2 is either zero or the unique element of $H^2(L^{2n+1};\mathbb{Z}_m) \approx \mathbb{Z}_m$ of order two. In fact it is the latter possibility which holds, since the 2-skeleton L^2 is the circle L^1 with a 2-cell attached by a map of degree m, and we computed the cup product structure in this 2-complex in Example 3.9. It does not seem to be possible to deduce the nontriviality of α^2 from Poincaré duality alone, except when $m = 2$.

The cup product structure for an infinite-dimensional lens space L^∞ follows from the finite-dimensional case since the restriction map $H^j(L^\infty;\mathbb{Z}_m) \to H^j(L^{2n+1};\mathbb{Z}_m)$ is an isomorphism for $j \le 2n+1$. As with $\mathbb{R}P^n$, the ring structure in $H^*(L^{2n+1};\mathbb{Z})$ is determined by the ring structure in $H^*(L^{2n+1};\mathbb{Z}_m)$, and likewise for L^∞, where one has the slightly simpler structure $H^*(L^\infty;\mathbb{Z}) \approx \mathbb{Z}[\alpha]/(m\alpha)$ with $|\alpha| = 2$. The case of L^{2n+1} is obtained from this by setting $\alpha^{n+1} = 0$ and adjoining the extra $\mathbb{Z} \approx H^{2n+1}(L^{2n+1};\mathbb{Z})$.

A different derivation of the cup product structure in lens spaces is given in Example 3E.2.

Using the ad hoc notation $H^k_{free}(M)$ for $H^k(M)$ modulo its torsion subgroup, the preceding proposition implies that for a closed orientable manifold M of dimension $2n$, the middle-dimensional cup product pairing $H^n_{free}(M)\times H^n_{free}(M) \to \mathbb{Z}$ is a nonsingular bilinear form on $H^n_{free}(M)$. This form is symmetric or skew-symmetric according to whether n is even or odd. The algebra in the skew-symmetric case is rather simple: With a suitable choice of basis, the matrix of a skew-symmetric nonsingular bilinear form over $\mathbb{Z}$ can be put into the standard form consisting of 2×2 blocks $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ along the diagonal and zeros elsewhere, according to an algebra exercise at the end of the section. In particular, the rank of $H^n(M^{2n})$ must be even when n is odd. We are already familiar with these facts in the case $n = 1$ by the explicit computations of cup products for surfaces in §3.2.

The symmetric case is much more interesting algebraically. There are only finitely many isomorphism classes of symmetric nonsingular bilinear forms over $\mathbb{Z}$ of a fixed rank, but this 'finitely many' grows rather rapidly, for example it is more than 80 million for rank 32; see [Serre 1973] for an exposition of this beautiful chapter of number theory. It is known that for each even $n \ge 2$, every symmetric nonsingular form actually occurs as the cup product pairing in some closed manifold M^{2n}. One can even take M^{2n} to be simply-connected and have the bare minimum of homology: $\mathbb{Z}$'s in dimensions 0 and $2n$ and a $\mathbb{Z}^k$ in dimension n. For $n = 2$ there are at most two nonhomeomorphic simply-connected closed 4-manifolds with the same bilinear form. Namely, there are two manifolds with the same form if the square $\alpha \smile \alpha$ of some $\alpha \in H^2(M^4)$ is an odd multiple of a generator of $H^4(M^4)$, for example for $\mathbb{C}P^2$, and otherwise the M^4 is unique, for example for S^4 or $S^2\times S^2$; see [Freedman & Quinn 1990]. In §4.C we take the first step in this direction by proving a classical result of J. H. C. Whitehead that the homotopy type of a simply-connected closed 4-manifold is uniquely determined by its cup product structure.