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# Chapter 4

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## Homotopy Theory

Homotopy theory begins with the homotopy groups  $\pi_n(X)$ , which are the natural higher-dimensional analogs of the fundamental group. These higher homotopy groups have certain formal similarities with homology groups. For example,  $\pi_n(X)$  turns out to be always abelian for  $n \geq 2$ , and there are relative homotopy groups fitting into a long exact sequence just like the long exact sequence of homology groups. However, the higher homotopy groups are much harder to compute than either homology groups or the fundamental group, due to the fact that neither the excision property for homology nor van Kampen's theorem for  $\pi_1$  holds for higher homotopy groups.

In spite of these computational difficulties, homotopy groups are of great theoretical significance. One reason for this is Whitehead's theorem that a map between CW complexes which induces isomorphisms on all homotopy groups is a homotopy equivalence. The stronger statement that two CW complexes with isomorphic homotopy groups are homotopy equivalent is usually false, however. One of the rare cases when a CW complex does have its homotopy type uniquely determined by its homotopy groups is when it has just a single nontrivial homotopy group. Such spaces, known as Eilenberg–MacLane spaces, turn out to play a fundamental role in algebraic topology for a variety of reasons. Perhaps the most important is their close connection with cohomology: Cohomology classes in a CW complex correspond bijectively with homotopy classes of maps from the complex into an Eilenberg–MacLane space.

Thus cohomology has a strictly homotopy-theoretic interpretation, and there is an analogous but more subtle homotopy-theoretic interpretation of homology, explained in §4.F.

A more elementary and direct connection between homotopy and homology is the Hurewicz theorem, asserting that the first nonzero homotopy group  $\pi_n(X)$  of a simply-connected space  $X$  is isomorphic to the first nonzero homology group  $\tilde{H}_n(X)$ . This result, along with its relative version, is one of the cornerstones of algebraic topology.

Though the excision property does not always hold for homotopy groups, in some important special cases there is a range of dimensions in which it does hold. This leads to the idea of stable homotopy groups, the beginning of stable homotopy theory. Perhaps the major unsolved problem in algebraic topology is the computation of the stable homotopy groups of spheres. Near the end of §4.2 we give some tables of known calculations that show quite clearly the complexity of the problem.

Included in §4.2 is a brief introduction to fiber bundles, which generalize covering spaces and play a somewhat analogous role for higher homotopy groups. It would easily be possible to devote a whole book to the subject of fiber bundles, even the special case of vector bundles, but here we use fiber bundles only to provide a few basic examples and to motivate their more flexible homotopy-theoretic generalization, fibrations, which play a large role in §4.3. Among other things, fibrations allow one to describe, in theory at least, how the homotopy type of an arbitrary CW complex is built up from its homotopy groups by an inductive procedure of forming ‘twisted products’ of Eilenberg–MacLane spaces. This is the notion of a Postnikov tower. In favorable cases, including all simply-connected CW complexes, the additional data beyond homotopy groups needed to determine a homotopy type can also be described, in the form of a sequence of cohomology classes called the  $k$ -invariants of a space. If these are all zero, the space is homotopy equivalent to a product of Eilenberg–MacLane spaces, and otherwise not. Unfortunately the  $k$ -invariants are cohomology classes in rather complicated spaces in general, so this is not a practical way of classifying homotopy types, but it is useful for various more theoretical purposes.

This chapter is arranged so that it begins with purely homotopy-theoretic notions, largely independent of homology and cohomology theory, whose roles gradually increase in later sections of the chapter. It should therefore be possible to read a good portion of this chapter immediately after reading Chapter 1, with just an occasional glimpse at Chapter 2 for algebraic definitions, particularly the notion of an exact sequence which is just as important in homotopy theory as in homology and cohomology theory.

## 4.1 Homotopy Groups

Perhaps the simplest noncontractible spaces are spheres, so to get a glimpse of the subtlety inherent in homotopy groups let us look at some of the calculations of the groups  $\pi_i(S^n)$  that have been made. A small sample is shown in the table below, extracted from [Toda 1962].

|              |   | $\pi_i(S^n)$    |              |              |                |                |                   |                                     |                                    |                                    |                                       |                   |                                    |
|--------------|---|-----------------|--------------|--------------|----------------|----------------|-------------------|-------------------------------------|------------------------------------|------------------------------------|---------------------------------------|-------------------|------------------------------------|
|              |   | $i \rightarrow$ |              |              |                |                |                   |                                     |                                    |                                    |                                       |                   |                                    |
|              |   | 1               | 2            | 3            | 4              | 5              | 6                 | 7                                   | 8                                  | 9                                  | 10                                    | 11                | 12                                 |
| $n$          | 1 | $\mathbb{Z}$    | 0            | 0            | 0              | 0              | 0                 | 0                                   | 0                                  | 0                                  | 0                                     | 0                 | 0                                  |
| $\downarrow$ | 2 | 0               | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$                      | $\mathbb{Z}_2$                     | $\mathbb{Z}_3$                     | $\mathbb{Z}_{15}$                     | $\mathbb{Z}_2$    | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|              | 3 | 0               | 0            | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_{12}$ | $\mathbb{Z}_2$                      | $\mathbb{Z}_2$                     | $\mathbb{Z}_3$                     | $\mathbb{Z}_{15}$                     | $\mathbb{Z}_2$    | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|              | 4 | 0               | 0            | 0            | $\mathbb{Z}$   | $\mathbb{Z}_2$ | $\mathbb{Z}_2$    | $\mathbb{Z} \times \mathbb{Z}_{12}$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_{24} \times \mathbb{Z}_3$ | $\mathbb{Z}_{15}$ | $\mathbb{Z}_2$                     |
|              | 5 | 0               | 0            | 0            | 0              | $\mathbb{Z}$   | $\mathbb{Z}_2$    | $\mathbb{Z}_2$                      | $\mathbb{Z}_{24}$                  | $\mathbb{Z}_2$                     | $\mathbb{Z}_2$                        | $\mathbb{Z}_2$    | $\mathbb{Z}_{30}$                  |
|              | 6 | 0               | 0            | 0            | 0              | 0              | $\mathbb{Z}$      | $\mathbb{Z}_2$                      | $\mathbb{Z}_2$                     | $\mathbb{Z}_{24}$                  | 0                                     | $\mathbb{Z}$      | $\mathbb{Z}_2$                     |
|              | 7 | 0               | 0            | 0            | 0              | 0              | 0                 | $\mathbb{Z}$                        | $\mathbb{Z}_2$                     | $\mathbb{Z}_2$                     | $\mathbb{Z}_{24}$                     | 0                 | 0                                  |
|              | 8 | 0               | 0            | 0            | 0              | 0              | 0                 | 0                                   | $\mathbb{Z}$                       | $\mathbb{Z}_2$                     | $\mathbb{Z}_2$                        | $\mathbb{Z}_{24}$ | 0                                  |

This is an intriguing mixture of pattern and chaos. The most obvious feature is the large region of zeros below the diagonal, and indeed  $\pi_i(S^n) = 0$  for all  $i < n$  as we show in Corollary 4.9. There is also the sequence of zeros in the first row, suggesting that  $\pi_i(S^1) = 0$  for all  $i > 1$ . This too is a fairly elementary fact, a special case of Proposition 4.1, following easily from covering space theory.

The coincidences in the second and third rows can hardly be overlooked. These are the case  $n = 1$  of isomorphisms  $\pi_i(S^{2n}) \approx \pi_{i-1}(S^{2n-1}) \times \pi_i(S^{4n-1})$  that hold for  $n = 1, 2, 4$  and all  $i$ . The next case  $n = 2$  says that each entry in the fourth row is the product of the entry diagonally above it to the left and the entry three units below it. Actually, these isomorphisms  $\pi_i(S^{2n}) \approx \pi_{i-1}(S^{2n-1}) \times \pi_i(S^{4n-1})$  hold for all  $n$  if one factors out 2-torsion, the elements of order a power of 2. This is a theorem of James that will be proved in [SSAT].

The next regular feature in the table is the sequence of  $\mathbb{Z}$ 's down the diagonal. This is an illustration of the Hurewicz theorem, which asserts that for a simply-connected space  $X$ , the first nonzero homotopy group  $\pi_n(X)$  is isomorphic to the first nonzero homology group  $H_n(X)$ .

One may observe that all the groups above the diagonal are finite except for  $\pi_3(S^2)$ ,  $\pi_7(S^4)$ , and  $\pi_{11}(S^6)$ . In §4.B we use cup products in cohomology to show that  $\pi_{4k-1}(S^{2k})$  contains a  $\mathbb{Z}$  direct summand for all  $k \geq 1$ . It is a theorem of Serre proved in [SSAT] that  $\pi_i(S^n)$  is finite for  $i > n$  except for  $\pi_{4k-1}(S^{2k})$ , which is the direct sum of  $\mathbb{Z}$  with a finite group. So all the complexity of the homotopy groups of spheres resides in finite abelian groups. The problem thus reduces to computing the  $p$ -torsion in  $\pi_i(S^n)$  for each prime  $p$ .

An especially interesting feature of the table is that along each diagonal the groups  $\pi_{n+k}(S^n)$  with  $k$  fixed and varying  $n$  eventually become independent of  $n$  for large enough  $n$ . This stability property is the Freudenthal suspension theorem, proved in §4.2 where we give more extensive tables of these stable homotopy groups of spheres.

## Definitions and Basic Constructions

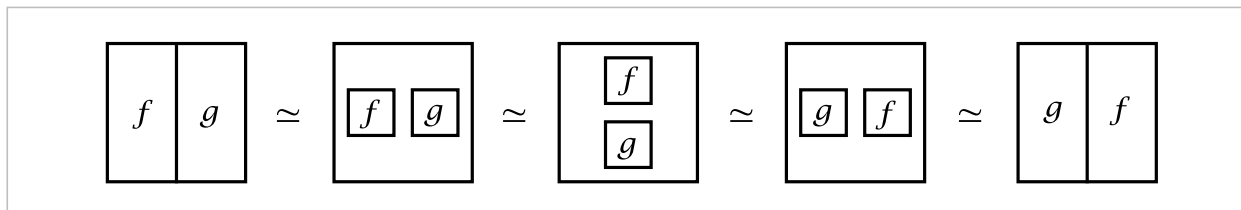
Let  $I^n$  be the  $n$ -dimensional unit cube, the product of  $n$  copies of the interval  $[0, 1]$ . The boundary  $\partial I^n$  of  $I^n$  is the subspace consisting of points with at least one coordinate equal to 0 or 1. For a space  $X$  with basepoint  $x_0 \in X$ , define  $\pi_n(X, x_0)$  to be the set of homotopy classes of maps  $f: (I^n, \partial I^n) \rightarrow (X, x_0)$ , where homotopies  $f_t$  are required to satisfy  $f_t(\partial I^n) = x_0$  for all  $t$ . The definition extends to the case  $n = 0$  by taking  $I^0$  to be a point and  $\partial I^0$  to be empty, so  $\pi_0(X, x_0)$  is just the set of path-components of  $X$ .

When  $n \geq 2$ , a sum operation in  $\pi_n(X, x_0)$ , generalizing the composition operation in  $\pi_1$ , is defined by

$$(f + g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n), & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \dots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

It is evident that this sum is well-defined on homotopy classes. Since only the first coordinate is involved in the sum operation, the same arguments as for  $\pi_1$  show that  $\pi_n(X, x_0)$  is a group, with identity element the constant map sending  $I^n$  to  $x_0$  and with inverses given by  $-f(s_1, s_2, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n)$ .

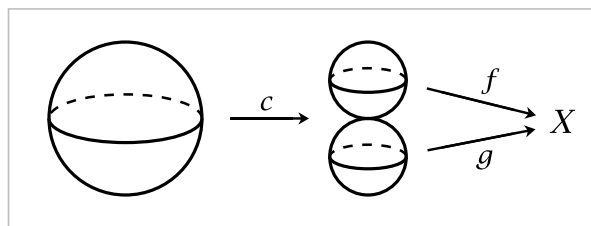
The additive notation for the group operation is used because  $\pi_n(X, x_0)$  is abelian for  $n \geq 2$ . Namely,  $f + g \simeq g + f$  via the homotopy indicated in the following figures.



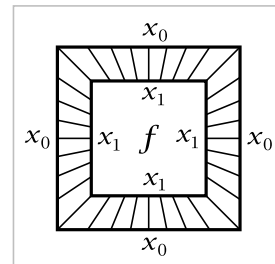
The homotopy begins by shrinking the domains of  $f$  and  $g$  to smaller subcubes of  $I^n$ , with the region outside these subcubes mapping to the basepoint. After this has been done, there is room to slide the two subcubes around anywhere in  $I^n$  as long as they stay disjoint, so if  $n \geq 2$  they can be slid past each other, interchanging their positions. Then to finish the homotopy, the domains of  $f$  and  $g$  can be enlarged back to their original size. If one likes, the whole process can be done using just the coordinates  $s_1$  and  $s_2$ , keeping the other coordinates fixed.

Maps  $(I^n, \partial I^n) \rightarrow (X, x_0)$  are the same as maps of the quotient  $I^n / \partial I^n = S^n$  to  $X$  taking the basepoint  $s_0 = \partial I^n / \partial I^n$  to  $x_0$ . This means that we can also view  $\pi_n(X, x_0)$  as homotopy classes of maps  $(S^n, s_0) \rightarrow (X, x_0)$ , where homotopies are through maps

of the same form  $(S^n, s_0) \rightarrow (X, x_0)$ . In this interpretation of  $\pi_n(X, x_0)$ , the sum  $f + g$  is the composition  $S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$  where  $c$  collapses the equator  $S^{n-1}$  in  $S^n$  to a point and we choose the basepoint  $s_0$  to lie in this  $S^{n-1}$ .



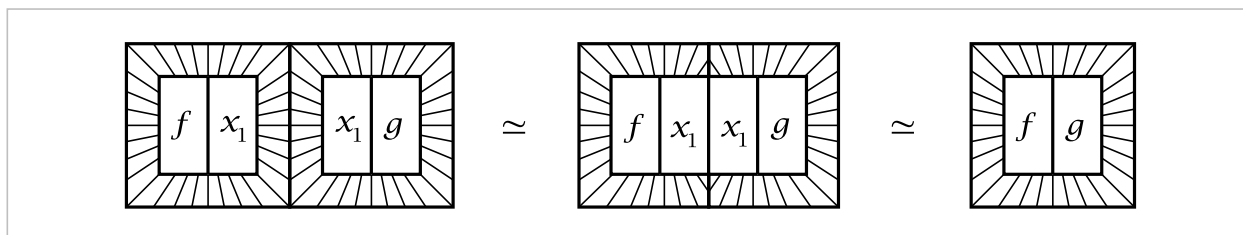
We will show next that if  $X$  is path-connected, different choices of the basepoint  $x_0$  always produce isomorphic groups  $\pi_n(X, x_0)$ , just as for  $\pi_1$ , so one is justified in writing  $\pi_n(X)$  for  $\pi_n(X, x_0)$  in these cases. Given a path  $\gamma: I \rightarrow X$  from  $x_0 = \gamma(0)$  to another basepoint  $x_1 = \gamma(1)$ , we may associate to each map  $f: (I^n, \partial I^n) \rightarrow (X, x_1)$  a new map  $\gamma f: (I^n, \partial I^n) \rightarrow (X, x_0)$  by shrinking the domain of  $f$  to a smaller concentric cube in  $I^n$ , then inserting the path  $\gamma$  on each radial segment in the shell between this smaller cube and  $\partial I^n$ . When  $n = 1$  the map  $\gamma f$  is the composition of the three paths  $\gamma$ ,  $f$ , and the inverse of  $\gamma$ , so the notation  $\gamma f$  conflicts with the notation for composition of paths. Since we are mainly interested in the cases  $n > 1$ , we leave it to the reader to make the necessary notational adjustments when  $n = 1$ .



A homotopy of  $\gamma$  or  $f$  through maps fixing  $\partial I$  or  $\partial I^n$ , respectively, yields a homotopy of  $\gamma f$  through maps  $(I^n, \partial I^n) \rightarrow (X, x_0)$ . Here are three other basic properties:

- (1)  $\gamma(f + g) \simeq \gamma f + \gamma g$ .
- (2)  $(\gamma \eta)f \simeq \gamma(\eta f)$ .
- (3)  $1f \simeq f$ , where  $1$  denotes the constant path.

The homotopies in (2) and (3) are obvious. For (1), we first deform  $f$  and  $g$  to be constant on the right and left halves of  $I^n$ , respectively, producing maps we may call  $f + 0$  and  $0 + g$ , then we excise a progressively wider symmetric middle slab of  $\gamma(f + 0) + \gamma(0 + g)$  until it becomes  $\gamma(f + g)$ :



An explicit formula for this homotopy is

$$h_t(s_1, s_2, \dots, s_n) = \begin{cases} \gamma(f + 0)((2 - t)s_1, s_2, \dots, s_n), & s_1 \in [0, 1/2] \\ \gamma(0 + g)((2 - t)s_1 + t - 1, s_2, \dots, s_n), & s_1 \in [1/2, 1] \end{cases}$$

Thus we have  $\gamma(f + g) \simeq \gamma(f + 0) + \gamma(0 + g) \simeq \gamma f + \gamma g$ .

If we define a change-of-basepoint transformation  $\beta_\gamma: \pi_n(X, x_1) \rightarrow \pi_n(X, x_0)$  by  $\beta_\gamma([f]) = [\gamma f]$ , then (1) shows that  $\beta_\gamma$  is a homomorphism, while (2) and (3) imply that  $\beta_\gamma$  is an isomorphism with inverse  $\beta_{\bar{\gamma}}$  where  $\bar{\gamma}$  is the inverse path of  $\gamma$ ,

$\bar{y}(s) = y(1-s)$ . Thus if  $X$  is path-connected, different choices of basepoint  $x_0$  yield isomorphic groups  $\pi_n(X, x_0)$ , which may then be written simply as  $\pi_n(X)$ .

Now let us restrict attention to loops  $y$  at the basepoint  $x_0$ . Since  $\beta_{y\eta} = \beta_y\beta_\eta$ , the association  $[y] \mapsto \beta_y$  defines a homomorphism from  $\pi_1(X, x_0)$  to  $\text{Aut}(\pi_n(X, x_0))$ , the group of automorphisms of  $\pi_n(X, x_0)$ . This is called the **action of  $\pi_1$  on  $\pi_n$** , each element of  $\pi_1$  acting as an automorphism  $[f] \mapsto [yf]$  of  $\pi_n$ . When  $n = 1$  this is the action of  $\pi_1$  on itself by inner automorphisms. When  $n > 1$ , the action makes the abelian group  $\pi_n(X, x_0)$  into a module over the group ring  $\mathbb{Z}[\pi_1(X, x_0)]$ . Elements of  $\mathbb{Z}[\pi_1]$  are finite sums  $\sum_i n_i y_i$  with  $n_i \in \mathbb{Z}$  and  $y_i \in \pi_1$ , multiplication being defined by distributivity and the multiplication in  $\pi_1$ . The module structure on  $\pi_n$  is given by  $(\sum_i n_i y_i)\alpha = \sum_i n_i (y_i \alpha)$  for  $\alpha \in \pi_n$ . For brevity one sometimes says  $\pi_n$  is a  $\pi_1$ -module rather than a  $\mathbb{Z}[\pi_1]$ -module.

In the literature, a space with trivial  $\pi_1$  action on  $\pi_n$  is called ‘ $n$ -simple’, and ‘simple’ means ‘ $n$ -simple for all  $n$ ’. In this book we will call a space **abelian** if it has trivial action of  $\pi_1$  on all homotopy groups  $\pi_n$ , since when  $n = 1$  this is the condition that  $\pi_1$  be abelian. This terminology is consistent with a long-established usage of the term ‘nilpotent’ to refer to spaces with nilpotent  $\pi_1$  and nilpotent action of  $\pi_1$  on all higher homotopy groups; see [Hilton, Mislin, & Roitberg 1975]. An important class of abelian spaces is H-spaces, as we show in Example 4A.3.

We next observe that  $\pi_n$  is a functor. Namely, a map  $\varphi: (X, x_0) \rightarrow (Y, y_0)$  induces  $\varphi_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)$  defined by  $\varphi_*([f]) = [\varphi f]$ . It is immediate from the definitions that  $\varphi_*$  is well-defined and a homomorphism for  $n \geq 1$ . The functor properties  $(\varphi\psi)_* = \varphi_*\psi_*$  and  $\mathbb{1}_* = \mathbb{1}$  are also evident, as is the fact that if  $\varphi_t: (X, x_0) \rightarrow (Y, y_0)$  is a homotopy then  $\varphi_{0*} = \varphi_{1*}$ .

In particular, a homotopy equivalence  $(X, x_0) \simeq (Y, y_0)$  in the basepointed sense induces isomorphisms on all homotopy groups  $\pi_n$ . This is true even if basepoints are not required to be stationary during homotopies. We showed this for  $\pi_1$  in Proposition 1.18, and the generalization to higher  $n$ ’s is an exercise at the end of this section.

Homotopy groups behave very nicely with respect to covering spaces:

**Proposition 4.1.** *A covering space projection  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  induces isomorphisms  $p_*: \pi_n(\tilde{X}, \tilde{x}_0) \rightarrow \pi_n(X, x_0)$  for all  $n \geq 2$ .*

**Proof:** For surjectivity of  $p_*$  we apply the lifting criterion in Proposition 1.33, which implies that every map  $(S^n, s_0) \rightarrow (X, x_0)$  lifts to  $(\tilde{X}, \tilde{x}_0)$  provided that  $n \geq 2$  so that  $S^n$  is simply-connected. Injectivity of  $p_*$  is immediate from the covering homotopy property, just as in Proposition 1.31 which treated the case  $n = 1$ .  $\square$

In particular,  $\pi_n(X, x_0) = 0$  for  $n \geq 2$  whenever  $X$  has a contractible universal cover. This applies for example to  $S^1$ , so we obtain the first row of the table of homotopy groups of spheres shown earlier. More generally, the  $n$ -dimensional torus  $T^n$ ,

the product of  $n$  circles, has universal cover  $\mathbb{R}^n$ , so  $\pi_i(T^n) = 0$  for  $i > 1$ . This is in marked contrast to the homology groups  $H_i(T^n)$  which are nonzero for all  $i \leq n$ . Spaces with  $\pi_n = 0$  for all  $n \geq 2$  are sometimes called **aspherical**.

The behavior of homotopy groups with respect to products is very simple:

**Proposition 4.2.** *For a product  $\prod_{\alpha} X_{\alpha}$  of an arbitrary collection of path-connected spaces  $X_{\alpha}$  there are isomorphisms  $\pi_n(\prod_{\alpha} X_{\alpha}) \approx \prod_{\alpha} \pi_n(X_{\alpha})$  for all  $n$ .*

**Proof:** A map  $f: Y \rightarrow \prod_{\alpha} X_{\alpha}$  is the same thing as a collection of maps  $f_{\alpha}: Y \rightarrow X_{\alpha}$ . Taking  $Y$  to be  $S^n$  and  $S^n \times I$  gives the result.  $\square$

Very useful generalizations of the homotopy groups  $\pi_n(X, x_0)$  are the **relative homotopy groups**  $\pi_n(X, A, x_0)$  for a pair  $(X, A)$  with a basepoint  $x_0 \in A$ . To define these, regard  $I^{n-1}$  as the face of  $I^n$  with the last coordinate  $s_n = 0$  and let  $J^{n-1}$  be the closure of  $\partial I^n - I^{n-1}$ , the union of the remaining faces of  $I^n$ . Then  $\pi_n(X, A, x_0)$  for  $n \geq 1$  is defined to be the set of homotopy classes of maps  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ , with homotopies through maps of the same form. There does not seem to be a completely satisfactory way of defining  $\pi_0(X, A, x_0)$ , so we shall leave this undefined (but see the exercises for one possible definition). Note that  $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$ , so absolute homotopy groups are a special case of relative homotopy groups.

A sum operation is defined in  $\pi_n(X, A, x_0)$  by the same formulas as for  $\pi_n(X, x_0)$ , except that the coordinate  $s_n$  now plays a special role and is no longer available for the sum operation. Thus  $\pi_n(X, A, x_0)$  is a group for  $n \geq 2$ , and this group is abelian for  $n \geq 3$ . For  $n = 1$  we have  $I^1 = [0, 1]$ ,  $I^0 = \{0\}$ , and  $J^0 = \{1\}$ , so  $\pi_1(X, A, x_0)$  is the set of homotopy classes of paths in  $X$  from a varying point in  $A$  to the fixed basepoint  $x_0 \in A$ . In general this is not a group in any natural way.

Just as elements of  $\pi_n(X, x_0)$  can be regarded as homotopy classes of maps  $(S^n, s_0) \rightarrow (X, x_0)$ , there is an alternative definition of  $\pi_n(X, A, x_0)$  as the set of homotopy classes of maps  $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$ , since collapsing  $J^{n-1}$  to a point converts  $(I^n, \partial I^n, J^{n-1})$  into  $(D^n, S^{n-1}, s_0)$ . From this viewpoint, addition is done via the map  $c: D^n \rightarrow D^n \vee D^n$  collapsing  $D^{n-1} \subset D^n$  to a point.

A useful and conceptually enlightening reformulation of what it means for an element of  $\pi_n(X, A, x_0)$  to be trivial is given by the following *compression criterion*:

- A map  $f: (D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$  represents zero in  $\pi_n(X, A, x_0)$  iff it is homotopic rel  $S^{n-1}$  to a map with image contained in  $A$ .

For if we have such a homotopy to a map  $g$ , then  $[f] = [g]$  in  $\pi_n(X, A, x_0)$ , and  $[g] = 0$  via the homotopy obtained by composing  $g$  with a deformation retraction of  $D^n$  onto  $s_0$ . Conversely, if  $[f] = 0$  via a homotopy  $F: D^n \times I \rightarrow X$ , then by restricting  $F$  to a family of  $n$ -disks in  $D^n \times I$  starting with  $D^n \times \{0\}$  and ending with the disk  $D^n \times \{1\} \cup S^{n-1} \times I$ , all the disks in the family having the same boundary, then we get a homotopy from  $f$  to a map into  $A$ , stationary on  $S^{n-1}$ .

A map  $\varphi: (X, A, x_0) \rightarrow (Y, B, y_0)$  induces maps  $\varphi_*: \pi_n(X, A, x_0) \rightarrow \pi_n(Y, B, y_0)$  which are homomorphisms for  $n \geq 2$  and have properties analogous to those in the absolute case:  $(\varphi\psi)_* = \varphi_*\psi_*$ ,  $\mathbb{1}_* = \mathbb{1}$ , and  $\varphi_* = \psi_*$  if  $\varphi \simeq \psi$  through maps  $(X, A, x_0) \rightarrow (Y, B, y_0)$ .

Probably the most useful feature of the relative groups  $\pi_n(X, A, x_0)$  is that they fit into a long exact sequence

$$\cdots \rightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \rightarrow \cdots \rightarrow \pi_0(X, x_0)$$

Here  $i$  and  $j$  are the inclusions  $(A, x_0) \hookrightarrow (X, x_0)$  and  $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$ . The map  $\partial$  comes from restricting maps  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$  to  $I^{n-1}$ , or by restricting maps  $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$  to  $S^{n-1}$ . The map  $\partial$ , called the *boundary map*, is a homomorphism when  $n > 1$ .

|| **Theorem 4.3.** *This sequence is exact.*

Near the end of the sequence, where group structures are not defined, exactness still makes sense: The image of one map is the kernel of the next, those elements mapping to the homotopy class of the constant map.

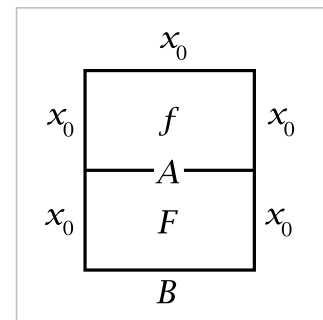
**Proof:** With only a little more effort we can derive the long exact sequence of a triple  $(X, A, B, x_0)$  with  $x_0 \in B \subset A \subset X$ :

$$\begin{aligned} \cdots \rightarrow \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \rightarrow \cdots \\ \rightarrow \pi_1(X, A, x_0) \end{aligned}$$

When  $B = x_0$  this reduces to the exact sequence for the pair  $(X, A, x_0)$ , though the latter sequence continues on two more steps to  $\pi_0(X, x_0)$ . The verification of exactness at these last two steps is left as a simple exercise.

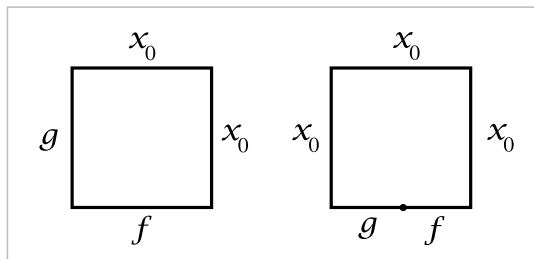
*Exactness at  $\pi_n(X, B, x_0)$ :* First note that the composition  $j_*i_*$  is zero since every map  $(I^n, \partial I^n, J^{n-1}) \rightarrow (A, B, x_0)$  represents zero in  $\pi_n(X, A, x_0)$  by the compression criterion. To see that  $\text{Ker } j_* \subset \text{Im } i_*$ , let  $f: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, B, x_0)$  represent zero in  $\pi_n(X, A, x_0)$ . Then by the compression criterion again,  $f$  is homotopic rel  $\partial I^n$  to a map with image in  $A$ , hence the class  $[f] \in \pi_n(X, B, x_0)$  is in the image of  $i_*$ .

*Exactness at  $\pi_n(X, A, x_0)$ :* The composition  $\partial j_*$  is zero since the restriction of a map  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, B, x_0)$  to  $I^{n-1}$  has image lying in  $B$ , and hence represents zero in  $\pi_{n-1}(A, B, x_0)$ . Conversely, suppose the restriction of  $f: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$  to  $I^{n-1}$  represents zero in  $\pi_{n-1}(A, B, x_0)$ . Then  $f|_{I^{n-1}}$  is homotopic to a map with image in  $B$  via a homotopy  $F: I^{n-1} \times I \rightarrow A$  rel  $\partial I^{n-1}$ . We can tack  $F$  onto  $f$  to get a new map  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, B, x_0)$  which, as a map  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ , is homotopic to  $f$  by the homotopy that tacks on increasingly longer initial segments of  $F$ . So  $[f] \in \text{Im } j_*$ .





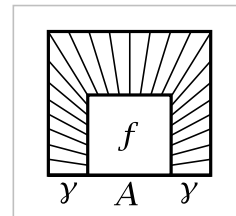
**Exactness at  $\pi_n(A, B, x_0)$ :** The composition  $i_* \partial$  is zero since the restriction of a map  $f: (I^{n+1}, \partial I^{n+1}, J^n) \rightarrow (X, A, x_0)$  to  $I^n$  is homotopic rel  $\partial I^n$  to a constant map via  $f$  itself. The converse is easy if  $B$  is a point, since a nullhomotopy  $f_t: (I^n, \partial I^n) \rightarrow (X, x_0)$  of  $f_0: (I^n, \partial I^n) \rightarrow (A, x_0)$  gives a map  $F: (I^{n+1}, \partial I^{n+1}, J^n) \rightarrow (X, A, x_0)$  with  $\partial([F]) = [f_0]$ . Thus the proof is finished in this case. For a general  $B$ , let  $F$  be a nullhomotopy of  $f: (I^n, \partial I^n, J^{n-1}) \rightarrow (A, B, x_0)$  through maps  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, B, x_0)$ , and let  $g$  be the restriction of  $F$  to  $I^{n-1} \times I$ , as in the first of the two pictures below. Reparametrizing the  $n^{\text{th}}$  and  $(n+1)^{\text{st}}$  coordinates as shown in the second picture, we see that  $f$  with  $g$  tacked on is in the image of  $\partial$ . But as we noted in the preceding paragraph, tacking  $g$  onto  $f$  gives the same element of  $\pi_n(A, B, x_0)$ .  $\square$



**Example 4.4.** Let  $CX$  be the cone on a path-connected space  $X$ , the quotient space of  $X \times I$  obtained by collapsing  $X \times \{0\}$  to a point. We can view  $X$  as the subspace  $X \times \{1\} \subset CX$ . Since  $CX$  is contractible, the long exact sequence of homotopy groups for the pair  $(CX, X)$  gives isomorphisms  $\pi_n(CX, X, x_0) \approx \pi_{n-1}(X, x_0)$  for all  $n \geq 1$ . Taking  $n = 2$ , we can thus realize any group  $G$ , abelian or not, as a relative  $\pi_2$  by choosing  $X$  to have  $\pi_1(X) \approx G$ .

The long exact sequence of homotopy groups is clearly natural: A map of basepointed triples  $(X, A, B, x_0) \rightarrow (Y, C, D, y_0)$  induces a map between the associated long exact sequences, with commuting squares.

There are change-of-basepoint isomorphisms  $\beta_\gamma$  for relative homotopy groups analogous to those in the absolute case. One starts with a path  $\gamma$  in  $A \subset X$  from  $x_0$  to  $x_1$ , and this induces  $\beta_\gamma: \pi_n(X, A, x_1) \rightarrow \pi_n(X, A, x_0)$  by setting  $\beta_\gamma([f]) = [\gamma f]$  where  $\gamma f$  is defined as in the picture, by placing a copy of  $f$  in a smaller cube with its face  $I^{n-1}$  centered in the corresponding face of the larger cube. This construction satisfies the same basic properties as in the absolute case, with very similar proofs that we leave to the exercises. Separate proofs must be given in the two cases since the definition of  $\gamma f$  in the relative case does not specialize to the definition of  $\gamma f$  in the absolute case.



The isomorphisms  $\beta_\gamma$  show that  $\pi_n(X, A, x_0)$  is independent of  $x_0$  when  $A$  is path-connected. In this case  $\pi_n(X, A, x_0)$  is often written simply as  $\pi_n(X, A)$ .

Restricting to loops at the basepoint, the association  $\gamma \mapsto \beta_\gamma$  defines an action of  $\pi_1(A, x_0)$  on  $\pi_n(X, A, x_0)$  analogous to the action of  $\pi_1(X, x_0)$  on  $\pi_n(X, x_0)$  in the absolute case. In fact, it is clear from the definitions that  $\pi_1(A, x_0)$  acts on the whole long exact sequence of homotopy groups for  $(X, A, x_0)$ , the action commuting with the various maps in the sequence.

A space  $X$  with basepoint  $x_0$  is said to be  **$n$ -connected** if  $\pi_i(X, x_0) = 0$  for  $i \leq n$ . Thus 0-connected means path-connected and 1-connected means simply-connected. Since  $n$ -connected implies 0-connected, the choice of the basepoint  $x_0$  is not significant. The condition of being  $n$ -connected can be expressed without mention of a basepoint since it is an easy exercise to check that the following three conditions are equivalent.

- (1) Every map  $S^i \rightarrow X$  is homotopic to a constant map.
- (2) Every map  $S^i \rightarrow X$  extends to a map  $D^{i+1} \rightarrow X$ .
- (3)  $\pi_i(X, x_0) = 0$  for all  $x_0 \in X$ .

Thus  $X$  is  $n$ -connected if any one of these three conditions holds for all  $i \leq n$ . Similarly, in the relative case it is not hard to see that the following four conditions are equivalent, for  $i > 0$ :

- (1) Every map  $(D^i, \partial D^i) \rightarrow (X, A)$  is homotopic rel  $\partial D^i$  to a map  $D^i \rightarrow A$ .
- (2) Every map  $(D^i, \partial D^i) \rightarrow (X, A)$  is homotopic through such maps to a map  $D^i \rightarrow A$ .
- (3) Every map  $(D^i, \partial D^i) \rightarrow (X, A)$  is homotopic through such maps to a constant map  $D^i \rightarrow A$ .
- (4)  $\pi_i(X, A, x_0) = 0$  for all  $x_0 \in A$ .

When  $i = 0$  we did not define the relative  $\pi_0$ , and (1)–(3) are each equivalent to saying that each path-component of  $X$  contains points in  $A$  since  $D^0$  is a point and  $\partial D^0$  is empty. The pair  $(X, A)$  is called  **$n$ -connected** if (1)–(4) hold for all  $i \leq n$ ,  $i > 0$ , and (1)–(3) hold for  $i = 0$ .

Note that  $X$  is  $n$ -connected iff  $(X, x_0)$  is  $n$ -connected for some  $x_0$  and hence for all  $x_0$ .

## Whitehead's Theorem

Since CW complexes are built using attaching maps whose domains are spheres, it is perhaps not too surprising that homotopy groups of CW complexes carry a lot of information. Whitehead's theorem makes this explicit:

**Theorem 4.5.** *If a map  $f: X \rightarrow Y$  between connected CW complexes induces isomorphisms  $f_*: \pi_n(X) \rightarrow \pi_n(Y)$  for all  $n$ , then  $f$  is a homotopy equivalence. In case  $f$  is the inclusion of a subcomplex  $X \hookrightarrow Y$ , the conclusion is stronger:  $X$  is a deformation retract of  $Y$ .*

The proof will follow rather easily from a more technical result that turns out to be very useful in quite a number of arguments. For convenient reference we call this the **compression lemma**.

**Lemma 4.6.** *Let  $(X, A)$  be a CW pair and let  $(Y, B)$  be any pair with  $B \neq \emptyset$ . For each  $n$  such that  $X - A$  has cells of dimension  $n$ , assume that  $\pi_n(Y, B, y_0) = 0$  for all  $y_0 \in B$ . Then every map  $f: (X, A) \rightarrow (Y, B)$  is homotopic rel  $A$  to a map  $X \rightarrow B$ .*

When  $n = 0$  the condition that  $\pi_n(Y, B, \gamma_0) = 0$  for all  $\gamma_0 \in B$  is to be regarded as saying that  $(Y, B)$  is 0-connected.

**Proof:** Assume inductively that  $f$  has already been homotoped to take the skeleton  $X^{k-1}$  to  $B$ . If  $\Phi$  is the characteristic map of a cell  $e^k$  of  $X - A$ , the composition  $f\Phi: (D^k, \partial D^k) \rightarrow (Y, B)$  can be homotoped into  $B$  rel  $\partial D^k$  in view of the hypothesis that  $\pi_k(Y, B, \gamma_0) = 0$  if  $k > 0$ , or that  $(Y, B)$  is 0-connected if  $k = 0$ . This homotopy of  $f\Phi$  induces a homotopy of  $f$  on the quotient space  $X^{k-1} \cup e^k$  of  $X^{k-1} \amalg D^k$ , a homotopy rel  $X^{k-1}$ . Doing this for all  $k$ -cells of  $X - A$  simultaneously, and taking the constant homotopy on  $A$ , we obtain a homotopy of  $f|X^k \cup A$  to a map into  $B$ . By the homotopy extension property in Proposition 0.16, this homotopy extends to a homotopy defined on all of  $X$ , and the induction step is completed.

Finitely many applications of the induction step finish the proof if the cells of  $X - A$  are of bounded dimension. In the general case we perform the homotopy of the induction step during the  $t$ -interval  $[1 - 1/2^k, 1 - 1/2^{k+1}]$ . Any finite skeleton  $X^k$  is eventually stationary under these homotopies, hence we have a well-defined homotopy  $f_t$ ,  $t \in [0, 1]$ , with  $f_1(X) \subset B$ .  $\square$

**Proof of Whitehead's Theorem:** In the special case that  $f$  is the inclusion of a subcomplex, consider the long exact sequence of homotopy groups for the pair  $(Y, X)$ . Since  $f$  induces isomorphisms on all homotopy groups, the relative groups  $\pi_n(Y, X)$  are zero. Applying the lemma to the identity map  $(Y, X) \rightarrow (Y, X)$  then yields a deformation retraction of  $Y$  onto  $X$ .

The general case can be proved using mapping cylinders. Recall that the mapping cylinder  $M_f$  of a map  $f: X \rightarrow Y$  is the quotient space of the disjoint union of  $X \times I$  and  $Y$  under the identifications  $(x, 1) \sim f(x)$ . Thus  $M_f$  contains both  $X = X \times \{0\}$  and  $Y$  as subspaces, and  $M_f$  deformation retracts onto  $Y$ . The map  $f$  becomes the composition of the inclusion  $X \hookrightarrow M_f$  with the retraction  $M_f \rightarrow Y$ . Since this retraction is a homotopy equivalence, it suffices to show that  $M_f$  deformation retracts onto  $X$  if  $f$  induces isomorphisms on homotopy groups, or equivalently, if the relative groups  $\pi_n(M_f, X)$  are all zero.

If the map  $f$  happens to be cellular, taking the  $n$ -skeleton of  $X$  to the  $n$ -skeleton of  $Y$  for all  $n$ , then  $(M_f, X)$  is a CW pair and so we are done by the first paragraph of the proof. If  $f$  is not cellular, we can either appeal to Theorem 4.8 which says that  $f$  is homotopic to a cellular map, or we can use the following argument. First apply the preceding lemma to obtain a homotopy rel  $X$  of the inclusion  $(X \cup Y, X) \hookrightarrow (M_f, X)$  to a map into  $X$ . Since the pair  $(M_f, X \cup Y)$  obviously satisfies the homotopy extension property, this homotopy extends to a homotopy from the identity map of  $M_f$  to a map  $g: M_f \rightarrow M_f$  taking  $X \cup Y$  into  $X$ . Then apply the lemma again to the composition  $(X \times I \amalg Y, X \times \partial I \amalg Y) \rightarrow (M_f, X \cup Y) \xrightarrow{g} (M_f, X)$  to finish the construction of a deformation retraction of  $M_f$  onto  $X$ .  $\square$

Whitehead's theorem does not say that two CW complexes  $X$  and  $Y$  with isomorphic homotopy groups are homotopy equivalent, since there is a big difference between saying that  $X$  and  $Y$  have isomorphic homotopy groups and saying that there is a map  $X \rightarrow Y$  inducing isomorphisms on homotopy groups. For example, consider  $X = \mathbb{R}P^2$  and  $Y = S^2 \times \mathbb{R}P^\infty$ . These both have fundamental group  $\mathbb{Z}_2$ , and Proposition 4.1 implies that their higher homotopy groups are isomorphic since their universal covers  $S^2$  and  $S^2 \times S^\infty$  are homotopy equivalent,  $S^\infty$  being contractible. But  $\mathbb{R}P^2$  and  $S^2 \times \mathbb{R}P^\infty$  are not homotopy equivalent since their homology groups are vastly different,  $S^2 \times \mathbb{R}P^\infty$  having nonvanishing homology in infinitely many dimensions since it retracts onto  $\mathbb{R}P^\infty$ . Another pair of CW complexes that are not homotopy equivalent but have isomorphic homotopy groups is  $S^2$  and  $S^3 \times \mathbb{C}P^\infty$ , as we shall see in Example 4.51. In fact it turns out to be quite rare that the homotopy type of a CW complex is determined by its homotopy groups.

One very special case when the homotopy type of a CW complex is determined by its homotopy groups is when all the homotopy groups are trivial, for then the inclusion map of a 0-cell into the complex induces an isomorphism on homotopy groups, so the complex deformation retracts to the 0-cell.

Somewhat similar in spirit to the compression lemma is the following rather basic **extension lemma**:

**Lemma 4.7.** *Given a CW pair  $(X, A)$  and a map  $f: A \rightarrow Y$  with  $Y$  path-connected, then  $f$  can be extended to a map  $X \rightarrow Y$  if  $\pi_{n-1}(Y) = 0$  for all  $n$  such that  $X - A$  has cells of dimension  $n$ .*

**Proof:** Assume inductively that  $f$  has been extended over the  $(n-1)$ -skeleton. Then an extension over an  $n$ -cell exists iff the composition of the cell's attaching map  $S^{n-1} \rightarrow X^{n-1}$  with  $f: X^{n-1} \rightarrow Y$  is nullhomotopic.  $\square$

## Cellular Approximation

An intuitively appealing strategy for proving that  $\pi_n(S^k) = 0$  for  $n < k$  would be to show first that every map  $S^n \rightarrow S^k$  can be deformed to make its image miss at least one point of  $S^k$ , and then use the fact that the complement of a point in  $S^k$  is contractible to finish the proof. One might think that the first step was unnecessary, that no continuous map  $S^n \rightarrow S^k$  could be surjective when  $n < k$ , but it is not hard to use space-filling curves from point-set topology to produce such maps. Thus to make this strategy into a valid proof some work must be done to construct homotopies eliminating the strange behavior of these dimension-raising maps.

For maps between CW complexes it turns out to be sufficient for this and many other purposes in homotopy theory to require just that cells map to cells of the same or lower dimension. Such a map  $f: X \rightarrow Y$ , satisfying  $f(X^n) \subset Y^n$  for all  $n$ , is called

a **cellular map**. It is a fundamental fact that arbitrary maps can always be deformed to be cellular. This is the **cellular approximation theorem**:

**Theorem 4.8.** *Every map  $f: X \rightarrow Y$  of CW complexes is homotopic to a cellular map. If  $f$  is already cellular on a subcomplex  $A \subset X$ , the homotopy may be taken to be stationary on  $A$ .*

**Corollary 4.9.**  $\pi_n(S^k) = 0$  for  $n < k$ .

**Proof:** If  $S^n$  and  $S^k$  are given their usual CW structures, with the 0-cells as basepoints, then every basepoint-preserving map  $S^n \rightarrow S^k$  can be homotoped, fixing the basepoint, to be cellular, and hence constant if  $n < k$ .  $\square$

Linear maps cannot increase dimension, so one might try to prove the theorem by showing that arbitrary maps between CW complexes can be homotoped to have some sort of linearity properties. For simplicial complexes the simplicial approximation theorem, Theorem 2C.1, achieves this, and cellular approximation can be regarded as a CW analog of simplicial approximation since simplicial maps are cellular. However, simplicial maps are much more rigid than cellular maps, which perhaps explains why subdivision of the domain is required for simplicial approximation but not for cellular approximation. The core of the proof of cellular approximation will be a weak form of simplicial approximation that can be proved by a rather elementary direct argument.

**Proof of 4.8:** Suppose inductively that  $f: X \rightarrow Y$  is already cellular on the skeleton  $X^{n-1}$ , and let  $e^n$  be an  $n$ -cell of  $X$ . The closure of  $e^n$  in  $X$  is compact, being the image of a characteristic map for  $e^n$ , so  $f$  takes the closure of  $e^n$  to a compact set in  $Y$ . Since a compact set in a CW complex can meet only finitely many cells by Proposition A.1 in the Appendix, it follows that  $f(e^n)$  meets only finitely many cells of  $Y$ . Let  $e^k \subset Y$  be a cell of highest dimension meeting  $f(e^n)$ . We may assume  $k > n$ , otherwise  $f$  is already cellular on  $e^n$ . We will show below that it is possible to deform  $f|X^{n-1} \cup e^n$ , staying fixed on  $X^{n-1}$ , so that  $f(e^n)$  misses some point  $p \in e^k$ . Then we can deform  $f|X^{n-1} \cup e^n \text{ rel } X^{n-1}$  so that  $f(e^n)$  misses the whole cell  $e^k$  by composing with a deformation retraction of  $Y^k - \{p\}$  onto  $Y^k - e^k$ . By finitely many iterations of this process we eventually make  $f(e^n)$  miss all cells of dimension greater than  $n$ . Doing this for all  $n$ -cells simultaneously, staying fixed on  $n$ -cells in  $A$  where  $f$  is already cellular, we obtain a homotopy of  $f|X^n \text{ rel } X^{n-1} \cup A^n$  to a cellular map. The induction step is then completed by appealing to the homotopy extension property in Proposition 0.16 to extend this homotopy, together with the constant homotopy on  $A$ , to a homotopy defined on all of  $X$ . Letting  $n$  go to  $\infty$ , the resulting possibly infinite string of homotopies can be realized as a single homotopy by performing the  $n^{\text{th}}$  homotopy during the  $t$ -interval  $[1 - 1/2^n, 1 - 1/2^{n+1}]$ . This makes sense since each point of  $X$  lies in some  $X^n$ , which is eventually stationary in the infinite chain of homotopies.

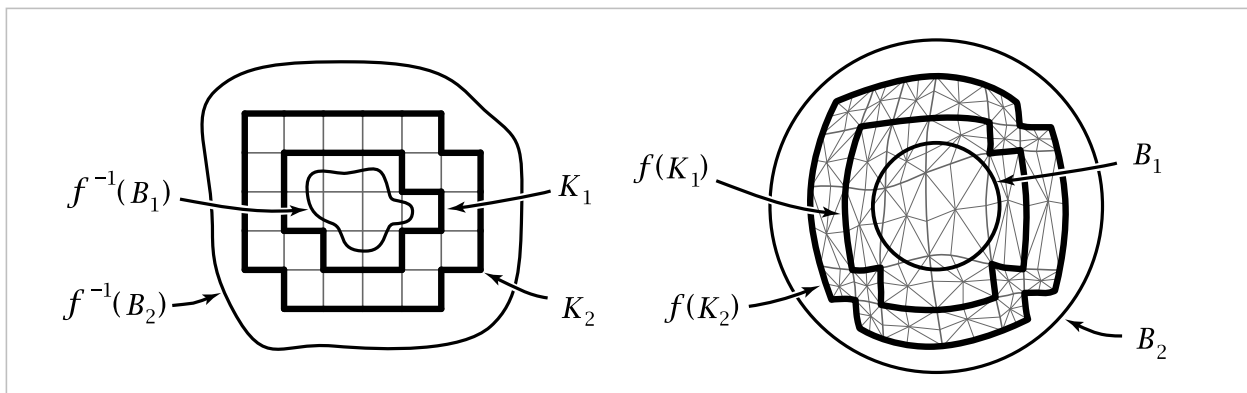
To fill in the missing step in this argument we will need a technical lemma about deforming maps to create some linearity. Define a **polyhedron** in  $\mathbb{R}^n$  to be a subspace that is the union of finitely many convex polyhedra, each of which is a compact set obtained by intersecting finitely many half-spaces defined by linear inequalities of the form  $\sum_i a_i x_i \leq b$ . By a **PL (piecewise linear) map** from a polyhedron to  $\mathbb{R}^k$  we shall mean a map which is linear when restricted to each convex polyhedron in some such decomposition of the polyhedron into convex polyhedra.

**Lemma 4.10.** *Let  $f: I^n \rightarrow Z$  be a map, where  $Z$  is obtained from a subspace  $W$  by attaching a cell  $e^k$ . Then there is a homotopy  $f_t: (I^n, f^{-1}(e^k)) \rightarrow (Z, e^k)$  rel  $f^{-1}(W)$  from  $f = f_0$  to a map  $f_1$  for which there is a polyhedron  $K \subset I^n$  such that:*

- (a)  $f_1(K) \subset e^k$  and  $f_1|_K$  is PL with respect to some identification of  $e^k$  with  $\mathbb{R}^k$ .
- (b)  $K \supset f_1^{-1}(U)$  for some nonempty open set  $U$  in  $e^k$ .

Before proving the lemma, let us see how it finishes the proof of the cellular approximation theorem. Composing the given map  $f: X^{n-1} \cup e^n \rightarrow Y^k$  with a characteristic map  $I^n \rightarrow X$  for  $e^n$ , we obtain a map  $f$  as in the lemma, with  $Z = Y^k$  and  $W = Y^k - e^k$ . The homotopy given by the lemma is fixed on  $\partial I^n$ , hence induces a homotopy  $f_t$  of  $f|_{X^{n-1} \cup e^n}$  fixed on  $X^{n-1}$ . The image of the resulting map  $f_1$  intersects the open set  $U$  in  $e^k$  in a set contained in the union of finitely many hyperplanes of dimension at most  $n$ , so if  $n < k$  there will be points  $p \in U$  not in the image of  $f_1$ .  $\square$

**Proof of 4.10:** Identifying  $e^k$  with  $\mathbb{R}^k$ , let  $B_1, B_2 \subset e^k$  be the closed balls of radius 1 and 2 centered at the origin. Since  $f^{-1}(B_2)$  is closed and therefore compact in  $I^n$ , it follows that  $f$  is uniformly continuous on  $f^{-1}(B_2)$ . Thus there exists  $\varepsilon > 0$  such that  $|x - y| < \varepsilon$  implies  $|f(x) - f(y)| < 1/2$  for all  $x, y \in f^{-1}(B_2)$ . Subdivide the interval  $I$  so that the induced subdivision of  $I^n$  into cubes has each cube lying in a ball of diameter less than  $\varepsilon$ . Let  $K_1$  be the union of all the cubes meeting  $f^{-1}(B_1)$ , and let  $K_2$  be the union of all the cubes meeting  $K_1$ . We may assume  $\varepsilon$  is chosen smaller than half the distance between the compact sets  $f^{-1}(B_1)$  and  $I^n - f^{-1}(\text{int}(B_2))$ , and then we will have  $K_2 \subset f^{-1}(B_2)$ .



Now we subdivide all the cubes of  $K_2$  into simplices. This can be done inductively. The boundary of each cube is a union of cubes of one lower dimension, so assuming these lower-dimensional cubes have already been subdivided into simplices, we obtain a subdivision of the cube itself by taking its center point as a new vertex and joining this by a cone to each simplex in the boundary of the cube.

Let  $g: K_2 \rightarrow e^k = \mathbb{R}^k$  be the map that equals  $f$  on all vertices of simplices of the subdivision and is linear on each simplex. Let  $\varphi: K_2 \rightarrow [0, 1]$  be the map that is linear on simplices and has the value 1 on vertices in  $K_1$  and 0 on vertices in  $K_2 - K_1$ . Thus  $\varphi(K_1) = 1$ . Define a homotopy  $f_t: K_2 \rightarrow e^k$  by the formula  $(1 - t\varphi)f + (t\varphi)g$ , so  $f_0 = f$  and  $f_1|_{K_1} = g|_{K_1}$ . Since  $f_t$  is the constant homotopy on simplices in  $K_2$  disjoint from  $K_1$ , and in particular on simplices in the closure of  $I^n - K_2$ , we may extend  $f_t$  to be the constant homotopy of  $f$  on  $I^n - K_2$ .

The map  $f_1$  takes the closure of  $I^n - K_1$  to a compact set  $C$  which, we claim, is disjoint from the centerpoint 0 of  $B_1$  and hence from a neighborhood  $U$  of 0. This will prove the lemma, with  $K = K_1$ , since we will then have  $f_1^{-1}(U) \subset K_1$  with  $f_1$  PL on  $K_1$  where it is equal to  $g$ .

The verification of the claim has two steps:

- (1) On  $I^n - K_2$  we have  $f_1 = f$ , and  $f$  takes  $I^n - K_2$  outside  $B_1$  since  $f^{-1}(B_1) \subset K_2$  by construction.
- (2) For a simplex  $\sigma$  of  $K_2$  not in  $K_1$  we have  $f(\sigma)$  contained in some ball  $B_\sigma$  of radius  $1/2$  by the choice of  $\varepsilon$  and the fact that  $K_2 \subset f^{-1}(B_2)$ . Since  $f(\sigma) \subset B_\sigma$  and  $B_\sigma$  is convex, we must have  $g(\sigma) \subset B_\sigma$ , hence also  $f_t(\sigma) \subset B_\sigma$  for all  $t$ , and in particular  $f_1(\sigma) \subset B_\sigma$ . We know that  $B_\sigma$  is not contained in  $B_1$  since  $\sigma$  contains points outside  $K_1$  hence outside  $f^{-1}(B_1)$ . The radius of  $B_\sigma$  is half that of  $B_1$ , so it follows that 0 is not in  $B_\sigma$ , and hence 0 is not in  $f_1(\sigma)$ .  $\square$

**Example 4.11: Cellular Approximation for Pairs.** Every map  $f: (X, A) \rightarrow (Y, B)$  of CW pairs can be deformed through maps  $(X, A) \rightarrow (Y, B)$  to a cellular map. This follows from the theorem by first deforming the restriction  $f: A \rightarrow B$  to be cellular, then extending this to a homotopy of  $f$  on all of  $X$ , then deforming the resulting map to be cellular staying fixed on  $A$ . As a further refinement, the homotopy of  $f$  can be taken to be stationary on any subcomplex of  $X$  where  $f$  is already cellular.

An easy consequence of this is:

**Corollary 4.12.** *A CW pair  $(X, A)$  is  $n$ -connected if all the cells in  $X - A$  have dimension greater than  $n$ . In particular the pair  $(X, X^n)$  is  $n$ -connected, hence the inclusion  $X^n \hookrightarrow X$  induces isomorphisms on  $\pi_i$  for  $i < n$  and a surjection on  $\pi_n$ .*

**Proof:** Applying cellular approximation to maps  $(D^i, \partial D^i) \rightarrow (X, A)$  with  $i \leq n$  gives the first statement. The last statement comes from the long exact sequence of the pair  $(X, X^n)$ .  $\square$

## CW Approximation

A map  $f: X \rightarrow Y$  is called a **weak homotopy equivalence** if it induces isomorphisms  $\pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$  for all  $n \geq 0$  and all choices of basepoint  $x_0$ . Whitehead's theorem can be restated as saying that a weak homotopy equivalence between CW complexes is a homotopy equivalence. It follows easily that this holds also for spaces homotopy equivalent to CW complexes. In general, however, weak homotopy equivalence is strictly weaker than homotopy equivalence. For example, there exist noncontractible spaces whose homotopy groups are all trivial, such as the 'quasi-circle' according to an exercise at the end of this section, and for such spaces a map to a point is a weak homotopy equivalence that is not a homotopy equivalence.

We will show that for every space  $X$  there is a CW complex  $Z$  and a weak homotopy equivalence  $f: Z \rightarrow X$ . Such a map  $f: Z \rightarrow X$  is called a **CW approximation** to  $X$ . A weak homotopy equivalence induces isomorphisms on all homology and cohomology groups, as we will see, so CW approximations allow many general statements in algebraic topology to be proved using cell-by-cell arguments for CW complexes.

The construction of a CW approximation  $f: Z \rightarrow X$  for a space  $X$  is inductive, so let us describe the induction step. Suppose given a CW complex  $A$  with a map  $f: A \rightarrow X$  and suppose we have chosen a basepoint 0-cell  $a_y$  in each component of  $A$ . Then for an integer  $k \geq 0$  we will attach  $k$ -cells to  $A$  to form a CW complex  $B$  with a map  $f: B \rightarrow X$  extending the given  $f$ , such that:

- (\*) The induced map  $f_*: \pi_i(B, a_y) \rightarrow \pi_i(X, f(a_y))$  is injective for  $i = k - 1$  and surjective for  $i = k$ , for all  $a_y$ .

There are two steps to the construction:

- (1) Choose maps  $\varphi_\alpha: (S^{k-1}, s_0) \rightarrow (A, a_y)$  representing a set of generators for the kernel of  $f_*: \pi_{k-1}(A, a_y) \rightarrow \pi_{k-1}(X, f(a_y))$  for all the basepoints  $a_y$ . We may assume the maps  $\varphi_\alpha$  are cellular, where  $S^{k-1}$  has its standard CW structure with  $s_0$  as 0-cell. Attaching cells  $e_\alpha^k$  to  $A$  via the maps  $\varphi_\alpha$  then produces a CW complex, and the map  $f$  extends over these cells using nullhomotopies of the compositions  $f\varphi_\alpha$ , which exist by the choice of the  $\varphi_\alpha$ 's.
- (2) Choose maps  $f_\beta: S^k \rightarrow X$  representing generators for the groups  $\pi_k(X, f(a_y))$ , attach cells  $e_\beta^k$  to  $A$  via the constant maps at the appropriate basepoints  $a_y$ , and extend  $f$  over the resulting spheres  $S_\beta^k$  via the  $f_\beta$ 's.

The surjectivity condition in (\*) then holds by construction. For the injectivity condition, an element of the kernel of  $f_*: \pi_{k-1}(B, a_y) \rightarrow \pi_{k-1}(X, f(a_y))$  can be represented by a cellular map  $h: S^{k-1} \rightarrow B$ . This has image in  $A$ , so is in the kernel of  $f_*: \pi_{k-1}(A, a_y) \rightarrow \pi_{k-1}(X, f(a_y))$  and hence is homotopic to a linear combination of the  $\varphi_\alpha$ 's, which are nullhomotopic in  $B$ , so  $h$  is nullhomotopic as well. When  $k = 1$  there is no group structure on  $\pi_{k-1}$  so injectivity on  $\pi_0$  does not follow from having a trivial kernel, and we modify the construction by choosing the cells  $e_\alpha^1$  to join each



pair of basepoints  $a_\gamma$  that map by  $f$  to the same path-component of  $X$ . The map  $f$  can then be extended over these 1-cells  $e_\alpha^1$ .

Note that if the given map  $f: A \rightarrow X$  happened to be injective or surjective on  $\pi_i$  for some  $i < k - 1$  or  $i < k$ , respectively, then this remains true after attaching the  $k$ -cells. This is because attaching  $k$ -cells does not affect  $\pi_i$  if  $i < k - 1$ , by cellular approximation, nor does it destroy surjectivity on  $\pi_{k-1}$  or indeed any  $\pi_i$ , obviously.

Now to construct a CW approximation  $f: Z \rightarrow X$  one can start with  $A$  consisting of one point for each path-component of  $X$ , with  $f: A \rightarrow X$  mapping each of these points to the corresponding path-component. Having now a bijection on  $\pi_0$ , attach 1-cells to  $A$  to create a surjection on  $\pi_1$  for each path-component, then 2-cells to improve this to an isomorphism on  $\pi_1$  and a surjection on  $\pi_2$ , and so on for each successive  $\pi_i$  in turn. After all cells have been attached one has a CW complex  $Z$  with a weak homotopy equivalence  $f: Z \rightarrow X$ . This proves:

**Proposition 4.13.** *Every space  $X$  has a CW approximation  $f: Z \rightarrow X$ . If  $X$  is path-connected,  $Z$  can be chosen to have a single 0-cell, with all other cells attached by basepoint-preserving maps. Thus every connected CW complex is homotopy equivalent to a CW complex with these additional properties.*  $\square$

**Example 4.14.** One can also apply this technique to produce a CW approximation to a pair  $(X, X_0)$ . First construct a CW approximation  $f_0: Z_0 \rightarrow X_0$ , then starting with the composition  $Z_0 \rightarrow X_0 \hookrightarrow X$ , attach cells to  $Z_0$  to create a weak homotopy equivalence  $f: Z \rightarrow X$  extending  $f_0$ . By the five-lemma, the map  $f: (Z, Z_0) \rightarrow (X, X_0)$  induces isomorphisms on relative as well as absolute homotopy groups.

Here is another application of the technique, giving a more geometric interpretation to the homotopy-theoretic notion of  $n$ -connectedness:

**Proposition 4.15.** *If  $(X, A)$  is an  $n$ -connected CW pair, then there exists a CW pair  $(Z, A) \simeq (X, A)$  rel  $A$  such that all cells of  $Z - A$  have dimension greater than  $n$ .*

**Proof:** Starting with the inclusion  $A \hookrightarrow X$ , attach cells to  $A$  of dimension  $n + 1$  and higher to produce a CW complex  $Z$  and a map  $f: Z \rightarrow X$  that is the identity on  $A$  and induces an injection on  $\pi_n$  and isomorphisms on all higher homotopy groups. The induced map on  $\pi_n$  is also surjective since this is true for the composition  $A \hookrightarrow Z \xrightarrow{f} X$  by the hypothesis that  $(X, A)$  is  $n$ -connected. In dimensions below  $n$ ,  $f$  induces isomorphisms on homotopy groups since both inclusions  $A \hookrightarrow Z$  and  $A \hookrightarrow X$  induce isomorphisms in these dimensions. Thus  $f$  is a weak homotopy equivalence, and hence a homotopy equivalence by Whitehead's theorem.

To see that  $f$  is a homotopy equivalence rel  $A$ , form a quotient space  $W$  of the mapping cylinder  $M_f$  by collapsing each segment  $\{a\} \times I$  to a point, for  $a \in A$ . Assuming  $f$  has been made cellular,  $W$  is a CW complex containing  $X$  and  $Z$  as subcomplexes, and  $W$  deformation retracts to  $X$  just as  $M_f$  does. Also,  $\pi_i(W, Z) = 0$  for all

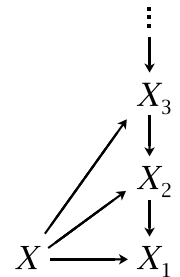
$i$  since  $f$  induces isomorphisms on all homotopy groups, so  $W$  deformation retracts onto  $Z$ . These two deformation retractions of  $W$  onto  $X$  and  $Z$  are stationary on  $A$ , hence give a homotopy equivalence  $X \simeq Z \text{ rel } A$ .  $\square$

**Example 4.16: Postnikov Towers.** We can also apply the technique to construct, for each connected CW complex  $X$  and each integer  $n \geq 1$ , a CW complex  $X_n$  containing  $X$  as a subcomplex such that:

- (a)  $\pi_i(X_n) = 0$  for  $i > n$ .
- (b) The inclusion  $X \hookrightarrow X_n$  induces an isomorphism on  $\pi_i$  for  $i \leq n$ .

To do this, all we have to do is apply the general construction to the constant map of  $X$  to a point, starting at the stage of attaching cells of dimension  $n + 2$ . Thus we attach  $(n + 2)$ -cells to  $X$  using cellular maps  $S^{n+1} \rightarrow X$  that generate  $\pi_{n+1}(X)$  to form a space with  $\pi_{n+1}$  trivial, then for this space we attach  $(n + 3)$ -cells to make  $\pi_{n+2}$  trivial, and so on. The result is a CW complex  $X_n$  with the desired properties.

The inclusion  $X \hookrightarrow X_n$  extends to a map  $X_{n+1} \rightarrow X_n$  since  $X_{n+1}$  is obtained from  $X$  by attaching cells of dimension  $n + 3$  and greater, and  $\pi_i(X_n) = 0$  for  $i > n$  so we can apply Lemma 4.7, the extension lemma. Thus we have a commutative diagram as at the right. This is called a *Postnikov tower* for  $X$ . One can regard the spaces  $X_n$  as truncations of  $X$  which provide successively better approximations to  $X$  as  $n$  increases. Postnikov towers turn out to be quite powerful tools for proving general theorems, and we will study them further in §4.3.



Now that we have seen several varied applications of the technique of attaching cells to make a map  $f: A \rightarrow X$  more nearly a weak homotopy equivalence, it might be useful to give a name to the properties that the construction can achieve. To simplify the description, we may assume without loss of generality that the given  $f$  is an inclusion  $A \hookrightarrow X$  by replacing  $X$  by the mapping cylinder of  $f$ . Thus, starting with a pair  $(X, A)$  where the subspace  $A \subset X$  is a nonempty CW complex, we define an  **$n$ -connected CW model for  $(X, A)$**  to be an  $n$ -connected CW pair  $(Z, A)$  and a map  $f: Z \rightarrow X$  with  $f|_A$  the identity, such that  $f_*: \pi_i(Z) \rightarrow \pi_i(X)$  is an isomorphism for  $i > n$  and an injection for  $i = n$ , for all choices of basepoint. Since  $(Z, A)$  is  $n$ -connected, the map  $\pi_i(A) \rightarrow \pi_i(Z)$  is an isomorphism for  $i < n$  and a surjection for  $i = n$ . In the critical dimension  $n$ , the maps  $A \hookrightarrow Z \xrightarrow{f} X$  induce a composition  $\pi_n(A) \rightarrow \pi_n(Z) \rightarrow \pi_n(X)$  factoring the map  $\pi_n(A) \rightarrow \pi_n(X)$  as a surjection followed by an injection, just as any homomorphism  $\varphi: G \rightarrow H$  can be factored (uniquely) as a surjection  $\varphi: G \rightarrow \text{Im } \varphi$  followed by an injection  $\text{Im } \varphi \hookrightarrow H$ . One can think of  $Z$  as a sort of homotopy-theoretic hybrid of  $A$  and  $X$ . As  $n$  increases, the hybrid looks more and more like  $A$ , and less and less like  $X$ .

Our earlier construction shows:

**Proposition 4.17.** *For every pair  $(X, A)$  with  $A$  a nonempty CW complex there exist  $n$ -connected CW models  $f: (Z, A) \rightarrow (X, A)$  for all  $n \geq 0$ , and these models can be chosen to have the additional property that  $Z$  is obtained from  $A$  by attaching cells of dimension greater than  $n$ .*  $\square$

The construction of  $n$ -connected CW models involves many arbitrary choices, so it may be somewhat surprising that they turn out to be unique up to homotopy equivalence. This will follow easily from the next proposition. Another application of the proposition will be to build a tower like the Postnikov tower from the various  $n$ -connected CW models for a given pair  $(X, A)$ .

**Proposition 4.18.** *Suppose we are given:*

- (i) *an  $n$ -connected CW model  $f: (Z, A) \rightarrow (X, A)$ ,*
- (ii) *an  $n'$ -connected CW model  $f': (Z', A') \rightarrow (X', A')$ ,*
- (iii) *a map  $g: (X, A) \rightarrow (X', A')$ .*

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ h \downarrow & & \downarrow g \\ Z' & \xrightarrow{f'} & X' \end{array}$$

*Then if  $n \geq n'$ , there is a map  $h: Z \rightarrow Z'$  such that  $h|_A = g$  and  $gf \simeq f'h \text{ rel } A$ , so the diagram above is commutative up to homotopy  $\text{rel } A$ . Furthermore, such a map  $h$  is unique up to homotopy  $\text{rel } A$ .*

**Proof:** By Proposition 4.15 we may assume all cells of  $Z - A$  have dimension greater than  $n$ . Let  $W$  be the quotient space of the mapping cylinder of  $f'$  obtained by collapsing each line segment  $\{a'\} \times I$  to a point, for  $a' \in A'$ . We can think of  $W$  as a relative mapping cylinder, and like the ordinary mapping cylinder,  $W$  contains copies of  $Z'$  and  $X'$ , the latter as a deformation retract. The assumption that  $(Z', A')$  is an  $n'$ -connected CW model for  $(X', A')$  implies that the relative groups  $\pi_i(W, Z')$  are zero for  $i > n'$ .

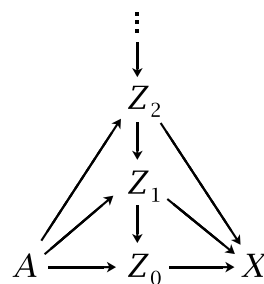
Via the inclusion  $X' \hookrightarrow W$  we can view  $gf$  as a map  $Z \rightarrow W$ . As a map of pairs  $(Z, A) \rightarrow (W, Z')$ ,  $gf$  is homotopic  $\text{rel } A$  to a map  $h$  with image in  $Z'$ , by the compression lemma and the hypothesis  $n \geq n'$ . This proves the first assertion. For the second, suppose  $h_0$  and  $h_1$  are two maps  $Z \rightarrow Z'$  whose compositions with  $f'$  are homotopic to  $gf \text{ rel } A$ . Thus if we regard  $h_0$  and  $h_1$  as maps to  $W$ , they are homotopic  $\text{rel } A$ . Such a homotopy gives a map  $(Z \times I, Z \times \partial I \cup A \times I) \rightarrow (W, Z')$ , and by the compression lemma again this map can be deformed  $\text{rel } Z \times \partial I \cup A \times I$  to a map with image in  $Z'$ , which gives the desired homotopy  $h_0 \simeq h_1 \text{ rel } A$ .  $\square$

**Corollary 4.19.** *An  $n$ -connected CW model for  $(X, A)$  is unique up to homotopy equivalence  $\text{rel } A$ . In particular, CW approximations to spaces are unique up to homotopy equivalence.*

**Proof:** Given two  $n$ -connected CW models  $(Z, A)$  and  $(Z', A)$  for  $(X, A)$ , we apply the proposition twice with  $g$  the identity map to obtain maps  $h: Z \rightarrow Z'$  and  $h': Z' \rightarrow Z$ . The uniqueness statement gives homotopies  $hh' \simeq \mathbb{1}$  and  $h'h \simeq \mathbb{1} \text{ rel } A$ .  $\square$

Taking  $n = n'$  in the proposition, we obtain also a functoriality property for  $n$ -connected CW models. For example, a map  $X \rightarrow X'$  induces a map of CW approximations  $Z \rightarrow Z'$ , which is unique up to homotopy.

The proposition allows us to relate  $n$ -connected CW models  $(Z_n, A)$  for  $(X, A)$  for varying  $n$ , by means of maps  $Z_n \rightarrow Z_{n-1}$  that form a tower as shown in the diagram, with commutative triangles on the left and homotopy-commutative triangles on the right. We can make the triangles on the right strictly commutative by replacing the maps  $Z_n \rightarrow X$  by the compositions through  $Z_0$ .



**Example 4.20: Whitehead Towers** . If we take  $X$  to be an arbitrary CW complex with the subspace  $A$  a point, then the resulting tower of  $n$ -connected CW models amounts to a sequence of maps

$$\cdots \rightarrow Z_2 \rightarrow Z_1 \rightarrow Z_0 \rightarrow X$$

with  $Z_n$   $n$ -connected and the map  $Z_n \rightarrow X$  inducing an isomorphism on all homotopy groups  $\pi_i$  with  $i > n$ . The space  $Z_0$  is path-connected and homotopy equivalent to the component of  $X$  containing  $A$ , so one may as well assume  $Z_0$  equals this component. The next space  $Z_1$  is simply-connected, and the map  $Z_1 \rightarrow X$  has the homotopy properties of the universal cover of the component  $Z_0$  of  $X$ . For larger values of  $n$  one can by analogy view the map  $Z_n \rightarrow X$  as an ' $n$ -connected cover' of  $X$ . For  $n > 1$  these do not seem to arise so frequently in nature as in the case  $n = 1$ . A rare exception is the Hopf map  $S^3 \rightarrow S^2$  defined in Example 4.45, which is a 2-connected cover.

Now let us show that CW approximations behave well with respect to homology and cohomology:

**Proposition 4.21.** *A weak homotopy equivalence  $f: X \rightarrow Y$  induces isomorphisms  $f_*: H_n(X; G) \rightarrow H_n(Y; G)$  and  $f^*: H^n(Y; G) \rightarrow H^n(X; G)$  for all  $n$  and all coefficient groups  $G$ .*

**Proof:** Replacing  $Y$  by the mapping cylinder  $M_f$  and looking at the long exact sequences of homotopy, homology, and cohomology groups for  $(M_f, X)$ , we see that it suffices to show:

- If  $(Z, X)$  is an  $n$ -connected pair of path-connected spaces, then  $H_i(Z, X; G) = 0$  and  $H^i(Z, X; G) = 0$  for all  $i \leq n$  and all  $G$ .

Let  $\alpha = \sum_j n_j \sigma_j$  be a relative cycle representing an element of  $H_k(Z, X; G)$ , for singular  $k$ -simplices  $\sigma_j: \Delta^k \rightarrow Z$ . Build a finite  $\Delta$ -complex  $K$  from a disjoint union of  $k$ -simplices, one for each  $\sigma_j$ , by identifying all  $(k-1)$ -dimensional faces of these  $k$ -simplices for which the corresponding restrictions of the  $\sigma_j$ 's are equal. Thus the  $\sigma_j$ 's induce a map  $\sigma: K \rightarrow Z$ . Since  $\alpha$  is a relative cycle,  $\partial\alpha$  is a chain in  $X$ . Let

$L \subset K$  be the subcomplex consisting of  $(k-1)$ -simplices corresponding to the singular  $(k-1)$ -simplices in  $\partial\alpha$ , so  $\sigma(L) \subset X$ . The chain  $\alpha$  is the image under the chain map  $\sigma_\#$  of a chain  $\tilde{\alpha}$  in  $K$ , with  $\partial\tilde{\alpha}$  a chain in  $L$ . In relative homology we then have  $\sigma_*[\tilde{\alpha}] = [\alpha]$ . If we assume  $\pi_i(Z, X) = 0$  for  $i \leq k$ , then  $\sigma: (K, L) \rightarrow (Z, X)$  is homotopic rel  $L$  to a map with image in  $X$ , by the compression lemma. Hence  $\sigma_*[\tilde{\alpha}]$  is in the image of the map  $H_k(X, X; G) \rightarrow H_k(Z, X; G)$ , and since  $H_k(X, X; G) = 0$  we conclude that  $[\alpha] = \sigma_*[\tilde{\alpha}] = 0$ . This proves the result for homology, and the result for cohomology then follows by the universal coefficient theorem.  $\square$

CW approximations can be used to reduce many statements about general spaces to the special case of CW complexes. For example, the cup product version of the Künneth formula in Theorem 3.15, asserting that  $H^*(X \times Y; R) \approx H^*(X; R) \otimes H^*(Y; R)$  under certain conditions, can now be extended to non-CW spaces since if  $X$  and  $Y$  are CW approximations to spaces  $Z$  and  $W$ , respectively, then  $X \times Y$  is a CW approximation to  $Z \times W$ . Here we are giving  $X \times Y$  the CW topology rather than the product topology, but this has no effect on homotopy groups since the two topologies have the same compact sets, as explained in the Appendix. Similarly, the general Künneth formula for homology in §3.B holds for arbitrary products  $X \times Y$ .

The condition for a map  $Y \rightarrow Z$  to be a weak homotopy equivalence involves only maps of spheres into  $Y$  and  $Z$ , but in fact weak homotopy equivalences  $Y \rightarrow Z$  behave nicely with respect to maps of arbitrary CW complexes into  $Y$  and  $Z$ , not just spheres. The following proposition gives a precise statement, using the notations  $[X, Y]$  for the set of homotopy classes of maps  $X \rightarrow Y$  and  $\langle X, Y \rangle$  for the set of basepoint-preserving-homotopy classes of basepoint-preserving maps  $X \rightarrow Y$ . (The notation  $\langle X, Y \rangle$  is not standard, but is intended to suggest ‘pointed homotopy classes’.)

**Proposition 4.22.** *A weak homotopy equivalence  $f: Y \rightarrow Z$  induces bijections  $[X, Y] \rightarrow [X, Z]$  and  $\langle X, Y \rangle \rightarrow \langle X, Z \rangle$  for all CW complexes  $X$ .*

**Proof:** Consider first  $[X, Y] \rightarrow [X, Z]$ . We may assume  $f$  is an inclusion by replacing  $Z$  by the mapping cylinder  $M_f$  as usual. The groups  $\pi_n(Z, Y, y_0)$  are then zero for all  $n$  and all basepoints  $y_0 \in Y$ , so the compression lemma implies that any map  $X \rightarrow Z$  can be homotoped to have image in  $Y$ . This gives surjectivity of  $[X, Y] \rightarrow [X, Z]$ . A relative version of this argument shows injectivity since we can deform a homotopy  $(X \times I, X \times \partial I) \rightarrow (Z, Y)$  to have image in  $Y$ .

In the case of  $\langle X, Y \rangle \rightarrow \langle X, Z \rangle$  the same argument applies if  $M_f$  is replaced by the reduced mapping cylinder, the quotient of  $M_f$  obtained by collapsing the segment  $\{y_0\} \times I$  to a point, for  $y_0$  the basepoint of  $Y$ . This collapsed segment then serves as the common basepoint of  $Y$ ,  $Z$ , and the reduced mapping cylinder. The reduced mapping cylinder deformation retracts to  $Z$  just as the unreduced one does, but with the advantage that the basepoint does not move.  $\square$

## Exercises

1. Suppose a sum  $f +' g$  of maps  $f, g: (I^n, \partial I^n) \rightarrow (X, x_0)$  is defined using a coordinate of  $I^n$  other than the first coordinate as in the usual sum  $f + g$ . Verify the formula  $(f + g) +' (h + k) = (f +' h) + (g +' k)$ , and deduce that  $f +' k \simeq f + k$  so the two sums agree on  $\pi_n(X, x_0)$ , and also that  $g +' h \simeq h + g$  so the addition is abelian.
2. Show that if  $\varphi: X \rightarrow Y$  is a homotopy equivalence, then the induced homomorphisms  $\varphi_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, \varphi(x_0))$  are isomorphisms for all  $n$ . [The case  $n = 1$  is Proposition 1.18.]
3. For an H-space  $(X, x_0)$  with multiplication  $\mu: X \times X \rightarrow X$ , show that the group operation in  $\pi_n(X, x_0)$  can also be defined by the rule  $(f + g)(x) = \mu(f(x), g(x))$ .
4. Let  $p: \tilde{X} \rightarrow X$  be the universal cover of a path-connected space  $X$ . Show that under the isomorphism  $\pi_n(X) \approx \pi_n(\tilde{X})$ , which holds for  $n \geq 2$ , the action of  $\pi_1(X)$  on  $\pi_n(X)$  corresponds to the action of  $\pi_1(X)$  on  $\pi_n(\tilde{X})$  induced by the action of  $\pi_1(X)$  on  $\tilde{X}$  as deck transformations. More precisely, prove a formula like  $\gamma p_*(\alpha) = p_*(\beta_{\tilde{\gamma}}(\gamma_*(\alpha)))$  where  $\gamma \in \pi_1(X, x_0)$ ,  $\alpha \in \pi_n(\tilde{X}, \tilde{x}_0)$ , and  $\gamma_*$  denotes the homomorphism induced by the action of  $\gamma$  on  $\tilde{X}$ .
5. For a pair  $(X, A)$  of path-connected spaces, show that  $\pi_1(X, A, x_0)$  can be identified in a natural way with the set of cosets  $\alpha H$  of the subgroup  $H \subset \pi_1(X, x_0)$  represented by loops in  $A$  at  $x_0$ .
6. If  $p: (\tilde{X}, \tilde{A}, \tilde{x}_0) \rightarrow (X, A, x_0)$  is a covering space with  $\tilde{A} = p^{-1}(A)$ , show that the map  $p_*: \pi_n(\tilde{X}, \tilde{A}, \tilde{x}_0) \rightarrow \pi_n(X, A, x_0)$  is an isomorphism for all  $n > 1$ .
7. Extend the results proved near the beginning of this section for the change-of-basepoint maps  $\beta_y$  to the case of relative homotopy groups.
8. Show the sequence  $\pi_1(X, x_0) \rightarrow \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \rightarrow \pi_0(X, x_0)$  is exact.
9. Suppose we define  $\pi_0(X, A, x_0)$  to be the quotient set  $\pi_0(X, x_0)/i^*(\pi_0(A, x_0))$ , so that the long exact sequence of homotopy groups for the pair  $(X, A)$  extends to  $\cdots \rightarrow \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0) \rightarrow \pi_0(X, A, x_0) \rightarrow 0$ .
  - (a) Show that with this extension, the five-lemma holds for the map of long exact sequences induced by a map  $(X, A, x_0) \rightarrow (Y, B, y_0)$ , in the following form: One of the maps between the two sequences is a bijection if the four surrounding maps are bijections for all choices of  $x_0$ .
  - (b) Show that the long exact sequence of a triple  $(X, A, B, x_0)$  can be extended only to the term  $\pi_0(A, B, x_0)$  in general, and that the five-lemma holds for this extension.
10. Show the 'quasi-circle' described in Exercise 7 in §1.3 has trivial homotopy groups but is not contractible, hence does not have the homotopy type of a CW complex.
11. Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes  $X_1 \subset X_2 \subset \cdots$  such that each inclusion  $X_i \hookrightarrow X_{i+1}$  is nullhomotopic, a condition sometimes expressed by saying  $X_i$  is contractible in  $X_{i+1}$ . An example is

$S^\infty$ , or more generally the infinite suspension  $S^\infty X$  of any CW complex  $X$ , the union of the iterated suspensions  $S^n X$ .

12. Show that an  $n$ -connected,  $n$ -dimensional CW complex is contractible.
13. Use the extension lemma to show that a CW complex retracts onto any contractible subcomplex.
14. Use cellular approximation to show that the  $n$ -skeletons of homotopy equivalent CW complexes without cells of dimension  $n + 1$  are also homotopy equivalent.
15. Show that every map  $f: S^n \rightarrow S^n$  is homotopic to a multiple of the identity map by the following steps.
  - (a) Use Lemma 4.10 (or simplicial approximation, Theorem 2C.1) to reduce to the case that there exists a point  $q \in S^n$  with  $f^{-1}(q) = \{p_1, \dots, p_k\}$  and  $f$  is an invertible linear map near each  $p_i$ .
  - (b) For  $f$  as in (a), consider the composition  $gf$  where  $g: S^n \rightarrow S^n$  collapses the complement of a small ball about  $q$  to the basepoint. Use this to reduce (a) further to the case  $k = 1$ .
  - (c) Finish the argument by showing that an invertible  $n \times n$  matrix can be joined by a path of such matrices to either the identity matrix or the matrix of a reflection. (Use Gaussian elimination, for example.)
16. Show that a map  $f: X \rightarrow Y$  between connected CW complexes factors as a composition  $X \rightarrow Z_n \rightarrow Y$  where the first map induces isomorphisms on  $\pi_i$  for  $i \leq n$  and the second map induces isomorphisms on  $\pi_i$  for  $i \geq n + 1$ .
17. Show that if  $X$  and  $Y$  are CW complexes with  $X$   $m$ -connected and  $Y$   $n$ -connected, then  $(X \times Y, X \vee Y)$  is  $(m + n + 1)$ -connected, as is the smash product  $X \wedge Y$ .
18. Give an example of a weak homotopy equivalence  $X \rightarrow Y$  for which there does not exist a weak homotopy equivalence  $Y \rightarrow X$ .
19. Consider the equivalence relation  $\simeq_w$  generated by weak homotopy equivalence:  $X \simeq_w Y$  if there are spaces  $X = X_1, X_2, \dots, X_n = Y$  with weak homotopy equivalences  $X_i \rightarrow X_{i+1}$  or  $X_i \leftarrow X_{i+1}$  for each  $i$ . Show that  $X \simeq_w Y$  iff  $X$  and  $Y$  have a common CW approximation.
20. Show that  $[X, Y]$  is finite if  $X$  is a finite connected CW complex and  $\pi_i(Y)$  is finite for  $i \leq \dim X$ .
21. For this problem it is convenient to use the notations  $X^n$  for the  $n^{\text{th}}$  stage in a Postnikov tower for  $X$  and  $X_m$  for an  $(m - 1)$ -connected covering of  $X$ , where  $X$  is a connected CW complex. Show that  $(X^n)_m \simeq (X_m)^n$ , so the notation  $X_m^n$  is unambiguous. Thus  $\pi_i(X_m^n) \approx \pi_i(X)$  for  $m \leq i \leq n$  and all other homotopy groups of  $X_m^n$  are zero.
 
$$\begin{array}{ccc} X_m & \longrightarrow & X_m^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^n \end{array}$$
22. Show that a path-connected space  $X$  has a CW approximation with countably many cells iff  $\pi_n(X)$  is countable for all  $n$ . [Use the results on simplicial approximations to maps and spaces in §2.C.]

23. If  $f: X \rightarrow Y$  is a map with  $X$  and  $Y$  homotopy equivalent to CW complexes, show that the pair  $(M_f, X)$  is homotopy equivalent to a CW pair, where  $M_f$  is the mapping cylinder. Deduce that the mapping cone  $C_f$  has the homotopy type of a CW complex.

## 4.2 Elementary Methods of Calculation

We have not yet computed any nonzero homotopy groups  $\pi_n(X)$  with  $n \geq 2$ . In Chapter 1 the two main tools we used for computing fundamental groups were van Kampen's theorem and covering spaces. In the present section we will study the higher-dimensional analogs of these: the excision theorem for homotopy groups, and fiber bundles. Both of these are quite a bit weaker than their fundamental group analogs, in that they do not directly compute homotopy groups but only give relations between the homotopy groups of different spaces. Their applicability is thus more limited, but suffices for a number of interesting calculations, such as  $\pi_n(S^n)$  and more generally the Hurewicz theorem relating the first nonzero homotopy and homology groups of a space. Another noteworthy application is the Freudenthal suspension theorem, which leads to stable homotopy groups and in fact the whole subject of stable homotopy theory.

### Excision for Homotopy Groups

What makes homotopy groups so much harder to compute than homology groups is the failure of the excision property. However, there is a certain dimension range, depending on connectivities, in which excision does hold for homotopy groups:

**Theorem 4.23.** *Let  $X$  be a CW complex decomposed as the union of subcomplexes  $A$  and  $B$  with nonempty connected intersection  $C = A \cap B$ . If  $(A, C)$  is  $m$ -connected and  $(B, C)$  is  $n$ -connected,  $m, n \geq 0$ , then the map  $\pi_i(A, C) \rightarrow \pi_i(X, B)$  induced by inclusion is an isomorphism for  $i < m + n$  and a surjection for  $i = m + n$ .*

This yields the **Freudenthal suspension theorem**:

**Corollary 4.24.** *The suspension map  $\pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$  is an isomorphism for  $i < 2n - 1$  and a surjection for  $i = 2n - 1$ . More generally this holds for the suspension  $\pi_i(X) \rightarrow \pi_{i+1}(SX)$  whenever  $X$  is an  $(n - 1)$ -connected CW complex.*

**Proof:** Decompose the suspension  $SX$  as the union of two cones  $C_+X$  and  $C_-X$  intersecting in a copy of  $X$ . The suspension map is the same as the map

$$\pi_i(X) \approx \pi_{i+1}(C_+X, X) \rightarrow \pi_{i+1}(SX, C_-X) \approx \pi_{i+1}(SX)$$



where the two isomorphisms come from long exact sequences of pairs and the middle map is induced by inclusion. From the long exact sequence of the pair  $(C_\pm X, X)$  we see that this pair is  $n$ -connected if  $X$  is  $(n-1)$ -connected. The preceding theorem then says that the middle map is an isomorphism for  $i+1 < 2n$  and surjective for  $i+1 = 2n$ .  $\square$

**Corollary 4.25.**  $\pi_n(S^n) \approx \mathbb{Z}$ , generated by the identity map, for all  $n \geq 1$ . In particular, the degree map  $\pi_n(S^n) \rightarrow \mathbb{Z}$  is an isomorphism.

**Proof:** From the preceding corollary we know that in the sequence of suspension maps  $\pi_1(S^1) \rightarrow \pi_2(S^2) \rightarrow \pi_3(S^3) \rightarrow \cdots$  the first map is surjective and all the subsequent maps are isomorphisms. Since  $\pi_1(S^1)$  is  $\mathbb{Z}$  generated by the identity map, it follows that  $\pi_n(S^n)$  for  $n \geq 2$  is a finite or infinite cyclic group independent of  $n$ , generated by the identity map. The fact that this cyclic group is infinite can be deduced from homology theory since there exist basepoint-preserving maps  $S^n \rightarrow S^n$  of arbitrary degree, and degree is a homotopy invariant. Alternatively, if one wants to avoid appealing to homology theory one can use the Hopf bundle  $S^1 \rightarrow S^3 \rightarrow S^2$  described in Example 4.45, whose long exact sequence of homotopy groups gives an isomorphism  $\pi_1(S^1) \approx \pi_2(S^2)$ .

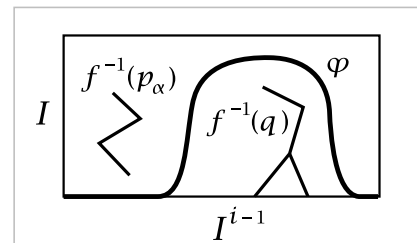
The degree map  $\pi_n(S^n) \rightarrow \mathbb{Z}$  is an isomorphism since the map  $z \mapsto z^k$  of  $S^1$  has degree  $k$ , as do its iterated suspensions by Proposition 2.33.  $\square$

**Proof of 4.23:** We proceed by proving successively more general cases. The first case contains the heart of the argument, and suffices for the calculation of  $\pi_n(S^n)$ .

**Case 1:**  $A$  is obtained from  $C$  by attaching cells  $e_\alpha^{m+1}$  and  $B$  is obtained from  $C$  by attaching a cell  $e^{n+1}$ . To show surjectivity of  $\pi_i(A, C) \rightarrow \pi_i(X, B)$  we start with a map  $f: (I^i, \partial I^i, J^{i-1}) \rightarrow (X, B, x_0)$ . This has compact image, meeting only finitely many of the cells  $e_\alpha^{m+1}$  and  $e^{n+1}$ . By Lemma 4.10 we may homotope  $f$  through maps  $(I^i, \partial I^i, J^{i-1}) \rightarrow (X, B, x_0)$  so that there are simplices  $\Delta_\alpha^{m+1} \subset e_\alpha^{m+1}$  and  $\Delta^{n+1} \subset e^{n+1}$  for which  $f^{-1}(\Delta_\alpha^{m+1})$  and  $f^{-1}(\Delta^{n+1})$  are finite unions of convex polyhedra, on each of which  $f$  is the restriction of a linear map from  $\mathbb{R}^i$  to  $\mathbb{R}^{m+1}$  or  $\mathbb{R}^{n+1}$ . We may assume these linear maps are surjections by rechoosing smaller simplices  $\Delta_\alpha^{m+1}$  and  $\Delta^{n+1}$  in the complement of the images of the nonsurjective linear maps.

**Claim:** If  $i \leq m+n$ , then there exist points  $p_\alpha \in \Delta_\alpha^{m+1}$ ,  $q \in \Delta^{n+1}$ , and a map  $\varphi: I^{i-1} \rightarrow [0, 1]$  such that:

- (a)  $f^{-1}(q)$  lies below the graph of  $\varphi$  in  $I^{i-1} \times I = I^i$ .
- (b)  $f^{-1}(p_\alpha)$  lies above the graph of  $\varphi$  for each  $\alpha$ .
- (c)  $\varphi = 0$  on  $\partial I^{i-1}$ .



Granting this, let  $f_t$  be a homotopy of  $f$  excising the region under the graph of  $\varphi$  by restricting  $f$  to the region above the graph of  $t\varphi$  for  $0 \leq t \leq 1$ . By (b),  $f_t(I^{i-1})$  is disjoint from  $P = \bigcup_\alpha \{p_\alpha\}$  for all  $t$ , and by (a),  $f_1(I^i)$  is disjoint from  $Q = \{q\}$ . This

means that in the commutative diagram at the right the given element  $[f]$  in the upper-right group, when regarded as an element of the lower-right group,

$$\begin{array}{ccc} \pi_i(A, C) & \longrightarrow & \pi_i(X, B) \\ \downarrow \approx & & \downarrow \approx \\ \pi_i(X - Q, X - Q - P) & \longrightarrow & \pi_i(X, X - P) \end{array}$$

is equal to the element  $[f_1]$  in the image of the lower horizontal map. Since the vertical maps are isomorphisms, this proves the surjectivity statement.

Now we prove the Claim. For any  $q \in \Delta^{n+1}$ ,  $f^{-1}(q)$  is a finite union of convex polyhedra of dimension  $\leq i - n - 1$  since  $f^{-1}(\Delta^{n+1})$  is a finite union of convex polyhedra on each of which  $f$  is the restriction of a linear surjection  $\mathbb{R}^i \rightarrow \mathbb{R}^{n+1}$ . We wish to choose the points  $p_\alpha \in \Delta_\alpha^{m+1}$  so that not only is  $f^{-1}(q)$  disjoint from  $f^{-1}(p_\alpha)$  for each  $\alpha$ , but also so that  $f^{-1}(q)$  and  $f^{-1}(p_\alpha)$  have disjoint images under the projection  $\pi: I^i \rightarrow I^{i-1}$ . This is equivalent to saying that  $f^{-1}(p_\alpha)$  is disjoint from  $T = \pi^{-1}(\pi(f^{-1}(q)))$ , the union of all segments  $\{x\} \times I$  meeting  $f^{-1}(q)$ . This set  $T$  is a finite union of convex polyhedra of dimension  $\leq i - n$  since  $f^{-1}(q)$  is a finite union of convex polyhedra of dimension  $\leq i - n - 1$ . Since linear maps cannot increase dimension,  $f(T) \cap \Delta_\alpha^{m+1}$  is also a finite union of convex polyhedra of dimension  $\leq i - n$ . Thus if  $m + 1 > i - n$ , there is a point  $p_\alpha \in \Delta_\alpha^{m+1}$  not in  $f(T)$ . This gives  $f^{-1}(p_\alpha) \cap T = \emptyset$  if  $i \leq m + n$ . Hence we can choose a neighborhood  $U$  of  $\pi(f^{-1}(q))$  in  $I^{i-1}$  disjoint from  $\pi(f^{-1}(p_\alpha))$  for all  $\alpha$ . Then there exists  $\varphi: I^{i-1} \rightarrow [0, 1)$  having support in  $U$ , with  $f^{-1}(q)$  lying under the graph of  $\varphi$ . This verifies the Claim, and so finishes the proof of surjectivity in Case 1.

For injectivity in Case 1 the argument is very similar. Suppose we have two maps  $f_0, f_1: (I^i, \partial I^i, J^{i-1}) \rightarrow (A, C, x_0)$  representing elements of  $\pi_i(A, C, x_0)$  having the same image in  $\pi_i(X, B, x_0)$ . Thus there is a homotopy from  $f_0$  to  $f_1$  in the form of a map  $F: (I^i, \partial I^i, J^{i-1}) \times [0, 1] \rightarrow (X, B, x_0)$ . After a preliminary deformation of  $F$  via Lemma 4.10, we construct a function  $\varphi: I^{i-1} \times I \rightarrow [0, 1)$  separating  $F^{-1}(q)$  from the sets  $F^{-1}(p_\alpha)$  as before. This allows us to excise  $F^{-1}(q)$  from the domain of  $F$ , from which it follows that  $f_0$  and  $f_1$  represent the same element of  $\pi_i(A, C, x_0)$ . Since  $I^i \times I$  now plays the role of  $I^i$ , the dimension  $i$  is replaced by  $i + 1$  and the dimension restriction  $i \leq m + n$  becomes  $i + 1 \leq m + n$ , or  $i < m + n$ .

**Case 2:**  $A$  is obtained from  $C$  by attaching  $(m + 1)$ -cells as in Case 1 and  $B$  is obtained from  $C$  by attaching cells of dimension  $\geq n + 1$ . To show surjectivity of  $\pi_i(A, C) \rightarrow \pi_i(X, B)$ , consider a map  $f: (I^i, \partial I^i, J^{i-1}) \rightarrow (X, B, x_0)$  representing an element of  $\pi_i(X, B)$ . The image of  $f$  is compact, meeting only finitely many cells, and by repeated applications of Case 1 we can push  $f$  off the cells of  $B - C$  one at a time, in order of decreasing dimension. Injectivity is quite similar, starting with a homotopy  $F: (I^i, \partial I^i, J^{i-1}) \times [0, 1] \rightarrow (X, B, x_0)$  and pushing this off cells of  $B - C$ .

**Case 3:**  $A$  is obtained from  $C$  by attaching cells of dimension  $\geq m + 1$  and  $B$  is as in Case 2. We may assume all cells of  $A - C$  have dimension  $\leq m + n + 1$  since higher-dimensional cells have no effect on  $\pi_i$  for  $i \leq m + n$ , by cellular approximation. Let

$A_k \subset A$  be the union of  $C$  with the cells of  $A$  of dimension  $\leq k$  and let  $X_k = A_k \cup B$ . We prove the result for  $\pi_i(A_k, C) \rightarrow \pi_i(X_k, B)$  by induction on  $k$ . The induction starts with  $k = m + 1$ , which is Case 2. For the induction step consider the following commutative diagram formed by the exact sequences of the triples  $(A_k, A_{k-1}, C)$  and  $(X_k, X_{k-1}, B)$ :

$$\begin{array}{ccccccccc} \pi_{i+1}(A_k, A_{k-1}) & \longrightarrow & \pi_i(A_{k-1}, C) & \longrightarrow & \pi_i(A_k, C) & \longrightarrow & \pi_i(A_k, A_{k-1}) & \longrightarrow & \pi_{i-1}(A_{k-1}, C) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi_{i+1}(X_k, X_{k-1}) & \longrightarrow & \pi_i(X_{k-1}, B) & \longrightarrow & \pi_i(X_k, B) & \longrightarrow & \pi_i(X_k, X_{k-1}) & \longrightarrow & \pi_{i-1}(X_{k-1}, B) \end{array}$$

When  $i < m + n$  the first and fourth vertical maps are isomorphisms by Case 2, while by induction the second and fifth maps are isomorphisms, so the middle map is an isomorphism by the five-lemma. Similarly, when  $i = m + n$  the second and fourth maps are surjective and the fifth map is injective, which is enough to imply the middle map is surjective by one half of the five-lemma. When  $i = 2$  the diagram may contain nonabelian groups and the two terms on the right may not be groups, but the five-lemma remains valid in this generality, with trivial modifications to the proof in §2.1. When  $i = 1$  the assertion about  $\pi_1(A, C) \rightarrow \pi_1(X, B)$  follows by a direct argument: If  $m \geq 1$  then both terms are trivial, while if  $m = 0$  then  $n \geq 1$  and the result follows by cellular approximation.

After these special cases we can now easily deal with the general case. The connectivity assumptions on the pairs  $(A, C)$  and  $(B, C)$  imply by Proposition 4.15 that they are homotopy equivalent to pairs  $(A', C)$  and  $(B', C)$  as in Case 3, via homotopy equivalences fixed on  $C$ , so these homotopy equivalences fit together to give a homotopy equivalence  $A \cup B \simeq A' \cup B'$ . Thus the general case reduces to Case 3.  $\square$

**Example 4.26.** The calculation of  $\pi_n(S^n)$  can be extended to show that  $\pi_n(\bigvee_{\alpha} S_{\alpha}^n)$  for  $n \geq 2$  is free abelian with basis the homotopy classes of the inclusions  $S_{\alpha}^n \hookrightarrow \bigvee_{\alpha} S_{\alpha}^n$ . Suppose first that there are only finitely many summands  $S_{\alpha}^n$ . We can regard  $\bigvee_{\alpha} S_{\alpha}^n$  as the  $n$ -skeleton of the product  $\prod_{\alpha} S_{\alpha}^n$ , where  $S_{\alpha}^n$  is given its usual CW structure and  $\prod_{\alpha} S_{\alpha}^n$  has the product CW structure. Since  $\prod_{\alpha} S_{\alpha}^n$  has cells only in dimensions a multiple of  $n$ , the pair  $(\prod_{\alpha} S_{\alpha}^n, \bigvee_{\alpha} S_{\alpha}^n)$  is  $(2n - 1)$ -connected. Hence from the long exact sequence of homotopy groups for this pair we see that the inclusion  $\bigvee_{\alpha} S_{\alpha}^n \hookrightarrow \prod_{\alpha} S_{\alpha}^n$  induces an isomorphism on  $\pi_n$  if  $n \geq 2$ . By Proposition 4.2 we have  $\pi_n(\prod_{\alpha} S_{\alpha}^n) \approx \bigoplus_{\alpha} \pi_n(S_{\alpha}^n)$ , a free abelian group with basis the inclusions  $S_{\alpha}^n \hookrightarrow \prod_{\alpha} S_{\alpha}^n$ , so the same is true for  $\bigvee_{\alpha} S_{\alpha}^n$ . This takes care of the case of finitely many  $S_{\alpha}^n$ 's.

To reduce the case of infinitely many summands  $S_{\alpha}^n$  to the finite case, consider the homomorphism  $\Phi: \bigoplus_{\alpha} \pi_n(S_{\alpha}^n) \rightarrow \pi_n(\bigvee_{\alpha} S_{\alpha}^n)$  induced by the inclusions  $S_{\alpha}^n \hookrightarrow \bigvee_{\alpha} S_{\alpha}^n$ . Then  $\Phi$  is surjective since any map  $f: S^n \rightarrow \bigvee_{\alpha} S_{\alpha}^n$  has compact image contained in the wedge sum of finitely many  $S_{\alpha}^n$ 's, so by the finite case already proved,  $[f]$  is in the image of  $\Phi$ . Similarly, a nullhomotopy of  $f$  has compact image contained in a finite wedge sum of  $S_{\alpha}^n$ 's, so the finite case also implies that  $\Phi$  is injective.

**Example 4.27.** Let us show that  $\pi_n(S^1 \vee S^n)$  for  $n \geq 2$  is free abelian on a countably infinite number of generators. By Proposition 4.1 we may compute  $\pi_i(S^1 \vee S^n)$  for  $i \geq 2$  by passing to the universal cover. This consists of a copy of  $\mathbb{R}$  with a sphere  $S_k^n$  attached at each integer point  $k \in \mathbb{R}$ , so it is homotopy equivalent to  $\bigvee_k S_k^n$ . The preceding Example 4.26 says that  $\pi_n(\bigvee_k S_k^n)$  is free abelian with basis represented by the inclusions of the wedge summands. So a basis for  $\pi_n$  of the universal cover of  $S^1 \vee S^n$  is represented by maps that lift the maps obtained from the inclusion  $S^n \hookrightarrow S^1 \vee S^n$  by the action of the various elements of  $\pi_1(S^1 \vee S^n) \approx \mathbb{Z}$ . This means that  $\pi_n(S^1 \vee S^n)$  is a free  $\mathbb{Z}[\pi_1(S^1 \vee S^n)]$ -module on a single basis element, the homotopy class of the inclusion  $S^n \hookrightarrow S^1 \vee S^n$ . Writing a generator of  $\pi_1(S^1 \vee S^n)$  as  $t$ , the group ring  $\mathbb{Z}[\pi_1(S^1 \vee S^n)]$  becomes  $\mathbb{Z}[t, t^{-1}]$ , the Laurent polynomials in  $t$  and  $t^{-1}$  with  $\mathbb{Z}$  coefficients, and we have  $\pi_n(S^1 \vee S^n) \approx \mathbb{Z}[t, t^{-1}]$ .

This example shows that the homotopy groups of a finite CW complex need not be finitely generated, in contrast to the homology groups. However, if we restrict attention to spaces with trivial action of  $\pi_1$  on all  $\pi_n$ 's, then a theorem of Serre, proved in [SSAT], says that the homotopy groups of such a space are finitely generated iff the homology groups are finitely generated.

In this example,  $\pi_n(S^1 \vee S^n)$  is finitely generated as a  $\mathbb{Z}[\pi_1]$ -module, but there are finite CW complexes where even this fails. This happens in fact for  $\pi_3(S^1 \vee S^2)$ , according to Exercise 38 at the end of this section. In §4.A we construct more complicated examples for each  $\pi_n$  with  $n > 1$ , in particular for  $\pi_2$ .

A useful tool for more complicated calculations is the following general result:

**Proposition 4.28.** *If a CW pair  $(X, A)$  is  $r$ -connected and  $A$  is  $s$ -connected, with  $r, s \geq 0$ , then the map  $\pi_i(X, A) \rightarrow \pi_i(X/A)$  induced by the quotient map  $X \rightarrow X/A$  is an isomorphism for  $i \leq r + s$  and a surjection for  $i = r + s + 1$ .*

**Proof:** Consider  $X \cup CA$ , the complex obtained from  $X$  by attaching a cone  $CA$  along  $A \subset X$ . Since  $CA$  is a contractible subcomplex of  $X \cup CA$ , the quotient map  $X \cup CA \rightarrow (X \cup CA)/CA = X/A$  is a homotopy equivalence by Proposition 0.17. So we have a commutative diagram

$$\begin{array}{ccccc} \pi_i(X, A) & \longrightarrow & \pi_i(X \cup CA, CA) & \longrightarrow & \pi_i(X \cup CA/CA) = \pi_i(X/A) \\ & & \uparrow \approx & \nearrow \approx & \\ & & \pi_i(X \cup CA) & & \end{array}$$

where the vertical isomorphism comes from a long exact sequence. Now apply the excision theorem to the first map in the diagram, using the fact that  $(CA, A)$  is  $(s + 1)$ -connected if  $A$  is  $s$ -connected, which comes from the exact sequence for the pair  $(CA, A)$ .  $\square$

**Example 4.29.** Suppose  $X$  is obtained from a wedge of spheres  $\bigvee_\alpha S_\alpha^n$  by attaching cells  $e_\beta^{n+1}$  via basepoint-preserving maps  $\varphi_\beta: S^n \rightarrow \bigvee_\alpha S_\alpha^n$ , with  $n \geq 2$ . By cellular

approximation we know that  $\pi_i(X) = 0$  for  $i < n$ , and we shall show that  $\pi_n(X)$  is the quotient of the free abelian group  $\pi_n(\bigvee_{\alpha} S_{\alpha}^n) \approx \bigoplus_{\alpha} \mathbb{Z}$  by the subgroup generated by the classes  $[\varphi_{\beta}]$ . Any subgroup can be realized in this way, by choosing maps  $\varphi_{\beta}$  to represent a set of generators for the subgroup, so it follows that every abelian group can be realized as  $\pi_n(X)$  for such a space  $X = (\bigvee_{\alpha} S_{\alpha}^n) \cup_{\beta} e_{\beta}^{n+1}$ . This is the higher-dimensional analog of the construction in Corollary 1.28 of a 2-dimensional CW complex with prescribed fundamental group.

To see that  $\pi_n(X)$  is as claimed, consider the following portion of the long exact sequence of the pair  $(X, \bigvee_{\alpha} S_{\alpha}^n)$ :

$$\pi_{n+1}(X, \bigvee_{\alpha} S_{\alpha}^n) \xrightarrow{\partial} \pi_n(\bigvee_{\alpha} S_{\alpha}^n) \longrightarrow \pi_n(X) \longrightarrow 0$$

The quotient  $X/\bigvee_{\alpha} S_{\alpha}^n$  is a wedge of spheres  $S_{\beta}^{n+1}$ , so the preceding proposition and Example 4.26 imply that  $\pi_{n+1}(X, \bigvee_{\alpha} S_{\alpha}^n)$  is free with basis the characteristic maps of the cells  $e_{\beta}^{n+1}$ . The boundary map  $\partial$  takes these to the classes  $[\varphi_{\beta}]$ , and the result follows.

### Eilenberg–MacLane Spaces

A space  $X$  having just one nontrivial homotopy group  $\pi_n(X) \approx G$  is called an **Eilenberg–MacLane space**  $K(G, n)$ . The case  $n = 1$  was considered in §1.B, where the condition that  $\pi_i(X) = 0$  for  $i > 1$  was replaced by the condition that  $X$  have a contractible universal cover, which is equivalent for spaces that have a universal cover of the homotopy type of a CW complex.

We can build a CW complex  $K(G, n)$  for arbitrary  $G$  and  $n$ , assuming  $G$  is abelian if  $n > 1$ , in the following way. To begin, let  $X$  be an  $(n-1)$ -connected CW complex of dimension  $n+1$  such that  $\pi_n(X) \approx G$ , as was constructed in Example 4.29 above when  $n > 1$  and in Corollary 1.28 when  $n = 1$ . Then we showed in Example 4.16 how to attach higher-dimensional cells to  $X$  to make  $\pi_i$  trivial for  $i > n$  without affecting  $\pi_n$  or the lower homotopy groups.

By taking products of  $K(G, n)$ 's for varying  $n$  we can then realize any sequence of groups  $G_n$ , abelian for  $n > 1$ , as the homotopy groups  $\pi_n$  of a space.

A fair number of  $K(G, 1)$ 's arise naturally in a variety of contexts, and a few of these are mentioned in §1.B. By contrast, naturally occurring  $K(G, n)$ 's for  $n \geq 2$  are rare. It seems the only real example is  $\mathbb{CP}^{\infty}$ , which is a  $K(\mathbb{Z}, 2)$  as we shall see in Example 4.50. One could of course trivially generalize this example by taking a product of  $\mathbb{CP}^{\infty}$ 's to get a  $K(G, 2)$  with  $G$  a product of  $\mathbb{Z}$ 's.

Actually there is a fairly natural construction of a  $K(\mathbb{Z}, n)$  for arbitrary  $n$ , the infinite symmetric product  $SP(S^n)$  defined in §3.C. In §4.K we prove that the functor  $SP$  has the surprising property of converting homology groups into homotopy groups, namely  $\pi_i(SP(X)) \approx H_i(X; \mathbb{Z})$  for all  $i > 0$  and all connected CW complexes  $X$ . Taking  $X$  to be a sphere, we deduce that  $SP(S^n)$  is a  $K(\mathbb{Z}, n)$ . More generally,  $SP(M(G, n))$  is a  $K(G, n)$  for each Moore space  $M(G, n)$ .

Having shown the existence of  $K(G, n)$ 's, we now consider the uniqueness question, which has the nicest possible answer:

**Proposition 4.30.** *The homotopy type of a CW complex  $K(G, n)$  is uniquely determined by  $G$  and  $n$ .*

The proof will be based on a more technical statement:

**Lemma 4.31.** *Let  $X$  be a CW complex of the form  $(\bigvee_{\alpha} S_{\alpha}^n) \cup_{\beta} e_{\beta}^{n+1}$  for some  $n \geq 1$ . Then for every homomorphism  $\psi: \pi_n(X) \rightarrow \pi_n(Y)$  with  $Y$  path-connected there exists a map  $f: X \rightarrow Y$  with  $f_* = \psi$ .*

**Proof:** To begin, let  $f$  send the natural basepoint of  $\bigvee_{\alpha} S_{\alpha}^n$  to a chosen basepoint  $y_0 \in Y$ . Extend  $f$  over each sphere  $S_{\alpha}^n$  via a map representing  $\psi([i_{\alpha}])$  where  $i_{\alpha}$  is the inclusion  $S_{\alpha}^n \hookrightarrow X$ . Thus for the map  $f: X^n \rightarrow Y$  constructed so far we have  $f_*([i_{\alpha}]) = \psi([i_{\alpha}])$  for all  $\alpha$ , hence  $f_*([\varphi]) = \psi([\varphi])$  for all basepoint-preserving maps  $\varphi: S^n \rightarrow X^n$  since the  $i_{\alpha}$ 's generate  $\pi_n(X^n)$ . To extend  $f$  over a cell  $e_{\beta}^{n+1}$  all we need is that the composition of the attaching map  $\varphi_{\beta}: S^n \rightarrow X^n$  for this cell with  $f$  be nullhomotopic in  $Y$ . But this composition  $f\varphi_{\beta}$  represents  $f_*([\varphi_{\beta}]) = \psi([\varphi_{\beta}])$ , and  $\psi([\varphi_{\beta}]) = 0$  because  $[\varphi_{\beta}]$  is zero in  $\pi_n(X)$  since  $\varphi_{\beta}$  is nullhomotopic in  $X$  via the characteristic map of  $e_{\beta}^{n+1}$ . Thus we obtain an extension  $f: X \rightarrow Y$ . This has  $f_* = \psi$  since the elements  $[i_{\alpha}]$  generate  $\pi_n(X^n)$  and hence also  $\pi_n(X)$  by cellular approximation.  $\square$

**Proof of 4.30:** Suppose  $K$  and  $K'$  are  $K(G, n)$  CW complexes. Since homotopy equivalence is an equivalence relation, there is no loss of generality if we assume  $K$  is a particular  $K(G, n)$ , namely one constructed from a space  $X$  as in the lemma by attaching cells of dimension  $n+2$  and greater. By the lemma there is a map  $f: X \rightarrow K'$  inducing an isomorphism on  $\pi_n$ . To extend this  $f$  over  $K$  we proceed inductively. For each cell  $e^{n+2}$ , the composition of its attaching map with  $f$  is nullhomotopic in  $K'$  since  $\pi_{n+1}(K') = 0$ , so  $f$  extends over this cell. The same argument applies for all the higher-dimensional cells in turn. The resulting  $f: K \rightarrow K'$  is a homotopy equivalence since it induces isomorphisms on all homotopy groups.  $\square$

## The Hurewicz Theorem

Using the calculations of homotopy groups done above we can easily prove the simplest and most often used cases of the Hurewicz theorem:

**Theorem 4.32.** *If a space  $X$  is  $(n-1)$ -connected,  $n \geq 2$ , then  $\tilde{H}_i(X) = 0$  for  $i < n$  and  $\pi_n(X) \approx H_n(X)$ . If a pair  $(X, A)$  is  $(n-1)$ -connected,  $n \geq 2$ , with  $A$  simply-connected and nonempty, then  $H_i(X, A) = 0$  for  $i < n$  and  $\pi_n(X, A) \approx H_n(X, A)$ .*

Thus the first nonzero homotopy and homology groups of a simply-connected space occur in the same dimension and are isomorphic. One cannot expect any nice

relationship between  $\pi_i(X)$  and  $H_i(X)$  beyond this. For example,  $S^n$  has trivial homology groups above dimension  $n$  but many nontrivial homotopy groups in this range when  $n \geq 2$ . In the other direction, Eilenberg-MacLane spaces such as  $\mathbb{C}P^\infty$  have trivial higher homotopy groups but many nontrivial homology groups.

The theorem can sometimes be used to compute  $\pi_2(X)$  if  $X$  is a path-connected space that is nice enough to have a universal cover. For if  $\tilde{X}$  is the universal cover, then  $\pi_2(X) \approx \pi_2(\tilde{X})$  and the latter group is isomorphic to  $H_2(\tilde{X})$  by the Hurewicz theorem. So if one can understand  $\tilde{X}$  well enough to compute  $H_2(\tilde{X})$ , one can compute  $\pi_2(X)$ .

In the part of the theorem dealing with relative groups, notice that  $X$  must be simply-connected as well as  $A$  since  $(X, A)$  is 1-connected by hypothesis. There is a more general version of the relative Hurewicz theorem given later in Theorem 4.37 that allows  $A$  and  $X$  to be nonsimply-connected, but this requires  $\pi_n(X, A)$  to be replaced by a certain quotient group.

**Proof:** We may assume  $X$  is a CW complex and  $(X, A)$  is a CW pair by taking CW approximations to  $X$  and  $(X, A)$ . For CW pairs the relative case then reduces to the absolute case since  $\pi_i(X, A) \approx \pi_i(X/A)$  for  $i \leq n$  by Proposition 4.28, while  $H_i(X, A) \approx \tilde{H}_i(X/A)$  for all  $i$  by Proposition 2.22.

In the absolute case we can apply Proposition 4.15 to replace  $X$  by a homotopy equivalent CW complex with  $(n-1)$ -skeleton a point, hence  $\tilde{H}_i(X) = 0$  for  $i < n$ . To show  $\pi_n(X) \approx H_n(X)$ , we can further simplify by throwing away cells of dimension greater than  $n+1$  since these have no effect on  $\pi_n$  or  $H_n$ . Thus  $X$  has the form  $(\bigvee_\alpha S_\alpha^n) \cup_\beta e_\beta^{n+1}$ . We may assume the attaching maps  $\varphi_\beta$  of the cells  $e_\beta^{n+1}$  are basepoint-preserving since this is what the proof of Proposition 4.15 gives. Example 4.29 then applies to compute  $\pi_n(X)$  as the cokernel of the boundary map  $\pi_{n+1}(X, X^n) \rightarrow \pi_n(X^n)$ , a map  $\bigoplus_\beta \mathbb{Z} \rightarrow \bigoplus_\alpha \mathbb{Z}$ . This is the same as the cellular boundary map  $d: H_{n+1}(X^{n+1}, X^n) \rightarrow H_n(X^n, X^{n-1})$  since for a cell  $e_\beta^{n+1}$ , the coefficients of  $de_\beta^{n+1}$  are the degrees of the compositions  $q_\alpha \varphi_\beta$  where  $q_\alpha$  collapses all  $n$ -cells except  $e_\alpha^n$  to a point, and the isomorphism  $\pi_n(S^n) \approx \mathbb{Z}$  in Corollary 4.25 is given by degree. Since there are no  $(n-1)$ -cells, we have  $H_n(X) \approx \text{Coker } d$ .  $\square$

Since homology groups are usually more computable than homotopy groups, the following version of Whitehead's theorem is often easier to apply:

**Corollary 4.33.** *A map  $f: X \rightarrow Y$  between simply-connected CW complexes is a homotopy equivalence if  $f_*: H_n(X) \rightarrow H_n(Y)$  is an isomorphism for each  $n$ .*

**Proof:** After replacing  $Y$  by the mapping cylinder  $M_f$  we may take  $f$  to be an inclusion  $X \hookrightarrow Y$ . Since  $X$  and  $Y$  are simply-connected, we have  $\pi_1(Y, X) = 0$ . The relative Hurewicz theorem then says that the first nonzero  $\pi_n(Y, X)$  is isomorphic to the first nonzero  $H_n(Y, X)$ . All the groups  $H_n(Y, X)$  are zero from the long exact sequence of homology, so all the groups  $\pi_n(Y, X)$  also vanish. This means that the inclusion

$X \hookrightarrow Y$  induces isomorphisms on all homotopy groups, and therefore this inclusion is a homotopy equivalence.  $\square$

**Example 4.34: Uniqueness of Moore Spaces.** Let us show that the homotopy type of a CW complex Moore space  $M(G, n)$  is uniquely determined by  $G$  and  $n$  if  $n > 1$ , so  $M(G, n)$  is simply-connected. Let  $X$  be an  $M(G, n)$  as constructed in Example 2.40 by attaching  $(n + 1)$ -cells to a wedge sum of  $n$ -spheres, and let  $Y$  be any other  $M(G, n)$  CW complex. By Lemma 4.31 there is a map  $f: X \rightarrow Y$  inducing an isomorphism on  $\pi_n$ . If we can show that  $f$  also induces an isomorphism on  $H_n$ , then the preceding corollary will imply the result.

One way to show that  $f$  induces an isomorphism on  $H_n$  would be to use a more refined version of the Hurewicz theorem giving an isomorphism between  $\pi_n$  and  $H_n$  that is natural with respect to maps between spaces, as in Theorem 4.37 below. However, here is a direct argument which avoids naturality questions. For the mapping cylinder  $M_f$  we know that  $\pi_i(M_f, X) = 0$  for  $i \leq n$ . If this held also for  $i = n + 1$  then the relative Hurewicz theorem would say that  $H_i(M_f, X) = 0$  for  $i \leq n + 1$  and hence that  $f_*$  would be an isomorphism on  $H_n$ . To make this argument work, let us temporarily enlarge  $Y$  by attaching  $(n + 2)$ -cells to make  $\pi_{n+1}$  zero. The new mapping cylinder  $M_f$  then has  $\pi_{n+1}(M_f, X) = 0$  from the long exact sequence of the pair. So for the enlarged  $Y$  the map  $f$  induces an isomorphism on  $H_n$ . But attaching  $(n + 2)$ -cells has no effect on  $H_n$ , so the original  $f: X \rightarrow Y$  had to be an isomorphism on  $H_n$ .

It is certainly possible for a map of nonsimply-connected spaces to induce isomorphisms on all homology groups but not on homotopy groups. Nonsimply-connected acyclic spaces, for which the inclusion of a point induces an isomorphism on homology, exhibit this phenomenon in its purest form. Perhaps the simplest nontrivial acyclic space is the 2-dimensional complex constructed in Example 2.38 with fundamental group  $\langle a, b \mid a^5 = b^3 = (ab)^2 \rangle$  of order 120.

It is also possible for a map between spaces with abelian fundamental groups to induce isomorphisms on homology but not on higher homotopy groups, as the next example shows.

**Example 4.35.** We construct a space  $X = (S^1 \vee S^n) \cup e^{n+1}$ , for arbitrary  $n > 1$ , such that the inclusion  $S^1 \hookrightarrow X$  induces an isomorphism on all homology groups and on  $\pi_i$  for  $i < n$ , but not on  $\pi_n$ . From Example 4.27 we have  $\pi_n(S^1 \vee S^n) \approx \mathbb{Z}[t, t^{-1}]$ . Let  $X$  be obtained from  $S^1 \vee S^n$  by attaching a cell  $e^{n+1}$  via a map  $S^n \rightarrow S^1 \vee S^n$  corresponding to  $2t - 1 \in \mathbb{Z}[t, t^{-1}]$ . By looking in the universal cover we see that  $\pi_n(X) \approx \mathbb{Z}[t, t^{-1}]/(2t - 1)$ , where  $(2t - 1)$  denotes the ideal in  $\mathbb{Z}[t, t^{-1}]$  generated by  $2t - 1$ . Note that setting  $t = 1/2$  embeds  $\mathbb{Z}[t, t^{-1}]/(2t - 1)$  in  $\mathbb{Q}$  as the subring  $\mathbb{Z}[1/2]$  consisting of rationals with denominator a power of 2. From the long exact sequence of homotopy groups for the  $(n - 1)$ -connected pair  $(X, S^1)$  we see that the inclusion



$S^1 \hookrightarrow X$  induces an isomorphism on  $\pi_i$  for  $i < n$ . The fact that this inclusion also induces isomorphisms on all homology groups can be deduced from cellular homology. The key point is that the cellular boundary map  $H_{n+1}(X^{n+1}, X^n) \rightarrow H_n(X^n, X^{n-1})$  is an isomorphism since the degree of the composition of the attaching map  $S^n \rightarrow S^1 \vee S^n$  of  $e^{n+1}$  with the collapse  $S^1 \vee S^n \rightarrow S^n$  is  $2 - 1 = 1$ .

This example relies heavily on the nontriviality of the action of  $\pi_1(X)$  on  $\pi_n(X)$ , so one might ask whether the simple-connectivity assumption in Corollary 4.33 can be weakened to trivial action of  $\pi_1$  on all  $\pi_n$ 's. This is indeed the case, as we will show in Proposition 4.74.

The form of the Hurewicz theorem given above asserts merely the existence of an isomorphism between homotopy and homology groups, but one might want a more precise statement which says that a particular map is an isomorphism. In fact, there are always natural maps from homotopy groups to homology groups, defined in the following way. Thinking of  $\pi_n(X, A, x_0)$  for  $n > 0$  as homotopy classes of maps  $f: (D^n, \partial D^n, s_0) \rightarrow (X, A, x_0)$ , the **Hurewicz map**  $h: \pi_n(X, A, x_0) \rightarrow H_n(X, A)$  is defined by  $h([f]) = f_*(\alpha)$  where  $\alpha$  is a fixed generator of  $H_n(D^n, \partial D^n) \approx \mathbb{Z}$  and  $f_*: H_n(D^n, \partial D^n) \rightarrow H_n(X, A)$  is induced by  $f$ . If we have a homotopy  $f \simeq g$  through maps  $(D^n, \partial D^n, s_0) \rightarrow (X, A, x_0)$ , or even through maps  $(D^n, \partial D^n) \rightarrow (X, A)$  not preserving the basepoint, then  $f_* = g_*$ , so  $h$  is well-defined.

**Proposition 4.36.** *The Hurewicz map  $h: \pi_n(X, A, x_0) \rightarrow H_n(X, A)$  is a homomorphism, assuming  $n > 1$  so that  $\pi_n(X, A, x_0)$  is a group.*

**Proof:** It suffices to show that for maps  $f, g: (D^n, \partial D^n) \rightarrow (X, A)$ , the induced maps on homology satisfy  $(f + g)_* = f_* + g_*$ , for if this is the case then  $h([f + g]) = (f + g)_*(\alpha) = f_*(\alpha) + g_*(\alpha) = h([f]) + h([g])$ . Our proof that  $(f + g)_* = f_* + g_*$  will in fact work for any homology theory.

Let  $c: D^n \rightarrow D^n \vee D^n$  be the map collapsing the equatorial  $D^{n-1}$  to a point, and let  $q_1, q_2: D^n \vee D^n \rightarrow D^n$  be the quotient maps onto the two summands, collapsing the other summand to a point. We then have a diagram

$$\begin{array}{ccccc} H_n(D^n, \partial D^n) & \xrightarrow{c_*} & H_n(D^n \vee D^n, \partial D^n \vee \partial D^n) & \xrightarrow{(f \vee g)_*} & H_n(X, A) \\ & & \downarrow q_{1*} \oplus q_{2*} \approx & & \\ & & H_n(D^n, \partial D^n) \oplus H_n(D^n, \partial D^n) & & \end{array}$$

The map  $q_{1*} \oplus q_{2*}$  is an isomorphism with inverse  $i_{1*} + i_{2*}$  where  $i_1$  and  $i_2$  are the inclusions of the two summands  $D^n \hookrightarrow D^n \vee D^n$ . Since  $q_1 c$  and  $q_2 c$  are homotopic to the identity through maps  $(D^n, \partial D^n) \rightarrow (D^n, \partial D^n)$ , the composition  $(q_{1*} \oplus q_{2*})c_*$  is the diagonal map  $x \mapsto (x, x)$ . From the equalities  $(f \vee g)i_1 = f$  and  $(f \vee g)i_2 = g$  we deduce that  $(f \vee g)_*(i_{1*} + i_{2*})$  sends  $(x, 0)$  to  $f_*(x)$  and  $(0, x)$  to  $g_*(x)$ , hence  $(x, x)$  to  $f_*(x) + g_*(x)$ . Thus the composition across the top of the diagram is

$x \mapsto f_*(x) + g_*(x)$ . On the other hand,  $f + g = (f \vee g)c$ , so this composition is also  $(f + g)_*$ .  $\square$

There is also an absolute Hurewicz map  $h : \pi_n(X, x_0) \rightarrow H_n(X)$  defined in a similar way by setting  $h([f]) = f_*(\alpha)$  for  $f : (S^n, s_0) \rightarrow (X, x_0)$  and  $\alpha$  a chosen generator of  $H_n(S^n)$ . For example, if  $X = S^n$  then  $f_*(\alpha)$  is  $(\deg f)\alpha$  by the definition of degree, so we can view  $h$  in this case as the degree map  $\pi_n(S^n) \rightarrow \mathbb{Z}$ , which we know is an isomorphism by Corollary 4.25. The proof of the preceding proposition is readily modified to show that the absolute  $h$  is a homomorphism for  $n \geq 1$ .

The absolute and relative Hurewicz maps can be combined in a diagram of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n(A, x_0) & \longrightarrow & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) \longrightarrow \cdots \\ & & \downarrow h & & \downarrow h & & \downarrow h \\ \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) \longrightarrow \cdots \end{array}$$

An easy definition check which we leave to the reader shows that this diagram commutes up to sign at least. With more care in the choice of the generators  $\alpha$  it can be made to commute exactly.

Another elementary property of Hurewicz maps is that they are natural: A map  $f : (X, x_0) \rightarrow (Y, y_0)$  induces a commutative diagram as at the right, and similarly in the relative case.

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{f_*} & \pi_n(Y, y_0) \\ \downarrow h & & \downarrow h \\ H_n(X) & \xrightarrow{f_*} & H_n(Y) \end{array}$$

It is easy to construct nontrivial elements of the kernel of the Hurewicz homomorphism  $h : \pi_n(X, x_0) \rightarrow H_n(X)$  if  $\pi_1(X, x_0)$  acts nontrivially on  $\pi_n(X, x_0)$ , namely elements of the form  $[y][f] - [f]$ . This is because  $yf$  and  $f$ , viewed as maps  $S^n \rightarrow X$ , are homotopic if we do not require the basepoint to be fixed during the homotopy, so  $(yf)_*(\alpha) = f_*(\alpha)$  for  $\alpha$  a generator of  $H_n(S^n)$ .

Similarly in the relative case the kernel of  $h : \pi_n(X, A, x_0) \rightarrow H_n(X, A)$  contains the elements of the form  $[y][f] - [f]$  for  $[y] \in \pi_1(A, x_0)$ . For example the Hurewicz map  $\pi_n(S^1 \vee S^n, S^1) \rightarrow H_n(S^1 \vee S^n, S^1)$  is the homomorphism  $\mathbb{Z}[t, t^{-1}] \rightarrow \mathbb{Z}$  sending all powers of  $t$  to 1. Since the pair  $(S^1 \vee S^n, S^1)$  is  $(n-1)$ -connected, this example shows that the hypothesis  $\pi_1(A, x_0) = 0$  in the relative form of the Hurewicz theorem proved earlier cannot be dropped.

If we define  $\pi'_n(X, A, x_0)$  to be the quotient group of  $\pi_n(X, A, x_0)$  obtained by factoring out the subgroup generated by all elements of the form  $[y][f] - [f]$ , or the normal subgroup generated by such elements in the case  $n = 2$  when  $\pi_2(X, A, x_0)$  may not be abelian, then  $h$  induces a homomorphism  $h' : \pi'_n(X, A, x_0) \rightarrow H_n(X, A)$ . The general form of the Hurewicz theorem deals with this homomorphism:

**Theorem 4.37.** *If  $(X, A)$  is an  $(n - 1)$ -connected pair of path-connected spaces with  $n \geq 2$  and  $A \neq \emptyset$ , then  $h' : \pi'_n(X, A, x_0) \rightarrow H_n(X, A)$  is an isomorphism and  $H_i(X, A) = 0$  for  $i < n$ .*

Note that this statement includes the absolute form of the theorem by taking  $A$  to be the basepoint.

Before starting the proof of this general Hurewicz theorem we have a preliminary step:

**Lemma 4.38.** *If  $X$  is a connected CW complex to which cells  $e_\alpha^n$  are attached for a fixed  $n \geq 2$ , forming a CW complex  $W = X \cup_\alpha e_\alpha^n$ , then  $\pi_n(W, X)$  is a free  $\pi_1(X)$ -module with basis the homotopy classes of the characteristic maps  $\Phi_\alpha$  of the cells  $e_\alpha^n$ , provided that the map  $\pi_1(X) \rightarrow \pi_1(W)$  induced by inclusion is an isomorphism. In particular, this is always the case if  $n \geq 3$ . In the general  $n = 2$  case,  $\pi_2(W, X)$  is generated by the classes of the characteristic maps of the cells  $e_\alpha^2$  together with their images under the action of  $\pi_1(X)$ .*

If the characteristic maps  $\Phi_\alpha : (D^n, \partial D^n) \rightarrow (W, X)$  do not take a basepoint  $s_0$  in  $\partial D^n$  to the basepoint  $x_0$  in  $X$ , then they will define elements of  $\pi_n(W, X, x_0)$  only after we choose change-of-basepoint paths from the points  $\Phi_\alpha(s_0)$  to  $x_0$ . Different choices of such paths yield elements of  $\pi_n(W, X, x_0)$  related by the action of  $\pi_1(X, x_0)$ , so the basis for  $\pi_n(W, X, x_0)$  is well-defined up to multiplication by invertible elements of  $\mathbb{Z}[\pi_1(X)]$ .

The situation when  $n = 2$  and the map  $\pi_1(X) \rightarrow \pi_1(W)$  is not an isomorphism is more complicated because the relative  $\pi_2$  can be nonabelian in this case. Whitehead analyzed what happens here and showed that  $\pi_2(W, X)$  has the structure of a ‘free crossed  $\pi_1(X)$ -module’. See [Whitehead 1949] or [Sieradski 1993].

**Proof:** Since  $W/X = \bigvee_\alpha S_\alpha^n$ , we have  $\pi_n(W, X) \approx \pi_n(\bigvee_\alpha S_\alpha^n)$  when  $X$  is simply-connected, by Proposition 4.28. The conclusion of the lemma in this case is then immediate from Example 4.26.

When  $X$  is not simply-connected but the inclusion  $X \hookrightarrow W$  induces an isomorphism on  $\pi_1$ , then the universal cover of  $W$  is obtained from the universal cover of  $X$  by attaching  $n$ -cells lifting the cells  $e_\alpha^n$ . If we choose one such lift  $\tilde{e}_\alpha^n$  of  $e_\alpha^n$ , then all the other lifts are the images  $\gamma \tilde{e}_\alpha^n$  of  $\tilde{e}_\alpha^n$  under the deck transformations  $\gamma \in \pi_1(X)$ . The special case proved in the preceding paragraph shows that the relative  $\pi_n$  for the universal cover is the free abelian group with basis corresponding to the cells  $\gamma \tilde{e}_\alpha^n$ . By the relative version of Proposition 4.1, the projection of the universal cover of  $W$  onto  $W$  induces an isomorphism on relative  $\pi_n$ ’s, so  $\pi_n(W, X)$  is free abelian with basis the classes  $[\gamma e_\alpha^n]$  as  $\gamma$  ranges over  $\pi_1(X)$ , or in other words the free  $\pi_1(X)$ -module with basis the cells  $e_\alpha^n$ .

It remains to consider the  $n = 2$  case in general. Since both of the pairs  $(W, X)$  and  $(X^1 \cup_\alpha e_\alpha^2, X^1)$  are 1-connected, the homotopy excision theorem implies that the

map  $\pi_2(X^1 \cup_{\alpha} e_{\alpha}^2, X^1) \rightarrow \pi_2(W, X)$  is surjective. This gives a reduction to the case that  $X$  is 1-dimensional. We may also assume the 2-cells  $e_{\alpha}^2$  are attached along loops passing through the basepoint 0-cell  $x_0$ , since this can be achieved by homotopy of the attaching maps, which does not affect the homotopy type of the pair  $(W, X)$ .

In the closure of each 2-cell  $e_{\alpha}^2$  choose an embedded disk  $D_{\alpha}^2$  which contains  $x_0$  but is otherwise contained entirely in the interior of  $e_{\alpha}^2$ . Let  $Y = X \cup_{\alpha} D_{\alpha}^2$ , the wedge sum of  $X$  with the disks  $D_{\alpha}^2$ , and let  $Z = W - \bigcup_{\alpha} \text{int}(D_{\alpha}^2)$ , so  $Y$  and  $Z$  are 2-dimensional CW complexes with a common 1-skeleton  $Y^1 = Z^1 = Y \cap Z = X \vee_{\alpha} \partial D_{\alpha}^2$ . The inclusion  $(W, X) \hookrightarrow (W, Z)$  is a homotopy equivalence of pairs. Homotopy excision gives a surjection  $\pi_2(Y, Y^1) \rightarrow \pi_2(W, Z)$ . The universal cover  $\tilde{Y}$  of  $Y$  is obtained from the universal cover  $\tilde{X}$  of  $X$  by taking the wedge sum with lifts  $\tilde{D}_{\alpha\beta}^2$  of the disks  $D_{\alpha}^2$ . Hence we have isomorphisms

$$\begin{aligned} \pi_2(Y, Y^1) &\approx \pi_2(\tilde{Y}, \tilde{Y}^1) \quad \text{where } \tilde{Y}^1 \text{ is the 1-skeleton of } \tilde{Y} \\ &\approx \pi_2(\bigvee_{\alpha\beta} \tilde{D}_{\alpha\beta}^2, \bigvee_{\alpha\beta} \partial \tilde{D}_{\alpha\beta}^2) \quad \text{since } \tilde{X} \text{ is contractible} \\ &\approx \pi_1(\bigvee_{\alpha\beta} \partial \tilde{D}_{\alpha\beta}^2) \quad \text{since } \bigvee_{\alpha\beta} \tilde{D}_{\alpha\beta}^2 \text{ is contractible} \end{aligned}$$

This last group is free with basis the loops  $\partial \tilde{D}_{\alpha\beta}^2$ , so the inclusions  $\tilde{D}_{\alpha\beta}^2 \hookrightarrow \bigvee_{\alpha\beta} \tilde{D}_{\alpha\beta}^2$  form a basis for  $\pi_2(\bigvee_{\alpha\beta} \tilde{D}_{\alpha\beta}^2, \bigvee_{\alpha\beta} \partial \tilde{D}_{\alpha\beta}^2)$ . This implies that  $\pi_2(Y, Y^1)$  is generated by the inclusions  $D_{\alpha}^2 \hookrightarrow Y$  and their images under the action of loops in  $X$ . The same is true for  $\pi_2(W, Z)$  via the surjection  $\pi_2(Y, Y^1) \rightarrow \pi_2(W, Z)$ . Using the isomorphism  $\pi_2(W, Z) \approx \pi_2(W, X)$ , we conclude that  $\pi_2(W, X)$  is generated by the characteristic maps of the cells  $e_{\alpha}^2$  and their images under the action of  $\pi_1(X)$ .  $\square$

**Proof of the general Hurewicz Theorem:** As in the earlier form of the theorem we may assume  $(X, A)$  is a CW pair such that the cells of  $X - A$  have dimension  $\geq n$ .

We first prove the theorem assuming that  $\pi_1(A) \rightarrow \pi_1(X)$  is an isomorphism. This is always the case if  $n \geq 3$ , so this case will finish the proof except when  $n = 2$ . We may assume also that  $X = X^{n+1}$  since higher-dimensional cells have no effect on  $\pi_n$  or  $H_n$ . Consider the commutative diagram

$$\begin{array}{ccccccc} \pi_{n+1}(X, X^n \cup A) & \xrightarrow{\partial} & \pi_n(X^n \cup A, A) & \xrightarrow{i_*} & \pi_n(X, A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi'_{n+1}(X, X^n \cup A) & \xrightarrow{\partial'} & \pi'_n(X^n \cup A, A) & \xrightarrow{i'_*} & \pi'_n(X, A) & \longrightarrow & 0 \\ \downarrow h' & & \downarrow h' & & \downarrow h' & & \\ H_{n+1}(X, X^n \cup A) & \longrightarrow & H_n(X^n \cup A, A) & \longrightarrow & H_n(X, A) & \longrightarrow & 0 \end{array}$$

The first and third rows are exact sequences for the triple  $(X, X^n \cup A, A)$ . The left-hand  $h'$  is an isomorphism since by the preceding lemma,  $\pi_{n+1}(X, X^n \cup A)$  is a free  $\pi_1$ -module with basis the characteristic maps of the  $(n+1)$ -cells of  $X - A$ , so  $\pi'_{n+1}(X, X^n \cup A)$  is a free abelian group with the same basis, and  $H_{n+1}(X, X^n \cup A)$  is also free with basis the  $(n+1)$ -cells of  $X - A$ . Similarly, the lemma implies that

the middle  $h'$  is an isomorphism since the assumption that  $\pi_1(A) \rightarrow \pi_1(X)$  is an isomorphism implies that  $\pi_1(A) \rightarrow \pi_1(X^n \cup A)$  is injective, hence an isomorphism if  $n \geq 2$ .

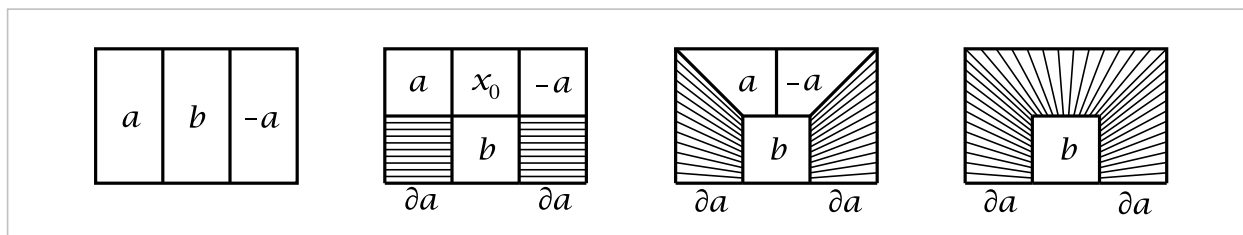
A simple diagram chase now shows that the right-hand  $h'$  is an isomorphism. Namely, surjectivity follows since  $H_n(X^n \cup A, A) \rightarrow H_n(X, A)$  is surjective and the middle  $h'$  is an isomorphism. For injectivity, take an element  $x \in \pi'_n(X, A)$  with  $h'(x) = 0$ . The map  $i'_*$  is surjective since  $i_*$  is, so  $x = i'_*(y)$  for some element  $y \in \pi'_n(X^n \cup A, A)$ . Since the first two maps  $h'$  are isomorphisms and the bottom row is exact, there is a  $z \in \pi'_{n+1}(X, X^n \cup A)$  with  $\partial'(z) = y$ . Hence  $x = 0$  since  $i_*\partial = 0$  implies  $i'_*\partial' = 0$ .

It remains to prove the theorem when  $n = 2$  and  $\pi_1(A) \rightarrow \pi_1(X)$  is not an isomorphism. The proof above will apply once we show that the middle  $h'$  in the diagram is an isomorphism. The preceding lemma implies that  $\pi'_2(X^2 \cup A, A)$  is generated by characteristic maps of the 2-cells of  $X - A$ . The images of these generators under  $h'$  form a basis for  $H_2(X^2 \cup A, A)$ . Thus  $h'$  is a homomorphism from a group which, by the lemma below, is abelian to a free abelian group taking a set of generators to a basis, hence  $h'$  is an isomorphism.  $\square$

**Lemma 4.39.** *For any  $(X, A, x_0)$ , the formula  $a + b - a = (\partial a)b$  holds for all  $a, b \in \pi_2(X, A, x_0)$ , where  $\partial: \pi_2(X, A, x_0) \rightarrow \pi_1(A, x_0)$  is the usual boundary map and  $(\partial a)b$  denotes the action of  $\partial a$  on  $b$ . Hence  $\pi'_2(X, A, x_0)$  is abelian.*

Here the '+' and '-' in  $a + b - a$  refer to the group operation in the nonabelian group  $\pi_2(X, A, x_0)$ .

**Proof:** The formula is obtained by constructing a homotopy from  $a + b - a$  to  $(\partial a)b$  as indicated in the pictures below.  $\square$



### The Plus Construction

There are quite a few situations in algebraic topology where having a nontrivial fundamental group complicates matters considerably. We shall next describe a construction which in certain circumstances allows one to modify a space so as to eliminate its fundamental group, or at least simplify it, without affecting homology or cohomology. Here is the simplest case:

**Proposition 4.40.** *Let  $X$  be a connected CW complex with  $H_1(X) = 0$ . Then there is a simply-connected CW complex  $X^+$  and a map  $X \rightarrow X^+$  inducing isomorphisms on all homology groups.*

**Proof:** Choose loops  $\varphi_\alpha: S^1 \rightarrow X^1$  generating  $\pi_1(X)$  and use these to attach cells  $e_\alpha^2$  to  $X$  to form a simply-connected CW complex  $X'$ . The homology exact sequence

$$0 \rightarrow H_2(X) \rightarrow H_2(X') \rightarrow H_2(X', X) \rightarrow 0 = H_1(X)$$

splits since  $H_2(X', X)$  is free with basis the cells  $e_\alpha^2$ . Thus we have an isomorphism  $H_2(X') \approx H_2(X) \oplus H_2(X', X)$ . Since  $X'$  is simply-connected, the Hurewicz theorem gives an isomorphism  $H_2(X') \approx \pi_2(X')$ , and so we may represent a basis for the free summand  $H_2(X', X)$  by maps  $\psi_\alpha: S^2 \rightarrow X'$ . We may assume these are cellular maps, and then use them to attach cells  $e_\alpha^3$  to  $X'$  forming a simply-connected CW complex  $X^+$ , with the inclusion  $X \hookrightarrow X^+$  an isomorphism on all homology groups.  $\square$

In the preceding proposition, the condition  $H_1(X) = 0$  means that  $\pi_1(X)$  is equal to its commutator subgroup, that is,  $\pi_1(X)$  is a **perfect** group. Suppose more generally that  $X$  is a connected CW complex and  $H \subset \pi_1(X)$  is a perfect subgroup. Let  $p: \tilde{X} \rightarrow X$  be the covering space corresponding to  $H$ , so  $\pi_1(\tilde{X}) \approx H$  is perfect and  $H_1(\tilde{X}) = 0$ . From the previous proposition we get an inclusion  $\tilde{X} \hookrightarrow \tilde{X}^+$ . Let  $X^+$  be obtained from the disjoint union of  $\tilde{X}^+$  and the mapping cylinder  $M_p$  by identifying the copies of  $\tilde{X}$  in these two spaces. Thus we have the commutative diagram of inclusion maps shown at the right. From the van Kampen theorem, the induced map  $\pi_1(X) \rightarrow \pi_1(X^+)$  is surjective with kernel the normal subgroup generated by  $H$ . Further, since  $X^+/M_p$  is homeomorphic to  $\tilde{X}^+/\tilde{X}$  we have  $H_*(X^+, M_p) = H_*(\tilde{X}^+, \tilde{X}) = 0$ , so the map  $X \rightarrow X^+$  induces an isomorphism on homology.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \tilde{X}^+ \\ \downarrow & & \downarrow \\ X \simeq M_p & \longrightarrow & X^+ \end{array}$$

This construction  $X \rightarrow X^+$ , killing a perfect subgroup of  $\pi_1(X)$  while preserving homology, is known as the Quillen **plus construction**. In some of the main applications  $X$  is a  $K(G, 1)$  where  $G$  has perfect commutator subgroup, so the map  $X \rightarrow X^+$  abelianizes  $\pi_1$  while preserving homology. The space  $X^+$  need no longer be a  $K(\pi, 1)$ , and in fact its homotopy groups can be quite interesting. The most striking example is  $G = \Sigma_\infty$ , the infinite symmetric group consisting of permutations of  $1, 2, \dots$  fixing all but finitely many  $n$ 's, with commutator subgroup the infinite alternating group  $A_\infty$ , which is perfect. In this case a famous theorem of Barratt, Priddy, and Quillen says that the homotopy groups  $\pi_i(K(\Sigma_\infty, 1)^+)$  are the stable homotopy groups of spheres!

There are limits, however, on which subgroups of  $\pi_1(X)$  can be killed without affecting the homology of  $X$ . For example, for  $X = S^1 \vee S^1$  it is impossible to kill the commutator subgroup of  $\pi_1(X)$  while preserving homology. In fact, by Exercise 23 at the end of this section every space with fundamental group  $\mathbb{Z} \times \mathbb{Z}$  must have  $H_2$  nontrivial.

## Fiber Bundles

A ‘short exact sequence of spaces’  $A \hookrightarrow X \rightarrow X/A$  gives rise to a long exact sequence of homology groups, but not to a long exact sequence of homotopy groups due to the failure of excision. However, there is a different sort of ‘short exact sequence of spaces’ that does give a long exact sequence of homotopy groups. This sort of short exact sequence  $F \rightarrow E \xrightarrow{p} B$ , called a *fiber bundle*, is distinguished from the type  $A \hookrightarrow X \rightarrow X/A$  in that it has more homogeneity: All the subspaces  $p^{-1}(b) \subset E$ , which are called **fibers**, are homeomorphic. For example,  $E$  could be the product  $F \times B$  with  $p: E \rightarrow B$  the projection. General fiber bundles can be thought of as twisted products. Familiar examples are the Möbius band, which is a twisted annulus with line segments as fibers, and the Klein bottle, which is a twisted torus with circles as fibers.

The topological homogeneity of all the fibers of a fiber bundle is rather like the algebraic homogeneity in a short exact sequence of groups  $0 \rightarrow K \rightarrow G \xrightarrow{p} H \rightarrow 0$  where the ‘fibers’  $p^{-1}(h)$  are the cosets of  $K$  in  $G$ . In a few fiber bundles  $F \rightarrow E \rightarrow B$  the space  $E$  is actually a group,  $F$  is a subgroup (though seldom a normal subgroup), and  $B$  is the space of left or right cosets. One of the nicest such examples is the Hopf bundle  $S^1 \rightarrow S^3 \rightarrow S^2$  where  $S^3$  is the group of quaternions of unit norm and  $S^1$  is the subgroup of unit complex numbers. For this bundle, the long exact sequence of homotopy groups takes the form

$$\cdots \rightarrow \pi_i(S^1) \rightarrow \pi_i(S^3) \rightarrow \pi_i(S^2) \rightarrow \pi_{i-1}(S^1) \rightarrow \pi_{i-1}(S^3) \rightarrow \cdots$$

In particular, the exact sequence gives an isomorphism  $\pi_2(S^2) \approx \pi_1(S^1)$  since the two adjacent terms  $\pi_2(S^3)$  and  $\pi_1(S^3)$  are zero by cellular approximation. Thus we have a direct homotopy-theoretic proof that  $\pi_2(S^2) \approx \mathbb{Z}$ . Also, since  $\pi_i(S^1) = 0$  for  $i > 1$  by Proposition 4.1, the exact sequence implies that there are isomorphisms  $\pi_i(S^3) \approx \pi_i(S^2)$  for all  $i \geq 3$ , so in particular  $\pi_3(S^2) \approx \pi_3(S^3)$ , and by Corollary 4.25 the latter group is  $\mathbb{Z}$ .

After these preliminary remarks, let us begin by defining the property that leads to a long exact sequence of homotopy groups. A map  $p: E \rightarrow B$  is said to have the **homotopy lifting property** with respect to a space  $X$  if, given a homotopy  $g_t: X \rightarrow B$  and a map  $\tilde{g}_0: X \rightarrow E$  lifting  $g_0$ , so  $p\tilde{g}_0 = g_0$ , then there exists a homotopy  $\tilde{g}_t: X \rightarrow E$  lifting  $g_t$ . From a formal point of view, this can be regarded as a special case of the *lift extension property for a pair*  $(Z, A)$ , which asserts that every map  $Z \rightarrow B$  has a lift  $Z \rightarrow E$  extending a given lift defined on the subspace  $A \subset Z$ . The case  $(Z, A) = (X \times I, X \times \{0\})$  is the homotopy lifting property.

A **fibration** is a map  $p: E \rightarrow B$  having the homotopy lifting property with respect to all spaces  $X$ . For example, a projection  $B \times F \rightarrow B$  is a fibration since we can choose lifts of the form  $\tilde{g}_t(x) = (g_t(x), h(x))$  where  $\tilde{g}_0(x) = (g_0(x), h(x))$ .

**Theorem 4.41.** Suppose  $p: E \rightarrow B$  has the homotopy lifting property with respect to disks  $D^k$  for all  $k \geq 0$ . Choose basepoints  $b_0 \in B$  and  $x_0 \in F = p^{-1}(b_0)$ . Then the map  $p_*: \pi_n(E, F, x_0) \rightarrow \pi_n(B, b_0)$  is an isomorphism for all  $n \geq 1$ . Hence if  $B$  is path-connected, there is a long exact sequence

$$\cdots \rightarrow \pi_n(F, x_0) \rightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, x_0) \rightarrow \cdots \rightarrow \pi_0(E, x_0) \rightarrow 0$$

The proof will use a relative form of the homotopy lifting property. The map  $p: E \rightarrow B$  is said to have the **homotopy lifting property for a pair**  $(X, A)$  if each homotopy  $f_t: X \rightarrow B$  lifts to a homotopy  $\tilde{g}_t: X \rightarrow E$  starting with a given lift  $\tilde{g}_0$  and extending a given lift  $\tilde{g}_t: A \rightarrow E$ . In other words, the homotopy lifting property for  $(X, A)$  is the lift extension property for  $(X \times I, X \times \{0\} \cup A \times I)$ .

The homotopy lifting property for  $D^k$  is equivalent to the homotopy lifting property for  $(D^k, \partial D^k)$  since the pairs  $(D^k \times I, D^k \times \{0\})$  and  $(D^k \times I, D^k \times \{0\} \cup \partial D^k \times I)$  are homeomorphic. This implies that the homotopy lifting property for disks is equivalent to the homotopy lifting property for all CW pairs  $(X, A)$ . For by induction over the skeleta of  $X$  it suffices to construct a lifting  $\tilde{g}_t$  one cell of  $X - A$  at a time. Composing with the characteristic map  $\Phi: D^k \rightarrow X$  of a cell then gives a reduction to the case  $(X, A) = (D^k, \partial D^k)$ . A map  $p: E \rightarrow B$  satisfying the homotopy lifting property for disks is sometimes called a *Serre fibration*.

**Proof:** First we show that  $p_*$  is onto. Represent an element of  $\pi_n(B, b_0)$  by a map  $f: (I^n, \partial I^n) \rightarrow (B, b_0)$ . The constant map to  $x_0$  provides a lift of  $f$  to  $E$  over the subspace  $J^{n-1} \subset I^n$ , so the relative homotopy lifting property for  $(I^{n-1}, \partial I^{n-1})$  extends this to a lift  $\tilde{f}: I^n \rightarrow E$ , and this lift satisfies  $\tilde{f}(\partial I^n) \subset F$  since  $f(\partial I^n) = b_0$ . Then  $\tilde{f}$  represents an element of  $\pi_n(E, F, x_0)$  with  $p_*([\tilde{f}]) = [f]$  since  $p\tilde{f} = f$ .

Injectivity of  $p_*$  is similar. Given  $\tilde{f}_0, \tilde{f}_1: (I^n, \partial I^n, J^{n-1}) \rightarrow (E, F, x_0)$  such that  $p_*([\tilde{f}_0]) = p_*([\tilde{f}_1])$ , let  $G: (I^n \times I, \partial I^n \times I) \rightarrow (B, b_0)$  be a homotopy from  $p\tilde{f}_0$  to  $p\tilde{f}_1$ . We have a partial lift  $\tilde{G}$  given by  $\tilde{f}_0$  on  $I^n \times \{0\}$ ,  $\tilde{f}_1$  on  $I^n \times \{1\}$ , and the constant map to  $x_0$  on  $J^{n-1} \times I$ . After permuting the last two coordinates of  $I^n \times I$ , the relative homotopy lifting property gives an extension of this partial lift to a full lift  $\tilde{G}: I^n \times I \rightarrow E$ . This is a homotopy  $\tilde{f}_t: (I^n, \partial I^n, J^{n-1}) \rightarrow (E, F, x_0)$  from  $\tilde{f}_0$  to  $\tilde{f}_1$ . So  $p_*$  is injective.

For the last statement of the theorem we plug  $\pi_n(B, b_0)$  in for  $\pi_n(E, F, x_0)$  in the long exact sequence for the pair  $(E, F)$ . The map  $\pi_n(E, x_0) \rightarrow \pi_n(E, F, x_0)$  in the exact sequence then becomes the composition  $\pi_n(E, x_0) \rightarrow \pi_n(E, F, x_0) \xrightarrow{p_*} \pi_n(B, b_0)$ , which is just  $p_*: \pi_n(E, x_0) \rightarrow \pi_n(B, b_0)$ . The 0 at the end of the sequence, surjectivity of  $\pi_0(F, x_0) \rightarrow \pi_0(E, x_0)$ , comes from the hypothesis that  $B$  is path-connected since a path in  $E$  from an arbitrary point  $x \in E$  to  $F$  can be obtained by lifting a path in  $B$  from  $p(x)$  to  $b_0$ .  $\square$

A **fiber bundle** structure on a space  $E$ , with fiber  $F$ , consists of a projection map  $p: E \rightarrow B$  such that each point of  $B$  has a neighborhood  $U$  for which there is a



homeomorphism  $h: p^{-1}(U) \rightarrow U \times F$  making the diagram at the right commute, where the unlabeled map is projection onto the first factor. Commutativity of the diagram means that  $h$  carries each fiber  $F_b = p^{-1}(b)$  homeomorphically

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & \searrow p & \swarrow \\ & U & \end{array}$$

onto the copy  $\{b\} \times F$  of  $F$ . Thus the fibers  $F_b$  are arranged locally as in the product  $B \times F$ , though not necessarily globally. An  $h$  as above is called a **local trivialization** of the bundle. Since the first coordinate of  $h$  is just  $p$ ,  $h$  is determined by its second coordinate, a map  $p^{-1}(U) \rightarrow F$  which is a homeomorphism on each fiber  $F_b$ .

The fiber bundle structure is determined by the projection map  $p: E \rightarrow B$ , but to indicate what the fiber is we sometimes write a fiber bundle as  $F \rightarrow E \rightarrow B$ , a ‘short exact sequence of spaces’. The space  $B$  is called the **base space** of the bundle, and  $E$  is the **total space**.

**Example 4.42.** A fiber bundle with fiber a discrete space is a covering space. Conversely, a covering space whose fibers all have the same cardinality, for example a covering space over a connected base space, is a fiber bundle with discrete fiber.

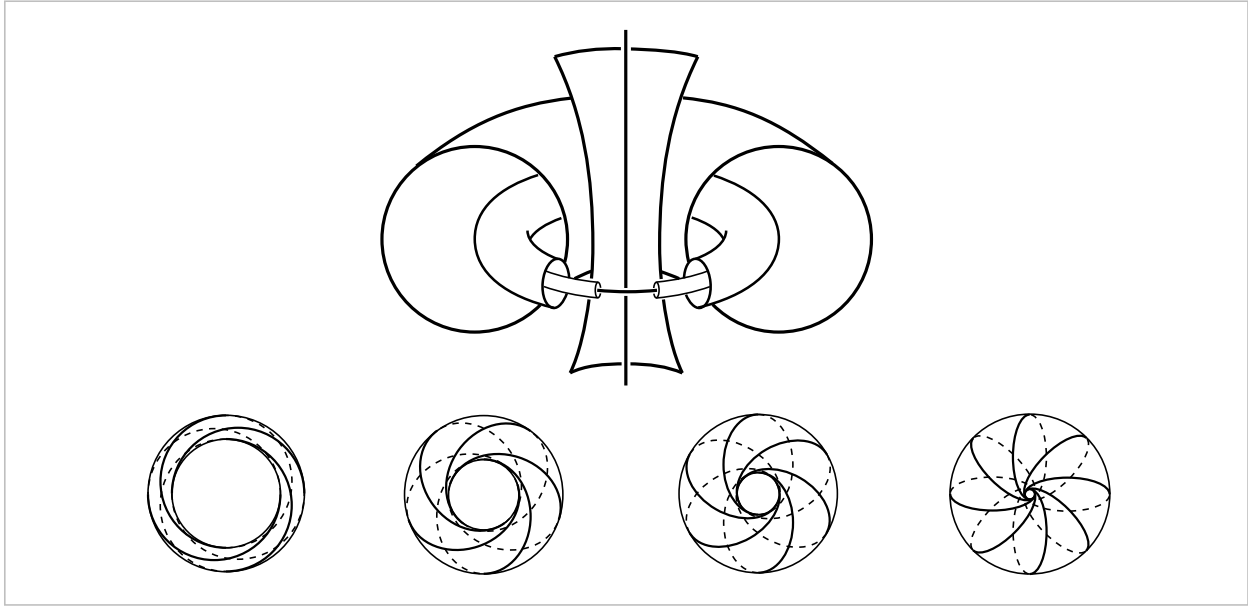
**Example 4.43.** One of the simplest nontrivial fiber bundles is the Möbius band, which is a bundle over  $S^1$  with fiber an interval. Specifically, take  $E$  to be the quotient of  $I \times [-1, 1]$  under the identifications  $(0, v) \sim (1, -v)$ , with  $p: E \rightarrow S^1$  induced by the projection  $I \times [-1, 1] \rightarrow I$ , so the fiber is  $[-1, 1]$ . Glueing two copies of  $E$  together by the identity map between their boundary circles produces a Klein bottle, a bundle over  $S^1$  with fiber  $S^1$ .

**Example 4.44.** Projective spaces yield interesting fiber bundles. In the real case we have the familiar covering spaces  $S^n \rightarrow \mathbb{RP}^n$  with fiber  $S^0$ . Over the complex numbers the analog of this is a fiber bundle  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ . Here  $S^{2n+1}$  is the unit sphere in  $\mathbb{C}^{n+1}$  and  $\mathbb{CP}^n$  is viewed as the quotient space of  $S^{2n+1}$  under the equivalence relation  $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$  for  $\lambda \in S^1$ , the unit circle in  $\mathbb{C}$ . The projection  $p: S^{2n+1} \rightarrow \mathbb{CP}^n$  sends  $(z_0, \dots, z_n)$  to its equivalence class  $[z_0, \dots, z_n]$ , so the fibers are copies of  $S^1$ . To see that the local triviality condition for fiber bundles is satisfied, let  $U_i \subset \mathbb{CP}^n$  be the open set of equivalence classes  $[z_0, \dots, z_n]$  with  $z_i \neq 0$ . Define  $h_i: p^{-1}(U_i) \rightarrow U_i \times S^1$  by  $h_i(z_0, \dots, z_n) = ([z_0, \dots, z_n], z_i/|z_i|)$ . This takes fibers to fibers, and is a homeomorphism since its inverse is the map  $([z_0, \dots, z_n], \lambda) \mapsto \lambda|z_i|z_i^{-1}(z_0, \dots, z_n)$ , as one checks by calculation.

The construction of the bundle  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$  also works when  $n = \infty$ , so there is a fiber bundle  $S^1 \rightarrow S^\infty \rightarrow \mathbb{CP}^\infty$ .

**Example 4.45.** The case  $n = 1$  is particularly interesting since  $\mathbb{CP}^1 = S^2$  and the bundle becomes  $S^1 \rightarrow S^3 \rightarrow S^2$  with fiber, total space, and base all spheres. This is known as the **Hopf bundle**, and is of low enough dimension to be seen explicitly. The projection  $S^3 \rightarrow S^2$  can be taken to be  $(z_0, z_1) \mapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2$ . In polar coordinates we have  $p(r_0 e^{i\theta_0}, r_1 e^{i\theta_1}) = (r_0/r_1) e^{i(\theta_0 - \theta_1)}$  where  $r_0^2 + r_1^2 = 1$ . For a

fixed ratio  $\rho = r_0/r_1 \in (0, \infty)$  the angles  $\theta_0$  and  $\theta_1$  vary independently over  $S^1$ , so the points  $(r_0 e^{i\theta_0}, r_1 e^{i\theta_1})$  form a torus  $T_\rho \subset S^3$ . Letting  $\rho$  vary, these disjoint tori  $T_\rho$  fill up  $S^3$ , if we include the limiting cases  $T_0$  and  $T_\infty$  where the radii  $r_0$  and  $r_1$  are zero, making the tori  $T_0$  and  $T_\infty$  degenerate to circles. These two circles are the unit circles in the two  $\mathbb{C}$  factors of  $\mathbb{C}^2$ , so under stereographic projection of  $S^3$  from the point  $(0, 1)$  onto  $\mathbb{R}^3$  they correspond to the unit circle in the  $xy$ -plane and the  $z$ -axis. The concentric tori  $T_\rho$  are then arranged as in the following figure.



Each torus  $T_\rho$  is a union of circle fibers, the pairs  $(\theta_0, \theta_1)$  with  $\theta_0 - \theta_1$  constant. These fiber circles have slope 1 on the torus, winding around once longitudinally and once meridionally. With respect to the ambient space it might be more accurate to say they have slope  $\rho$ . As  $\rho$  goes to 0 or  $\infty$  the fiber circles approach the circles  $T_0$  and  $T_\infty$ , which are also fibers. The figure shows four of the tori decomposed into fibers.

**Example 4.46.** Replacing the field  $\mathbb{C}$  by the quaternions  $\mathbb{H}$ , the same constructions yield fiber bundles  $S^3 \rightarrow S^{4n+3} \rightarrow \mathbb{H}P^n$  over quaternionic projective spaces  $\mathbb{H}P^n$ . Here the fiber  $S^3$  is the unit quaternions, and  $S^{4n+3}$  is the unit sphere in  $\mathbb{H}^{n+1}$ . Taking  $n = 1$  gives a second Hopf bundle  $S^3 \rightarrow S^7 \rightarrow S^4 = \mathbb{H}P^1$ .

**Example 4.47.** Another Hopf bundle  $S^7 \rightarrow S^{15} \rightarrow S^8$  can be defined using the octonion algebra  $\mathbb{O}$ . Elements of  $\mathbb{O}$  are pairs of quaternions  $(a_1, a_2)$  with multiplication given by  $(a_1, a_2)(b_1, b_2) = (a_1 b_1 - \bar{b}_2 a_2, a_2 \bar{b}_1 + b_2 a_1)$ . Regarding  $S^{15}$  as the unit sphere in the 16-dimensional vector space  $\mathbb{O}^2$ , the projection map  $p: S^{15} \rightarrow S^8 = \mathbb{O} \cup \{\infty\}$  is  $(z_0, z_1) \mapsto z_0 z_1^{-1}$ , just as for the other Hopf bundles, but because  $\mathbb{O}$  is not associative, a little care is needed to show this is a fiber bundle with fiber  $S^7$ , the unit octonions. Let  $U_0$  and  $U_1$  be the complements of  $\infty$  and 0 in the base space  $\mathbb{O} \cup \{\infty\}$ . Define  $h_i: p^{-1}(U_i) \rightarrow U_i \times S^7$  and  $g_i: U_i \times S^7 \rightarrow p^{-1}(U_i)$  by

$$\begin{aligned} h_0(z_0, z_1) &= (z_0 z_1^{-1}, z_1 / |z_1|), & g_0(z, w) &= (zw, w) / |(zw, w)| \\ h_1(z_0, z_1) &= (z_0 z_1^{-1}, z_0 / |z_0|), & g_1(z, w) &= (w, z^{-1}w) / |(w, z^{-1}w)| \end{aligned}$$

If one assumes the known fact that any subalgebra of  $\mathbb{O}$  generated by two elements is associative, then it is a simple matter to check that  $g_i$  and  $h_i$  are inverse homeomorphisms, so we have a fiber bundle  $S^7 \rightarrow S^{15} \rightarrow S^8$ . Actually, the calculation that  $g_i$  and  $h_i$  are inverses needs only the following more elementary facts about octonions  $z, w$ , where the conjugate  $\bar{z}$  of  $z = (a_1, a_2)$  is defined by the expected formula  $\bar{z} = (\bar{a}_1, -a_2)$ :

- (1)  $rz = zr$  for all  $r \in \mathbb{R}$  and  $z \in \mathbb{O}$ , where  $\mathbb{R} \subset \mathbb{O}$  as the pairs  $(r, 0)$ .
- (2)  $|z|^2 = z\bar{z} = \bar{z}z$ , hence  $z^{-1} = \bar{z}/|z|^2$ .
- (3)  $|zw| = |z||w|$ .
- (4)  $\overline{zw} = \bar{w}\bar{z}$ , hence  $(zw)^{-1} = w^{-1}z^{-1}$ .
- (5)  $z(\bar{z}w) = (z\bar{z})w$  and  $(zw)\bar{w} = z(w\bar{w})$ , hence  $z(z^{-1}w) = w$  and  $(zw)w^{-1} = z$ .

These facts can be checked by somewhat tedious direct calculation. More elegant derivations can be found in Chapter 8 of [Ebbinghaus 1991].

There is an octonion projective plane  $\mathbb{O}P^2$  obtained by attaching a cell  $e^{16}$  to  $S^8$  via the Hopf map  $S^{15} \rightarrow S^8$ , just as  $\mathbb{C}P^2$  and  $\mathbb{H}P^2$  are obtained from the other Hopf maps. However, there is no octonion analog of  $\mathbb{R}P^n$ ,  $\mathbb{C}P^n$ , and  $\mathbb{H}P^n$  for  $n > 2$  since associativity of multiplication is needed for the relation  $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$  to be an equivalence relation.

There are no fiber bundles with fiber, total space, and base space spheres of other dimensions than in these Hopf bundle examples. This is discussed in an exercise for §4.D, which reduces the question to the famous ‘Hopf invariant one’ problem.

**Proposition 4.48.** *A fiber bundle  $p: E \rightarrow B$  has the homotopy lifting property with respect to all CW pairs  $(X, A)$ .*

A theorem of Huebsch and Hurewicz proved in §2.7 of [Spanier 1966] says that fiber bundles over paracompact base spaces are fibrations, having the homotopy lifting property with respect to all spaces. This stronger result is not often needed in algebraic topology, however.

**Proof:** As noted earlier, the homotopy lifting property for CW pairs is equivalent to the homotopy lifting property for disks, or equivalently, cubes. Let  $G: I^n \times I \rightarrow B$ ,  $G(x, t) = g_t(x)$ , be a homotopy we wish to lift, starting with a given lift  $\tilde{g}_0$  of  $g_0$ . Choose an open cover  $\{U_\alpha\}$  of  $B$  with local trivializations  $h_\alpha: p^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ . Using compactness of  $I^n \times I$ , we may subdivide  $I^n$  into small cubes  $C$  and  $I$  into intervals  $I_j = [t_j, t_{j+1}]$  so that each product  $C \times I_j$  is mapped by  $G$  into a single  $U_\alpha$ . We may assume by induction on  $n$  that  $\tilde{g}_t$  has already been constructed over  $\partial C$  for each of the subcubes  $C$ . To extend this  $\tilde{g}_t$  over a cube  $C$  we may proceed in stages, constructing  $\tilde{g}_t$  for  $t$  in each successive interval  $I_j$ . This in effect reduces us to the case that no subdivision of  $I^n \times I$  is necessary, so  $G$  maps all of  $I^n \times I$  to a single  $U_\alpha$ . Then we have  $\tilde{G}(I^n \times \{0\} \cup \partial I^n \times I) \subset p^{-1}(U_\alpha)$ , and composing  $\tilde{G}$  with the local trivialization

$h_\alpha$  reduces us to the case of a product bundle  $U_\alpha \times F$ . In this case the first coordinate of a lift  $\tilde{g}_t$  is just the given  $g_t$ , so only the second coordinate needs to be constructed. This can be obtained as a composition  $I^n \times I \rightarrow I^n \times \{0\} \cup \partial I^n \times I \rightarrow F$  where the first map is a retraction and the second map is what we are given.  $\square$

**Example 4.49.** Applying this theorem to a covering space  $p: E \rightarrow B$  with  $E$  and  $B$  path-connected, and discrete fiber  $F$ , the resulting long exact sequence of homotopy groups yields Proposition 4.1 that  $p_*: \pi_n(E) \rightarrow \pi_n(B)$  is an isomorphism for  $n \geq 2$ . We also obtain a short exact sequence  $0 \rightarrow \pi_1(E) \rightarrow \pi_1(B) \rightarrow \pi_0(F) \rightarrow 0$ , consistent with the covering space theory facts that  $p_*: \pi_1(E) \rightarrow \pi_1(B)$  is injective and that the fiber  $F$  can be identified, via path-lifting, with the set of cosets of  $p_*\pi_1(E)$  in  $\pi_1(B)$ .

**Example 4.50.** From the bundle  $S^1 \rightarrow S^\infty \rightarrow \mathbb{CP}^\infty$  we obtain  $\pi_i(\mathbb{CP}^\infty) \approx \pi_{i-1}(S^1)$  for all  $i$  since  $S^\infty$  is contractible. Thus  $\mathbb{CP}^\infty$  is a  $K(\mathbb{Z}, 2)$ . In similar fashion the bundle  $S^3 \rightarrow S^\infty \rightarrow \mathbb{HP}^\infty$  gives  $\pi_i(\mathbb{HP}^\infty) \approx \pi_{i-1}(S^3)$  for all  $i$ , but these homotopy groups are far more complicated than for  $\mathbb{CP}^\infty$  and  $S^1$ . In particular,  $\mathbb{HP}^\infty$  is not a  $K(\mathbb{Z}, 4)$ .

**Example 4.51.** The long exact sequence for the Hopf bundle  $S^1 \rightarrow S^3 \rightarrow S^2$  gives isomorphisms  $\pi_2(S^2) \approx \pi_1(S^1)$  and  $\pi_n(S^3) \approx \pi_n(S^2)$  for all  $n \geq 3$ . Taking  $n = 3$ , we see that  $\pi_3(S^2)$  is infinite cyclic, generated by the Hopf map  $S^3 \rightarrow S^2$ .

From this example and the preceding one we see that  $S^2$  and  $S^3 \times \mathbb{CP}^\infty$  are simply-connected CW complexes with isomorphic homotopy groups, though they are not homotopy equivalent since they have quite different homology groups.

**Example 4.52: Whitehead Products.** Let us compute  $\pi_3(\bigvee_\alpha S_\alpha^2)$ , showing that it is free abelian with basis consisting of the Hopf maps  $S^3 \rightarrow S_\alpha^2 \subset \bigvee_\alpha S_\alpha^2$  together with the attaching maps  $S^3 \rightarrow S_\alpha^2 \vee S_\beta^2 \subset \bigvee_\alpha S_\alpha^2$  of the cells  $e_\alpha^2 \times e_\beta^2$  in the products  $S_\alpha^2 \times S_\beta^2$  for all unordered pairs  $\alpha \neq \beta$ .

Suppose first that there are only finitely many summands  $S_\alpha^2$ . For a finite product  $\prod_\alpha X_\alpha$  of path-connected spaces, the map  $\pi_n(\bigvee_\alpha X_\alpha) \rightarrow \pi_n(\prod_\alpha X_\alpha)$  induced by inclusion is surjective since the group  $\pi_n(\prod_\alpha X_\alpha) \approx \bigoplus_\alpha \pi_n(X_\alpha)$  is generated by the subgroups  $\pi_n(X_\alpha)$ . Thus the long exact sequence of homotopy groups for the pair  $(\prod_\alpha X_\alpha, \bigvee_\alpha X_\alpha)$  breaks up into short exact sequences

$$0 \rightarrow \pi_{n+1}(\prod_\alpha X_\alpha, \bigvee_\alpha X_\alpha) \rightarrow \pi_n(\bigvee_\alpha X_\alpha) \rightarrow \pi_n(\prod_\alpha X_\alpha) \rightarrow 0$$

These short exact sequences split since the inclusions  $X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$  induce maps  $\pi_n(X_\alpha) \rightarrow \pi_n(\bigvee_\alpha X_\alpha)$  and hence a splitting homomorphism  $\bigoplus_\alpha \pi_n(X_\alpha) \rightarrow \pi_n(\bigvee_\alpha X_\alpha)$ . Taking  $X_\alpha = S_\alpha^2$  and  $n = 3$ , we get an isomorphism

$$\pi_3(\bigvee_\alpha S_\alpha^2) \approx \pi_4(\prod_\alpha S_\alpha^2, \bigvee_\alpha S_\alpha^2) \oplus (\bigoplus_\alpha \pi_3(S_\alpha^2))$$

The factor  $\bigoplus_\alpha \pi_3(S_\alpha^2)$  is free with basis the Hopf maps  $S^3 \rightarrow S_\alpha^2$  by the preceding example. For the other factor we have  $\pi_4(\prod_\alpha S_\alpha^2, \bigvee_\alpha S_\alpha^2) \approx \pi_4(\prod_\alpha S_\alpha^2 / \bigvee_\alpha S_\alpha^2)$  by Proposition 4.28. The quotient  $\prod_\alpha S_\alpha^2 / \bigvee_\alpha S_\alpha^2$  has 5-skeleton a wedge of spheres  $S_{\alpha\beta}^4$  for  $\alpha \neq \beta$ ,

so  $\pi_4(\prod_\alpha S_\alpha^2 / \vee_\alpha S_\alpha^2) \approx \pi_4(\vee_{\alpha\beta} S_{\alpha\beta}^4)$  is free with basis the inclusions  $S_{\alpha\beta}^4 \hookrightarrow \vee_{\alpha\beta} S_{\alpha\beta}^4$ . Hence  $\pi_4(\prod_\alpha S_\alpha^2, \vee_\alpha S_\alpha^2)$  is free with basis the characteristic maps of the 4-cells  $e_\alpha^2 \times e_\beta^2$ . Via the injection  $\partial: \pi_4(\prod_\alpha S_\alpha^2, \vee_\alpha S_\alpha^2) \rightarrow \pi_3(\vee_\alpha S_\alpha^2)$  this means that the attaching maps of the cells  $e_\alpha^2 \times e_\beta^2$  form a basis for the summand  $\text{Im } \partial$  of  $\pi_3(\vee_\alpha S_\alpha^2)$ . This finishes the proof for the case of finitely many summands  $S_\alpha^2$ . The case of infinitely many  $S_\alpha^2$ 's follows immediately since any map  $S^3 \rightarrow \vee_\alpha S_\alpha^2$  has compact image, lying in a finite union of summands, and similarly for any homotopy between such maps.

The maps  $S^3 \rightarrow S_\alpha^2 \vee S_\beta^2$  in this example are expressible in terms of a product in homotopy groups called the **Whitehead product**, defined as follows. Given basepoint-preserving maps  $f: S^k \rightarrow X$  and  $g: S^\ell \rightarrow X$ , let  $[f, g]: S^{k+\ell-1} \rightarrow X$  be the composition  $S^{k+\ell-1} \rightarrow S^k \vee S^\ell \xrightarrow{f \vee g} X$  where the first map is the attaching map of the  $(k + \ell)$ -cell of  $S^k \times S^\ell$  with its usual CW structure. Since homotopies of  $f$  or  $g$  give rise to homotopies of  $[f, g]$ , we have a well-defined product  $\pi_k(X) \times \pi_\ell(X) \rightarrow \pi_{k+\ell-1}(X)$ . The notation  $[f, g]$  is used since for  $k = \ell = 1$  this is just the commutator product in  $\pi_1(X)$ . It is an exercise to show that when  $k = 1$  and  $\ell > 1$ ,  $[f, g]$  is the difference between  $g$  and its image under the  $\pi_1$ -action of  $f$ .

In these terms the map  $S^3 \rightarrow S_\alpha^2 \vee S_\beta^2$  in the preceding example is the Whitehead product  $[i_\alpha, i_\beta]$  of the two inclusions of  $S^2$  into  $S_\alpha^2 \vee S_\beta^2$ . Another example of a Whitehead product we have encountered previously is  $[\mathbb{1}, \mathbb{1}]: S^{2n-1} \rightarrow S^n$ , which is the attaching map of the  $2n$ -cell of the space  $J(S^n)$  considered in §3.2.

The calculation of  $\pi_3(\vee_\alpha S_\alpha^2)$  is the first nontrivial case of a more general theorem of Hilton calculating all the homotopy groups of any wedge sum of spheres in terms of homotopy groups of spheres, using Whitehead products. A further generalization by Milnor extends this to wedge sums of suspensions of arbitrary connected CW complexes. See [Whitehead 1978] for an exposition of these results and further information on Whitehead products.

**Example 4.53: Stiefel and Grassmann Manifolds.** The fiber bundles with total space a sphere and base space a projective space considered above are the cases  $n = 1$  of families of fiber bundles in each of the real, complex, and quaternionic cases:

$$\begin{array}{ll} O(n) \rightarrow V_n(\mathbb{R}^k) \rightarrow G_n(\mathbb{R}^k) & O(n) \rightarrow V_n(\mathbb{R}^\infty) \rightarrow G_n(\mathbb{R}^\infty) \\ U(n) \rightarrow V_n(\mathbb{C}^k) \rightarrow G_n(\mathbb{C}^k) & U(n) \rightarrow V_n(\mathbb{C}^\infty) \rightarrow G_n(\mathbb{C}^\infty) \\ Sp(n) \rightarrow V_n(\mathbb{H}^k) \rightarrow G_n(\mathbb{H}^k) & Sp(n) \rightarrow V_n(\mathbb{H}^\infty) \rightarrow G_n(\mathbb{H}^\infty) \end{array}$$

Taking the real case first, the Stiefel manifold  $V_n(\mathbb{R}^k)$  is the space of  $n$ -frames in  $\mathbb{R}^k$ , that is,  $n$ -tuples of orthonormal vectors in  $\mathbb{R}^k$ . This is topologized as a subspace of the product of  $n$  copies of the unit sphere in  $\mathbb{R}^k$ . The Grassmann manifold  $G_n(\mathbb{R}^k)$  is the space of  $n$ -dimensional vector subspaces of  $\mathbb{R}^k$ . There is a natural surjection  $p: V_n(\mathbb{R}^k) \rightarrow G_n(\mathbb{R}^k)$  sending an  $n$ -frame to the subspace it spans, and  $G_n(\mathbb{R}^k)$  is topologized as a quotient space of  $V_n(\mathbb{R}^k)$  via this projection. The fibers of the map

$p$  are the spaces of  $n$ -frames in a fixed  $n$ -plane in  $\mathbb{R}^k$  and so are homeomorphic to  $V_n(\mathbb{R}^n)$ . An  $n$ -frame in  $\mathbb{R}^n$  is the same as an orthogonal  $n \times n$  matrix, regarding the columns of the matrix as an  $n$ -frame, so the fiber can also be described as the orthogonal group  $O(n)$ . There is no difficulty in allowing  $k = \infty$  in these definitions, and in fact  $V_n(\mathbb{R}^\infty) = \bigcup_k V_n(\mathbb{R}^k)$  and  $G_n(\mathbb{R}^\infty) = \bigcup_k G_n(\mathbb{R}^k)$ .

The complex and quaternionic Stiefel manifolds and Grassmann manifolds are defined in the same way using the usual Hermitian inner products in  $\mathbb{C}^k$  and  $\mathbb{H}^k$ . The unitary group  $U(n)$  consists of  $n \times n$  matrices whose columns form orthonormal bases for  $\mathbb{C}^n$ , and the symplectic group  $Sp(n)$  is the quaternionic analog of this.

We should explain why the various projection maps  $V_n \rightarrow G_n$  are fiber bundles. Let us take the real case for concreteness, though the argument is the same in all cases. If we fix an  $n$ -plane  $P \in G_n(\mathbb{R}^k)$  and choose an orthonormal basis for  $P$ , then we obtain continuously varying orthonormal bases for all  $n$ -planes  $P'$  in a neighborhood  $U$  of  $P$  by projecting the basis for  $P$  orthogonally onto  $P'$  to obtain a nonorthonormal basis for  $P'$ , then applying the Gram-Schmidt process to this basis to make it orthonormal. The formulas for the Gram-Schmidt process show that it is continuous. Having orthonormal bases for all  $n$ -planes in  $U$ , we can use these to identify these  $n$ -planes with  $\mathbb{R}^n$ , hence  $n$ -frames in these  $n$ -planes are identified with  $n$ -frames in  $\mathbb{R}^n$ , and so  $p^{-1}(U)$  is identified with  $U \times V_n(\mathbb{R}^n)$ . This argument works for  $k = \infty$  as well as for finite  $k$ .

In the case  $n = 1$  the total spaces  $V_1$  are spheres, which are highly connected, and the same is true in general:

- $V_n(\mathbb{R}^k)$  is  $(k - n - 1)$ -connected.
- $V_n(\mathbb{C}^k)$  is  $(2k - 2n)$ -connected.
- $V_n(\mathbb{H}^k)$  is  $(4k - 4n + 2)$ -connected.
- $V_n(\mathbb{R}^\infty)$ ,  $V_n(\mathbb{C}^\infty)$ , and  $V_n(\mathbb{H}^\infty)$  are contractible.

The first three statements will be proved in the next example. For the last statement the argument is the same in the three cases, so let us consider the real case. Define a homotopy  $h_t: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$  by  $h_t(x_1, x_2, \dots) = (1-t)(x_1, x_2, \dots) + t(0, x_1, x_2, \dots)$ . This is linear for each  $t$ , and its kernel is easily checked to be trivial. So if we apply  $h_t$  to an  $n$ -frame we get an  $n$ -tuple of independent vectors, which can be made orthonormal by the Gram-Schmidt formulas. Thus we have a deformation retraction, in the weak sense, of  $V_n(\mathbb{R}^\infty)$  onto the subspace of  $n$ -frames with first coordinate zero. Iterating this  $n$  times, we deform into the subspace of  $n$ -frames with first  $n$  coordinates zero. For such an  $n$ -frame  $(v_1, \dots, v_n)$  define a homotopy  $(1-t)(v_1, \dots, v_n) + t(e_1, \dots, e_n)$  where  $e_i$  is the  $i^{\text{th}}$  standard basis vector in  $\mathbb{R}^\infty$ . This homotopy preserves linear independence, so after again applying Gram-Schmidt we have a deformation through  $n$ -frames, which finishes the construction of a contraction of  $V_n(\mathbb{R}^\infty)$ .

Since  $V_n(\mathbb{R}^\infty)$  is contractible, we obtain isomorphisms  $\pi_i O(n) \approx \pi_{i+1} G_n(\mathbb{R}^\infty)$  for all  $i$  and  $n$ , and similarly in the complex and quaternionic cases.

**Example 4.54.** For  $m < n \leq k$  there are fiber bundles

$$V_{n-m}(\mathbb{R}^{k-m}) \rightarrow V_n(\mathbb{R}^k) \xrightarrow{p} V_m(\mathbb{R}^k)$$

where the projection  $p$  sends an  $n$ -frame onto the  $m$ -frame formed by its first  $m$  vectors, so the fiber consists of  $(n-m)$ -frames in the  $(k-m)$ -plane orthogonal to a given  $m$ -frame. Local trivializations can be constructed as follows. For an  $m$ -frame  $F$ , choose an orthonormal basis for the  $(k-m)$ -plane orthogonal to  $F$ . This determines orthonormal bases for the  $(k-m)$ -planes orthogonal to all nearby  $m$ -frames by orthogonal projection and Gram-Schmidt, as in the preceding example. This allows us to identify these  $(k-m)$ -planes with  $\mathbb{R}^{k-m}$ , and in particular the fibers near  $p^{-1}(F)$  are identified with  $V_{n-m}(\mathbb{R}^{k-m})$ , giving a local trivialization.

There are analogous bundles in the complex and quaternionic cases as well, with local triviality shown in the same way.

Restricting to the case  $m = 1$ , we have bundles  $V_{n-1}(\mathbb{R}^{k-1}) \rightarrow V_n(\mathbb{R}^k) \rightarrow S^{k-1}$  whose associated long exact sequence of homotopy groups allows us deduce that  $V_n(\mathbb{R}^k)$  is  $(k-n-1)$ -connected by induction on  $n$ . In the complex and quaternionic cases the same argument yields the other connectivity statements in the preceding example.

Taking  $k = n$  we obtain fiber bundles  $O(k-m) \rightarrow O(k) \rightarrow V_m(\mathbb{R}^k)$ . The fibers are in fact just the cosets  $\alpha O(k-m)$  for  $\alpha \in O(k)$ , where  $O(k-m)$  is regarded as the subgroup of  $O(k)$  fixing the first  $m$  standard basis vectors. So we see that  $V_m(\mathbb{R}^k)$  is identifiable with the coset space  $O(k)/O(k-m)$ , or in other words the orbit space for the free action of  $O(k-m)$  on  $O(k)$  by right-multiplication. In similar fashion one can see that  $G_m(\mathbb{R}^k)$  is the coset space  $O(k)/(O(m) \times O(k-m))$  where the subgroup  $O(m) \times O(k-m) \subset O(k)$  consists of the orthogonal transformations taking the  $m$ -plane spanned by the first  $m$  standard basis vectors to itself. The corresponding observations apply also in the complex and quaternionic cases, with the unitary and symplectic groups.

**Example 4.55: Bott Periodicity.** Specializing the preceding example by taking  $m = 1$  and  $k = n$  we obtain bundles

$$\begin{aligned} O(n-1) &\rightarrow O(n) \xrightarrow{p} S^{n-1} \\ U(n-1) &\rightarrow U(n) \xrightarrow{p} S^{2n-1} \\ Sp(n-1) &\rightarrow Sp(n) \xrightarrow{p} S^{4n-1} \end{aligned}$$

The map  $p$  can be described as evaluation of an orthogonal, unitary, or symplectic transformation on a fixed unit vector. These bundles show that computing homotopy groups of  $O(n)$ ,  $U(n)$ , and  $Sp(n)$  should be at least as difficult as computing homotopy groups of spheres. For example, if one knew the homotopy groups of  $O(n)$  and  $O(n-1)$ , then from the long exact sequence of homotopy groups for the first bundle one could say quite a bit about the homotopy groups of  $S^{n-1}$ .

The bundles above imply a very interesting stability property. In the real case, the inclusion  $O(n-1) \hookrightarrow O(n)$  induces an isomorphism on  $\pi_i$  for  $i < n-2$ , from the long exact sequence of the first bundle. Hence the groups  $\pi_i O(n)$  are independent of  $n$  if  $n$  is sufficiently large, and the same is true for the groups  $\pi_i U(n)$  and  $\pi_i Sp(n)$  via the other two bundles. One of the most surprising results in all of algebraic topology is the Bott Periodicity Theorem which asserts that these stable groups repeat periodically, with a period of eight for  $O$  and  $Sp$  and a period of two for  $U$ . Their values are given in the following table:

| $i \bmod 8$   | 0              | 1              | 2 | 3            | 4              | 5              | 6 | 7            |
|---------------|----------------|----------------|---|--------------|----------------|----------------|---|--------------|
| $\pi_i O(n)$  | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0              | 0              | 0 | $\mathbb{Z}$ |
| $\pi_i U(n)$  | 0              | $\mathbb{Z}$   | 0 | $\mathbb{Z}$ | 0              | $\mathbb{Z}$   | 0 | $\mathbb{Z}$ |
| $\pi_i Sp(n)$ | 0              | 0              | 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ |

## Stable Homotopy Groups

We showed in Corollary 4.24 that for an  $n$ -connected CW complex  $X$ , the suspension map  $\pi_i(X) \rightarrow \pi_{i+1}(SX)$  is an isomorphism for  $i < 2n+1$ . In particular this holds for  $i \leq n$  so  $SX$  is  $(n+1)$ -connected. This implies that in the sequence of iterated suspensions

$$\pi_i(X) \rightarrow \pi_{i+1}(SX) \rightarrow \pi_{i+2}(S^2X) \rightarrow \dots$$

all maps are eventually isomorphisms, even without any connectivity assumption on  $X$  itself. The resulting **stable homotopy group** is denoted  $\pi_i^S(X)$ .

An especially interesting case is the group  $\pi_i^S(S^0)$ , which equals  $\pi_{i+n}(S^n)$  for  $n > i+1$ . This stable homotopy group is often abbreviated to  $\pi_i^S$  and called the **stable  $i$ -stem**. It is a theorem of Serre which we prove in [SSAT] that  $\pi_i^S$  is always finite for  $i > 0$ .

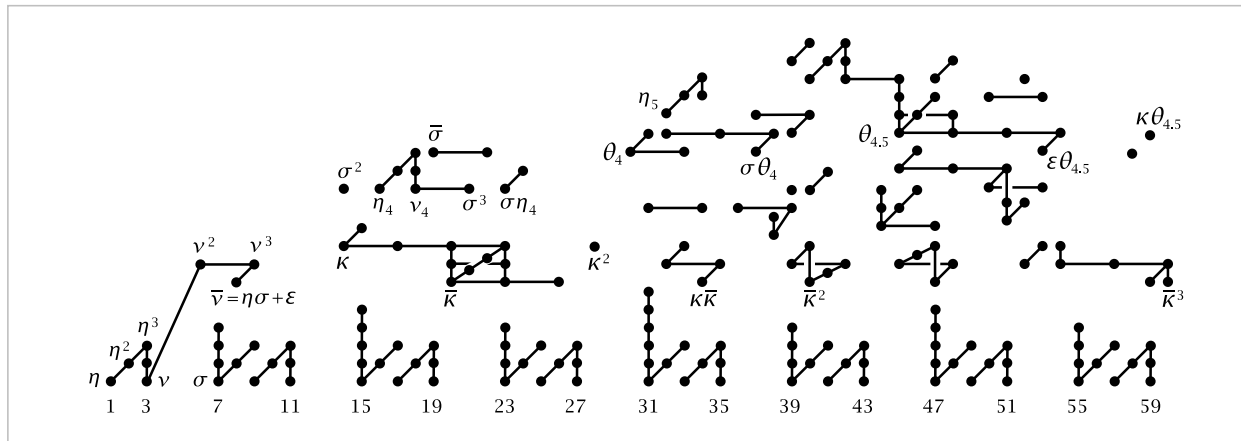
These stable homotopy groups of spheres are among the most fundamental objects in topology, and much effort has gone into their calculation. At the present time, complete calculations are known only for  $i$  up to around 60 or so. Here is a table for  $i \leq 19$ , taken from [Toda 1962]:

| $i$       | 0              | 1                                  | 2              | 3                 | 4                                      | 5 | 6                                  | 7                  | 8  | 9  | 10                                 | 11                 | 12                                     |
|-----------|----------------|------------------------------------|----------------|-------------------|--|---|------------------------------------|--------------------|--|--|------------------------------------|--------------------|--|
| $\pi_i^S$ | $\mathbb{Z}$   | $\mathbb{Z}_2$                     | $\mathbb{Z}_2$ | $\mathbb{Z}_{24}$ | 0                                      | 0 | $\mathbb{Z}_2$                     | $\mathbb{Z}_{240}$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$   | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_6$                     | $\mathbb{Z}_{504}$ | 0                                      |
|           | 13             | 14                                 |                |                   | 15                                     |   | 16                                 |                    | 17   |  | 18                                 |                    | 19                                     |
|           | $\mathbb{Z}_3$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |                |                   | $\mathbb{Z}_{480} \times \mathbb{Z}_2$ |   | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |                    | $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ |  | $\mathbb{Z}_8 \times \mathbb{Z}_2$ |                    | $\mathbb{Z}_{264} \times \mathbb{Z}_2$ |

Patterns in this apparent chaos begin to emerge only when one projects  $\pi_i^S$  onto its  **$p$ -components**, the quotient groups obtained by factoring out all elements of order relatively prime to the prime  $p$ . For  $i > 0$  the  $p$ -component  ${}_p\pi_i^S$  is of course isomorphic to the subgroup of  $\pi_i^S$  consisting of elements of order a power of  $p$ , but the quotient viewpoint is in some ways preferable.



The figure below is a schematic diagram of the 2-components of  $\pi_i^S$  for  $i \leq 60$ . A vertical chain of  $n$  dots in the  $i^{\text{th}}$  column represents a  $\mathbb{Z}_{2^n}$  summand of  $\pi_i^S$ . The bottom dot of such a chain denotes a generator of this summand, and the vertical segments denote multiplication by 2, so the second dot up is twice a generator, the next dot is four times a generator, and so on. The three generators  $\eta$ ,  $\nu$ , and  $\sigma$  in dimensions 1, 3, and 7 are represented by the Hopf bundle maps  $S^3 \rightarrow S^2$ ,  $S^7 \rightarrow S^4$ ,  $S^{15} \rightarrow S^8$  defined in Examples 4.45, 4.46, and 4.47. Some of the other elements also have standard names indicated by the Greek letter labels.



The other line segments in the diagram provide some information about compositions of maps between spheres. Namely, there are products  $\pi_i^S \times \pi_j^S \rightarrow \pi_{i+j}^S$  defined by compositions  $S^{i+j+k} \rightarrow S^{j+k} \rightarrow S^k$ .

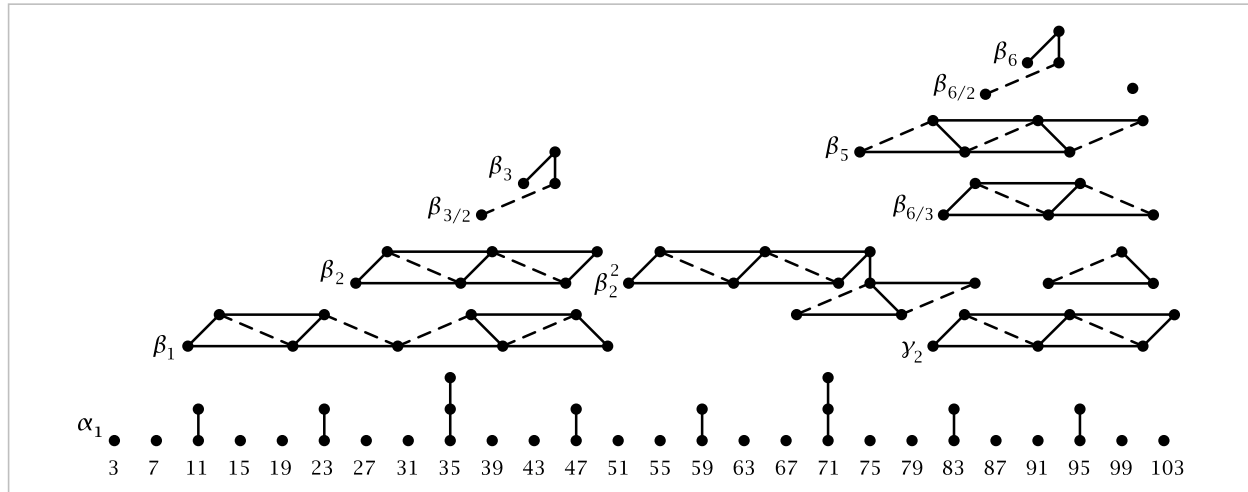
**Proposition 4.56.** *The composition products  $\pi_i^S \times \pi_j^S \rightarrow \pi_{i+j}^S$  induce a graded ring structure on  $\pi_*^S = \bigoplus_i \pi_i^S$  satisfying the commutativity relation  $\alpha\beta = (-1)^{ij}\beta\alpha$  for  $\alpha \in \pi_i^S$  and  $\beta \in \pi_j^S$ .*

This will be proved at the end of this subsection. It follows that  ${}_p\pi_*^S$ , the direct sum of the  $p$ -components  ${}_p\pi_i^S$ , is also a graded ring satisfying the same commutativity property. In  ${}_2\pi_*^S$  many of the compositions with suspensions of the Hopf maps  $\eta$  and  $\nu$  are nontrivial, and these nontrivial compositions are indicated in the diagram by segments extending 1 unit to the right and diagonally upward for  $\eta$  or 3 units to the right, usually horizontally, for  $\nu$ . Thus for example we see the relation  $\eta^3 = 4\nu$  in  ${}_2\pi_3^S$ . Remember that  ${}_2\pi_3^S \approx \mathbb{Z}_8$  is a quotient of  $\pi_3^S \approx \mathbb{Z}_{24}$ , where the actual relation is  $\eta^3 = 12\nu$  since  $2\eta = 0$  implies  $2\eta^3 = 0$ , so  $\eta^3$  is the unique element of order two in this  $\mathbb{Z}_{24}$ .

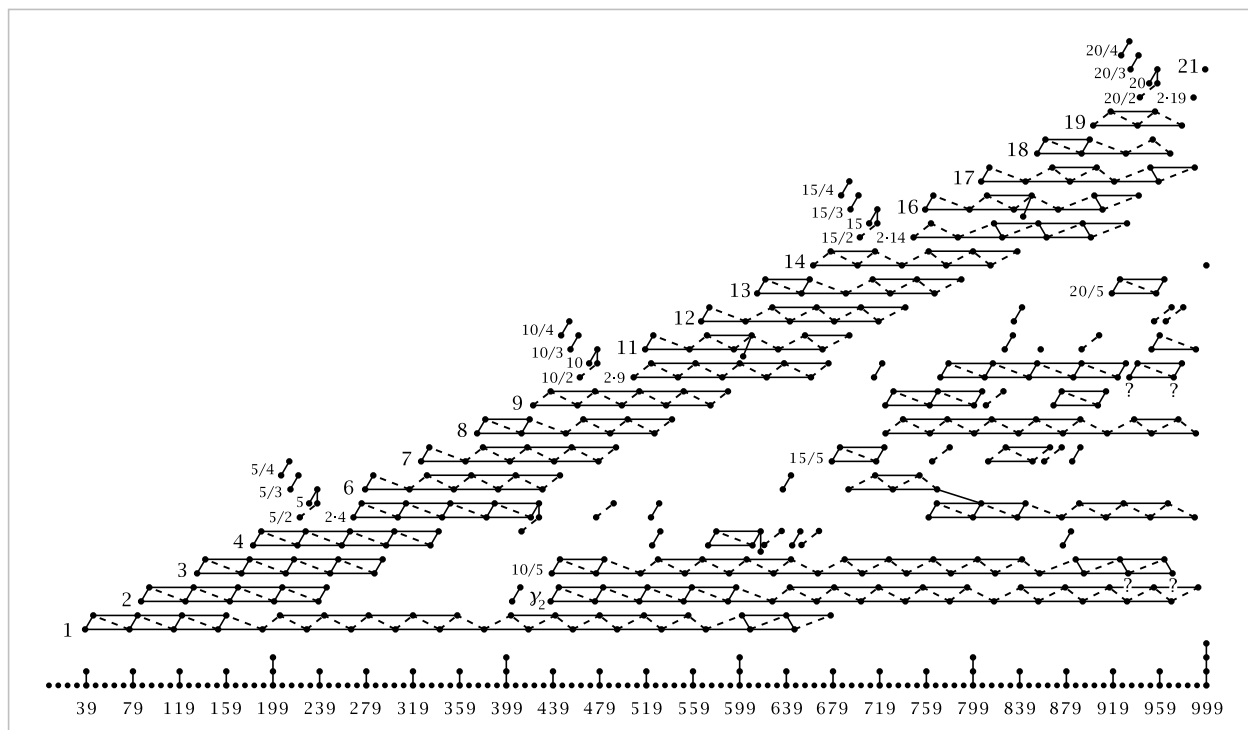
Across the bottom of the diagram there is a repeated pattern of pairs of ‘teeth’. This pattern continues to infinity, though with the spikes in dimensions  $8k - 1$  not all of the same height, namely, the spike in dimension  $2^m(2n + 1) - 1$  has height  $m + 1$ .

The next diagram shows the 3-components of  $\pi_i^S$  for  $i \leq 100$ . Here vertical segments denote multiplication by 3 and the other solid segments denote composition with elements  $\alpha_1 \in {}_3\pi_3^S$  and  $\beta_1 \in {}_3\pi_{10}^S$ . The meaning of the dashed lines will be

explained later. The most regular part of the diagram is the ‘triadic ruler’ across the bottom. This continues in the same pattern forever, with spikes of height  $m + 1$  in dimension  $4k - 1$  for  $3^m$  the highest power of 3 dividing  $4k$ . Looking back at the  $p = 2$  diagram, one can see that the vertical segments of the ‘teeth’ form a ‘dyadic ruler’.



The case  $p = 5$  is shown in the next diagram. Again one has the infinite ruler, this time a ‘pentadic’ ruler. The four dots with question marks below them near the right edge of the diagram are hypothetical since it is still undecided whether these potential elements of  ${}_5\pi_i^s$  for  $i = 932, 933, 970$ , and  $971$  actually exist.



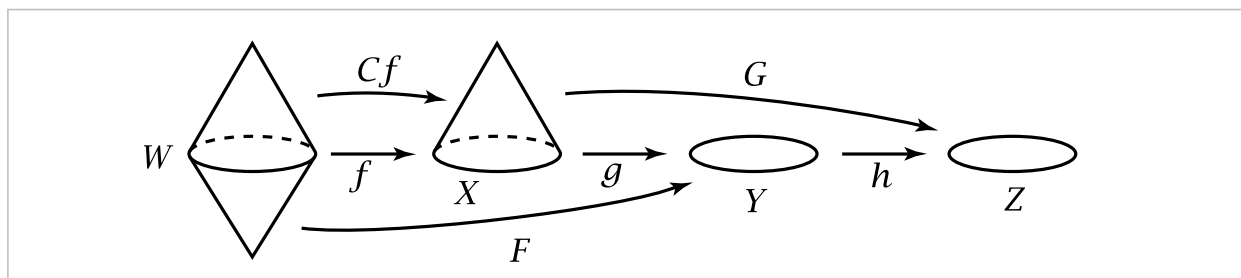
These three diagrams are based on calculations described in [Isaksen 2014] for  $p = 2$  and [Ravenel 1986] for  $p = 3, 5$ .

For each  $p$  there is a similar infinite ‘ $p$ -adic ruler’, corresponding to cyclic subgroups of order  $p^{m+1}$  in  ${}_p\pi_{2j(p-1)-1}^S$  for all  $j$ , where  $p^m$  is the highest power of  $p$  dividing  $j$ . These subgroups are the  $p$ -components of a certain cyclic subgroup of  $\pi_{4k-1}^S$  known as  $\text{Im } J$ , the image of a homomorphism  $J: \pi_{4k-1}(O) \rightarrow \pi_{4k-1}^S$ . There are also  $\mathbb{Z}_2$  subgroups of  $\pi_i^S$  for  $i = 8k, 8k + 1$  forming  $\text{Im } J$  in these dimensions. In the diagram of  ${}_2\pi_*^S$  these are the parts of the teeth connected to the spike in dimension  $8k - 1$ . The  $J$ -homomorphism will be studied in some detail in [VBKT].

The other known infinite families in  $\pi_*^S$  include classes  $\eta_n \in {}_2\pi_{2n}^S$  for  $n \geq 4$ ,  $\beta_n \in {}_p\pi_{2(p^2-1)n-2p}^S$  for  $p \geq 5$ , and  $\gamma_n \in {}_p\pi_{2(p^3-1)n-2p^2-2p+1}^S$  for  $p \geq 7$ . The element  $\beta_n$  appears in the diagram for  $p = 5$  as the dot in the upper part of the diagram labeled by the number  $n$ . These  $\beta_n$ ’s generate the strips along the upward diagonal, except when  $n$  is a multiple of 5 and the strip is generated by  $\beta_2\beta_{n-1}$  rather than  $\beta_n$ . There are also elements  $\beta_n$  for certain fractional values of  $n$ . The element  $\gamma_2$  generates the long strip starting in dimension 437, but  $\gamma_3 = 0$ . The element  $\gamma_4$  in dimension 933 is one of the question marks.

In  $\pi_*^S$  there are many compositions which are zero. One can get some idea of this from the diagrams above, where all sequences of edges break off after a short time. As a special instance of the vanishing of products, the commutativity formula in Proposition 4.56 implies that the square of an odd-dimensional element of odd order is zero. More generally, a theorem of Nishida says that every positive-dimensional element  $\alpha \in \pi_*^S$  is nilpotent, with  $\alpha^n = 0$  for some  $n$ . For example, for the element  $\beta_1 \in {}_5\pi_{38}^S$  the smallest such  $n$  is 18.

The widespread vanishing of products in  $\pi_*^S$  can be seen as limiting their usefulness in describing the structure of  $\pi_*^S$ . But it can also be used to construct new elements of  $\pi_*^S$ . Suppose one has maps  $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$  such that the compositions  $gf$  and  $hg$  are both homotopic to constant maps. A nullhomotopy of  $gf$  gives an extension of  $gf$  to a map  $F: CW \rightarrow Y$ , and a nullhomotopy of  $hg$  gives an extension of  $hg$  to a map  $G: CX \rightarrow Z$ . Regarding the suspension  $SW$  as the union of two cones  $CW$ , define the **Toda bracket**  $\langle f, g, h \rangle: SW \rightarrow Z$  to be the composition  $G(Cf)$  on one cone and  $hF$  on the other.



The map  $\langle f, g, h \rangle$  is not uniquely determined by  $f$ ,  $g$ , and  $h$  since it depends on the choices of the nullhomotopies. In the case of  $\pi_*^S$ , the various choices of  $\langle f, g, h \rangle$  range over a coset of a certain subgroup, described in an exercise at the end of the section.

There are also higher-order Toda brackets  $\langle f_1, \dots, f_n \rangle$  defined in somewhat similar fashion. The dashed lines in the diagrams of  ${}_3\pi_*^S$  and  ${}_5\pi_*^S$  join an element  $x$  to a bracket element  $\langle \alpha_1, \dots, \alpha_1, x \rangle$ . Many of the elements in all three diagrams can be expressed in terms of brackets. For example, in  ${}_2\pi_*^S$  the 8-dimensional element  $\bar{v}$  is  $\langle v, \eta, v \rangle$ . This element is also equal to  $\eta\sigma + \varepsilon$  where  $\varepsilon = \langle v^2, 2, \eta \rangle = \langle 2, \eta, v, \eta^2 \rangle$ . Some other bracket formulas in  ${}_2\pi_*^S$  are  $\eta_4 = \langle \sigma^2, 2, \eta \rangle$ ,  $v_4 = \langle 2\sigma, \sigma, v \rangle = -\langle \sigma, v, \sigma \rangle$ ,  $\bar{\sigma} = \langle v, \sigma, \eta\sigma \rangle$ ,  $\theta_4 = \langle \sigma, 2\sigma, \sigma, 2\sigma \rangle$ , and  $\eta_5 = \langle \eta, 2, \theta_4 \rangle$ .

**Proof of 4.56:** Only distributivity and commutativity need to be checked. One distributivity law is easy: Given  $f, g: S^{i+j+k} \rightarrow S^{j+k}$  and  $h: S^{j+k} \rightarrow S^k$ , then  $h(f + g) = hf + hg$  since both expressions equal  $hf$  and  $hg$  on the two hemispheres of  $S^{i+j+k}$ . The other distributivity law will follow from this one and the commutativity relation.

To prove the commutativity relation it will be convenient to express suspension in terms of smash product. The smash product  $S^n \wedge S^1$  can be regarded as the quotient space of  $S^n \times I$  with  $S^n \times \partial I \cup \{x_0\} \times I$  collapsed to a point. This is the same as the quotient of the suspension  $S^{n+1}$  of  $S^n$  obtained by collapsing to a point the suspension of  $x_0$ . Collapsing this arc in  $S^{n+1}$  to a point again yields  $S^{n+1}$ , so we obtain in this way a homeomorphism identifying  $S^n \wedge S^1$  with  $S^{n+1}$ . Under this identification the suspension  $Sf$  of a basepoint-preserving map  $f: S^n \rightarrow S^n$  becomes the smash product  $f \wedge \mathbb{1}: S^n \wedge S^1 \rightarrow S^n \wedge S^1$ . By iteration, the  $k$ -fold suspension  $S^k f$  then corresponds to  $f \wedge \mathbb{1}: S^n \wedge S^k \rightarrow S^n \wedge S^k$ .

Now we verify the commutativity relation. Let  $f: S^{i+k} \rightarrow S^k$  and  $g: S^{j+k} \rightarrow S^k$  be given. We may assume  $k$  is even. Consider the commutative diagram below, where  $\sigma$  and  $\tau$  transpose the two factors.

Thinking of  $S^{j+k}$  and  $S^k$  as smash products of circles,  $\sigma$  is the composition of  $k(j+k)$  transpositions of adjacent circle

$$\begin{array}{ccccc} S^{i+k} \wedge S^{j+k} & \xrightarrow{f \wedge \mathbb{1}} & S^k \wedge S^{j+k} & \xrightarrow{\mathbb{1} \wedge g} & S^k \wedge S^k \\ \downarrow \sigma & & \downarrow \sigma & & \downarrow \tau \\ S^{j+k} \wedge S^k & \xrightarrow{g \wedge \mathbb{1}} & S^k \wedge S^k & & S^k \wedge S^k \end{array}$$

factors. Such a transposition has degree  $-1$  since it is realized as a reflection of the  $S^2 = S^1 \wedge S^1$  involved. Hence  $\sigma$  has degree  $(-1)^{k(j+k)}$ , which is  $+1$  since  $k$  is even. Thus  $\sigma$  is homotopic to the identity. Similarly,  $\tau$  is homotopic to the identity. Hence  $f \wedge g = (\mathbb{1} \wedge g)(f \wedge \mathbb{1})$  is homotopic to the composition  $(g \wedge \mathbb{1})(f \wedge \mathbb{1})$ , which is stably equivalent to the composition  $gf$ . Symmetrically,  $fg$  is stably homotopic to  $g \wedge f$ .

So it suffices to show  $f \wedge g \simeq (-1)^{ij} g \wedge f$ . This we do by the commutative diagram at the right, where  $\sigma$  and  $\tau$  are again the transpositions of the two factors. As before,  $\tau$  is homotopic to the identity, but now  $\sigma$  has

$$\begin{array}{ccc} S^{i+k} \wedge S^{j+k} & \xrightarrow{f \wedge g} & S^k \wedge S^k \\ \downarrow \sigma & & \downarrow \tau \\ S^{j+k} \wedge S^{i+k} & \xrightarrow{g \wedge f} & S^k \wedge S^k \end{array}$$

degree  $(-1)^{(i+k)(j+k)}$ , which equals  $(-1)^{ij}$  since  $k$  is even. The composition  $(g \wedge f)\sigma$  is homotopic to  $(-1)^{ij}(g \wedge f)$  since additive inverses in homotopy groups are obtained by precomposing with a reflection, of degree  $-1$ . Thus from the commutativity of the diagram we obtain the relation  $f \wedge g \simeq (-1)^{ij} g \wedge f$ .  $\square$

## Exercises

1. Use homotopy groups to show there is no retraction  $\mathbb{R}P^n \rightarrow \mathbb{R}P^k$  if  $n > k > 0$ .
2. Show the action of  $\pi_1(\mathbb{R}P^n)$  on  $\pi_n(\mathbb{R}P^n) \approx \mathbb{Z}$  is trivial for  $n$  odd and nontrivial for  $n$  even.
3. Let  $X$  be obtained from a lens space of dimension  $2n + 1$  by deleting a point. Compute  $\pi_{2n}(X)$  as a module over  $\mathbb{Z}[\pi_1(X)]$ .
4. Let  $X \subset \mathbb{R}^{n+1}$  be the union of the infinite sequence of spheres  $S_k^n$  of radius  $1/k$  and center  $(1/k, 0, \dots, 0)$ . Show that  $\pi_i(X) = 0$  for  $i < n$  and construct a homomorphism from  $\pi_n(X)$  onto  $\prod_k \pi_n(S_k^n)$ .
5. Let  $f: S_\alpha^2 \vee S_\beta^2 \rightarrow S_\alpha^2 \vee S_\beta^2$  be the map which is the identity on the  $S_\alpha^2$  summand and which on the  $S_\beta^2$  summand is the sum of the identity map and a homeomorphism  $S_\beta^2 \rightarrow S_\alpha^2$ . Let  $X$  be the mapping torus of  $f$ , the quotient space of  $(S_\alpha^2 \vee S_\beta^2) \times I$  under the identifications  $(x, 0) \sim (f(x), 1)$ . The mapping torus of the restriction of  $f$  to  $S_\alpha^2$  forms a subspace  $A = S^1 \times S_\alpha^2 \subset X$ . Show that the maps  $\pi_2(A) \rightarrow \pi_2(X) \rightarrow \pi_2(X, A)$  form a short exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$ , and compute the action of  $\pi_1(A)$  on these three groups. In particular, show the action of  $\pi_1(A)$  is trivial on  $\pi_2(A)$  and  $\pi_2(X, A)$  but is nontrivial on  $\pi_2(X)$ .
6. Show that the relative form of the Hurewicz theorem in dimension  $n$  implies the absolute form in dimension  $n - 1$  by considering the pair  $(CX, X)$  where  $CX$  is the cone on  $X$ .
7. Construct a CW complex  $X$  with prescribed homotopy groups  $\pi_i(X)$  and prescribed actions of  $\pi_1(X)$  on the  $\pi_i(X)$ 's.
8. Show the suspension of an acyclic CW complex is contractible.
9. Show that a map between simply-connected CW complexes is a homotopy equivalence if its mapping cone is contractible. Use the preceding exercise to give an example where this fails in the nonsimply-connected case.
10. Let the CW complex  $X$  be obtained from  $S^1 \vee S^n$ ,  $n \geq 2$ , by attaching a cell  $e^{n+1}$  by a map representing the polynomial  $p(t) \in \mathbb{Z}[t, t^{-1}] \approx \pi_n(S^1 \vee S^n)$ , so  $\pi_n(X) \approx \mathbb{Z}[t, t^{-1}]/(p(t))$ . Show  $\pi'_n(X)$  is cyclic and compute its order in terms of  $p(t)$ . Give examples showing that the group  $\pi_n(X)$  can be finitely generated or not, independently of whether  $\pi'_n(X)$  is finite or infinite.
11. Let  $X$  be a connected CW complex with 1-skeleton  $X^1$ . Show that  $\pi_2(X, X^1) \approx \pi_2(X) \times K$  where  $K$  is the kernel of  $\pi_1(X^1) \rightarrow \pi_1(X)$ , a free group. Show also that the map  $\pi'_2(X) \rightarrow \pi'_2(X, X^1)$  need not be injective by considering the case  $X = \mathbb{R}P^2$  with its standard CW structure.
12. Show that a map  $f: X \rightarrow Y$  of connected CW complexes is a homotopy equivalence if it induces an isomorphism on  $\pi_1$  and if a lift  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  to the universal covers induces an isomorphism on homology. [The latter condition can be restated in terms of

homology with local coefficients as saying that  $f_* : H_*(X; \mathbb{Z}[\pi_1 X]) \rightarrow H_*(Y; \mathbb{Z}[\pi_1 Y])$  is an isomorphism; see §3.H.]

13. Show that a map between connected  $n$ -dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on  $\pi_i$  for  $i \leq n$ . [Pass to universal covers and use homology.]

14. If an  $n$ -dimensional CW complex  $X$  contains a subcomplex  $Y$  homotopy equivalent to  $S^n$ , show that the map  $\pi_n(Y) \rightarrow \pi_n(X)$  induced by inclusion is injective. [Use the Hurewicz homomorphism.]

15. Show that a closed simply-connected 3-manifold is homotopy equivalent to  $S^3$ . [Use Poincaré duality, and also the fact that closed manifolds are homotopy equivalent to CW complexes, from Corollary A.12 in the Appendix. The stronger statement that a closed simply-connected 3-manifold is homeomorphic to  $S^3$  is the *Poincaré conjecture*, finally proved by Perelman. The higher-dimensional analog, that a closed  $n$ -manifold homotopy equivalent to  $S^n$  is homeomorphic to  $S^n$ , had been proved earlier for all  $n \geq 4$ .]

16. Show that the closed surfaces with infinite fundamental group are  $K(\pi, 1)$ 's by showing that their universal covers are contractible, via the Hurewicz theorem and results of §3.3.

17. Show that the map  $\langle X, Y \rangle \rightarrow \text{Hom}(\pi_n(X), \pi_n(Y))$ ,  $[f] \mapsto f_*$ , is a bijection if  $X$  is an  $(n-1)$ -connected CW complex and  $Y$  is a path-connected space with  $\pi_i(Y) = 0$  for  $i > n$ . Deduce that CW complex  $K(G, n)$ 's are uniquely determined, up to homotopy type, by  $G$  and  $n$ .

18. If  $X$  and  $Y$  are simply-connected CW complexes such that  $\tilde{H}_i(X)$  and  $\tilde{H}_j(Y)$  are finite and of relatively prime orders for all pairs  $(i, j)$ , show that the inclusion  $X \vee Y \hookrightarrow X \times Y$  is a homotopy equivalence and  $X \wedge Y$  is contractible. [Use the Künneth formula.]

19. If  $X$  is a  $K(G, 1)$  CW complex, show that  $\pi_n(X^n)$  is free abelian for  $n \geq 2$ .

20. Let  $G$  be a group and  $X$  a simply-connected space. Show that for the product  $K(G, 1) \times X$  the action of  $\pi_1$  on  $\pi_n$  is trivial for all  $n > 1$ .

21. Given a sequence of CW complexes  $K(G_n, n)$ ,  $n = 1, 2, \dots$ , let  $X_n$  be the CW complex formed by the product of the first  $n$  of these  $K(G_n, n)$ 's. Via the inclusions  $X_{n-1} \subset X_n$  coming from regarding  $X_{n-1}$  as the subcomplex of  $X_n$  with  $n^{\text{th}}$  coordinate equal to a basepoint 0-cell of  $K(G_n, n)$ , we can then form the union of all the  $X_n$ 's, a CW complex  $X$ . Show  $\pi_n(X) \approx G_n$  for all  $n$ .

22. Show that  $H_{n+1}(K(G, n); \mathbb{Z}) = 0$  if  $n > 1$ . [Build a  $K(G, n)$  from a Moore space  $M(G, n)$  by attaching cells of dimension  $> n + 1$ .]

23. Extend the Hurewicz theorem by showing that if  $X$  is an  $(n-1)$ -connected CW complex, then the Hurewicz homomorphism  $h : \pi_{n+1}(X) \rightarrow H_{n+1}(X)$  is surjective

when  $n > 1$ , and when  $n = 1$  show there is an isomorphism  $H_2(X)/h(\pi_2(X)) \approx H_2(K(\pi_1(X), 1))$ . [Build a  $K(\pi_n(X), n)$  from  $X$  by attaching cells of dimension  $n + 2$  and greater, and then consider the homology sequence of the pair  $(Y, X)$  where  $Y$  is  $X$  with the  $(n + 2)$ -cells of  $K(\pi_n(X), n)$  attached. Note that the image of the boundary map  $H_{n+2}(Y, X) \rightarrow H_{n+1}(X)$  coincides with the image of  $h$ , and  $H_{n+1}(Y) \approx H_{n+1}(K(\pi_n(X), n))$ . The previous exercise is needed for the case  $n > 1$ .]

24. Show there is a Moore space  $M(G, 1)$  with  $\pi_1(M(G, 1)) \approx G$  iff  $H_2(K(G, 1); \mathbb{Z}) = 0$ . [Use the preceding problem. Build such an  $M(G, 1)$  from the 2-skeleton  $K^2$  of a  $K(G, 1)$  by attaching 3-cells according to a basis for the free group  $H_2(K^2; \mathbb{Z})$ .] In particular, there is no  $M(\mathbb{Z}^n, 1)$  with fundamental group  $\mathbb{Z}^n$ , free abelian of rank  $n$ , if  $n \geq 2$ .

25. For  $X$  a connected CW complex with  $\pi_i(X) = 0$  for  $1 < i < n$  for some  $n \geq 2$ , show that  $H_n(X)/h(\pi_n(X)) \approx H_n(K(\pi_1(X), 1))$ , where  $h$  is the Hurewicz map.

26. Generalizing the example of  $\mathbb{RP}^2$  and  $S^2 \times \mathbb{RP}^\infty$ , show that if  $X$  is a connected finite-dimensional CW complex with universal cover  $\tilde{X}$ , then  $X$  and  $\tilde{X} \times K(\pi_1(X), 1)$  have isomorphic homotopy groups but are not homotopy equivalent if  $\pi_1(X)$  contains elements of finite order.

27. Show that the image of the map  $\pi_2(X, x_0) \rightarrow \pi_2(X, A, x_0)$  lies in the center of  $\pi_2(X, A, x_0)$ . (This exercise should be in §4.1.)

28. Show that the group  $\mathbb{Z}_p \times \mathbb{Z}_p$  with  $p$  prime cannot act freely on any sphere  $S^n$ , by filling in details of the following argument. Such an action would define a covering space  $S^n \rightarrow M$  with  $M$  a closed manifold. When  $n > 1$ , build a  $K(\mathbb{Z}_p \times \mathbb{Z}_p, 1)$  from  $M$  by attaching a single  $(n + 1)$ -cell and then cells of higher dimension. Deduce that  $H^{n+1}(K(\mathbb{Z}_p \times \mathbb{Z}_p, 1); \mathbb{Z}_p)$  is  $\mathbb{Z}_p$  or 0, a contradiction. (The case  $n = 1$  is more elementary.)

29. Finish the homotopy classification of lens spaces begun in Exercise 2 of §3.E by showing that two lens spaces  $L_m(\ell_1, \dots, \ell_n)$  and  $L_m(\ell'_1, \dots, \ell'_n)$  are homotopy equivalent if  $\ell_1 \cdots \ell_n \equiv \pm k^n \ell'_1 \cdots \ell'_n \pmod{m}$  for some integer  $k$ , via the following steps:

- Reduce to the case  $k = 1$  by showing that  $L_m(\ell'_1, \dots, \ell'_n) = L_m(k\ell'_1, \dots, k\ell'_n)$  if  $k$  is relatively prime to  $m$ . [Rechoose the generator of the  $\mathbb{Z}_m$  action on  $S^{2n-1}$ .]
- Let  $f: L \rightarrow L'$  be a map constructed as in part (b) of the exercise in §3.E. Construct a map  $g: L \rightarrow L'$  as a composition  $L \rightarrow L \vee S^{2n-1} \rightarrow L \vee S^{2n-1} \rightarrow L'$  where the first map collapses the boundary of a small ball to a point, the second map is the wedge of the identity on  $L$  and a map of some degree  $d$  on  $S^{2n-1}$ , and the third map is  $f$  on  $L$  and the projection  $S^{2n-1} \rightarrow L'$  on  $S^{2n-1}$ . Show that  $g$  has degree  $k_1 \cdots k_n + dm$ , that is,  $g$  induces multiplication by  $k_1 \cdots k_n + dm$  on  $H_{2n-1}(-; \mathbb{Z})$ . [Show first that a lift of  $g$  to the universal cover  $S^{2n-1}$  has this degree.]

- (c) If  $\ell_1 \cdots \ell_n \equiv \pm \ell'_1 \cdots \ell'_n \pmod{m}$ , choose  $d$  so that  $k_1 \cdots k_n + dm = \pm 1$  and show this implies that  $g$  induces an isomorphism on all homotopy groups, hence is a homotopy equivalence. [For  $\pi_i$  with  $i > 1$ , consider a lift of  $g$  to the universal cover.]

30. Let  $E$  be a subspace of  $\mathbb{R}^2$  obtained by deleting a subspace of  $\{0\} \times \mathbb{R}$ . For which such spaces  $E$  is the projection  $E \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto x$ , a fiber bundle?

31. For a fiber bundle  $F \rightarrow E \rightarrow B$  such that the inclusion  $F \hookrightarrow E$  is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks up into split short exact sequences giving isomorphisms  $\pi_n(B) \approx \pi_n(E) \oplus \pi_{n-1}(F)$ . In particular, for the Hopf bundles  $S^3 \rightarrow S^7 \rightarrow S^4$  and  $S^7 \rightarrow S^{15} \rightarrow S^8$  this yields isomorphisms

$$\begin{aligned}\pi_n(S^4) &\approx \pi_n(S^7) \oplus \pi_{n-1}(S^3) \\ \pi_n(S^8) &\approx \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)\end{aligned}$$

Thus  $\pi_7(S^4)$  and  $\pi_{15}(S^8)$  contain  $\mathbb{Z}$  summands.

32. Show that if  $S^k \rightarrow S^m \rightarrow S^n$  is a fiber bundle, then  $k = n - 1$  and  $m = 2n - 1$ . [Look at the long exact sequence of homotopy groups.]

33. Show that if there were fiber bundles  $S^{n-1} \rightarrow S^{2n-1} \rightarrow S^n$  for all  $n$ , then the groups  $\pi_i(S^n)$  would be finitely generated free abelian groups computable by induction, and nonzero for  $i \geq n \geq 2$ .

34. Let  $p: S^3 \rightarrow S^2$  be the Hopf bundle and let  $q: T^3 \rightarrow S^3$  be the quotient map collapsing the complement of a ball in the 3-dimensional torus  $T^3 = S^1 \times S^1 \times S^1$  to a point. Show that  $pq: T^3 \rightarrow S^2$  induces the trivial map on  $\pi_*$  and  $\tilde{H}_*$ , but is not homotopic to a constant map.

35. Show that the fiber bundle  $S^3 \rightarrow S^{4n+3} \rightarrow \mathbb{H}P^n$  gives rise to a quotient fiber bundle  $S^2 \rightarrow \mathbb{C}P^{2n+1} \rightarrow \mathbb{H}P^n$  by factoring out the action of  $S^1$  on  $S^{4n+3}$  by complex scalar multiplication.

36. For basepoint-preserving maps  $f: S^1 \rightarrow X$  and  $g: S^n \rightarrow X$  with  $n > 1$ , show that the Whitehead product  $[f, g]$  is  $\pm(g - fg)$ , where  $fg$  denotes the action of  $f$  on  $g$ .

37. Show that all Whitehead products in a path-connected H-space are trivial.

38. Show  $\pi_3(S^1 \vee S^2)$  is not finitely generated as a module over  $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$  by considering Whitehead products in the universal cover, using the results in Example 4.52. Generalize this to  $\pi_{i+j-1}(S^1 \vee S^i \vee S^j)$  for  $i, j > 1$ .

39. Show that the indeterminacy of a Toda bracket  $\langle f, g, h \rangle$  with  $f \in \pi_i^S$ ,  $g \in \pi_j^S$ ,  $h \in \pi_k^S$  is the subgroup  $f \cdot \pi_{j+k+1}^S + h \cdot \pi_{i+j+1}^S$  of  $\pi_{i+j+k+1}^S$ .



## 4.3 Connections with Cohomology

The Hurewicz theorem provides a strong link between homotopy groups and homology, and hence also an indirect relation with cohomology. But there is a more direct connection with cohomology of a quite different sort. We will show that for every CW complex  $X$  there is a natural bijection between  $H^n(X; G)$  and the set  $\langle X, K(G, n) \rangle$  of basepoint-preserving homotopy classes of maps from  $X$  to a  $K(G, n)$ . We will also define a natural group structure on  $\langle X, K(G, n) \rangle$  that makes the bijection a group isomorphism. The mere fact that there is any connection at all between cohomology and homotopy classes of maps is the first surprise here, and the second is that Eilenberg–MacLane spaces are involved, since their definition is entirely in terms of homotopy groups, which on the face of it have nothing to do with cohomology.

After proving this basic isomorphism  $H^n(X; G) \approx \langle X, K(G, n) \rangle$  and describing a few of its immediate applications, the later parts of this section aim toward a further study of Postnikov towers, which were introduced briefly in §4.1. These provide a general theoretical method for realizing an arbitrary CW complex as a sort of twisted product of Eilenberg–MacLane spaces, up to homotopy equivalence. The most geometric interpretation of the phrase ‘twisted product’ is the notion of fiber bundle introduced in the previous section, but here we need the more homotopy-theoretic notion of a *fibration*, so before we begin the discussion of Postnikov towers we first take a few pages to present some basic constructions and results about fibrations.

As we shall see, Postnikov towers can be expressed as sequences of fibrations with fibers Eilenberg–MacLane spaces, so we can again expect close connections with cohomology. One such connection is provided by *k-invariants*, which describe, at least in principle, how Postnikov towers for a broad class of spaces are determined by a sequence of cohomology classes. Another application of these ideas, described at the end of the section, is a technique for factoring basic extension and lifting problems in homotopy theory into a sequence of smaller problems whose solutions are equivalent to the vanishing of certain cohomology classes. This technique goes under the somewhat grandiose title of Obstruction Theory, though it is really quite a simple idea when expressed in terms of Postnikov towers.

### The Homotopy Construction of Cohomology

The main result of this subsection is the following fundamental relationship between singular cohomology and Eilenberg–MacLane spaces:

**Theorem 4.57.** *There are natural bijections  $T: \langle X, K(G, n) \rangle \rightarrow H^n(X; G)$  for all CW complexes  $X$  and all  $n > 0$ , with  $G$  any abelian group. Such a  $T$  has the form  $T([f]) = f^*(\alpha)$  for a certain distinguished class  $\alpha \in H^n(K(G, n); G)$ .*

In the course of the proof we will define a natural group structure on  $\langle X, K(G, n) \rangle$  such that the transformation  $T$  is an isomorphism.

A class  $\alpha \in H^n(K(G, n); G)$  with the property stated in the theorem is called a **fundamental class**. The proof of the theorem will yield an explicit fundamental class, namely the element of  $H^n(K(G, n); G) \approx \text{Hom}(H_n(K; \mathbb{Z}), G)$  given by the inverse of the Hurewicz isomorphism  $G = \pi_n(K(G, n)) \rightarrow H_n(K; \mathbb{Z})$ . Concretely, if we choose  $K(G, n)$  to be a CW complex with  $(n-1)$ -skeleton a point, then a fundamental class is represented by the cellular cochain assigning to each  $n$ -cell of  $K(G, n)$  the element of  $\pi_n(K(G, n))$  defined by a characteristic map for the  $n$ -cell.

The theorem also holds with  $\langle X, K(G, n) \rangle$  replaced by  $[X, K(G, n)]$ , the non-basepointed homotopy classes. This is easy to see when  $n > 1$  since every map  $X \rightarrow K(G, n)$  can be homotoped to take basepoint to basepoint, and every homotopy between basepoint-preserving maps can be homotoped to be basepoint-preserving since the target space  $K(G, n)$  is simply-connected. When  $n = 1$  it is still true that  $[X, K(G, n)] = \langle X, K(G, n) \rangle$  for abelian  $G$  according to an exercise for §4.A. For  $n = 0$  it is elementary that  $H^0(X; G) = [X, K(G, 0)]$  and  $\tilde{H}^0(X; G) = \langle X, K(G, 0) \rangle$ .

It is possible to give a direct proof of the theorem, constructing maps and homotopies cell by cell. This provides geometric insight into why the result is true, but unfortunately the technical details of this proof are rather tedious. So we shall take a different approach, one that has the advantage of placing the result in its natural context via general machinery that turns out to be quite useful in other situations as well. The two main steps will be the following assertions.

- (1) The functors  $h^n(X) = \langle X, K(G, n) \rangle$  define a reduced cohomology theory on the category of basepointed CW complexes.
- (2) If a reduced cohomology theory  $h^*$  defined on CW complexes has coefficient groups  $h^n(S^0)$  which are zero for  $n \neq 0$ , then there are natural isomorphisms  $h^n(X) \approx \tilde{H}^n(X; h^0(S^0))$  for all CW complexes  $X$  and all  $n$ .

Towards proving (1) we will study a more general question: When does a sequence of spaces  $K_n$  define a cohomology theory by setting  $h^n(X) = \langle X, K_n \rangle$ ? Note that this will be a reduced cohomology theory since  $\langle X, K_n \rangle$  is trivial when  $X$  is a point.

The first question to address is putting a group structure on the set  $\langle X, K \rangle$ . This requires that either  $X$  or  $K$  have some special structure. When  $X = S^n$  we have  $\langle S^n, K \rangle = \pi_n(K)$ , which has a group structure when  $n > 0$ . The definition of this group structure works more generally whenever  $S^n$  is replaced by a suspension  $SX$ , with the sum of maps  $f, g: SX \rightarrow K$  defined as the composition  $SX \rightarrow SX \vee SX \rightarrow K$  where the first map collapses an ‘equatorial’  $X \subset SX$  to a point and the second map consists of  $f$  and  $g$  on the two summands. However, for this to make sense we must be talking about basepoint-preserving maps, and there is a problem with where to choose the basepoint in  $SX$ . If  $x_0$  is a basepoint of  $X$ , the basepoint of  $SX$  should be somewhere along the segment  $\{x_0\} \times I \subset SX$ , most likely either an endpoint or the

midpoint, but no single choice of such a basepoint gives a well-defined sum. The sum would be well-defined if we restricted attention to maps sending the whole segment  $\{x_0\} \times I$  to the basepoint. This is equivalent to considering basepoint-preserving maps  $\Sigma X \rightarrow K$  where  $\Sigma X = SX/(\{x_0\} \times I)$  and the image of  $\{x_0\} \times I$  in  $\Sigma X$  is taken to be the basepoint. If  $X$  is a CW complex with  $x_0$  a 0-cell, the quotient map  $SX \rightarrow \Sigma X$  is a homotopy equivalence since it collapses a contractible subcomplex of  $SX$  to a point, so we can identify  $\langle SX, K \rangle$  with  $\langle \Sigma X, K \rangle$ . The space  $\Sigma X$  is called the **reduced suspension** of  $X$  when we want to distinguish it from the ordinary suspension  $SX$ .

It is easy to check that  $\langle \Sigma X, K \rangle$  is a group with respect to the sum defined above, inverses being obtained by reflecting the  $I$  coordinate in the suspension. However, what we would really like to have is a group structure on  $\langle X, K \rangle$  arising from a special structure on  $K$  rather than on  $X$ . This can be obtained using the following basic **adjoint relation**:

- $\langle \Sigma X, K \rangle = \langle X, \Omega K \rangle$  where  $\Omega K$  is the space of loops in  $K$  at its chosen basepoint and the constant loop is taken as the basepoint of  $\Omega K$ .

The space  $\Omega K$ , called the **loopspace** of  $K$ , is topologized as a subspace of the space  $K^I$  of all maps  $I \rightarrow K$ , where  $K^I$  is given the compact-open topology; see the Appendix for the definition and basic properties of this topology. The adjoint relation  $\langle \Sigma X, K \rangle = \langle X, \Omega K \rangle$  holds because basepoint-preserving maps  $\Sigma X \rightarrow K$  are exactly the same as basepoint-preserving maps  $X \rightarrow \Omega K$ , the correspondence being given by associating to  $f: \Sigma X \rightarrow K$  the family of loops obtained by restricting  $f$  to the images of the segments  $\{x\} \times I$  in  $\Sigma X$ .

Taking  $X = S^n$  in the adjoint relation, we see that  $\pi_{n+1}(K) = \pi_n(\Omega K)$  for all  $n \geq 0$ . Thus passing from a space to its loop space has the effect of shifting homotopy groups down a dimension. In particular we see that  $\Omega K(G, n)$  is a  $K(G, n-1)$ . This fact will turn out to be important in what follows.

Note that the association  $X \mapsto \Omega X$  is a functor: A basepoint-preserving map  $f: X \rightarrow Y$  induces a map  $\Omega f: \Omega X \rightarrow \Omega Y$  by composition with  $f$ . A homotopy  $f \simeq g$  induces a homotopy  $\Omega f \simeq \Omega g$ , so it follows formally that  $X \simeq Y$  implies  $\Omega X \simeq \Omega Y$ .

It is a theorem of [Milnor 1959] that the loop space of a CW complex has the homotopy type of a CW complex. This may be a bit surprising since loop spaces are usually quite large spaces, though of course CW complexes can be quite large too, in terms of the number of cells. What often happens in practice is that if a CW complex  $X$  has only finitely many cells in each dimension, then  $\Omega X$  is homotopy equivalent to a CW complex with the same property. We will see explicitly how this happens for  $X = S^n$  in §4.J.

Composition of loops defines a map  $\Omega K \times \Omega K \rightarrow \Omega K$ , and this gives a sum operation in  $\langle X, \Omega K \rangle$  by setting  $(f + g)(x) = f(x) \cdot g(x)$ , the composition of the loops  $f(x)$  and  $g(x)$ . Under the adjoint relation this is the same as the sum in  $\langle \Sigma X, K \rangle$  defined previously. If we take the composition of loops as the sum operation then it

is perhaps somewhat easier to see that  $\langle X, \Omega K \rangle$  is a group since the same reasoning which shows that  $\pi_1(K)$  is a group can be applied.

Since cohomology groups are abelian, we would like the group  $\langle X, \Omega K \rangle$  to be abelian. This can be achieved by iterating the operation of forming loopspaces. One has a double loop space  $\Omega^2 K = \Omega(\Omega K)$  and inductively an  $n$ -fold loop space  $\Omega^n K = \Omega(\Omega^{n-1} K)$ . The evident bijection  $K^{Y \times Z} \approx (K^Y)^Z$  is a homeomorphism for locally compact Hausdorff spaces  $Y$  and  $Z$ , as shown in Proposition A.16 in the Appendix, and from this it follows by induction that  $\Omega^n K$  can be regarded as the space of maps  $I^n \rightarrow K$  sending  $\partial I^n$  to the basepoint. Taking  $n = 2$ , we see that the argument that  $\pi_2(K)$  is abelian shows more generally that  $\langle X, \Omega^2 K \rangle$  is an abelian group. Iterating the adjoint relation gives  $\langle \Sigma^n X, K \rangle = \langle X, \Omega^n K \rangle$ , so this is an abelian group for all  $n \geq 2$ .

Thus for a sequence of spaces  $K_n$  to define a cohomology theory  $h^n(X) = \langle X, K_n \rangle$  we have been led to the assumption that each  $K_n$  should be a loop space and in fact a double loop space. Actually we do not need  $K_n$  to be literally a loop space since it would suffice for it to be homotopy equivalent to a loop space, as  $\langle X, K_n \rangle$  depends only on the homotopy type of  $K_n$ . In fact it would suffice to have just a weak homotopy equivalence  $K_n \rightarrow \Omega L_n$  for some space  $L_n$  since this would induce a bijection  $\langle X, K_n \rangle = \langle X, \Omega L_n \rangle$  by Proposition 4.22. In the special case that  $K_n = K(G, n)$  for all  $n$ , we can take  $L_n = K_{n+1} = K(G, n+1)$  by the earlier observation that  $\Omega K(G, n+1)$  is a  $K(G, n)$ . Thus if we take the  $K(G, n)$ 's to be CW complexes, the map  $K_n \rightarrow \Omega K_{n+1}$  is just a CW approximation  $K(G, n) \rightarrow \Omega K(G, n+1)$ .

There is another reason to look for weak homotopy equivalences  $K_n \rightarrow \Omega K_{n+1}$ . For a reduced cohomology theory  $h^n(X)$  there are natural isomorphisms  $h^n(X) \approx h^{n+1}(\Sigma X)$  coming from the long exact sequence of the pair  $(CX, X)$  with  $CX$  the cone on  $X$ , so if  $h^n(X) = \langle X, K_n \rangle$  for all  $n$  then the isomorphism  $h^n(X) \approx h^{n+1}(\Sigma X)$  translates into a bijection  $\langle X, K_n \rangle \approx \langle \Sigma X, K_{n+1} \rangle = \langle X, \Omega K_{n+1} \rangle$  and the most natural thing would be for this to come from a weak equivalence  $K_n \rightarrow \Omega K_{n+1}$ . Weak equivalences of this form would give also weak equivalences  $K_n \rightarrow \Omega K_{n+1} \rightarrow \Omega^2 K_{n+2}$  and so we would automatically obtain an abelian group structure on  $\langle X, K_n \rangle \approx \langle X, \Omega^2 K_{n+2} \rangle$ .

These observations lead to the following definition. An  **$\Omega$ -spectrum** is a sequence of CW complexes  $K_1, K_2, \dots$  together with weak homotopy equivalences  $K_n \rightarrow \Omega K_{n+1}$  for all  $n$ . By using the theorem of Milnor mentioned above it would be possible to replace 'weak homotopy equivalence' by 'homotopy equivalence' in this definition. However it does not noticeably simplify matters to do this, except perhaps psychologically.

Notice that if we discard a finite number of spaces  $K_n$  from the beginning of an  $\Omega$ -spectrum  $K_1, K_2, \dots$ , then these omitted terms can be reconstructed from the remaining  $K_n$ 's since each  $K_n$  determines  $K_{n-1}$  as a CW approximation to  $\Omega K_n$ . So it is not important that the sequence start with  $K_1$ . By the same token, this allows us

to extend the sequence of  $K_n$ 's to all negative values of  $n$ . This is significant because a general cohomology theory  $h^n(X)$  need not vanish for negative  $n$ .

**Theorem 4.58.** *If  $\{K_n\}$  is an  $\Omega$ -spectrum, then the functors  $X \mapsto h^n(X) = \langle X, K_n \rangle$ ,  $n \in \mathbb{Z}$ , define a reduced cohomology theory on the category of basepointed CW complexes and basepoint-preserving maps.*

Rather amazingly, the converse is also true: Every reduced cohomology theory on CW complexes arises from an  $\Omega$ -spectrum in this way. This is the Brown representability theorem which will be proved in §4.E.

A space  $K_n$  in an  $\Omega$ -spectrum is sometimes called an *infinite loopspace* since there are weak homotopy equivalences  $K_n \rightarrow \Omega^k K_{n+k}$  for all  $k$ . A number of important spaces in algebraic topology turn out to be infinite loopspaces. Besides Eilenberg–MacLane spaces, two other examples are the infinite-dimensional orthogonal and unitary groups  $O$  and  $U$ , for which there are weak homotopy equivalences  $O \rightarrow \Omega^8 O$  and  $U \rightarrow \Omega^2 U$  by a strong form of the Bott periodicity theorem, as we will show in [VBKT]. So  $O$  and  $U$  give periodic  $\Omega$ -spectra, hence periodic cohomology theories known as real and complex K-theory. For a more in-depth introduction to the theory of infinite loopspaces, the book [Adams 1978] can be much recommended.

**Proof:** Two of the three axioms for a cohomology theory, the homotopy axiom and the wedge sum axiom, are quite easy to check. For the homotopy axiom, a basepoint-preserving map  $f: X \rightarrow Y$  induces  $f^*: \langle Y, K_n \rangle \rightarrow \langle X, K_n \rangle$  by composition, sending a map  $Y \rightarrow K_n$  to  $X \xrightarrow{f} Y \rightarrow K_n$ . Clearly  $f^*$  depends only on the basepoint-preserving homotopy class of  $f$ , and it is obvious that  $f^*$  is a homomorphism if we replace  $K_n$  by  $\Omega K_{n+1}$  and use the composition of loops to define the group structure. The wedge sum axiom holds since in the realm of basepoint-preserving maps, a map  $\bigvee_{\alpha} X_{\alpha} \rightarrow K_n$  is the same as a collection of maps  $X_{\alpha} \rightarrow K_n$ .

The bulk of the proof involves associating a long exact sequence to each CW pair  $(X, A)$ . As a first step we build the following diagram:

$$(1) \quad \begin{array}{ccccccc} A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA) \\ \parallel \quad \parallel \quad \simeq \downarrow \quad \quad \simeq \downarrow \uparrow \quad \quad \quad \subset \quad \quad \simeq \downarrow \uparrow \\ A \hookrightarrow X \longrightarrow X/A \longrightarrow SA \longrightarrow SX \end{array}$$

The first row is obtained from the inclusion  $A \hookrightarrow X$  by iterating the rule, ‘attach a cone on the preceding subspace’, as shown in the pictures below.



The three downward arrows in the diagram (1) are quotient maps collapsing the most recently attached cone to a point. Since cones are contractible, these downward maps

are homotopy equivalences. The second and third of them have homotopy inverses the evident inclusion maps, indicated by the upward arrows. In the lower row of the diagram the maps are the obvious ones, except for the map  $X/A \rightarrow SA$  which is the composition of a homotopy inverse of the quotient map  $X \cup CA \rightarrow X/A$  followed by the maps  $X \cup CA \rightarrow (X \cup CA) \cup CX \rightarrow SA$ . Thus the square containing this map commutes up to homotopy. It is easy to check that the same is true of the right-hand square as well.

The whole construction can now be repeated with  $SA \hookrightarrow SX$  in place of  $A \hookrightarrow X$ , then with double suspensions, and so on. The resulting infinite sequence can be written in either of the following two forms:

$$A \rightarrow X \rightarrow X \cup CA \rightarrow SA \rightarrow SX \rightarrow S(X \cup CA) \rightarrow S^2 A \rightarrow S^2 X \rightarrow \dots$$

$$A \rightarrow X \rightarrow X/A \rightarrow SA \rightarrow SX \rightarrow SX/SA \rightarrow S^2 A \rightarrow S^2 X \rightarrow \dots$$

In the first version we use the obvious equality  $SX \cup CSA = S(X \cup CA)$ . The first version has the advantage that the map  $X \cup CA \rightarrow SA$  is easily described and canonical, whereas in the second version the corresponding map  $X/A \rightarrow SA$  is only defined up to homotopy since it depends on choosing a homotopy inverse to the quotient map  $X \cup CA \rightarrow X/A$ . The second version does have the advantage of conciseness, however.

When basepoints are important it is generally more convenient to use reduced cones and reduced suspensions, obtained from ordinary cones and suspensions by collapsing the segment  $\{x_0\} \times I$  where  $x_0$  is the basepoint. The image point of this segment in the reduced cone or suspension then serves as a natural basepoint in the quotient. Assuming  $x_0$  is a 0-cell, these collapses of  $\{x_0\} \times I$  are homotopy equivalences. Using reduced cones and suspensions in the preceding construction yields a sequence

$$(2) \quad A \hookrightarrow X \rightarrow X/A \rightarrow \Sigma A \hookrightarrow \Sigma X \rightarrow \Sigma(X/A) \rightarrow \Sigma^2 A \hookrightarrow \Sigma^2 X \rightarrow \dots$$

where we identify  $\Sigma X/\Sigma A$  with  $\Sigma(X/A)$ , and all the later maps in the sequence are suspensions of the first three maps. This sequence, or its unreduced version, is called the **cofibration sequence** or **Puppe sequence** of the pair  $(X, A)$ . It has an evident naturality property, namely, a map  $(X, A) \rightarrow (Y, B)$  induces a map between the cofibration sequences of these two pairs, with homotopy-commutative squares:

$$\begin{array}{ccccccccccc} A & \longrightarrow & X & \longrightarrow & X/A & \longrightarrow & \Sigma A & \longrightarrow & \Sigma X & \longrightarrow & \Sigma(X/A) & \longrightarrow & \Sigma^2 A & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ B & \longrightarrow & Y & \longrightarrow & Y/B & \longrightarrow & \Sigma B & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma(Y/B) & \longrightarrow & \Sigma^2 B & \longrightarrow & \dots \end{array}$$

Taking basepoint-preserving homotopy classes of maps from the spaces in (2) to a fixed space  $K$  gives a sequence

$$(3) \quad \langle A, K \rangle \leftarrow \langle X, K \rangle \leftarrow \langle X/A, K \rangle \leftarrow \langle \Sigma A, K \rangle \leftarrow \langle \Sigma X, K \rangle \leftarrow \dots$$

whose maps are defined by composition with those in (2). For example, the map  $\langle X, K \rangle \rightarrow \langle A, K \rangle$  sends a map  $X \rightarrow K$  to  $A \rightarrow X \rightarrow K$ . The sets in (3) are groups starting

with  $\langle \Sigma A, K \rangle$ , and abelian groups from  $\langle \Sigma^2 A, K \rangle$  onward. It is easy to see that the maps between these groups are homomorphisms since the maps in (2) are suspensions from  $\Sigma A \rightarrow \Sigma X$  onward. In general the first three terms of (3) are only sets with distinguished ‘zero’ elements, the constant maps.

A key observation is that the sequence (3) is exact. To see this, note first that the diagram (1) shows that, up to homotopy equivalence, each term in (2) is obtained from its two predecessors by the same procedure of forming a mapping cone, so it suffices to show that  $\langle A, K \rangle \leftarrow \langle X, K \rangle \leftarrow \langle X \cup CA, K \rangle$  is exact. This is easy: A map  $f: X \rightarrow K$  goes to zero in  $\langle A, K \rangle$  iff its restriction to  $A$  is nullhomotopic, fixing the basepoint, and this is equivalent to  $f$  extending to a map  $X \cup CA \rightarrow K$ .

If we have a weak homotopy equivalence  $K \rightarrow \Omega K'$  for some space  $K'$ , then the sequence (3) can be continued three steps to the left via the commutative diagram

$$\begin{array}{ccccccc}
 \langle A, K \rangle & \leftarrow & \langle X, K \rangle & \leftarrow & \langle X/A, K \rangle & \leftarrow & \cdots \\
 \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\
 \langle A, \Omega K' \rangle & \leftarrow & \langle X, \Omega K' \rangle & \leftarrow & \langle X/A, \Omega K' \rangle & \leftarrow & \cdots \\
 \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\
 \langle A, K' \rangle & \leftarrow & \langle X, K' \rangle & \leftarrow & \langle X/A, K' \rangle & \leftarrow & \langle \Sigma A, K' \rangle \leftarrow \langle \Sigma X, K' \rangle \leftarrow \langle \Sigma(X/A), K' \rangle \leftarrow \cdots
 \end{array}$$

Thus if we have a sequence of spaces  $K_n$  together with weak homotopy equivalences  $K_n \rightarrow \Omega K_{n+1}$ , we can extend the sequence (3) to the left indefinitely, producing a long exact sequence

$$\cdots \leftarrow \langle A, K_n \rangle \leftarrow \langle X, K_n \rangle \leftarrow \langle X/A, K_n \rangle \leftarrow \langle A, K_{n-1} \rangle \leftarrow \langle X, K_{n-1} \rangle \leftarrow \cdots$$

All the terms here are abelian groups and the maps homomorphisms. This long exact sequence is natural with respect to maps  $(X, A) \rightarrow (Y, B)$  since cofibration sequences are natural.  $\square$

There is no essential difference between cohomology theories on basepointed CW complexes and cohomology theories on nonbasepointed CW complexes. Given a reduced basepointed cohomology theory  $\tilde{h}^*$ , one gets an unreduced theory by setting  $h^n(X, A) = \tilde{h}^n(X/A)$ , where  $X/\emptyset = X_+$ , the union of  $X$  with a disjoint basepoint. This is a nonbasepointed theory since an arbitrary map  $X \rightarrow Y$  induces a basepoint-preserving map  $X_+ \rightarrow Y_+$ . Furthermore, a nonbasepointed unreduced theory  $h^*$  gives a nonbasepointed reduced theory by setting  $\tilde{h}^n(X) = \text{Coker}(h^n(\text{point}) \rightarrow h^n(X))$ , where the map is induced by the constant map  $X \rightarrow \text{point}$ . One could also give an argument using suspension, which is always an isomorphism for reduced theories, and which takes one from the nonbasepointed to the basepointed category.

**Theorem 4.59.** *If  $h^*$  is an unreduced cohomology theory on the category of CW pairs and  $h^n(\text{point}) = 0$  for  $n \neq 0$ , then there are natural isomorphisms  $h^n(X, A) \approx H^n(X, A; h^0(\text{point}))$  for all CW pairs  $(X, A)$  and all  $n$ . The corresponding statement for homology theories is also true.*

**Proof:** The case of homology is slightly simpler, so let us consider this first. For CW complexes, relative homology groups reduce to absolute groups, so it suffices to deal with the latter. For a CW complex  $X$  the long exact sequences of  $h_*$  homology groups for the pairs  $(X^n, X^{n-1})$  give rise to a cellular chain complex

$$\cdots \longrightarrow h_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} h_n(X^n, X^{n-1}) \xrightarrow{d_n} h_{n-1}(X^{n-1}, X^{n-2}) \longrightarrow \cdots$$

just as for ordinary homology. The hypothesis that  $h_n(\text{point}) = 0$  for  $n \neq 0$  implies that this chain complex has homology groups  $h_n(X)$  by the same argument as for ordinary homology. The main thing to verify now is that this cellular chain complex is isomorphic to the cellular chain complex in ordinary homology with coefficients in the group  $G = h_0(\text{point})$ . Certainly the cellular chain groups in the two cases are isomorphic, being direct sums of copies of  $G$  with one copy for each cell, so we have only to check that the cellular boundary maps are the same.

It is not really necessary to treat the cellular boundary map  $d_1$  from 1-chains to 0-chains since one can always pass from  $X$  to  $\Sigma X$ , suspension being a natural isomorphism in any homology theory, and the double suspension  $\Sigma^2 X$  has no 1-cells.

The calculation of cellular boundary maps  $d_n$  for  $n > 1$  in terms of degrees of certain maps between spheres works equally well for the homology theory  $h_*$ , where ‘degree’ now means degree with respect to the  $h_*$  theory, so what is needed is the fact that a map  $S^n \rightarrow S^n$  of degree  $m$  in the usual sense induces multiplication by  $m$  on  $h_n(S^n) \approx G$ . This is obviously true for degrees 0 and 1, represented by a constant map and the identity map. Since  $\pi_n(S^n) \approx \mathbb{Z}$ , every map  $S^n \rightarrow S^n$  is homotopic to some multiple of the identity, so the general case will follow if we know that degree in the  $h_*$  theory is additive with respect to the sum operation in  $\pi_n(S^n)$ . This is a special case of the following more general assertion:

**Lemma 4.60.** *If a functor  $h$  from basepointed CW complexes to abelian groups satisfies the homotopy and wedge axioms, then for any two basepoint-preserving maps  $f, g: \Sigma X \rightarrow K$ , we have  $(f + g)_* = f_* + g_*$  if  $h$  is covariant and  $(f + g)^* = f^* + g^*$  if  $h$  is contravariant.*

**Proof:** The map  $f + g$  is the composition  $\Sigma X \xrightarrow{c} \Sigma X \vee \Sigma X \xrightarrow{f \vee g} K$  where  $c$  is the quotient map collapsing an equatorial copy of  $X$ . In the covariant case consider the diagram at the right, where  $i_1$  and  $i_2$  are the inclusions  $\Sigma X \hookrightarrow \Sigma X \vee \Sigma X$ . Let  $q_1, q_2: \Sigma X \vee \Sigma X \rightarrow \Sigma X$  be the quotient maps restricting to the identity on the summand indicated by the subscript and collapsing the other summand to a point. Then  $q_{1*} \oplus q_{2*}$  is an inverse to  $i_{1*} \oplus i_{2*}$  since  $q_j i_k$  is the identity map for  $j = k$  and the constant map for  $j \neq k$ .

$$\begin{array}{ccccc} h(\Sigma X) & \xrightarrow{c_*} & h(\Sigma X \vee \Sigma X) & \xrightarrow{(f \vee g)_*} & h(K) \\ & & \uparrow i_{1*} \oplus i_{2*} & \approx & \\ & & h(\Sigma X) \oplus h(\Sigma X) & & \end{array}$$

An element  $x$  in the left-hand group  $h(\Sigma X)$  in the diagram is sent by the composition  $(q_{1*} \oplus q_{2*})c_*$  to the element  $(x, x)$  in the lower group  $h(\Sigma X) \oplus h(\Sigma X)$  since



$q_1c$  and  $q_2c$  are homotopic to the identity. The composition  $(f \vee g)_*(i_{1*} \oplus i_{2*})$  sends  $(x, 0)$  to  $f_*(x)$  and  $(0, y)$  to  $g_*(y)$  since  $(f \vee g)i_1 = f$  and  $(f \vee g)i_2 = g$ . Hence  $(x, y)$  is sent to  $f_*(x) + g_*(y)$ . Combining these facts, we see that the composition across the top of the diagram is  $x \mapsto f_*(x) + g_*(x)$ . But this composition is also  $(f + g)_*$  since  $f + g = (f \vee g)c$ . This finishes the proof in the covariant case.

The contravariant case is similar, using the corresponding diagram with arrows reversed. The inverse of  $i_1^* \oplus i_2^*$  is  $q_1^* \oplus q_2^*$  by the same reasoning. An element  $u$  in the right-hand group  $h(K)$  maps to the element  $(f^*(u), g^*(u))$  in the lower group  $h(\Sigma X) \oplus h(\Sigma X)$  since  $(f \vee g)i_1 = f$  and  $(f \vee g)i_2 = g$ . An element  $(x, 0)$  in the lower group in the diagram maps to the element  $x$  in the left-hand group since  $q_1c$  is homotopic to the identity, and similarly  $(0, y)$  maps to  $y$ . Hence  $(x, y)$  maps to  $x + y$  in the left-hand group. We conclude that  $u \in h(K)$  maps by the composition across the top of the diagram to  $f^*(u) + g^*(u)$  in  $h(\Sigma X)$ . But this composition is  $(f + g)^*$  by definition.  $\square$

Returning to the proof of the theorem, we see that the cellular chain complexes for  $h_*(X)$  and  $H_*(X; G)$  are isomorphic, so we obtain isomorphisms  $h_n(X) \approx H_n(X; G)$  for all  $n$ . To verify that these isomorphisms are natural with respect to maps  $f: X \rightarrow Y$  we may first deform such a map  $f$  to be cellular. Then  $f$  takes each pair  $(X^n, X^{n-1})$  to the pair  $(Y^n, Y^{n-1})$ , hence  $f$  induces a chain map of cellular chain complexes in the  $h_*$  theory, as well as for  $H_*(-; G)$ . To compute these chain maps we may pass to the quotient maps  $X^n/X^{n-1} \rightarrow Y^n/Y^{n-1}$ . These are maps of the form  $\bigvee_\alpha S_\alpha^n \rightarrow \bigvee_\beta S_\beta^n$ , so the induced maps  $f_*$  on  $h_n$  are determined by their component maps  $f_*: S_\alpha^n \rightarrow S_\beta^n$ . This is exactly the same situation as with the cellular boundary maps before, where we saw that the degree of a map  $S^n \rightarrow S^n$  determines the induced map on  $h_n$ . We conclude that the cellular chain map induced by  $f$  in the  $h_*$  theory agrees exactly with the cellular chain map for  $H_*(-; G)$ . This implies that the isomorphism between the two theories is natural.

The situation for cohomology is quite similar, but there is one point in the argument where a few more words are needed. For cohomology theories the cellular cochain groups are the direct product, rather than the direct sum, of copies of the coefficient group  $G = h^0(\text{point})$ , with one copy per cell. This means that when there are infinitely many cells in a given dimension, it is not automatically true that the cellular coboundary maps are uniquely determined by how they map factors of one direct product to factors of the other direct product. To be precise, consider the cellular coboundary map  $d_n: h^n(X^n, X^{n-1}) \rightarrow h^{n+1}(X^{n+1}, X^n)$ . Decomposing the latter group as a product of copies of  $G$  for the  $(n+1)$ -cells, we see that  $d_n$  is determined by the maps  $h^n(X^n/X^{n-1}) \rightarrow h^n(S_\alpha^n)$  associated to the attaching maps  $\varphi_\alpha$  of the cells  $e_\alpha^{n+1}$ . The thing to observe is that since  $\varphi_\alpha$  has compact image, meeting only finitely many  $n$ -cells, this map  $h^n(X^n/X^{n-1}) \rightarrow h^n(S_\alpha^n)$  is *finitely supported* in the sense that

there is a splitting of the domain into a product of finitely many factors and a product of the remaining possibly infinite number of factors, such that the map is zero on the latter product. Finitely supported maps have the good property that they are determined by their restrictions to the  $G$  factors of  $h^n(X^n/X^{n-1})$ . From this we deduce, using the lemma, that the cellular coboundary maps in the  $h^*$  theory agree with those in ordinary cohomology with  $G$  coefficients. This extra argument is also needed to prove naturality of the isomorphisms  $h^n(X) \approx H^n(X; G)$ .

This completes the proof of Theorem 4.59.  $\square$

**Proof of Theorem 4.57:** The functors  $h^n(X) = \langle X, K(G, n) \rangle$  define a reduced cohomology theory, and the coefficient groups  $h^n(S^i) = \pi_i(K(G, n))$  are the same as  $\tilde{H}^n(S^i; G)$ , so Theorem 4.59, translated into reduced cohomology, gives natural isomorphisms  $T: \langle X, K(G, n) \rangle \rightarrow \tilde{H}^n(X; G)$  for all CW complexes  $X$ .

It remains to see that  $T([f]) = f^*(\alpha)$  for some  $\alpha \in \tilde{H}^n(K(G, n); G)$ , independent of  $f$ . This is purely formal: Take  $\alpha = T(\mathbb{1})$  for  $\mathbb{1}$  the identity map of  $K(G, n)$ , and then naturality gives  $T([f]) = T(f^*(\mathbb{1})) = f^*T(\mathbb{1}) = f^*(\alpha)$ , where the first  $f^*$  refers to induced homomorphisms for the functor  $h^n$ , which means composition with  $f$ .  $\square$

The fundamental class  $\alpha = T(\mathbb{1})$  can be made more explicit if we choose for  $K(G, n)$  a CW complex  $K$  with  $(n-1)$ -skeleton a point. Denoting  $\langle X, K(G, n) \rangle$  by  $h^n(X)$ , then we have

$$h^n(K) \approx h^n(K^{n+1}) \approx \text{Ker } d: h^n(K^n) \rightarrow h^{n+1}(K^{n+1}, K^n)$$

The map  $d$  is the cellular coboundary in  $h^*$  cohomology since we have  $h^n(K^n) = h^n(K^n, K^{n-1})$  because  $K^{n-1}$  is a point and  $h^*$  is a reduced theory. The isomorphism of  $h^n(K)$  with  $\text{Ker } d$  is given by restriction of maps  $K \rightarrow K$  to  $K^n$ , so the element  $\mathbb{1} \in h^n(K)$  defining the fundamental class  $T(\mathbb{1})$  corresponds, under the isomorphism  $h^n(K) \approx \text{Ker } d$ , to the inclusion  $K^n \hookrightarrow K$  viewed as an element of  $h^n(K^n)$ . As a cellular cocycle this element assigns to each  $n$ -cell of  $K$  the element of the coefficient group  $G = \pi_n(K)$  given by the inclusion of the closure of this cell into  $K$ . This means that the fundamental class  $\alpha \in H^n(K; G)$  is represented by the cellular cocycle assigning to each  $n$ -cell the element of  $\pi_n(K)$  given by a characteristic map for the cell.

By naturality of  $T$  it follows that for a cellular map  $f: X \rightarrow K$ , the corresponding element of  $H^n(X; G)$  is represented by the cellular cocycle sending each  $n$ -cell of  $X$  to the element of  $G = \pi_n(K)$  represented by the composition of  $f$  with a characteristic map for the cell.

The natural isomorphism  $H^n(X; G) \approx \langle X, K(G, n) \rangle$  leads to a basic principle which reappears many places in algebraic topology, the idea that the occurrence or nonoccurrence of a certain phenomenon is governed by what happens in a single special case, the *universal example*. To illustrate, let us prove the following special fact:

- The map  $H^1(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$ ,  $\alpha \mapsto \alpha^2$ , is identically zero for all spaces  $X$ .

By taking a CW approximation to  $X$  we are reduced to the case that  $X$  is a CW complex. Then every element of  $H^1(X; \mathbb{Z})$  has the form  $f^*(\alpha)$  for some  $f: X \rightarrow K(\mathbb{Z}, 1)$ , with  $\alpha$  a fundamental class in  $H^1(K(\mathbb{Z}, 1); \mathbb{Z})$ , further reducing us to verifying the result for this single  $\alpha$ , the ‘universal example’. And for this universal  $\alpha$  it is evident that  $\alpha^2 = 0$  since  $S^1$  is a  $K(\mathbb{Z}, 1)$  and  $H^2(S^1; \mathbb{Z}) = 0$ .

Does this fact generalize? It certainly does not hold if we replace the coefficient ring  $\mathbb{Z}$  by  $\mathbb{Z}_2$  since  $H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[x]$ . Indeed, the example of  $\mathbb{RP}^\infty$  shows more generally that the fundamental class  $\alpha \in H^n(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$  generates a polynomial subalgebra  $\mathbb{Z}_2[\alpha] \subset H^*(K(\mathbb{Z}_2, n); \mathbb{Z}_2)$  for each  $n \geq 1$ , since there is a map  $f: \mathbb{RP}^\infty \rightarrow K(\mathbb{Z}_2, n)$  with  $f^*(\alpha) = x^n$  and all the powers of  $x^n$  are nonzero, hence also all the powers of  $\alpha$ . By the same reasoning, the example of  $\mathbb{CP}^\infty$  shows that the fundamental class  $\alpha \in H^{2n}(K(\mathbb{Z}, 2n); \mathbb{Z})$  generates a polynomial subalgebra  $\mathbb{Z}[\alpha]$  in  $H^*(K(\mathbb{Z}, 2n); \mathbb{Z})$ . As we shall see in [SSAT],  $H^*(K(\mathbb{Z}, 2n); \mathbb{Z})/\text{torsion}$  is exactly this polynomial algebra  $\mathbb{Z}[\alpha]$ .

A little more subtle is the question of identifying the subalgebra of  $H^*(K(\mathbb{Z}, n); \mathbb{Z})$  generated by the fundamental class  $\alpha$  for odd  $n \geq 3$ . By the commutativity property of cup products we know that  $\alpha^2$  is either zero or of order two. To see that  $\alpha^2$  is nonzero it suffices to find a single space  $X$  with an element  $y \in H^n(X; \mathbb{Z})$  such that  $y^2 \neq 0$ . The first place to look might be  $\mathbb{RP}^\infty$ , but its cohomology with  $\mathbb{Z}$  coefficients is concentrated in even dimensions. Instead, consider  $X = \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ . This has  $\mathbb{Z}_2$  cohomology  $\mathbb{Z}_2[x, y]$  and Example 3E.5 shows that its  $\mathbb{Z}$  cohomology is the  $\mathbb{Z}_2[x^2, y^2]$ -submodule generated by 1 and  $x^2y + xy^2$ , except in dimension zero of course, where 1 generates a  $\mathbb{Z}$  rather than a  $\mathbb{Z}_2$ . In particular we can take  $y = x^{2k}(x^2y + xy^2)$  for any  $k \geq 0$ , and then all powers  $y^m$  are nonzero since we are inside the polynomial ring  $\mathbb{Z}_2[x, y]$ . It follows that the subalgebra of  $H^*(K(\mathbb{Z}, n); \mathbb{Z})$  generated by  $\alpha$  is  $\mathbb{Z}[\alpha]/(2\alpha^2)$  for odd  $n \geq 3$ .

These examples lead one to wonder just how complicated the cohomology of  $K(G, n)$ ’s is. The general construction of a  $K(G, n)$  is not very helpful in answering this question. Consider the case  $G = \mathbb{Z}$  for example. Here one would start with  $S^n$  and attach  $(n+2)$ -cells to kill  $\pi_{n+1}(S^n)$ . Since  $\pi_{n+1}(S^n)$  happens to be cyclic, only one  $(n+2)$ -cell is needed. To continue, one would have to compute generators for  $\pi_{n+2}$  of the resulting space  $S^n \cup e^{n+2}$ , use these to attach  $(n+3)$ -cells, then compute the resulting  $\pi_{n+3}$ , and so on for each successive dimension. When  $n = 2$  this procedure happens to work out very neatly, and the resulting  $K(\mathbb{Z}, 2)$  is  $\mathbb{CP}^\infty$  with its usual CW structure having one cell in each even dimension, according to an exercise at the end of the section. However, for larger  $n$  it quickly becomes impractical to make this procedure explicit since homotopy groups are so hard to compute. One can get some idea of the difficulties of the next case  $n = 3$  by considering the homology groups of  $K(\mathbb{Z}, 3)$ . Using techniques in [SSAT], the groups  $H_i(K(\mathbb{Z}, 3); \mathbb{Z})$  for  $0 \leq i \leq 12$  can be

computed to be

$$\mathbb{Z}, 0, 0, \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_{10}, \mathbb{Z}_2$$

To get this sequence of homology groups would require quite a few cells, and the situation only gets worse in higher dimensions, where the homology groups are not always cyclic.

Indeed, one might guess that computing the homology groups of  $K(\mathbb{Z}, n)$ 's would be of the same order of difficulty as computing the homotopy groups of spheres, but by some miracle this is not the case. The calculations are indeed complicated, but they were completely done by Serre and Cartan in the 1950s, not just for  $K(\mathbb{Z}, n)$ 's, but for all  $K(G, n)$ 's with  $G$  finitely generated abelian. For example,  $H^*(K(\mathbb{Z}, 3); \mathbb{Z}_2)$  is the polynomial algebra  $\mathbb{Z}_2[x_3, x_5, x_9, x_{17}, x_{33}, \dots]$  with generators of dimensions  $2^i + 1$ , indicated by the subscripts. And in general, for  $G$  finitely generated abelian,  $H^*(K(G, n); \mathbb{Z}_p)$  is a polynomial algebra on generators of specified dimensions if  $p$  is 2, while for  $p$  an odd prime one gets the tensor product of a polynomial ring on generators of specified even dimensions and an exterior algebra on generators of specified odd dimensions. With  $\mathbb{Z}$  coefficients the description of the cohomology is not nearly so neat, however. We will study these questions in some detail in [SSAT].

There is a good reason for being interested in the cohomology of  $K(G, n)$ 's, arising from the equivalence  $H^n(X; G) \approx \langle X, K(G, n) \rangle$ . Taking  $\mathbb{Z}$  coefficients for simplicity, an element of  $H^m(K(\mathbb{Z}, n); \mathbb{Z})$  corresponds to a map  $\theta: K(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}, m)$ . We can compose  $\theta$  with any map  $f: X \rightarrow K(\mathbb{Z}, n)$  to get a map  $\theta f: X \rightarrow K(\mathbb{Z}, m)$ . Letting  $f$  vary and keeping  $\theta$  fixed, this gives a function  $H^n(X; \mathbb{Z}) \rightarrow H^m(X; \mathbb{Z})$ , depending only on  $\theta$ . This is the idea of *cohomology operations*, which we study in more detail in §4.L.

The equivalence  $H^n(X; G) \approx \langle X, K(G, n) \rangle$  also leads to a new viewpoint toward cup products. Taking  $G$  to be a ring  $R$  and setting  $K_n = K(R, n)$ , then if we are given maps  $f: X \rightarrow K_m$  and  $g: Y \rightarrow K_n$ , we can define the cross product of the corresponding cohomology classes by the composition

$$X \times Y \xrightarrow{f \times g} K_m \times K_n \longrightarrow K_m \wedge K_n \xrightarrow{\mu} K_{m+n}$$

where the middle map is the quotient map and  $\mu$  can be defined in the following way. The space  $K_m \wedge K_n$  is  $(m + n - 1)$ -connected, so by the Hurewicz theorem and the Künneth formula for reduced homology we have isomorphisms  $\pi_{m+n}(K_m \wedge K_n) \approx H_{m+n}(K_m \wedge K_n) \approx H_m(K_m) \otimes H_n(K_n) \approx R \otimes R$ . By Lemmas 4.7 and 4.31 there is then a map  $\mu: K_m \wedge K_n \rightarrow K_{m+n}$  inducing the multiplication map  $R \otimes R \rightarrow R$  on  $\pi_{m+n}$ . Or we could use the isomorphism  $H^{m+n}(K_m \wedge K_n; R) \approx \text{Hom}(H_{m+n}(K_m \wedge K_n), R)$  and let  $\mu$  be the map corresponding to the cohomology class given by the multiplication homomorphism  $R \otimes R \rightarrow R$ .

The case  $R = \mathbb{Z}$  is particularly simple. We can take  $S^m$  as the  $(m+1)$ -skeleton of  $K_m$ , and similarly for  $K_n$ , so  $K_m \wedge K_n$  has  $S^m \wedge S^n$  as its  $(m+n+1)$ -skeleton and we can obtain  $\mu$  by extending the inclusion  $S^m \wedge S^n = S^{m+n} \hookrightarrow K_{m+n}$ .

It is not hard to prove the basic properties of cup product using this definition, and in particular the commutativity property becomes somewhat more transparent from this viewpoint. For example, when  $R = \mathbb{Z}$ , commutativity just comes down to the fact that the map  $S^m \wedge S^n \rightarrow S^n \wedge S^m$  switching the factors has degree  $(-1)^{mn}$  when regarded as a map of  $S^{m+n}$ .

## Fibrations

Recall from §4.2 that a fibration is a map  $p:E \rightarrow B$  having the homotopy lifting property with respect to all spaces. In a fiber bundle all the fibers are homeomorphic by definition, but this need not be true for fibrations. An example is the linear projection of a 2-simplex onto one of its edges, which is a fibration according to an exercise at the end of the section. The following result gives some evidence that fibrations should be thought of as a homotopy-theoretic analog of fiber bundles:

**Proposition 4.61.** *For a fibration  $p:E \rightarrow B$ , the fibers  $F_b = p^{-1}(b)$  over each path component of  $B$  are all homotopy equivalent.*

**Proof:** A path  $\gamma:I \rightarrow B$  gives rise to a homotopy  $g_t:F_{\gamma(0)} \rightarrow B$  with  $g_t(F_{\gamma(0)}) = \gamma(t)$ . The inclusion  $F_{\gamma(0)} \hookrightarrow E$  provides a lift  $\tilde{g}_0$ , so by the homotopy lifting property we have a homotopy  $\tilde{g}_t:F_{\gamma(0)} \rightarrow E$  with  $\tilde{g}_t(F_{\gamma(0)}) \subset F_{\gamma(t)}$  for all  $t$ . In particular,  $\tilde{g}_1$  gives a map  $L_\gamma:F_{\gamma(0)} \rightarrow F_{\gamma(1)}$ . The association  $\gamma \mapsto L_\gamma$  has the following basic properties:

- (a) If  $\gamma \simeq \gamma' \text{ rel } \partial I$ , then  $L_\gamma \simeq L_{\gamma'}$ . In particular the homotopy class of  $L_\gamma$  is independent of the choice of the lifting  $\tilde{g}_t$  of  $g_t$ .
- (b) For a composition of paths  $\gamma\gamma'$ ,  $L_{\gamma\gamma'}$  is homotopic to the composition  $L_{\gamma'}L_\gamma$ .

From these statements it follows that  $L_\gamma$  is a homotopy equivalence with homotopy inverse  $L_{\bar{\gamma}}$ , where  $\bar{\gamma}$  is the inverse path of  $\gamma$ .

Before proving (a), note that a fibration has the homotopy lifting property for pairs  $(X \times I, X \times \partial I)$  since the pairs  $(I \times I, I \times \{0\} \cup \partial I \times I)$  and  $(I \times I, I \times \{0\})$  are homeomorphic, hence the same is true after taking products with  $X$ .

To prove (a), let  $\gamma(s, t)$  be a homotopy from  $\gamma(t)$  to  $\gamma'(t)$ ,  $(s, t) \in I \times I$ . This determines a family  $g_{st}:F_{\gamma(0)} \rightarrow B$  with  $g_{st}(F_{\gamma(0)}) = \gamma(s, t)$ . Let  $\tilde{g}_{0,t}$  and  $\tilde{g}_{1,t}$  be lifts defining  $L_\gamma$  and  $L_{\gamma'}$ , and let  $\tilde{g}_{s,0}$  be the inclusion  $F_{\gamma(0)} \hookrightarrow E$  for all  $s$ . Using the homotopy lifting property for the pair  $(F_{\gamma(0)} \times I, F_{\gamma(0)} \times \partial I)$ , we can extend these lifts to lifts  $\tilde{g}_{st}$  for  $(s, t) \in I \times I$ . Restricting to  $t = 1$  then gives a homotopy  $L_\gamma \simeq L_{\gamma'}$ .

Property (b) holds since for lifts  $\tilde{g}_t$  and  $\tilde{g}'_t$  defining  $L_\gamma$  and  $L_{\gamma'}$  we obtain a lift defining  $L_{\gamma\gamma'}$  by taking  $\tilde{g}_{2t}$  for  $0 \leq t \leq 1/2$  and  $\tilde{g}'_{2t-1}L_{\gamma'}$  for  $1/2 \leq t \leq 1$ .  $\square$

One may ask whether fibrations satisfy a homotopy analog of the local triviality property of fiber bundles. Observe first that for a fibration  $p:E \rightarrow B$ , the restriction

$p: p^{-1}(A) \rightarrow A$  is a fibration for any subspace  $A \subset B$ . So we can ask whether every point of  $B$  has a neighborhood  $U$  for which the fibration  $p^{-1}(U) \rightarrow U$  is equivalent in some homotopy-theoretic sense to a projection  $U \times F \rightarrow U$ . The natural notion of equivalence for fibrations is defined in the following way. Given fibrations  $p_1: E_1 \rightarrow B$  and  $p_2: E_2 \rightarrow B$ , a map  $f: E_1 \rightarrow E_2$  is called **fiber-preserving** if  $p_1 = p_2 f$ , or in other words,  $f(p_1^{-1}(b)) \subset p_2^{-1}(b)$  for all  $b \in B$ . A fiber-preserving map  $f: E_1 \rightarrow E_2$  is a **fiber homotopy equivalence** if there is a fiber-preserving map  $g: E_2 \rightarrow E_1$  such that both compositions  $fg$  and  $gf$  are homotopic to the identity through fiber-preserving maps. A fiber homotopy equivalence can be thought of as a family of homotopy equivalences between corresponding fibers of  $E_1$  and  $E_2$ . An interesting fact is that a fiber-preserving map that is a homotopy equivalence is a fiber homotopy equivalence; this is an exercise for §4.H.

We will show that a fibration  $p: E \rightarrow B$  is locally fiber-homotopically trivial in the sense described above if  $B$  is locally contractible. In order to do this we first digress to introduce another basic concept.

Given a fibration  $p: E \rightarrow B$  and a map  $f: A \rightarrow B$ , there is a **pullback** or **induced** fibration  $f^*(E) \rightarrow A$  obtained by setting  $f^*(E) = \{(a, e) \in A \times E \mid f(a) = p(e)\}$ , with the projections of  $f^*(E)$  onto  $A$  and  $E$  giving a commutative diagram as shown at the right. The homotopy lifting property holds for  $f^*(E) \rightarrow A$  since a homotopy  $g_t: X \rightarrow A$  gives the first coordinate of a lift  $\tilde{g}_t: X \rightarrow f^*(E)$ , the second coordinate being a lifting to  $E$  of the composed homotopy  $f g_t$ .

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

**Proposition 4.62.** *Given a fibration  $p: E \rightarrow B$  and a homotopy  $f_t: A \rightarrow B$ , the pullback fibrations  $f_0^*(E) \rightarrow A$  and  $f_1^*(E) \rightarrow A$  are fiber homotopy equivalent.*

**Proof:** Let  $F: A \times I \rightarrow B$  be the homotopy  $f_t$ . The fibration  $F^*(E) \rightarrow A \times I$  contains  $f_0^*(E)$  and  $f_1^*(E)$  over  $A \times \{0\}$  and  $A \times \{1\}$ . So it suffices to prove the following: For a fibration  $p: E \rightarrow B \times I$ , the restricted fibrations  $E_s = p^{-1}(B \times \{s\}) \rightarrow B$  are all fiber homotopy equivalent for  $s \in [0, 1]$ .

To prove this assertion the idea is to imitate the construction of the homotopy equivalences  $L_y$  in the proof of Proposition 4.61. A path  $y: [0, 1] \rightarrow I$  gives rise to a fiber-preserving map  $L_y: E_{y(0)} \rightarrow E_{y(1)}$  by lifting the homotopy  $g_t: E_{y(0)} \rightarrow B \times I$ ,  $g_t(x) = (p(x), y(t))$ , starting with the inclusion  $E_{y(0)} \hookrightarrow E$ . As before, one shows the two basic properties (a) and (b), noting that in (a) the homotopy  $L_y \simeq L_{y'}$  is fiber-preserving since it is obtained by lifting a homotopy  $h_t: E_{y(0)} \times [0, 1] \rightarrow B \times I$  of the form  $h_t(x, u) = (p(x), -)$ . From (a) and (b) it follows that  $L_y$  is a fiber homotopy equivalence with inverse  $L_{\bar{y}}$ .  $\square$

**Corollary 4.63.** *A fibration  $E \rightarrow B$  over a contractible base  $B$  is fiber homotopy equivalent to a product fibration  $B \times F \rightarrow B$ .*

**Proof:** The pullback of  $E$  by the identity map  $B \rightarrow B$  is  $E$  itself, while the pullback by a constant map  $B \rightarrow B$  is a product  $B \times F$ .  $\square$

Thus we see that if  $B$  is locally contractible then any fibration over  $B$  is locally fiber homotopy equivalent to a product fibration.

### Pathspace Constructions

There is a simple but extremely useful way to turn arbitrary mappings into fibrations. Given a map  $f: A \rightarrow B$ , let  $E_f$  be the space of pairs  $(a, \gamma)$  where  $a \in A$  and  $\gamma: I \rightarrow B$  is a path in  $B$  with  $\gamma(0) = f(a)$ . We topologize  $E_f$  as a subspace of  $A \times B^I$ , where  $B^I$  is the space of mappings  $I \rightarrow B$  with the compact-open topology; see the Appendix for the definition and basic properties of this topology, in particular Proposition A.14 which we will be using shortly.

**Proposition 4.64.** *The map  $p: E_f \rightarrow B$ ,  $p(a, \gamma) = \gamma(1)$ , is a fibration.*

**Proof:** Continuity of  $p$  follows from (a) of Proposition A.14 in the Appendix which says that the evaluation map  $B^I \times I \rightarrow B$ ,  $(\gamma, s) \mapsto \gamma(s)$ , is continuous.

To verify the fibration property, let a homotopy  $g_t: X \rightarrow B$  and a lift  $\tilde{g}_0: X \rightarrow E_f$  of  $g_0$  be given. Write  $\tilde{g}_0(x) = (h(x), \gamma_x)$  for  $h: X \rightarrow A$  and  $\gamma_x: I \rightarrow B$ . Define a lift  $\tilde{g}_t: X \rightarrow E_f$  by  $\tilde{g}_t(x) = (h(x), \gamma_x \cdot g_{[0,t]}(x))$ , the second coordinate being the path  $\gamma_x$  followed by the path traced out by  $g_s(x)$  for  $0 \leq s \leq t$ . This composition of paths is defined since  $g_0(x) = p\tilde{g}_0(x) = \gamma_x(1)$ . To check that  $\tilde{g}_t$  is a continuous homotopy we regard it as a map  $X \times I \rightarrow E_f \subset A \times B^I$  and then apply (b) of Proposition A.14 which in the current context asserts that continuity of a map  $X \times I \rightarrow A \times B^I$  is equivalent to continuity of the associated map  $X \times I \times I \rightarrow A \times B$ .  $\square$

We can regard  $A$  as the subspace of  $E_f$  consisting of pairs  $(a, \gamma)$  with  $\gamma$  the constant path at  $f(a)$ , and  $E_f$  deformation retracts onto this subspace by restricting all the paths  $\gamma$  to shorter and shorter initial segments. The map  $p: E_f \rightarrow B$  restricts to  $f$  on the subspace  $A$ , so we have factored an arbitrary map  $f: A \rightarrow B$  as the composition  $A \hookrightarrow E_f \rightarrow B$  of a homotopy equivalence and a fibration. We can also think of this construction as extending  $f$  to a fibration  $E_f \rightarrow B$  by enlarging its domain to a homotopy equivalent space. The fiber  $F_f$  of  $E_f \rightarrow B$  is called the **homotopy fiber** of  $f$ . It consists of all pairs  $(a, \gamma)$  with  $a \in A$  and  $\gamma$  a path in  $B$  from  $f(a)$  to a basepoint  $b_0 \in B$ .

If  $f: A \rightarrow B$  is the inclusion of a subspace, then  $E_f$  is the space of paths in  $B$  starting at points of  $A$ . In this case a map  $(I^{i+1}, \partial I^{i+1}, J^i) \rightarrow (B, A, x_0)$  is the same as a map  $(I^i, \partial I^i) \rightarrow (F_f, \gamma_0)$  where  $\gamma_0$  is the constant path at  $x_0$  and  $F_f$  is the fiber of  $E_f$  over  $x_0$ . This means that  $\pi_{i+1}(B, A, x_0)$  can be identified with  $\pi_i(F_f, \gamma_0)$ , hence the long exact sequences of homotopy groups of the pair  $(B, A)$  and of the fibration  $E_f \rightarrow B$  can be identified.

An important special case is when  $f$  is the inclusion of the basepoint  $b_0$  into  $B$ . Then  $E_f$  is the space  $PB$  of paths in  $B$  starting at  $b_0$ , and  $p: PB \rightarrow B$  sends each path to its endpoint. The fiber  $p^{-1}(b_0)$  is the loop space  $\Omega B$  consisting of all loops in  $B$  based at  $b_0$ . Since  $PB$  is contractible by progressively truncating paths, the long exact sequence of homotopy groups for the path fibration  $PB \rightarrow B$  yields another proof that  $\pi_n(X, x_0) \approx \pi_{n-1}(\Omega X, x_0)$  for all  $n$ .

As we mentioned in the discussion of loopspaces earlier in this section, it is a theorem of [Milnor 1959] that the loop space of a CW complex is homotopy equivalent to a CW complex. Milnor's theorem is actually quite a bit more general than this, and implies in particular that the homotopy fiber of an arbitrary map between CW complexes has the homotopy type of a CW complex. One can usually avoid quoting these results by using CW approximations, though it is reassuring to know they are available if needed, or if one does not want to bother with CW approximations.

If the fibration construction  $f \mapsto E_f$  is applied to a map  $p: E \rightarrow B$  that is already a fibration, one might expect the resulting fibration  $E_p \rightarrow B$  to be closely related to the original fibration  $E \rightarrow B$ . This is indeed the case:

**Proposition 4.65.** *If  $p: E \rightarrow B$  is a fibration, then the inclusion  $E \hookrightarrow E_p$  is a fiber homotopy equivalence. In particular, the homotopy fibers of  $p$  are homotopy equivalent to the actual fibers.*

**Proof:** We apply the homotopy lifting property to the homotopy  $g_t: E_p \rightarrow B$ ,  $g_t(e, \gamma) = \gamma(t)$ , with initial lift  $\tilde{g}_0: E_p \rightarrow E$ ,  $\tilde{g}_0(e, \gamma) = e$ . The lifting  $\tilde{g}_t: E_p \rightarrow E$  is then the first coordinate of a homotopy  $h_t: E_p \rightarrow E_p$  whose second coordinate is the restriction of the paths  $\gamma$  to the interval  $[t, 1]$ . Since the endpoints of the paths  $\gamma$  are unchanged,  $h_t$  is fiber-preserving. We have  $h_0 = \mathbb{1}$ ,  $h_1(E_p) \subset E$ , and  $h_t(E) \subset E$  for all  $t$ . If we let  $i$  denote the inclusion  $E \hookrightarrow E_p$ , then  $ih_1 \simeq \mathbb{1}$  via  $h_t$  and  $h_1 i \simeq \mathbb{1}$  via  $h_t|_E$ , so  $i$  is a fiber homotopy equivalence.  $\square$

We have seen that loopspaces occur as fibers of fibrations  $PB \rightarrow B$  with contractible total space  $PB$ . Here is something of a converse:

**Proposition 4.66.** *If  $F \rightarrow E \rightarrow B$  is a fibration or fiber bundle with  $E$  contractible, then there is a weak homotopy equivalence  $F \rightarrow \Omega B$ .*

**Proof:** If we compose a contraction of  $E$  with the projection  $p: E \rightarrow B$  then we have for each point  $x \in E$  a path  $\gamma_x$  in  $B$  from  $p(x)$  to a basepoint  $b_0 = p(x_0)$ , where  $x_0$  is the point to which  $E$  contracts. This yields a map  $E \rightarrow PB$ ,  $x \mapsto \overline{\gamma}_x$ , whose composition with the fibration  $PB \rightarrow B$  is  $p$ . By restriction this gives a map  $F \rightarrow \Omega B$  where  $F = p^{-1}(b_0)$ , and the long exact sequence of homotopy groups for  $F \rightarrow E \rightarrow B$  maps to the long exact sequence for  $\Omega B \rightarrow PB \rightarrow B$ . Since  $E$  and  $PB$  are contractible, the five-lemma implies that the map  $F \rightarrow \Omega B$  is a weak homotopy equivalence.  $\square$



Examples arising from fiber bundles constructed earlier in the chapter are  $O(n) \simeq \Omega G_n(\mathbb{R}^\infty)$ ,  $U(n) \simeq \Omega G_n(\mathbb{C}^\infty)$ , and  $Sp(n) \simeq \Omega G_n(\mathbb{H}^\infty)$ . In particular, taking  $n = 1$  in the latter two examples, we have  $S^1 \simeq \Omega \mathbb{C}P^\infty$  and  $S^3 \simeq \Omega \mathbb{H}P^\infty$ . Note that in all these examples it is a topological group that is homotopy equivalent to a loop space. In [Milnor 1956] this is shown to hold in general: For each topological group  $G$  there is a fiber bundle  $G \rightarrow EG \rightarrow BG$  with  $EG$  contractible, hence by the proposition there is a weak equivalence  $G \simeq \Omega BG$ . There is also a converse statement: The loop space of a CW complex is homotopy equivalent to a topological group.

The relationship between  $X$  and  $\Omega X$  has been much studied, particularly the case that  $\Omega X$  has the homotopy type of a finite CW complex, which is of special interest because of the examples of the classical Lie groups such as  $O(n)$ ,  $U(n)$ , and  $Sp(n)$ . See [Kane 1988] for an introduction to this subject.

It is interesting to see what happens when the process of forming homotopy fibers is iterated. Given a fibration  $p: E \rightarrow B$  with fiber  $F = p^{-1}(b_0)$ , we know that the inclusion of  $F$  into the homotopy fiber  $F_p$  is a homotopy equivalence. Recall that  $F_p$  consists of pairs  $(e, \gamma)$  with  $e \in E$  and  $\gamma$  a path in  $B$  from  $p(e)$  to  $b_0$ . The inclusion  $F \hookrightarrow E$  extends to a map  $i: F_p \rightarrow E$ ,  $i(e, \gamma) = e$ , and this map is obviously a fibration. In fact it is the pullback via  $p$  of the path fibration  $PB \rightarrow B$ . This allows us to iterate, taking the homotopy fiber  $F_i$  with its map to  $F_p$ , and so on, as in the first row of the following diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & F_j & \longrightarrow & F_i & \xrightarrow{j} & F_p & \xrightarrow{i} & E & \xrightarrow{p} & B \\ & & \simeq \updownarrow & & \simeq \updownarrow & & \simeq \up & & \parallel & & \parallel \\ \cdots & \longrightarrow & \Omega E & \xrightarrow{\Omega p} & \Omega B & \longrightarrow & F & \longrightarrow & E & \xrightarrow{p} & B \end{array}$$

The actual fiber of  $i$  over a point  $e_0 \in p^{-1}(b_0)$  consists of pairs  $(e_0, \gamma)$  with  $\gamma$  a loop in  $B$  at the basepoint  $b_0$ , so this fiber is just  $\Omega B$ , and the inclusion  $\Omega B \hookrightarrow F_i$  is a homotopy equivalence. In the second row of the diagram the map  $\Omega B \rightarrow F$  is the composition  $\Omega B \hookrightarrow F_i \rightarrow F_p \rightarrow F$  where the last map is a homotopy inverse to the inclusion  $F \hookrightarrow F_p$ , so the square in the diagram containing these maps commutes up to homotopy. The homotopy fiber  $F_i$  consists of pairs  $(\gamma, \eta)$  where  $\eta$  is a path in  $E$  ending at  $e_0$  and  $\gamma$  is a path in  $B$  from  $p(\eta(0))$  to  $b_0$ . A homotopy inverse to the inclusion  $\Omega B \hookrightarrow F_i$  is the retraction  $F_i \rightarrow \Omega B$  sending  $(\gamma, \eta)$  to the loop obtained by composing the inverse path of  $p\eta$  with  $\gamma$ . These constructions can now be iterated indefinitely.

Thus we produce a sequence

$$\cdots \rightarrow \Omega^2 B \rightarrow \Omega F \rightarrow \Omega E \rightarrow \Omega B \rightarrow F \rightarrow E \rightarrow B$$

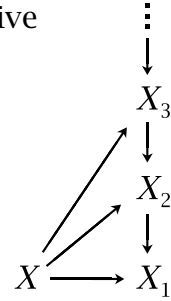
where any two consecutive maps form a fibration, up to homotopy equivalence, and all the maps to the left of  $\Omega B$  are obtained by applying the functor  $\Omega$  to the later maps. The long exact sequence of homotopy groups for any fibration in the sequence coincides with the long exact sequence for  $F \rightarrow E \rightarrow B$ , as the reader can check.

## Postnikov Towers

A Postnikov tower for a path-connected space  $X$  is a commutative diagram as at the right, such that:

- (1) The map  $X \rightarrow X_n$  induces an isomorphism on  $\pi_i$  for  $i \leq n$ .
- (2)  $\pi_i(X_n) = 0$  for  $i > n$ .

As we saw in Example 4.16, every connected CW complex  $X$  has a Postnikov tower, and this is unique up to homotopy equivalence by Corollary 4.19.



If we convert the map  $X_n \rightarrow X_{n-1}$  into a fibration, its fiber  $F_n$  is a  $K(\pi_n X, n)$ , as is apparent from a brief inspection of the long exact sequence of homotopy groups for the fibration:

$$\pi_{i+1}(X_n) \rightarrow \pi_{i+1}(X_{n-1}) \rightarrow \pi_i(F_n) \rightarrow \pi_i(X_n) \rightarrow \pi_i(X_{n-1})$$

We can replace each map  $X_n \rightarrow X_{n-1}$  by a fibration  $X'_n \rightarrow X'_{n-1}$  in succession, starting with  $X_2 \rightarrow X_1$  and working upward. For the inductive step we convert the composition  $X_n \rightarrow X_{n-1} \hookrightarrow X'_{n-1}$  into a fibration  $X'_n \rightarrow X'_{n-1}$  fitting into the commutative diagram at the right. Thus we obtain a Postnikov tower satisfying also the condition

$$\begin{array}{ccc} X_n & \hookrightarrow & X'_n \\ \downarrow & & \downarrow \\ X_{n-1} & \hookrightarrow & X'_{n-1} \end{array}$$

- (3) The map  $X_n \rightarrow X_{n-1}$  is a fibration with fiber a  $K(\pi_n X, n)$ .

To the extent that fibrations can be regarded as twisted products, up to homotopy equivalence, the spaces  $X_n$  in a Postnikov tower for  $X$  can be thought of as twisted products of Eilenberg-MacLane spaces  $K(\pi_n X, n)$ .

For many purposes, a CW complex  $X$  can be replaced by one of the stages  $X_n$  in a Postnikov tower for  $X$ , for example if one is interested in homotopy or homology groups in only a finite range of dimensions. However, to determine the full homotopy type of  $X$  from its Postnikov tower, some sort of limit process is needed. Let us investigate this question in somewhat greater generality.

Given a sequence of maps  $\cdots \rightarrow X_2 \rightarrow X_1$ , define their **inverse limit**  $\varprojlim X_n$  to be the subspace of the product  $\prod_n X_n$  consisting of sequences of points  $x_n \in X_n$  with  $x_n$  mapping to  $x_{n-1}$  under the map  $X_n \rightarrow X_{n-1}$ . The corresponding algebraic notion is the inverse limit  $\varprojlim X_n$  of a sequence of group homomorphisms  $\cdots \rightarrow G_2 \rightarrow G_1$ , which is the subgroup of  $\prod_n G_n$  consisting of sequences of elements  $g_n \in G_n$  with  $g_n$  mapping to  $g_{n-1}$  under the homomorphism  $G_n \rightarrow G_{n-1}$ .

**Proposition 4.67.** *For an arbitrary sequence of fibrations  $\cdots \rightarrow X_2 \rightarrow X_1$  the natural map  $\lambda: \pi_i(\varprojlim X_n) \rightarrow \varprojlim \pi_i(X_n)$  is surjective, and  $\lambda$  is injective if the maps  $\pi_{i+1}(X_n) \rightarrow \pi_{i+1}(X_{n-1})$  are surjective for  $n$  sufficiently large.*

**Proof:** Represent an element of  $\varprojlim \pi_i(X_n)$  by maps  $f_n: (S^i, s_0) \rightarrow (X_n, x_n)$ . Since the projection  $p_n: X_n \rightarrow X_{n-1}$  takes  $[f_n]$  to  $[f_{n-1}]$ , by applying the homotopy lifting

property for the pair  $(S^i, s_0)$  we can homotope  $f_n$ , fixing  $s_0$ , so that  $p_n f_n = f_{n-1}$ . Doing this inductively for  $n = 2, 3, \dots$ , we get  $p_n f_n = f_{n-1}$  for all  $n$  simultaneously, which gives surjectivity of  $\lambda$ .

For injectivity, note first that inverse limits are unaffected by throwing away a finite number of terms at the end of the sequence of spaces or groups, so we may assume the maps  $\pi_{i+1}(X_n) \rightarrow \pi_{i+1}(X_{n-1})$  are surjective for all  $n$ . Given a map  $f: S^i \rightarrow \varprojlim X_n$ , suppose we have nullhomotopies  $F_n: D^{i+1} \rightarrow X_n$  of the coordinate functions  $f_n: S^i \rightarrow X_n$  of  $f$ . We have  $p_n F_n = F_{n-1}$  on  $S^i$ , so  $p_n F_n$  and  $F_{n-1}$  are the restrictions to the two hemispheres of  $S^{i+1}$  of a map  $g_{n-1}: S^{i+1} \rightarrow X_{n-1}$ . If the map  $\pi_{i+1}(X_n) \rightarrow \pi_{i+1}(X_{n-1})$  is surjective, we can rechoose  $F_n$  so that the new  $g_{n-1}$  is nullhomotopic, that is, so that  $p_n F_n \simeq F_{n-1} \text{ rel } S^i$ . Applying the homotopy lifting property for  $(D^{i+1}, S^i)$ , we can make  $p_n F_n = F_{n-1}$ . Doing this inductively for  $n = 2, 3, \dots$ , we see that  $f: S^i \rightarrow \varprojlim X_n$  is nullhomotopic and  $\lambda$  is injective.  $\square$

One might wish to have a description of the kernel of  $\lambda$  in the case of an arbitrary sequence of fibrations  $\dots \rightarrow X_2 \rightarrow X_1$ , though for our present purposes this question is not relevant. In fact,  $\text{Ker } \lambda$  is naturally isomorphic to  $\varprojlim^1 \pi_{i+1}(X_n)$ , where  $\varprojlim^1$  is the functor defined in §3.F. Namely, if  $f: S^i \rightarrow \varprojlim X_n$  determines an element of  $\text{Ker } \lambda$ , then the sequence of maps  $g_n: S^{i+1} \rightarrow X_n$  constructed above gives an element of  $\prod_n \pi_{i+1}(X_n)$ , well-defined up to the choice of the nullhomotopies  $F_n$ . Any new choice of  $F_n$  is obtained by adding a map  $G_n: S^{i+1} \rightarrow X_n$  to  $F_n$ . The effect of this is to change  $g_n$  to  $g_n + G_n$  and  $g_{n-1}$  to  $g_{n-1} - p_n G_n$ . Since  $\varprojlim^1 \pi_{i+1}(X_n)$  is the quotient of  $\prod_n \pi_{i+1}(X_n)$  under exactly these identifications, we get  $\text{Ker } \lambda \approx \varprojlim^1 \pi_{i+1}(X_n)$ . Thus for each  $i > 0$  there is a natural exact sequence

$$0 \rightarrow \varprojlim^1 \pi_{i+1}(X_n) \rightarrow \pi_i(\varprojlim X_n) \rightarrow \varprojlim \pi_i(X_n) \rightarrow 0$$

The proposition says that the  $\varprojlim^1$  term vanishes if the maps  $\pi_{i+1}(X_n) \rightarrow \pi_{i+1}(X_{n-1})$  are surjective for sufficiently large  $n$ .

**Corollary 4.68.** *For the Postnikov tower of a connected CW complex  $X$  the natural map  $X \rightarrow \varprojlim X_n$  is a weak homotopy equivalence, so  $X$  is a CW approximation to  $\varprojlim X_n$ .*

**Proof:** The composition  $\pi_i(X) \rightarrow \pi_i(\varprojlim X_n) \xrightarrow{\lambda} \varprojlim \pi_i(X_n)$  is an isomorphism since  $\pi_i(X) \rightarrow \pi_i(X_n)$  is an isomorphism for large  $n$ .  $\square$

Having seen how to decompose a space  $X$  into the terms in its Postnikov tower, we consider now the inverse process of building a Postnikov tower, starting with  $X_1$  as a  $K(\pi, 1)$  and inductively constructing  $X_n$  from  $X_{n-1}$ . It would be very nice if the fibration  $K(\pi, n) \rightarrow X_n \rightarrow X_{n-1}$  could be extended another term to the right, to form a fibration sequence

$$K(\pi, n) \rightarrow X_n \rightarrow X_{n-1} \rightarrow K(\pi, n+1)$$

for this would say that  $X_n$  is the homotopy fiber of a map  $X_{n-1} \rightarrow K(\pi, n+1)$ , and homotopy classes of such maps are in one-to-one correspondence with elements of  $H^{n+1}(X_{n-1}; \pi)$  by Theorem 4.57. Since the homotopy fiber of  $X_{n-1} \rightarrow K(\pi, n+1)$  is the same as the pullback of the path fibration  $PK(\pi, n+1) \rightarrow K(\pi, n+1)$ , its homotopy type depends only on the homotopy class of the map  $X_{n-1} \rightarrow K(\pi, n+1)$ , by Proposition 4.62. Note that the last term  $K(\pi, n+1)$  in the fibration sequence above cannot be anything else but a  $K(\pi, n+1)$  since its loop space must be homotopy equivalent to the first term in the sequence, a  $K(\pi, n)$ .

In general, a fibration  $F \rightarrow E \rightarrow B$  is called **principal** if there is a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \Omega B' & \longrightarrow & F' & \longrightarrow & E' \longrightarrow B' \end{array}$$

where the second row is a fibration sequence and the vertical maps are weak homotopy equivalences. Thus if all the fibrations in a Postnikov tower for  $X$  happen to be principal, we have a diagram as at the right, where each  $X_{n+1}$  is, up to weak homotopy equivalence, the homotopy fiber of the map  $k_n: X_n \rightarrow K(\pi_{n+1}X, n+2)$ . The map  $k_n$  is equivalent to a class in  $H^{n+2}(X_n; \pi_{n+1}X)$  called the  $n^{\text{th}}$   **$k$ -invariant** of  $X$ . These classes specify how to construct  $X$  inductively from Eilenberg–MacLane spaces. For example, if all the  $k_n$ 's are zero,  $X$  is just the product of the spaces  $K(\pi_n X, n)$ , and in the general case  $X$  is some sort of twisted product of  $K(\pi_n X, n)$ 's.

$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow & & \\ K(\pi_3 X, 3) & \longrightarrow & X_3 & \xrightarrow{k_3} & K(\pi_4 X, 5) \\ & & \downarrow & & \\ K(\pi_2 X, 2) & \longrightarrow & X_2 & \xrightarrow{k_2} & K(\pi_3 X, 4) \\ & & \downarrow & & \\ K(\pi_1 X, 1) = X_1 & \xrightarrow{k_1} & & & K(\pi_2 X, 3) \end{array}$$

To actually build a space from its  $k$ -invariants is usually too unwieldy a procedure to be carried out in practice, but as a theoretical tool this procedure can be quite useful. The next result tells us when this tool is available:

**Theorem 4.69.** *A connected CW complex  $X$  has a Postnikov tower of principal fibrations iff  $\pi_1(X)$  acts trivially on  $\pi_n(X)$  for all  $n > 1$ .*

Notice that in the definition of a principal fibration, the map  $F \rightarrow \Omega B'$  automatically exists and is a homotopy weak equivalence once one has the right-hand square of the commutative diagram with its vertical maps weak homotopy equivalences. Thus the question of whether a fibration is principal can be rephrased in the following way: Given a map  $A \rightarrow X$ , which one can always replace by an equivalent fibration if one likes, does there exist a fibration  $F \rightarrow E \rightarrow B$  and a commutative square as at the right, with the vertical maps weak homotopy equivalences? By replacing  $A$  and  $X$  with CW approximations and converting the resulting map  $A \rightarrow X$  into an inclusion via a mapping cylinder, the question becomes whether a CW pair  $(X, A)$  is equivalent to a fibration pair  $(E, F)$ , that

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ F & \longrightarrow & E \end{array}$$

is, whether there is a fibration  $F \rightarrow E \rightarrow B$  and a map  $(X, A) \rightarrow (E, F)$  for which both  $X \rightarrow E$  and  $A \rightarrow F$  are weak homotopy equivalences. In general the answer will rarely be yes, since the homotopy fiber of  $A \hookrightarrow X$  would have to have the weak homotopy type of a loop space, which is a rather severe restriction. However, in the situation of Postnikov towers, the homotopy fiber is a  $K(\pi, n)$  with  $\pi$  abelian since  $n \geq 2$ , so it is a loop space. But there is another requirement: The action of  $\pi_1(A)$  on  $\pi_n(X, A)$  must be trivial for all  $n \geq 1$ . This is equivalent to the action of  $\pi_1(F)$  on  $\pi_n(E, F)$  being trivial, which is always the case in a fibration since under the isomorphism  $p_* : \pi_n(E, F) \rightarrow \pi_n(B, x_0)$  an element  $\gamma\alpha - \alpha$ , with  $\gamma \in \pi_1(F)$  and  $\alpha \in \pi_n(E, F)$ , maps to  $p_*(\gamma)p_*(\alpha) - p_*(\alpha)$  which is zero since  $p_*(\gamma)$  lies in the trivial group  $\pi_1(x_0)$ .

The relative group  $\pi_n(X, A)$  is always isomorphic to  $\pi_{n-1}$  of the homotopy fiber of the inclusion  $A \hookrightarrow X$ , so in the case at hand when the homotopy fiber is a  $K(\pi, n)$ , the only nontrivial relative homotopy group is  $\pi_{n+1}(X, A) \approx \pi$ . In this case the necessary condition of trivial action is also sufficient:

**Lemma 4.70.** *Let  $(X, A)$  be a CW pair with both  $X$  and  $A$  connected, such that the homotopy fiber of the inclusion  $A \hookrightarrow X$  is a  $K(\pi, n)$ ,  $n \geq 1$ . Then there exists a fibration  $F \rightarrow E \rightarrow B$  and a map  $(X, A) \rightarrow (E, F)$  inducing weak homotopy equivalences  $X \rightarrow E$  and  $A \rightarrow F$  iff the action of  $\pi_1(A)$  on  $\pi_{n+1}(X, A)$  is trivial.*

**Proof:** It remains only to prove the ‘if’ implication. As we noted just before the statement of the lemma, the groups  $\pi_i(X, A)$  are zero except for  $\pi_{n+1}(X, A) \approx \pi$ . If the action of  $\pi_1(A)$  on  $\pi_{n+1}(X, A)$  is trivial, the relative Hurewicz theorem gives an isomorphism  $\pi_{n+1}(X, A) \approx H_{n+1}(X, A)$ . Since  $(X, A)$  is  $n$ -connected, we may assume  $A$  contains the  $n$ -skeleton of  $X$ , so  $X/A$  is  $n$ -connected and the absolute Hurewicz theorem gives  $\pi_{n+1}(X/A) \approx H_{n+1}(X/A)$ . Hence the quotient map  $X \rightarrow X/A$  induces an isomorphism  $\pi_{n+1}(X, A) \approx \pi_{n+1}(X/A)$  since the analogous statement for homology is certainly true.

Since  $\pi_{n+1}(X/A) \approx \pi$ , we can build a  $K(\pi, n+1)$  from  $X/A$  by attaching cells of dimension  $n+3$  and greater. This leads to the commutative diagram at the right, where the vertical maps are inclusions and the lower row is obtained by converting the map  $k$  into a fibration. The map  $A \rightarrow F_k$  is a weak homotopy equivalence by the five-lemma applied to the map between the long exact sequences of homotopy groups for the pairs  $(X, A)$  and  $(E_k, F_k)$ , since the only nontrivial relative groups are  $\pi_{n+1}$ , both of which map isomorphically to  $\pi_{n+1}(K(\pi, n+1))$ .  $\square$

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & X/A \\ \downarrow & & \downarrow \simeq & \searrow k & \downarrow \\ F_k & \longrightarrow & E_k & \longrightarrow & K(\pi, n+1) \end{array}$$

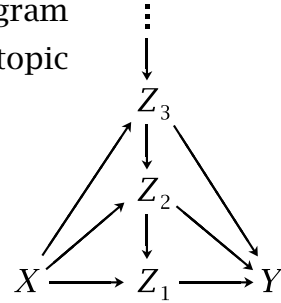
**Proof of 4.69:** In view of the lemma, all that needs to be done is identify the action of  $\pi_1(X)$  on  $\pi_n(X)$  with the action of  $\pi_1(X_n)$  on  $\pi_{n+1}(X_{n-1}, X_n)$  for  $n \geq 2$ , thinking of the map  $X_n \rightarrow X_{n-1}$  as an inclusion. From the exact sequence

$$0 = \pi_{n+1}(X_{n-1}) \rightarrow \pi_{n+1}(X_{n-1}, X_n) \xrightarrow{\partial} \pi_n(X_n) \rightarrow \pi_n(X_{n-1}) = 0$$

we have an isomorphism  $\pi_{n+1}(X_{n-1}, X_n) \approx \pi_n(X_n)$  respecting the action of  $\pi_1(X_n)$ . And the map  $X \rightarrow X_n$  induces isomorphisms on  $\pi_1$  and  $\pi_n$ , so we are done.  $\square$

Let us consider now a natural generalization of Postnikov towers, in which one starts with a map  $f: X \rightarrow Y$  between path-connected spaces rather than just a single space  $X$ . A **Moore–Postnikov tower** for  $f$  is a commutative diagram as shown at the right, with each composition  $X \rightarrow Z_n \rightarrow Y$  homotopic to  $f$ , and such that:

- (1) The map  $X \rightarrow Z_n$  induces an isomorphism on  $\pi_i$  for  $i < n$  and a surjection for  $i = n$ .
- (2) The map  $Z_n \rightarrow Y$  induces an isomorphism on  $\pi_i$  for  $i > n$  and an injection for  $i = n$ .
- (3) The map  $Z_{n+1} \rightarrow Z_n$  is a fibration with fiber a  $K(\pi_n F, n)$  where  $F$  is the homotopy fiber of  $f$ .



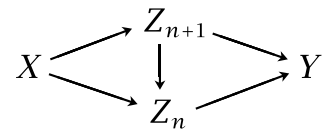
A Moore–Postnikov tower specializes to a Postnikov tower by taking  $Y$  to be a point and then setting  $X_n = Z_{n+1}$ , discarding the space  $Z_1$  which has trivial homotopy groups.

**Theorem 4.71.** *Every map  $f: X \rightarrow Y$  between connected CW complexes has a Moore–Postnikov tower, which is unique up to homotopy equivalence. A Moore–Postnikov tower of principal fibrations exists iff  $\pi_1(X)$  acts trivially on  $\pi_n(M_f, X)$  for all  $n > 1$ , where  $M_f$  is the mapping cylinder of  $f$ .*

**Proof:** The existence and uniqueness of a diagram satisfying (1) and (2) and commutative at least up to homotopy follows from Propositions 4.17 and 4.18 applied to the pair  $(M_f, X)$  with  $M_f$  the mapping cylinder of  $f$ . Having such a diagram, we proceed as in the earlier case of Postnikov towers, replacing each map  $Z_n \rightarrow Z_{n-1}$  by a homotopy equivalent fibration, starting with  $Z_2 \rightarrow Z_1$  and working upward. We can then apply the homotopy lifting property to make all the triangles in the left half of the tower strictly commutative. After these steps the triangles in the right half of the diagram commute up to homotopy, and to make them strictly commute we can just replace each map to  $Y$  by the composition through  $Z_1$ .

To see that the fibers of the maps  $Z_{n+1} \rightarrow Z_n$  are Eilenberg–MacLane spaces as in condition (3), consider two successive levels of the tower.

We may arrange that the maps  $X \rightarrow Z_{n+1} \rightarrow Z_n \rightarrow Y$  are inclusions by taking mapping cylinders, first of  $X \rightarrow Z_{n+1}$ , then



of the new  $Z_{n+1} \rightarrow Z_n$ , and then of the new  $Z_n \rightarrow Y$ . From the left-hand triangle we see that  $Z_{n+1} \rightarrow Z_n$  induces an isomorphism on  $\pi_i$  for  $i < n$  and a surjection for  $i = n$ , hence  $\pi_i(Z_n, Z_{n+1}) = 0$  for  $i < n + 1$ . Similarly, the other triangle gives  $\pi_i(Z_n, Z_{n+1}) = 0$  for  $i > n + 1$ . To show that  $\pi_{n+1}(Z_n, Z_{n+1}) \approx \pi_{n+1}(Y, X)$  we use the following diagram:

$$\begin{array}{ccccccccc}
\pi_{n+1}(Z_{n+1}) & \longrightarrow & \pi_{n+1}(Z_n) & \longrightarrow & \pi_{n+1}(Z_n, Z_{n+1}) & \longrightarrow & \pi_n(Z_{n+1}) & \longrightarrow & \pi_n(Z_n) \\
\downarrow = & & \downarrow \approx & & \downarrow & & \downarrow = & & \downarrow \\
\pi_{n+1}(Z_{n+1}) & \longrightarrow & \pi_{n+1}(Y) & \longrightarrow & \pi_{n+1}(Y, Z_{n+1}) & \longrightarrow & \pi_n(Z_{n+1}) & \longrightarrow & \pi_n(Y) \\
\uparrow & & \uparrow = & & \uparrow & & \uparrow \approx & & \uparrow = \\
\pi_{n+1}(X) & \longrightarrow & \pi_{n+1}(Y) & \longrightarrow & \pi_{n+1}(Y, X) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(Y)
\end{array}$$

The upper-right vertical map is injective and the lower-left vertical map is surjective, so the five-lemma implies that the two middle vertical maps are isomorphisms. Since the homotopy fiber of an inclusion  $A \hookrightarrow B$  has  $\pi_i$  equal to  $\pi_{i+1}(B, A)$ , we see that condition (3) is satisfied.

The statement about a tower of principal fibrations can be obtained as an application of Lemma 4.70. As we saw in the previous paragraph, there are isomorphisms  $\pi_{n+1}(Y, X) \approx \pi_{n+1}(Z_n, Z_{n+1})$ , and these respect the action of  $\pi_1(X) \approx \pi_1(Z_{n+1})$ , so Lemma 4.70 gives the result.  $\square$

Besides the case that  $Y$  is a point, which yields Postnikov towers, another interesting special case of Moore–Postnikov towers is when  $X$  is a point. In this case the space  $Z_n$  is an  $n$ -connected covering of  $Y$ , as in Example 4.20. The  $n$ -connected covering of  $Y$  can also be obtained as the homotopy fiber of the  $n^{\text{th}}$  stage  $Y \rightarrow Y_n$  of a Postnikov tower for  $Y$ . The tower of  $n$ -connected coverings of  $Y$  can be realized by principal fibrations by taking  $Z_n$  to be the homotopy fiber of the map  $Z_{n-1} \rightarrow K(\pi_n Y, n)$  that is the first nontrivial stage in a Postnikov tower for  $Z_{n-1}$ .

$$\begin{array}{c}
\vdots \\
\downarrow \\
Z_2 \rightarrow K(\pi_3 Y, 3) \\
\downarrow \\
Z_1 \rightarrow K(\pi_2 Y, 2) \\
\downarrow \\
Y \rightarrow K(\pi_1 Y, 1)
\end{array}$$

A generalization of the preceding theory allowing nontrivial actions of  $\pi_1$  can be found in [Robinson 1972].

## Obstruction Theory

It is very common in algebraic topology to encounter situations where one would like to extend or lift a given map. Obvious examples are the homotopy extension and homotopy lifting properties. In their simplest forms, extension and lifting questions can often be phrased in one of the following two ways:

**The Extension Problem.** Given a CW pair  $(W, A)$  and a map  $A \rightarrow X$ , does this extend to a map  $W \rightarrow X$ ?

$$\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & \nearrow & \\
W & & 
\end{array}$$

**The Lifting Problem.** Given a fibration  $X \rightarrow Y$  and a map  $W \rightarrow Y$ , is there a lift  $W \rightarrow X$ ?

$$\begin{array}{ccc}
& & X \\
& \nearrow & \downarrow \\
W & \longrightarrow & Y
\end{array}$$

In order for the lifting problem to include things like the homotopy lifting property, it should be generalized to a relative form:

**The Relative Lifting Problem.** Given a CW pair  $(W, A)$ , a fibration  $X \rightarrow Y$ , and a map  $W \rightarrow Y$ , does there exist a lift  $W \rightarrow X$  extending a given lift on  $A$ ?

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ W & \longrightarrow & Y \end{array}$$

Besides reducing to the absolute lifting problem when  $A = \emptyset$ , this includes the extension problem by taking  $Y$  to be a point. Of course, one could broaden these questions by dropping the requirements that  $(W, A)$  be a CW pair and that the map  $X \rightarrow Y$  be a fibration. However, these conditions are often satisfied in cases of interest, and they make the task of finding solutions much easier.

The term ‘obstruction theory’ refers to a procedure for defining a sequence of cohomology classes that are the obstructions to finding a solution to the extension, lifting, or relative lifting problem. In the most favorable cases these obstructions lie in cohomology groups that are all zero, so the problem has a solution. But even when the obstructions are nonzero it can be very useful to have the problem expressed in cohomological terms.

There are two ways of developing obstruction theory, which produce essentially the same result in the end. In the more elementary approach one tries to construct the extension or lifting one cell of  $W$  at a time, proceeding inductively over skeleta of  $W$ . This approach has an appealing directness, but the technical details of working at the level of cochains are perhaps a little tedious. Instead of pursuing this direct line we shall follow the second approach, which is slightly more sophisticated but has the advantage that the theory becomes an almost trivial application of Postnikov towers for the extension problem, or Moore–Postnikov towers for the lifting problem. The cellular viewpoint is explained in [VBKT], where it appears in the study of characteristic classes of vector bundles.

Let us consider the extension problem first, where we wish to extend a map  $A \rightarrow X$  to the larger complex  $W$ . Suppose that  $X$  has a Postnikov tower of principal fibrations. Then we have a commutative diagram as shown below, where we have enlarged the tower by adjoining the space  $X_0$ , which is just a point, at the bottom. The map  $X_1 \rightarrow X_0$  is then a fibration, and to say it is principal says that  $X_1$ , which in any case is a  $K(\pi_1 X, 1)$ , is the loopspace of  $K(\pi_1 X, 2)$ , hence  $\pi_1(X)$  must be abelian. Conversely, if  $\pi_1(X)$  is abelian and acts trivially on all the higher homotopy groups of  $X$ , then there is an extended Postnikov tower of principal fibrations as shown.

$$\begin{array}{ccccc} & & & & \vdots \\ & & & & \downarrow \\ & & & & X_3 \xrightarrow{k_3} K(\pi_4 X, 5) \\ & & & & \downarrow \\ & & & & X_2 \xrightarrow{k_2} K(\pi_3 X, 4) \\ & & & & \downarrow \\ & & & & X_1 \xrightarrow{k_1} K(\pi_2 X, 3) \\ & & & & \downarrow \\ & & & & X_0 \xrightarrow{k_0} K(\pi_1 X, 2) \\ A \longrightarrow X & \nearrow & \nearrow & \nearrow & \\ \downarrow & & & & \\ W \longrightarrow & & & & \end{array}$$

Our strategy will be to try to lift the constant map  $W \rightarrow X_0$  to maps  $W \rightarrow X_n$  for  $n = 1, 2, \dots$  in succession, extending the given maps  $A \rightarrow X_n$ . If we are able to find all these lifts  $W \rightarrow X_n$ , there will then be no difficulty in constructing the desired extension  $W \rightarrow X$ .



For the inductive step we have a commutative diagram as at the right. Since  $X_n$  is the pullback, its points are pairs consist-

$$\begin{array}{ccccc} A & \longrightarrow & X_n & \longrightarrow & PK \\ \downarrow & & \downarrow & & \downarrow \\ W & \longrightarrow & X_{n-1} & \longrightarrow & K = K(\pi_n X, n+1) \end{array}$$

ing of a point in  $X_{n-1}$  and a path from its image in  $K$  to the basepoint. A lift  $W \rightarrow X_n$  therefore amounts to a nullhomotopy of the composition  $W \rightarrow X_{n-1} \rightarrow K$ . We already have such a lift defined on  $A$ , hence a nullhomotopy of  $A \rightarrow K$ , and we want a nullhomotopy of  $W \rightarrow K$  extending this nullhomotopy on  $A$ .

The map  $W \rightarrow K$  together with the nullhomotopy on  $A$  gives a map  $W \cup CA \rightarrow K$ , where  $CA$  is the cone on  $A$ . Since  $K$  is a  $K(\pi_n X, n+1)$ , the map  $W \cup CA \rightarrow K$  determines an **obstruction** class  $\omega_n \in H^{n+1}(W \cup CA; \pi_n X) \approx H^{n+1}(W, A; \pi_n X)$ .

|| **Proposition 4.72.** *A lift  $W \rightarrow X_n$  extending the given  $A \rightarrow X_n$  exists iff  $\omega_n = 0$ .*

**Proof:** We need to show that the map  $W \cup CA \rightarrow K$  extends to a map  $CW \rightarrow K$  iff  $\omega_n = 0$ , or in other words, iff  $W \cup CA \rightarrow K$  is homotopic to a constant map.

Suppose that  $g_t: W \cup CA \rightarrow K$  is such a homotopy. The constant map  $g_1$  then extends to the constant map  $g_1: CW \rightarrow K$ , so by the homotopy extension property for the pair  $(CW, W \cup CA)$ , applied to the reversed homotopy  $g_{1-t}$ , we have a homotopy  $g_t: CW \rightarrow K$  extending the previous homotopy  $g_t: W \cup CA \rightarrow K$ . The map  $g_0: CW \rightarrow K$  then extends the given map  $W \cup CA \rightarrow K$ .

Conversely, if we have an extension  $CW \rightarrow K$ , then this is nullhomotopic since the cone  $CW$  is contractible, and we may restrict such a nullhomotopy to  $W \cup CA$ .  $\square$

If we succeed in extending the lifts  $A \rightarrow X_n$  to lifts  $W \rightarrow X_n$  for all  $n$ , then we obtain a map  $W \rightarrow \varinjlim X_n$  extending the given  $A \rightarrow X \rightarrow \varinjlim X_n$ . Let  $M$  be the mapping cylinder of  $X \rightarrow \varinjlim X_n$ . Since the restriction of  $W \rightarrow \varinjlim X_n \subset M$  to  $A$  factors through  $X$ , this gives a homotopy of this restriction to the map  $A \rightarrow X \subset M$ . Extend this to a homotopy of  $W \rightarrow M$ , producing a map  $(W, A) \rightarrow (M, X)$ . Since the map  $X \rightarrow \varinjlim X_n$  is a weak homotopy equivalence,  $\pi_i(M, X) = 0$  for all  $i$ , so by Lemma 4.6, the compression lemma, the map  $(W, A) \rightarrow (M, X)$  can be homotoped to a map  $W \rightarrow X$  extending the given  $A \rightarrow X$ , and we have solved the extension problem.

Thus if it happens that at each stage of the inductive process of constructing lifts  $W \rightarrow X_n$  the obstruction  $\omega_n \in H^{n+1}(W, A; \pi_n X)$  vanishes, then the extension problem has a solution. In particular, this yields:

|| **Corollary 4.73.** *If  $X$  is a connected abelian CW complex and  $(W, A)$  is a CW pair such that  $H^{n+1}(W, A; \pi_n X) = 0$  for all  $n$ , then every map  $A \rightarrow X$  can be extended to a map  $W \rightarrow X$ .*  $\square$

This is a considerable improvement on the more elementary result that extensions exist if  $\pi_n(X) = 0$  for all  $n$  such that  $W - A$  has cells of dimension  $n + 1$ , which is Lemma 4.7.

We can apply the Hurewicz theorem and obstruction theory to extend the homology version of Whitehead's theorem to CW complexes with trivial action of  $\pi_1$  on all homotopy groups:

**Proposition 4.74.** *If  $X$  and  $Y$  are connected abelian CW complexes, then a map  $f: X \rightarrow Y$  inducing isomorphisms on all homology groups is a homotopy equivalence.*

**Proof:** Taking the mapping cylinder of  $f$  reduces us to the case of an inclusion  $X \hookrightarrow Y$  of a subcomplex. If we can show that  $\pi_1(X)$  acts trivially on  $\pi_n(Y, X)$  for all  $n$ , then the relative Hurewicz theorem will imply that  $\pi_n(Y, X) = 0$  for all  $n$ , so  $X \rightarrow Y$  will be a weak homotopy equivalence. The assumptions guarantee that  $\pi_1(X) \rightarrow \pi_1(Y)$  is an isomorphism, so we know at least that  $\pi_1(Y, X) = 0$ .

We can use obstruction theory to extend the identity map  $X \rightarrow X$  to a retraction  $Y \rightarrow X$ . To apply the theory we need  $\pi_1(X)$  acting trivially on  $\pi_n(X)$ , which holds by hypothesis. Since the inclusion  $X \hookrightarrow Y$  induces isomorphisms on homology, we have  $H_*(Y, X) = 0$ , hence  $H^{n+1}(Y, X; \pi_n(X)) = 0$  for all  $n$  by the universal coefficient theorem. So there are no obstructions, and a retraction  $Y \rightarrow X$  exists. This implies that the maps  $\pi_n(Y) \rightarrow \pi_n(Y, X)$  are onto, so trivial action of  $\pi_1(X)$  on  $\pi_n(Y)$  implies trivial action on  $\pi_n(Y, X)$  by naturality of the action.  $\square$

The generalization of the preceding analysis of the extension problem to the relative lifting problem is straightforward. Assuming the fibration  $p: X \rightarrow Y$  in the statement of the relative lifting problem has a Moore–Postnikov tower of principal fibrations, we have the diagram at the right, where  $F$  is the fiber of the fibration  $X \rightarrow Y$ . The first step is to lift the map  $W \rightarrow Y$  to  $Z_1$ , extending the given lift on  $A$ . We may take  $Z_1$  to be the covering space of  $Y$  corresponding to the subgroup  $p_*(\pi_1(X))$  of  $\pi_1(Y)$ , so covering space theory tells us when we can lift  $W \rightarrow Y$  to  $Z_1$ , and the unique lifting property for covering spaces can be used to see whether a lift can be chosen to agree with the lift on  $A$  given by the diagram; this could only be a problem when  $A$  has more than one component.

$$\begin{array}{ccccc}
 & & & \vdots & \\
 & & & \downarrow & \\
 & & & Z_2 & \longrightarrow K(\pi_2 F, 3) \\
 & & \nearrow & \downarrow & \\
 A & \longrightarrow & X & \longrightarrow & Z_1 & \longrightarrow K(\pi_1 F, 2) \\
 \downarrow & & \searrow p & \downarrow & \\
 W & \longrightarrow & & Y & 
 \end{array}$$

Having a lift to  $Z_1$ , the analysis proceeds exactly as before. One finds a sequence of obstructions  $\omega_n \in H^{n+1}(W, A; \pi_n F)$ , assuming  $\pi_1 F$  is abelian in the case  $n = 1$ . A lift to  $X$  exists, extending the given lift on  $A$ , if each successive  $\omega_n$  is zero.

One can ask the converse question: If a lift exists, must the obstructions  $\omega_n$  all be zero? Since Proposition 4.72 is an if and only if statement, one might expect the answer to be yes, but upon closer inspection the matter becomes less clear. The difficulty is that, even if at some stage the obstruction  $\omega_n$  is zero, so a lift to  $Z_{n+1}$  exists, there may be many choices of such a lift, and different choices could lead to different  $\omega_{n+1}$ 's, some zero and others nonzero. Examples of such ambiguities are not hard to produce, for both the lifting and the extension problems, and the

ambiguities only become worse with each subsequent choice of a lift. So it is only in rather special circumstances that one can say that there are well-defined obstructions. A simple case is when  $\pi_i(F) = 0$  for  $i < n$ , so the Moore-Postnikov factorization begins with  $Z_n$  as in the diagram at the right. In this case the composition across the bottom of the diagram gives a well-defined **primary obstruction**  $\omega_n \in H^{n+1}(W, A; \pi_n F)$ .

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
 & & Z_{n+2} & \longrightarrow & K(\pi_{n+2}F, n+3) \\
 & \nearrow & \downarrow & & \\
 A & \longrightarrow & X & \longrightarrow & Z_{n+1} \longrightarrow K(\pi_{n+1}F, n+2) \\
 \downarrow & & \searrow p & & \downarrow \\
 W & \longrightarrow & Y = Z_n & \longrightarrow & K(\pi_n F, n+1)
 \end{array}$$

## Exercises

1. Show there is a map  $\mathbb{R}P^\infty \rightarrow \mathbb{C}P^\infty = K(\mathbb{Z}, 2)$  which induces the trivial map on  $\tilde{H}_*(-; \mathbb{Z})$  but a nontrivial map on  $\tilde{H}^*(-; \mathbb{Z})$ . How is this consistent with the universal coefficient theorem?
2. Show that the group structure on  $S^1$  coming from multiplication in  $\mathbb{C}$  induces a group structure on  $\langle X, S^1 \rangle$  such that the bijection  $\langle X, S^1 \rangle \rightarrow H^1(X; \mathbb{Z})$  of Theorem 4.57 is an isomorphism.
3. Suppose that a CW complex  $X$  contains a subcomplex  $S^1$  such that the inclusion  $S^1 \hookrightarrow X$  induces an injection  $H_1(S^1; \mathbb{Z}) \rightarrow H_1(X; \mathbb{Z})$  with image a direct summand of  $H_1(X; \mathbb{Z})$ . Show that  $S^1$  is a retract of  $X$ .
4. Given abelian groups  $G$  and  $H$  and CW complexes  $K(G, n)$  and  $K(H, n)$ , show that the map  $\langle K(G, n), K(H, n) \rangle \rightarrow \text{Hom}(G, H)$  sending a homotopy class  $[f]$  to the induced homomorphism  $f_*: \pi_n(K(G, n)) \rightarrow \pi_n(K(H, n))$  is a bijection.
5. Show that  $[X, S^n] \approx H^n(X; \mathbb{Z})$  if  $X$  is an  $n$ -dimensional CW complex. [Build a  $K(\mathbb{Z}, n)$  from  $S^n$  by attaching cells of dimension  $\geq n+2$ .]
6. Use Exercise 4 to construct a multiplication map  $\mu: K(G, n) \times K(G, n) \rightarrow K(G, n)$  for any abelian group  $G$ , making a CW complex  $K(G, n)$  into an H-space whose multiplication is commutative and associative up to homotopy and has a homotopy inverse. Show also that the H-space multiplication  $\mu$  is unique up to homotopy.
7. Using an H-space multiplication  $\mu$  on  $K(G, n)$ , define an addition in  $\langle X, K(G, n) \rangle$  by  $[f] + [g] = [\mu(f, g)]$  and show that under the bijection  $H^n(X; G) \approx \langle X, K(G, n) \rangle$  this addition corresponds to the usual addition in cohomology.
8. Show that a map  $p: E \rightarrow B$  is a fibration iff the map  $\pi: E^I \rightarrow E_p$ ,  $\pi(\gamma) = (\gamma(0), p\gamma)$ , has a section, that is, a map  $s: E_p \rightarrow E^I$  such that  $\pi s = \mathbb{1}$ .
9. Show that a linear projection of a 2-simplex onto one of its edges is a fibration but not a fiber bundle. [Use the preceding problem.]
10. Given a fibration  $F \rightarrow E \rightarrow B$ , use the homotopy lifting property to define an action of  $\pi_1(E)$  on  $\pi_n(F)$ , a homomorphism  $\pi_1(E) \rightarrow \text{Aut}(\pi_n(F))$ , such that the composition  $\pi_1(F) \rightarrow \pi_1(E) \rightarrow \text{Aut}(\pi_n(F))$  is the usual action of  $\pi_1(F)$  on  $\pi_n(F)$ . Deduce that if  $\pi_1(E) = 0$ , then the action of  $\pi_1(F)$  on  $\pi_n(F)$  is trivial.

11. For a space  $B$ , let  $\mathcal{F}(B)$  be the set of fiber homotopy equivalence classes of fibrations  $E \rightarrow B$ . Show that a map  $f: B_1 \rightarrow B_2$  induces  $f^*: \mathcal{F}(B_2) \rightarrow \mathcal{F}(B_1)$  depending only on the homotopy class of  $f$ , with  $f^*$  a bijection if  $f$  is a homotopy equivalence.
12. Show that for homotopic maps  $f, g: A \rightarrow B$  the fibrations  $E_f \rightarrow B$  and  $E_g \rightarrow B$  are fiber homotopy equivalent.
13. Given a map  $f: A \rightarrow B$  and a homotopy equivalence  $g: C \rightarrow A$ , show that the fibrations  $E_f \rightarrow B$  and  $E_{fg} \rightarrow B$  are fiber homotopy equivalent. [One approach is to use Corollary 0.21 to reduce to the case of deformation retractions.]
14. For a space  $B$ , let  $\mathcal{M}(B)$  denote the set of equivalence classes of maps  $f: A \rightarrow B$  where  $f_1: A_1 \rightarrow B$  is equivalent to  $f_2: A_2 \rightarrow B$  if there exists a homotopy equivalence  $g: A_1 \rightarrow A_2$  such that  $f_1 \simeq f_2 g$ . Show the natural map  $\mathcal{F}(B) \rightarrow \mathcal{M}(B)$  is a bijection. [See Exercises 11 and 13.]
15. If the fibration  $p: E \rightarrow B$  is a homotopy equivalence, show that  $p$  is a fiber homotopy equivalence of  $E$  with the trivial fibration  $\mathbb{1}: B \rightarrow B$ .
16. Show that a map  $f: X \rightarrow Y$  of connected CW complexes is a homotopy equivalence if it induces an isomorphism on  $\pi_1$  and its homotopy fiber  $F_f$  has  $\tilde{H}_*(F_f; \mathbb{Z}) = 0$ .
17. Show that  $\Omega X$  is an H-space with multiplication the composition of loops.
18. Show that a fibration sequence  $\cdots \rightarrow \Omega B \rightarrow F \rightarrow E \rightarrow B$  induces a long exact sequence  $\cdots \rightarrow \langle X, \Omega B \rangle \rightarrow \langle X, F \rangle \rightarrow \langle X, E \rangle \rightarrow \langle X, B \rangle$ , with groups and group homomorphisms except for the last three terms, abelian groups except for the last six terms.
19. Given a fibration  $F \rightarrow E \xrightarrow{p} B$ , define a natural action of  $\Omega B$  on the homotopy fiber  $F_p$  and use this to show that exactness at  $\langle X, F \rangle$  in the long exact sequence in the preceding problem can be improved to the statement that two elements of  $\langle X, F \rangle$  have the same image in  $\langle X, E \rangle$  iff they are in the same orbit of the induced action of  $\langle X, \Omega B \rangle$  on  $\langle X, F \rangle$ .
20. Show that by applying the loop space functor to a Postnikov tower for  $X$  one obtains a Postnikov tower of principal fibrations for  $\Omega X$ .
21. Show that in the Postnikov tower of an H-space, all the spaces are H-spaces and the maps are H-maps, commuting with the multiplication, up to homotopy.
22. Show that a principal fibration  $\Omega C \rightarrow E \xrightarrow{p} B$  is fiber homotopy equivalent to the product  $\Omega C \times B$  iff it has a section, a map  $s: B \rightarrow E$  with  $ps = \mathbb{1}$ .
23. Prove the following uniqueness result for the Quillen plus construction: Given a connected CW complex  $X$ , if there is an abelian CW complex  $Y$  and a map  $X \rightarrow Y$  inducing an isomorphism  $H_*(X; \mathbb{Z}) \approx H_*(Y; \mathbb{Z})$ , then such a  $Y$  is unique up to homotopy equivalence. [Use Corollary 4.73 with  $W$  the mapping cylinder of  $X \rightarrow Y$ .]
24. In the situation of the relative lifting problem, suppose one has two different lifts  $W \rightarrow X$  that agree on the subspace  $A \subset W$ . Show that the obstructions to finding a homotopy rel  $A$  between these two lifts lie in the groups  $H^n(W, A; \pi_n F)$ .