CW Complexes with Simplicial Structures

A **Δ-complex** can be defined as a CW complex X in which each cell e_α^n is provided with a distinguished characteristic map $\sigma_\alpha : \Delta^n \to X$ such that the restriction of σ_α to each face Δ^{n-1} of Δ^n is the distinguished σ_β for some $(n-1)$-cell e_β^{n-1}. It is understood that the simplices Δ^n and Δ^{n-1} have a specified ordering of their vertices, and the ordering of the vertices of Δ^n induces an ordering of the vertices of each face, which allows each face to be identified canonically with Δ^{n-1}. Intuitively, one thinks of the vertices of each n-cell of X as ordered by attaching the labels $0, 1, \cdots, n$ near the vertices, just inside the cell. The vertices themselves do not have to be distinct points of X.

If we no longer pay attention to orderings of vertices of simplices, we obtain a weaker structure which could be called an **unordered Δ-complex**. Here each cell e_α^n has a distinguished characteristic map $\sigma_\alpha : \Delta^n \to X$, but the restriction of σ_α to a face of Δ^n is allowed to be the composition of $\sigma_\beta : \Delta^{n-1} \to X$ with a symmetry of Δ^{n-1} permuting its vertices. Alternatively, we could say that each cell e_α^n has a family of $(n+1)!$ distinguished characteristic maps $\Delta^n \to X$ differing only by symmetries of Δ^n, such that the restrictions of these characteristic maps to faces give the distinguished characteristic maps for $(n-1)$-cells. The barycentric subdivision of any unordered Δ-complex is an ordered Δ-complex since the vertices of the barycentric subdivision are the barycenters of the simplices of the original complex, hence have a canonical ordering according to the dimensions of these simplices. The simplest example of an unordered Δ-complex that cannot be made into an ordered Δ-complex without subdivision is Δ^2 with its three edges identified by a one-third rotation of Δ^2 permuting the three vertices cyclically.

In the literature unordered Δ-complex structures are sometimes called generalized triangulations. They can be useful in situations where orderings of vertices are not needed. One disadvantage of unordered Δ-complexes is that they do not behave as well with respect to products. The product of two ordered simplices has a canonical subdivision into ordered simplices using the shuffling operation described in §3.B, and this allows the product of two ordered Δ-complexes to be given a canonical ordered Δ-complex structure. Without orderings this no longer works.

A CW complex is called **regular** if its characteristic maps can be chosen to be embeddings. The closures of the cells are then homeomorphic to closed balls, and so it makes sense to speak of closed cells in a regular CW complex. The closed cells can be regarded as cones on their boundary spheres, and these cone structures can be used to subdivide a regular CW complex into a regular Δ-complex, by induction over skeleta. In particular, regular CW complexes are homeomorphic to Δ-complexes. The barycentric subdivision of an unordered Δ-complex is a regular Δ-complex. A simplicial complex is a regular unordered Δ-complex in which each simplex is uniquely determined by its vertices. In the literature a regular unordered Δ-complex is sometimes called a simplicial multicomplex, or just a multicomplex, to convey the idea that there can be many simplices with the same set of vertices. The barycentric subdivision of a regular unordered Δ-complex is a simplicial complex. Hence barycentrically subdividing an unordered Δ-complex twice produces a simplicial complex.

A major disadvantage of Δ-complexes is that they do not allow quotient constructions. The quotient X/A of a Δ-complex X by a subcomplex A is not usually a Δ-complex. More generally, attaching a Δ-complex X to a Δ-complex Y via a simplicial map from a subcomplex $A \subset X$ to Y is not usually a Δ-complex. Here a simplicial map $f : A \to Y$ is one that sends each cell e^n_α of A onto a cell e^k_β of Y so that the square at the right commutes, with q a linear surjection sending vertices to vertices, preserving order. To fix this problem we need to broaden the definition of a Δ-complex to allow cells to be attached by arbitrary simplicial maps. Thus we define a **singular Δ-complex**, or $s\Delta$-complex, to be a CW complex with distinguished characteristic maps $\sigma_\alpha : \Delta^n \to X$ whose restrictions to faces are compositions $\sigma_\beta q : \Delta^{n-1} \to \Delta^k \to X$ for q a linear surjection taking vertices to vertices, preserving order. Simplicial maps between $s\Delta$-complexes are defined just as for Δ-complexes. With $s\Delta$-complexes one can perform attaching constructions in the same way as for CW complexes, using simplicial maps instead of cellular maps to specify the attachments. In particular one can form quotients, mapping cylinders, and mapping cones. One can also take products by the same subdivision procedure as for Δ-complexes.

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\sigma_\alpha} & A \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^k & \xrightarrow{\sigma_\beta} & Y \end{array}$$

We can view any $s\Delta$-complex X as being constructed inductively, skeleton by skeleton, where the skeleton X^n is obtained from X^{n-1} by attaching simplices Δ^n via simplicial maps $\partial\Delta^n \to X^{n-1}$ that preserve the ordering of vertices in each face of Δ^n. Conversely, any CW complex built in this way is an $s\Delta$-complex. For example, the usual CW structure on S^n consisting of one 0-cell and one n-cell is an $s\Delta$-complex structure since the attaching map of the n-cell, the constant map, is a simplicial map from $\partial\Delta^n$ to a point. One can regard this $s\Delta$-complex structure as assigning barycentric coordinates to all points of S^n other than the 0-cell. In fact, an arbitrary $s\Delta$-complex structure can be regarded as just a way of putting barycentric coordinates in all the open cells, subject to a compatibility condition on how the coordinates change when one passes from a cell to the cells in its boundary.

Combinatorial Descriptions

The data which specifies a Δ-complex is combinatorial in nature and can be formulated quite naturally in the language of categories. To see how this is done, let X be a Δ-complex and let X_n be its set of n-simplices. The way in which simplices of X fit together is determined by a 'face function' which assigns to each element of X_n and each $(n-1)$-dimensional face of Δ^n an element of X_{n-1}. Thinking of the n-simplex Δ^n combinatorially as its set of vertices, which we view as the ordered set $\Delta_n = \{0, 1, \cdots, n\}$, the face-function for X assigns to each order-preserving injection $\Delta_{n-1} \to \Delta_n$ a map $X_n \to X_{n-1}$. By composing these maps we get, for each order-preserving injection $g:\Delta_k \to \Delta_n$ a map $g^*:X_n \to X_k$ specifying how the k-simplices of X are arranged in the boundary of each n-simplex. The association $g \mapsto g^*$ satisfies $(gh)^* = h^* g^*$, and we can set $\mathbb{1}^* = \mathbb{1}$, so X determines a contravariant functor from the category whose objects are the ordered sets Δ_n, $n \geq 0$, and whose morphisms are the order-preserving injections, to the category of sets, namely the functor sending Δ_n to X_n and the injection g to g^*. Such a functor is exactly equivalent to a Δ-complex. Explicitly, we can reconstruct the Δ-complex X from the functor by setting

$$X = \coprod_n (X_n \times \Delta^n)/(g^*(x), y) \sim (x, g_*(y))$$

for $(x,y) \in X_n \times \Delta^k$, where g_* is the linear inclusion $\Delta^k \to \Delta^n$ sending the i^{th} vertex of Δ^k to the $g(i)^{th}$ vertex of Δ^n, and we perform the indicated identifications letting g range over all order-preserving injections $\Delta_k \to \Delta_n$.

If we wish to generalize this to $s\Delta$-complexes, we will have to consider surjective linear maps $\Delta^k \to \Delta^n$ as well as injections. This corresponds to considering order-preserving surjections $\Delta_k \to \Delta_n$ in addition to injections. Every map of sets decomposes canonically as a surjection followed by an injection, so we may as well consider arbitrary order-preserving maps $\Delta_k \to \Delta_n$. These form the morphisms in a category Δ_*, with objects the Δ_n's. We are thus led to consider contravariant functors from Δ_* to the category of sets. Such a functor is called a **simplicial set**. This terminology has the virtue that one can immediately define, for example, a simplicial group to be a contravariant functor from Δ_* to the category of groups, and similarly for simplicial rings, simplicial modules, and so on. One can even define simplicial spaces as contravariant functors from Δ_* to the category of topological spaces and continuous maps.

For any space X there is an associated rather large simplicial set $S(X)$, the singular complex of X, whose n-simplices are all the continuous maps $\Delta^n \to X$. For a morphism $g:\Delta_k \to \Delta_n$ the induced map g^* from n-simplices of $S(X)$ to k-simplices of $S(X)$ is obtained by composition with $g_*:\Delta^k \to \Delta^n$. We introduced $S(X)$ in §2.1 in connection with the definition of singular homology and described it as a Δ-complex, but in fact it has the additional structure of a simplicial set.

In a similar but more restricted way, an $s\Delta$-complex X gives rise to a simplicial set $\Delta(X)$ whose k-simplices are all the simplicial maps $\Delta^k \to X$. These are uniquely expressible as compositions $\sigma_\alpha q:\Delta^k \to \Delta^n \to X$ of simplicial surjections q (preserving orderings of vertices) with characteristic maps of simplices of X. The maps g^* are obtained just as for $S(X)$, by composition with the maps $g_*:\Delta^k \to \Delta^n$. These examples $\Delta(X)$ in fact account for all simplicial sets:

Proposition A.18. *Every simplicial set is isomorphic to one of the form $\Delta(X)$ for some $s\Delta$-complex X which is unique up to isomorphism.*

Here an isomorphism of simplicial sets means an isomorphism in the category of simplicial sets, where the morphisms are natural transformations between contravariant functors from Δ_* to the category of sets. This translates into just what one would expect, maps sending n-simplices to n-simplices that commute with the maps g^*. Note that the proposition implies in particular that a nonempty simplicial set contains simplices of all dimensions since this is evidently true for $\Delta(X)$. This is also easy to deduce directly from the definition of a simplicial set. Thus simplicial sets are in a certain sense large infinite objects, but the proposition says that their essential geometrical core, an $s\Delta$-complex, can be much smaller.

Proof: Let Y be a simplicial set, with Y_n its set of n-simplices. A simplex τ in Y_n is called **degenerate** if it is in the image of $g^*: Y_k \to Y_n$ for some noninjective $g: \Delta_n \to \Delta_k$. Since g can be factored as a surjection followed by an injection, there is no loss in requiring g to be surjective. For example, in $\Delta(X)$ the degenerate simplices are those that are the simplicial maps $\Delta^n \to X$ that are not injective on the interior of Δ^n. Thus the main difference between X and $\Delta(X)$ is the degenerate simplices.

Every degenerate simplex of Y has the form $g^*(\tau)$ for some nondegenerate simplex τ and surjection $g: \Delta_n \to \Delta_k$. We claim that such a g and τ are unique. For suppose we have $g_1^*(\tau_1) = g_2^*(\tau_2)$ with τ_1 and τ_2 nondegenerate and $g_1: \Delta_n \to \Delta_{k_1}$ and $g_2: \Delta_n \to \Delta_{k_2}$ surjective. Choose order-preserving injections $h_1: \Delta_{k_1} \to \Delta_n$ and $h_2: \Delta_{k_2} \to \Delta_n$ with $g_1 h_1 = \mathbb{1}$ and $g_2 h_2 = \mathbb{1}$. Then $g_1^*(\tau_1) = g_2^*(\tau_2)$ implies that $h_2^* g_1^*(\tau_1) = h_2^* g_2^*(\tau_2) = \tau_2$ and $h_1^* g_2^*(\tau_2) = h_1^* g_1^*(\tau_1) = \tau_1$, so the nondegeneracy of τ_1 and τ_2 implies that $g_1 h_2$ and $g_2 h_1$ are injective. This in turn implies that $k_1 = k_2$ and $g_1 h_2 = \mathbb{1} = g_2 h_1$, hence $\tau_1 = \tau_2$. If $g_1 \neq g_2$ then $g_1(i) \neq g_2(i)$ for some i, and if we choose h_1 so that $h_1 g_1(i) = i$, then $g_2 h_1 g_1(i) = g_2(i) \neq g_1(i)$, contradicting $g_2 h_1 = \mathbb{1}$ and finishing the proof of the claim.

Just as we reconstructed a Δ-complex from its categorical description, we can associate to the simplicial set Y an $s\Delta$-complex $|Y|$, its **geometric realization**, by setting

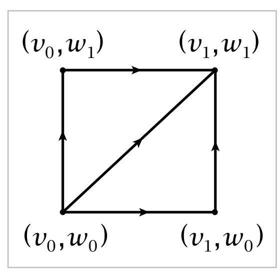
$$|Y| = \coprod_n (Y_n \times \Delta^n) / (g^*(y), z) \sim (y, g_*(z))$$

for $(y, z) \in Y_n \times \Delta^k$ and $g: \Delta_k \to \Delta_n$. Since every g factors canonically as a surjection followed by an injection, it suffices to perform the indicated identifications just when g is a surjection or an injection. Letting g range over surjections amounts to collapsing each simplex onto a unique nondegenerate simplex by a unique projection, by the claim in the preceding paragraph, so after performing the identifications just for surjections we obtain a collection of disjoint simplices, with one n-simplex for each nondegenerate n-simplex of Y. Then doing the identifications as g varies over injections attaches these nondegenerate simplices together to form an $s\Delta$-complex, which is $|Y|$. The quotient map from the collection of disjoint simplices to $|Y|$ gives the collection of distinguished characteristic maps for the cells of $|Y|$.

If we start with an $s\Delta$-complex X and form $|\Delta(X)|$, then this is clearly the same as X. In the other direction, if we start with a simplicial set Y and form $\Delta(|Y|)$ then there is an evident bijection between the n-simplices of these two simplicial sets, and this commutes with the maps g^* so the two simplicial sets are equivalent. ◻

As we observed in the preceding proof, the geometric realization $|Y|$ of a simplicial set Y can be built in two stages, by first collapsing all degenerate simplices by making the identifications $(g^*(y),z) \sim (y,g_*(z))$ as g ranges over surjections, and then glueing together these nondegenerate simplices by letting g range over injections. We could equally well perform these two types of indentifications in the opposite order. If we first do the identifications for injections, this amounts to regarding Y as a category-theoretic Δ-complex Y_Δ by restricting Y, regarded as a functor from Δ_* to sets, to the subcategory of Δ_* consisting of injective maps, and then taking the geometric realization $|Y_\Delta|$ to produce a geometric Δ-complex. After doing this, if we perform the identifications for surjections g we obtain a natural quotient map $|Y_\Delta| \to |Y|$. This is a homotopy equivalence, but we will not prove this fact here. The Δ-complex $|Y_\Delta|$ is sometimes called the thick geometric realization of Y.

Since simplicial sets are very combinatorial objects, many standard constructions can be performed on them. A good example is products. For simplicial sets X and Y there is an easily-defined product simplicial set $X \times Y$, having $(X\times Y)_n = X_n \times Y_n$ and $g^*(x,y) = (g^*(x), g^*(y))$. The nice surprise about this definition is that it is compatible with geometric realization: the realization $|X\times Y|$ turns out to be homeomorphic to $|X|\times|Y|$, the product of the CW complexes $|X|$ and $|Y|$ (with the compactly generated CW topology). The homeomorphism is just the product of the maps $|X\times Y| \to |X|$ and $|X\times Y|\to|Y|$ induced by the projections of $X\times Y$ onto its two factors. As a very simple example, consider the case that X and Y are both $\Delta(\Delta^1)$. Letting $[v_0,v_1]$ and $[w_0,w_1]$ be the two copies of Δ^1, the product $X\times Y$ has two nondegenerate 2-simplices:



$$([v_0,v_1,v_1],[w_0,w_0,w_1]) = [(v_0,w_0),(v_1,w_0),(v_1,w_1)]$$
$$([v_0,v_0,v_1],[w_0,w_1,w_1]) = [(v_0,w_0),(v_0,w_1),(v_1,w_1)]$$

These subdivide the square $\Delta^1\times\Delta^1$ into two 2-simplices. There are five nondegenerate 1-simplices in $X\times Y$, as shown in the figure. One of these, the diagonal of the square, is the pair $([v_0,v_1],[w_0,w_1])$ formed by the two nondegenerate 1-simplices $[v_0,v_1]$ and $[w_0,w_1]$, while the other four are pairs like $([v_0,v_0],[w_0,w_1])$ where one factor is a degenerate 1-simplex and the other is a nondegenerate 1-simplex. Obviously there are no nondegenerate n-simplices in $X\times Y$ for $n > 2$.

It is not hard to see how this example generalizes to the product $\Delta^p\times\Delta^q$. Here one obtains the subdivision of the product into $(p+q)$-simplices described in §3.B in terms of the shuffling operation. Once one understands the case of a product of simplices, the general case easily follows.

One could also define unordered $s\Delta$-complexes in a similar way to unordered Δ-complexes, and then work out the 'simplicial set' description of these objects. However, this sort of structure is more cumbersome to work with and has not been used much.