Solution to Exercise 1 in Section 3.C.

The CW complex hypothesis will be used only to have the homotopy extension property available when needed.

We are given a multiplication map $\mu : X \times X \to X$ with an element $e \in X$ such that the maps $x \mapsto \mu(e,x)$ and $x \mapsto \mu(x,e)$ are homotopic to the identity map of X. By an application of the homotopy extension property we can deform μ so that one of these maps, say $x \mapsto \mu(e,x)$, is actually equal to the identity. We can thus assume that $\mu(e,x) = x$ for all x from now on.

By assumption we have a homotopy $f_t : X \to X$ from $x \mapsto \mu(x,e)$ to the identity. Let $\eta : I \to X$ be defined by $\eta(t) = f_t(e)$. This is a loop at the basepoint e. If η represents the trivial element of $\pi_1(X,e)$, then we can use the homotopy extension property to homotope f_t to be a basepoint-preserving homotopy from $x \mapsto \mu(x,e)$ to the identity. Once f_t is basepoint-preserving, we can extend it to a homotopy $F_t : X \vee X \to X$ with $F_t(e,x) = x$ and $F_t(x,e) = f_t(x)$ for all x. Extending F_t to a homotopy defined on all of $X \times X$ with $F_0 = \mu$, we then have a homotopy from μ to a new μ with the desired property that $\mu(e,x) = x = \mu(x,e)$ for all x.

It remains to arrange that the loop η is nullhomotopic. For γ an arbitrary loop at the basepoint e, consider the homotopy $g_t : X \to X$, $g_t(x) = \mu(\gamma(t),x)$. This has g_0 and g_1 equal to the identity, and $g_t(e)$ traces out a loop γ'. The formula $f_s(\gamma(t))$ defines a homotopy from $f_0(\gamma(t)) = \mu(\gamma(t),e) = g_t(e) = \gamma'(t)$ to $f_1(\gamma(t)) = \gamma(t)$ with $f_s(\gamma(0)) = f_s(\gamma(1)) = f_s(e) = \eta(s)$, so we have $[\gamma'] = [\eta][\gamma][\eta]^{-1}$ in $\pi_1(X,e)$. Thus by a suitable choice of γ we can construct a homotopy g_t from the identity to itself such that $g_t(e)$ traces out a loop representing any given element of $\pi_1(X,e)$. In particular we can realize the element $[\eta]^{-1}$ by taking γ to be the inverse loop of η. Then if we follow the homotopy f_t by the homotopy g_t we obtain a new homotopy f_t that still goes from $x \mapsto \mu(x,e)$ to the identity and has $f_t(e)$ representing the trivial element of $\pi_1(X,e)$. By the previous paragraph this finishes the proof.