The Classification of 3-Manifolds — A Brief Overview

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Our aim here is to sketch the very nice conjectural picture of the classification of closed orientable 3-manifolds that emerged around 1980 as a consequence of geometric work of Thurston and more topological results of Jaco-Shalen and Johannson at the end of a long chain of topological developments going back 50 years or more.

A pleasant feature of 3-manifolds, in contrast to higher dimensions, is that there is no essential difference between smooth, piecewise linear, and topological manifolds. It was shown by Bing and Moise in the 1950s that every topological 3-manifold can be triangulated as a simplicial complex whose combinatorial type is unique up to subdivision. And every triangulation of a 3-manifold can be taken to be a smooth triangulation in some differential structure on the manifold, unique up to diffeomorphism. Thus every topological 3-manifold has a unique smooth structure, and the classifications up to diffeomorphism and homeomorphism coincide. In what follows we will deal with smooth manifolds and diffeomorphisms between them.

When we say “manifold" we will always mean “connected manifold". For the sake of simplicity we restrict attention to orientable manifolds, although with more trouble the nonorientable case could be covered as well. The primary focus will be on manifolds that are closed, that is, compact and without bounday, but from time to time it will be natural to consider also compact manifolds with nonempty boundary.

The most powerful of the standard invariants of algebraic topology for distinguishing 3-manifolds is the fundamental group. This determines all the homology groups of a closed orientable 3-manifold M. Namely, $H_1(M)$ is the abelianization of $\pi_1(M)$, and by Poincaré duality $H_2(M)$ is isomorphic to $H^1(M)$ which is $H_1(M)$ mod torsion by the universal coefficient theorem. Since M is closed and orientable, $H_3(M)$ is $\mathbb{Z}$. All higher homology groups are zero, of course.

In particular, if M is a simply-connected closed 3-manifold, then M has the same homology groups as the 3-sphere S^3. In fact M is homotopy equivalent to S^3. For by the Hurewicz theorem $\pi_2(M) = H_2(M) = 0$ and $\pi_3(M) = H_3(M) = \mathbb{Z}$. A generator of $\pi_3(M)$ is represented by a map $S^3 \to M^3$ of degree one, inducing an isomorphism on $H_3 = \pi_3$. This means we have a map $S^3 \to M$ of simply-connected simplicial complexes inducing isomorphisms on all homology groups, so by Whitehead's theorem the map is a homotopy equivalence. Thus a simply-connected closed 3-manifold is a homotopy sphere. The Poincaré conjecture asserts that S^3 is the only such manifold. (It seems that Poincaré himself did not explicitly formulate this as a conjecture.) As we shall see shortly, in the nonsimply-connected case there are many examples of closed 3-manifolds with isomorphic fundamental groups that are not diffeomorphic.

Prime Decomposition

The first reduction of the classification problem is due to Kneser around 1930. If a 3-manifold M contains an embedded sphere S^2 (disjoint from the boundary of M, if M has a nonempty boundary) separating M into two components, we can split M along this S^2 into manifolds M_1 and M_2 each having this sphere as a component of its boundary. We can then fill in these two boundary spheres with balls to produce manifolds N_1 and N_2 that are closed if M was closed. One says M is the *connected sum* of N_1 and N_2, or in symbols, $M = N_1 \sharp N_2$. This splitting operation is commutative by definition, and it is not hard to check that it is also associative. One rather trivial possibility for the splitting 2-sphere is as the boundary of a ball in M, and this gives the decomposition $M = M \sharp S^3$. If this is the only way that M splits as a connected sum, then M is said to be *prime*. This is equivalent to saying that every separating 2-sphere in M bounds a ball in M.

A fundamental theorem of Alexander from 1924 says that every 2-sphere in S^3 bounds a ball on each side. (We are assuming smoothness here, which rules out local pathology as in the Alexander horned sphere.) Hence S^3 is prime. Note that if this were not true then there would be no prime 3-manifolds, although perhaps something could be salvaged by redefining primeness to allow for units, as in algebra. Fortunately this is not necessary. Of course, if S^3 were not prime, then the Poincaré conjecture would be false since a connected summand of a simply-connected manifold is also simply-connected, as $\pi_1(M \sharp N)$ is the free product $\pi_1(M) * \pi_1(N)$ by van Kampen's theorem.

KNESER'S THEOREM. *Every compact orientable* 3*-manifold* M *factors as a connected sum of primes,* $M = P_1 \sharp \cdots \sharp P_n$*, and this decomposition is unique up to insertion or deletion of* S^3 *summands.*

If one assumes the Poincaré conjecture, then the existence of a prime decomposition follows easily from the algebra fact that a finitely-generated group cannot be split as a free product of an arbitrarily large number of nontrivial factors. Kneser's proof is noteworthy for being independent of the Poincaré conjecture. If there were homotopy spheres other than S^3, they too could be decomposed uniquely as sums of prime homotopy spheres.

At first glance one might think that the unique prime decomposition immediately reduced the classification of closed oriented 3-manifolds to the classification of the prime manifolds, but there is a small subtlety here: The prime factors P_i are uniquely determined by M, but the converse need not be true. Given two manifolds P and Q

there are are two potentially different ways of forming their connected sum since after removing the interior of a ball from each of P and Q, one could glue the two resulting boundary spheres together by a diffeomorphism that either preserves or reverses chosen orientations of the two 2-spheres. If either P or Q has a self-diffeomorphism that reverses orientation, then the two gluings produce diffeomorphic connected sums, but otherwise they will not. One way to avoid this ambiguity it to talk about manifolds that are not just orientable but oriented. Then there is a unique way to form connected sums respecting orientations, and Kneser's theorem remains true for oriented manifolds. This reduces the classification problem to classifying oriented prime manifolds, which in particular involves deciding for each prime orientable manifold P whether there is a diffeomorphism $P \to P$ that reverses orientation.

It turns out that there exist many prime 3-manifolds that do not have orientation-reversing self-diffeomorphisms. The two ways of forming the connected sum of two such manifolds will produce nondiffeomorphic manifolds with isomorphic fundamental groups since the isomorphism $\pi_1(P \sharp Q) \approx \pi_1(P) * \pi_1(Q)$ is valid no matter how the two summands are glued together.

Prime 3-manifolds that are closed and orientable can be lumped broadly into three classes:

Type I: finite fundamental group. For such a manifold M the universal cover $\widetilde{M}$ is simply-connected and closed, hence a homotopy sphere. All the known examples are *spherical* 3-manifolds, of the form $M = S^3/\Gamma$ for Γ a finite subgroup of $SO(4)$ acting freely on S^3 by rotations. Thus S^3 is the universal cover of M and $\Gamma = \pi_1(M)$. Spherical 3-manifolds were explicitly classified in the 1930s, using the fact that $SO(4)$ is a 2-sheeted covering group of $SO(3) \times SO(3)$, so the finite subgroups of $SO(4)$ can be determined from the well-known finite subgroups of $SO(3)$. The examples with Γ cyclic are known as *lens spaces*, and there are also a few infinite families with Γ noncyclic, including Poincaré's famous homology sphere, which can be defined as the coset space $SO(3)/I$ where I is the group of rotational symmetries of the icosahedron, of order 60. Since the universal cover of $SO(3)$ is S^3, a 2-sheeted cover, the quotient $SO(3)/I$ is S^3/Γ for Γ the preimage of I in S^3, a group of order 120.

There is a lens space $L_{p/q}$ for each fraction p/q between 0 and 1. The fundamental group of $L_{p/q}$ is the cyclic group $\mathbb{Z}_q$, and two lens spaces $L_{p/q}$ and $L_{p'/q'}$ are diffeomorphic if and only if $q = q'$ and p' is congruent to $\pm p^{\pm 1}$ mod q. For example, if q is prime one obtains at least $(q-1)/4$ nondiffeomorphic lens spaces with the same fundamental group $\mathbb{Z}_q$. If orientations are taken into account, then the condition $p' \equiv \pm p^{\pm 1}$ mod q becomes $p' \equiv p^{\pm 1}$ mod q, so there are many lens spaces without orientation-reversing self-diffeomorphisms, for example $L_{1/3}$. The spherical manifolds that are not lens spaces are determined up to diffeomorphism by their fundamental group.

It is an old conjecture that spherical 3-manifolds are the only closed 3-manifolds with finite π_1. This conjecture can be broken into two parts. The first is to show that the universal cover of such a manifold is S^3, which is the Poincaré conjecture, and then the second part is to show that any free action of a finite group on S^3 is equivalent to an action by isometrics of S^3 with its standard metric. In particular this involves showing that the group is a subgroup of $SO(4)$. There have been many partial results on this second part over the years, but it has proved to be a very difficult problem.

Type II: infinite cyclic fundamental group. It turns out that there is only one prime closed 3-manifold satisfying this condition, and that is $S^1 \times S^2$. This is also the only orientable 3-manifold that is prime but not irreducible, where a 3-manifold M is *irreducible* if every 2-sphere in M bounds a ball in M. For if M is reducible but prime it must contain a nonseparating S^2. This has a product neighborhood $S^2 \times I$, and the union of this neighborhood with a tubular neighborhood of an arc joining $S^2 \times \{0\}$ to $S^2 \times \{1\}$ in the complement of $S^2 \times I$ is diffeomorphic to the complement of a ball in $S^1 \times S^2$. This says that M has $S^1 \times S^2$ as a connected summand, so by primeness $M = S^1 \times S^2$.

Another special feature of $S^1 \times S^2$ is that it is the only prime orientable 3-manifold with nontrivial π_2. This is a consequence of the Sphere Theorem, which says that for an orientable 3-manifold M, if $\pi_2(M)$ is nonzero then there is an embedded sphere in M that represents a nontrivial element of $\pi_2(M)$. This sphere cannot bound a ball, so M is reducible, hence if it is prime it must be $S^1 \times S^2$.

Type III: infinite noncyclic fundamental group. Such a manifold M is a $K(\pi, 1)$, or in other words, its universal cover is contractible. More generally, any irreducible 3-manifold M, not necessarily closed, with $\pi_1(M)$ infinite is a $K(\pi, 1)$. For the universal cover $\widetilde{M}$ is simply-connected and has trivial homology groups: By the Hurewicz theorem $H_2(\widetilde{M}) = \pi_2(\widetilde{M})$ and $\pi_2(\widetilde{M}) = \pi_2(M) = 0$ by the Sphere Theorem and the irreducibility of M. Finally, $H_3(\widetilde{M}) = 0$ since $\widetilde{M}$ is noncompact, and all higher homology groups vanish since $\widetilde{M}$ is a 3-manifold. So Whitehead's theorem implies that $\widetilde{M}$ is contractible.

The homotopy type of a $K(\pi, 1)$ is uniquely determined by its fundamental group, so for a closed 3-manifold M to be a $K(\pi, 1)$ imposes strong restrictions on the group $\pi_1(M) = \pi$. For example, it implies that $\pi_1(M)$ must be torsionfree, since the covering space of M corresponding to a nontrivial finite cyclic subgroup $\mathbb{Z}_n$ would be a finite-dimensional $K(\mathbb{Z}_n, 1)$ CW complex, but this cannot exist since the homology groups of a $K(\mathbb{Z}_n, 1)$ are nonzero in infinitely many dimensions, as one can choose an infinite-dimensional lens space as a $K(\mathbb{Z}_n, 1)$. One can also see that the only free abelian group $\mathbb{Z}^n$ that can occur as $\pi_1(M)$ is $\mathbb{Z}^3$, the fundamental group of the 3-torus, since the n-torus is a $K(\mathbb{Z}^n, 1)$ and this has H_n nonzero and H_i zero for $i > n$. In particular, there is no closed orientable 3-manifold $K(\pi, 1)$ with infinite cyclic fundamental group, which shows that $S^1 \times S^2$ is indeed the only prime closed orientable 3-manifold with π_1 infinite cyclic.

Since the homotopy type of an irreducible closed $K(\pi,1)$ 3-manifold is determined by its fundamental group, one may then ask whether the fundamental group in fact determines the manifold completely. This is the 3-dimensional case of the Borel conjecture, that a closed n-manifold that is a $K(\pi,1)$ is determined up to homeomorphism by its fundamental group. (In high dimensions it is important to say "homeomorphism" here rather than "diffeomorphism".) No counterexamples are known in any dimension. In the 3-dimensional case, Waldhausen proved the conjecture for a large class of manifolds known as *Haken manifolds*. These are the irreducible compact orientable 3-manifolds M that contain an embedded nonsimply-connected compact orientable surface S for which the map $\pi_1(S) \to \pi_1(M)$ induced by the inclusion $S \hookrightarrow M$ is injective. Such a surface S is said to be *incompressible*. In case S has a nonempty boundary ∂S one requires that S is properly embedded in M, meaning that $S \cap \partial M = \partial S$. (A properly embedded disk that does not split off a ball from M is usually considered to be incompressible as well.) It turns out that for a Haken manifold M it is possible to perform a finite sequence of splitting operations along incompressible surfaces, so that in the end the manifold M has been reduced to a finite collection of disjoint balls. This sequence of splittings is called a *hierarchy* for M. In favorable cases a hierarchy can be used to construct proofs by induction over the successive steps in the hierarchy, and in particular this is how Waldhausen's theorem is proved.

It is still not clear whether a "typical" closed irreducible 3-manifold with infinite π_1 is a Haken manifold. Some manifolds that are Haken manifolds are:

- Products $F \times S^1$ where F is a closed orientable surface.
- More generally, fiber bundles over S^1 with fiber a closed orientable surface.
- More generally still, irreducible M with $H_1(M)$ infinite.

There are various infinite families of closed irreducible 3-manifolds with infinite π_1 that have been shown not to be Haken manifolds, but they all seem to be manifolds that are in some sense "small". Perhaps all manifolds that are sufficiently large are Haken. (Indeed, the original name for Haken manifolds was "sufficiently large", before Haken became famous for the Four-Color Theorem.) It is still an open question whether closed irreducible 3-manifolds with infinite π_1 always have a finite-sheeted covering space that is Haken.

Even if the Borel conjecture were proved for Type III prime 3-manifolds, this would still be far from an explicit classification. One would want to know exactly which groups occur as fundamental groups of these manifolds, and one would want to have an efficient way of distinguishing one such group from another. But perhaps there are just too many different manifolds to make such an explicit classification feasible.

Seifert Manifolds

There is a special family of 3-manifolds called *Seifert manifolds* for which an explicit classification was made early in the history of 3-manifolds, in the 1930s. These manifolds are “singular fiber bundles" with base space a compact surface and fibers circles. The fibering is locally trivial, as in an ordinary fiber bundle, except for a finite number of isolated “multiple" fibers where the local model is the following. Start with a product $D^2 \times I$ fibered by the intervals $\{x\} \times I$. If we glue the two ends $D^2 \times \{0\}$ and $D^1 \times \{1\}$ together by the identity map of D^2 we would then have the standard product fibering of the solid torus $D^2 \times S^1$. Instead of doing this, we glue the two ends together using a rotation of D^2 through an angle of $2\pi p/q$ for some pair of relatively prime integers p and q with $q \geq 2$. The resulting quotient manifold is still $D^2 \times S^1$ and the core interval $\{0\} \times I$ closes up to form a circle, but all the other intervals $\{x\} \times I$ do not immediately close to form circles. Instead one has to follow along q of these intervals before they close up to a circle. Thus we have the solid torus decomposed into disjoint circles which are all approximately parallel locally, but the central circle has multiplicty q in the sense that the projection of each nearby circle onto this core circle is a q-to-1 covering space. If the core circle is deleted one would have an actual fiber bundle, so there is just a single isolated singular fiber in this fibering of $D^2 \times S^1$ by circles. In a compact Seifert manifold M one can have a finite number of such singular fibers, all disjoint from ∂M, which, if it is nonempty, consists of tori with product fiberings by circles. The base space of the fibering is the quotient space of M obtained by identifying each circle fiber to a point. This is still a compact surface, even at the images of the singular fibers. We are assuming that M is orientable, but the base surface can be orientable or not. Since the singular fibers are isolated and project to isolated points in the base surface, they have no effect on orientability of the base.

The data specifying a Seifert fibering consists of:

- The topological type of the base surface.
- The number of multiple fibers and the fractions p/q specifying the local fibering near each multiple fiber. In fact the value of p/q modulo 1 is enough to specify the local fibering.
- In case the base surface is closed, an “Euler number" specifying how twisted the fibering is. As with ordinary circle bundles, this can be defined as the obstruction to a section of the bundle.

Thus one obtains an explicit classification of all the different Seifert fiberings. It turns out that in most cases the Seifert fibering of a Seifert manifold is unique up to diffeomorphism, and in fact up to isotopy. The exceptions can be listed quite explicitly, so one has a very concrete classification of all Seifert manifolds up to diffeomorphism. This includes also the information of which ones admit orientation-reversing diffeomorphisms.

It is a happy accident that all the spherical 3-manifolds are Seifert manifolds. The base surface is S^2 in each case, and there are at most three multiple fibers. The Type II manifold $S^1 \times S^2$ is of course also a Seifert manifold, via the product fibering. Most Seifert manifolds are of Type III. The only Seifert manifold that is not prime is $\mathbb{R}P^3 \sharp \mathbb{R}P^3$, the sum of two copies of real projective 3-space. (This manifold happens to have $S^1 \times S^2$ as a 2-sheeted covering space, the only instance of a prime manifold covering a nonprime manifold.)

Torus Decomposition

After the prime decomposition, it suffices to classify irreducible 3-manifolds. These cannot be simplified by splitting along spheres, but one may ask whether they can be simplified by splitting along the next-simplest surfaces, embedded tori. Splitting along a torus that lies inside a ball or bounds a solid torus is not likely to produce a simpler manifold, as one can see already in the case of knotted tori in S^3. Such tori are obviously compressible, however, so it seems more promising to try splitting along incompressible tori. In fact, in an irreducible manifold M any embedded torus that does not lie in a ball in M and does not bound a solid torus in M must be incompressible. This is an easy consequence of the Loop Theorem, which says that for a properly embedded compact orientable surface $S \subset M$, if the map $\pi_1(S) \to \pi_1(M)$ is not injective then there is an embedded disk $D \subset M$ with $D \cap S = \partial D$ such that the circle ∂D represents a nontrivial element of the kernel of the map $\pi_1(S) \to \pi_1(M)$. (Often this theorem is stated in the equivalent form that S is equal to ∂M or a subsurface of ∂M.)

Splitting along incompressible tori turns out to work very nicely:

> TORUS DECOMPOSITION THEOREM (JACO-SHALEN, JOHANNSON). *If M is an irreducible compact orientable manifold, then there is a collection of disjoint incompressible tori $T_1, \cdots, T_n$ in M such that splitting M along the union of these tori produces manifolds M_i which are either Seifert-fibered or atoroidal — every incompressible torus in M_i is isotopic to a torus component of ∂M_i. Furthermore, a minimal such collection of tori T_j is unique up to isotopy in M.*

The collection of tori T_j could be empty. This will happen if M itself is Seifert-fibered or atoroidal. It is possible to characterize the tori T_j intrinsically as the incompressible tori $T \subset M$ that are *isolated* in the sense that every other incompressible torus in M can be isotoped to be disjoint from T. Notice the strength of the uniqueness statement: up to isotopy, not just up to diffeomorphism of M. This differs from the prime decomposition where the spheres giving a splitting into primes are not at all unique, even up to diffeomorphism of the manifold. Only the prime factors are unique.

The M_i's that are Seifert-fibered could be further split along incompressible tori into atoroidal pieces, but the resulting atoroidal pieces are usually not unique. To get uniqueness it is essential to choose the collection of T_j's to be minimal. This subtlety in the uniqueness statement is probably the reason why this theorem was discovered only in the 1970s, since in hindsight one can see that it could have been proved in the 1930s when Seifert manifolds were first studied.

If the collection of tori T_j is nonempty, the manifolds M_i will have torus boundary components. Unlike in the prime decomposition where we split along spheres and there was a canonical way to fill in the newly-created boundary spheres with balls, when we split along tori there is no canonical way to fill in the resulting boundary tori and thereby stay within the realm of closed manifolds if the original M was a closed manifold. The natural thing to try is to fill in the new boundary tori with solid tori $S^1 \times D^2$, but there are infinitely many essentially different ways to do this since the glueing is achieved by a diffeomorphism of a torus and the group of isotopy classes of diffeomorphisms of a torus is $GL_2(\mathbb{Z})$, and only a relatively small number of these diffeomorphisms extend over a solid torus. (We are essentially talking about Dehn surgery here, which will be discussed in more detail later.) So it is best just to leave the M_i as manifolds with boundary tori. This means that even if one is primarily interested in closed manifolds, one is really forced to broaden one's domain to include manifolds with boundary tori.

The manifold M determines the M_i's uniquely, but there are many choices for how to glue the M_i's together to reconstruct M. In the prime decomposition one had only to worry about orientations to specify how to glue the pieces together, but here the glueings are by diffeomorphisms of tori, so again there is a wide choice of elements of $GL_2(\mathbb{Z})$, or $SL_2(\mathbb{Z})$ if orientations are specified, determining how to glue the pieces M_i together. For a complete classification of the various manifolds that can be obtained by glueing together a fixed collection of M_i's one needs to know which collections of diffeomorphisms of the boundary tori of each M_i extend to diffeomorphisms of M_i. In case all the M_i's happen to be Seifert manifolds this can be figured out, so the classification of the resulting manifolds M is known explicitly. This was first worked out by Waldhausen, who called these manifolds *graph manifolds*, in reference to the graph that describes the combinatorial pattern for glueing together the M_i's, the graph having a vertex for each M_i and an edge for each T_j.

After the Torus Decomposition Theorem, the remaining big problem is to classify "general" prime 3-manifolds, those which are atoroidal and not Seifert-fibered. Examples of such manifolds can be found among the manifolds M which fiber over S^1 with fiber a closed orientable surface F of genus at least 2. Such a bundle is determined by a diffeomorphism $\varphi : F \to F$. If φ is isotopic to a diffeomorphism of finite order, then M has an evident Seifert fibering, with circle fibers transverse to the surface fibers F. Another special case is if φ leaves invariant some finite collection of disjoint nontrivial circles in F, since such a collection gives rises to a set of incompressible tori in M transverse to the fibers. In all remaining cases it turns out that M is atoroidal and not a Seifert manifold. These “general" φ's are the subject of Thurston's theory of pseudo-Anosov surface diffeomorphisms.

There are many more examples. For example, the complement of an open tubular neighborhood of a knot in S^3 is an irreducible manifold with torus boundary, and this manifold is atoroidal and not Seifert-fibered for perhaps 99 percent of the first million knots.

Geometric examples, analogous to spherical manifolds, are the *hyperbolic manifolds*, the quotients of hyperbolic 3-space $\mathbb{H}^3$ obtained by factoring out the action of a group Γ of isometries of $\mathbb{H}^3$, where the action is free and the quotient space $\mathbb{H}^3/\Gamma$ is compact, hence a closed manifold. More generally, one can allow the quotient $\mathbb{H}^3/\Gamma$ to be noncompact but have finite volume, since in this case the quotient is the interior of a compact manifold whose boundary consists of tori, such as would arise in a nontrivial torus decomposition of a closed manifold.

Hyperbolic manifolds are always irreducible and atoroidal, and no hyperbolic manifold can be Seifert-fibered. Amazingly, Thurston conjectured that the converse is also true:

HYPERBOLIZATION CONJECTURE: *Every irreducible atoroidal closed* 3-*manifold that is not Seifert-fibered is hyperbolic. Furthermore, the interior of every compact irreducible atoroidal nonSeifert-fibered* 3-*manifold whose boundary consists of tori is hyperbolic.*

Thurston proved this in many cases, for example for nonclosed manifolds, surface bundles, and more generally all Haken manifolds. In addition to these general theorems many concrete examples have been worked out using computer programs that have been developed for this purpose. The evidence in favor of the conjecture seems quite strong.

Hyperbolic manifolds must have infinite fundamental group since they have finite volume but their universal cover $\mathbb{H}^3$ has infinite volume. This means that the hyperbolization conjecture implies the Poincaré conjecture, as there are no counterexamples to the Poincaré conjecture among Seifert manifolds, and manifolds that contain incompressible tori have infinite fundamental group. One could make a more restricted form of the hyperbolization conjecture that would not imply the Poincaré conjecture by adding the hypothesis of infinite fundamental group.

If two hyperbolic 3-manifolds have isomorphic fundamental groups, then they are in fact isometric, according to the Mostow Rigidity Theorem. In particular, this says that hyperbolic structures are unique, if they exist. It also means that the hyperbolization conjecture implies the Borel conjecture in dimension 3, that $K(\pi, 1)$ 3-manifolds are determined up to homeomorphism by their fundamental groups.

Much is known about hyperbolic manifolds, so the hyperbolization conjecture, if true, would give a great deal of information about individual 3-manifolds. However, even if the hyperbolization conjecture were proved, we would still be some distance away from a really explicit classification of 3-manifolds since the number of hyperbolic manifolds is so large.

Dehn Surgery

It is tempting to try to get a more complete picture of the set of all closed orientable 3-manifolds by putting some sort of global structure on this set. The number of compact 3-manifolds is countable since there are just countably many finite simplicial complexes, so perhaps there is some kind of "variety" whose rational points correspond bijectively, in some meaningful way, with all the diffeomorphism classes of closed orientable 3-manifolds.

One possible way to implement this vague idea is to use Dehn surgery, which is defined in the following way. Fix a closed orientable manifold M and choose a link L in M consisting of n disjoint embedded circles $L_1, \cdots, L_n$. These have disjoint tubular neighborhoods $N(L_i)$ that are solid tori $S^1 \times D^2$. Remove the interiors of these solid tori from M and then glue the solid tori back in by means of diffeomorphisms $\partial N(L_i) \to \partial(M - int(N(L_i)))$. One can think of glueing in a solid torus $S^1 \times D^2$ as first glueing in a meridian disk $\{x\} \times D^2$ and then glueing in a ball. All that matters is how the meridian disk is glued in, since glueing in a ball is canonical. To specify how the disk is glued in it suffices to specify the curve its boundary attaches to, and in a torus this curve will be, up to isotopy, p times a meridian plus q times a longitude for some pair (p, q) of relatively prime integers, which can conveniently be regarded as a fraction, or slope, p/q, possibly $1/0$. Doing this for each solid torus $N(L_i)$ in turn, we obtain a manifold $M_L(p_1/q_1, \cdots, p_n/q_n)$ which is said to be obtained from M by *Dehn surgery* on L. This gives a set of manifolds parametrized by the rational n-torus $(\mathbb{Q} \cup \{1/0\})^n$. To give this set of manifolds a name, let us call it a "Dehn variety" $V(M, L)$. (There is an ambiguity in the choice of longitudes, but this just corresponds to a simple change of coordinates in $V(M, L)$.)

As an example, if we take M to be a product bundle $F \times S^1$ with base F a compact orientable surface and we choose L to be a collection of n fibers $\{x\} \times S^1$, then the Seifert manifolds obtained by replacing these fibers by possibly multiple fibers are almost exactly the same as the manifolds obtained by Dehn surgery on L. The only exceptions are the Dehn surgeries in which a meridian disk is glued in along a curve isotopic to a circle fiber of $F \times S^1$. These exceptional surgeries produce connected sums of Seifert manifolds with $S^1 \times S^2$. In a similar fashion, the Seifert manifolds over a nonorientable base surface can be obtained by Dehn surgery on fibers of a twisted circle bundle.

The first theorem about Dehn surgery is that if one fixes M and varies the link L and the slopes p_i/q_i, then one obtains all closed orientable 3-manifolds. In particular this happens when $M = S^3$ and one is doing Dehn surgery on links in the usual sense. No single $V(M, L)$ can contain all 3-manifolds since the homology groups of manifolds in $V(M, L)$ have bounded ranks. Any two $V(M, L)$'s are contained in a third, so the union of all Dehn varieties can be thought of as a sort of an infinite-dimensional Dehn variety which consists of all closed orientable 3-manifolds.

The complicating factor with Dehn varieties is of course that the surjections $(\mathbb{Q} \cup \{1/0\})^n \to V(M,L)$, $(p_1/q_1, \cdots, p_n/q_n) \mapsto M_L(p_1/q_1, \cdots, p_n/q_n)$, need not be injective, so Dehn varieties are not just rational tori but some kind of singular objects.

A special case is Dehn curves, the varieties $V(M,L)$ when L has just one component. Two manifolds in the same Dehn curve have a lot in common: the complement of a knot, in fact. Here are some interesting theorems and conjectures:

- No Dehn curve passes through S^3 more than once. (Theorem of Gordon-Luecke)
- More generally, no Dehn curve through S^3 passes through a homotopy sphere at another point of the curve. (The Property P conjecture)
- There is a unique Dehn curve though S^3 and $S^1 \times S^2$, the Dehn curve for the trivial knot in S^3. (Property R, proved by Gabai. The other points on this curve are lens spaces.)
- Every other Dehn curve through S^3 passes through manifolds with finite π_1 at most three times. (Theorem of Culler-Gordon-Luecke-Shalen. There are examples showing the number “three" is best possible.)

Not much seems to have been done yet with Dehn varieties of higher dimension.

References for further reading

As a primary source for the more basic topological aspects of the theory, including prime decomposition, torus decomposition, and Seifert manifolds, we suggest

- A Hatcher. *Basic Topology of 3-Manifolds*. Unpublished notes available online at http://www.math.cornell.edu/˜hatcher

This does not include Waldhausen’s theorem about Haken manifolds, which can be found in

- J Hempel. *3-Manifolds*. Annals of Math Studies 86. Princeton University Press, 1976.

A book that contains an exposition of the classification of spherical 3-manifolds is

- W Thurston. *Three-Dimensional Geometry and Topology*. Princeton University Press, 1997.