# Stable Homology of Spaces of Graphs

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**Starting point**: Galatius' theorem, the analog of the Madsen-Weiss theorem for $\mathrm{Aut}(F_n)$:

$$\Sigma_n \hookrightarrow \mathrm{Aut}(F_n) \text{ induces } H_i(\Sigma_n) \cong H_i(\mathrm{Aut}(F_n)) \text{ for } n >> i.$$

Equivalent form:

$$\lim_n B\mathrm{Aut}(F_n)^+ \simeq \Omega_0^\infty S^\infty, \quad \text{one component of } \Omega^\infty S^\infty$$

Will talk about two extensions:

(I) Relative version, for relative graphs — attach 0-cells and 1-cells to a fixed base space $X$.

(II) Handlebody version, for $d$-dimensional thickenings of graphs, $d \geq 3$.

Application: Analogs of Madsen-Weiss for certain 3-manifolds.

## Relative Graphs

For a fixed space $X$ consider graphs on $X$ — attach a finite graph to $X$ by identifying some of its vertices with points in $X$.

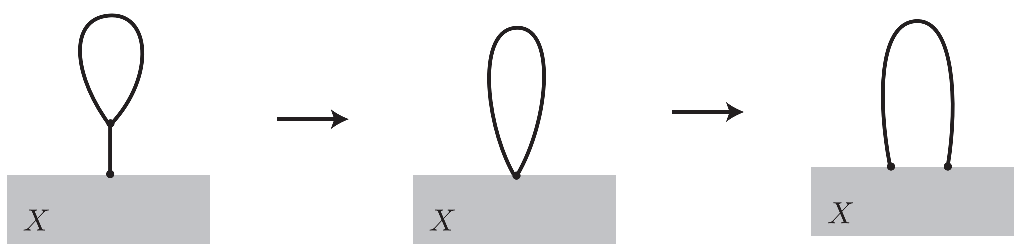
Special case: When $X$ is a point, graphs on $X$ are basepointed graphs.

Graphs on $X$ are the objects in a category $G(X)$ whose morphisms are compositions of collapsing subtrees and graph isomorphisms fixing $X$. Thus morphisms fix $X$.

Subcategory $G_n(X)$: restrict the objects to graphs $\simeq X \vee_n S^1 \operatorname{rel} X$, same morphisms. (Assume $X$ path-connected for simplicity.)

Topological category: attachments to $X$ can vary continuously.

<u>Example</u>: a morphism followed by a path of objects.




<u>Conjecture</u>: $BG_n(X) \simeq B\mathrm{HomEq}(X \vee_n S^1 \operatorname{rel} X)$.

True when $X = point$ via contractibility of basepointed version of Outer Space.

Galatius theorem: $\lim_n BG_n(point)^+ \simeq \Omega_0^\infty S^\infty$.

**<u>Theorem</u>**: $\lim_n BG_n(X)^+ \simeq \Omega_0^\infty S^\infty$.

Independent of $X$!

Special case: $X = K(\Gamma, 1)$, so $\mathrm{HomEq}(X \vee_n S^1 \,\mathrm{rel}\, X) \simeq \mathrm{Aut}(\Gamma * F_n \,\mathrm{rel}\, \Gamma)$. The Conjecture would then yield

$$\lim_n H_i(\mathrm{Aut}(\Gamma * F_n \,\mathrm{rel}\, \Gamma)) \cong \lim_n H_i(\Sigma_n)$$

**Sketch of proof**:

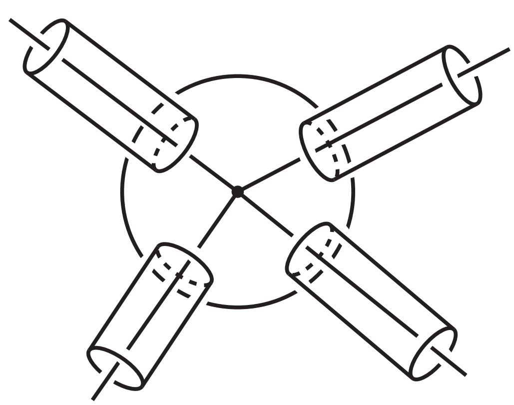
Follows general plan of Galatius' proof and later improvements by Galatius and Randal-Williams.

Start with a nice geometric model for $BG_n(X)$, a space of finite graphs in $\mathbb{R}^\infty$ with data on attaching to $X$.

Want the graphs to be embedded in $\mathbb{R}^k$ or $\mathbb{R}^\infty$ in such a way that they easily thicken to handlebodies:

- Graphs have smooth edges.
- Edges are linear near vertices.

Such graphs give *round handlebodies*: Thicken vertices to 0-handles which are round balls, truncated along disjoint disks where the 1-handles attach. Thicken the edges to 1-handles with round cross-sectional disks.



Allow variation of graphs by smooth isotopy, but also want to allow subtrees to shrink continuously to points.

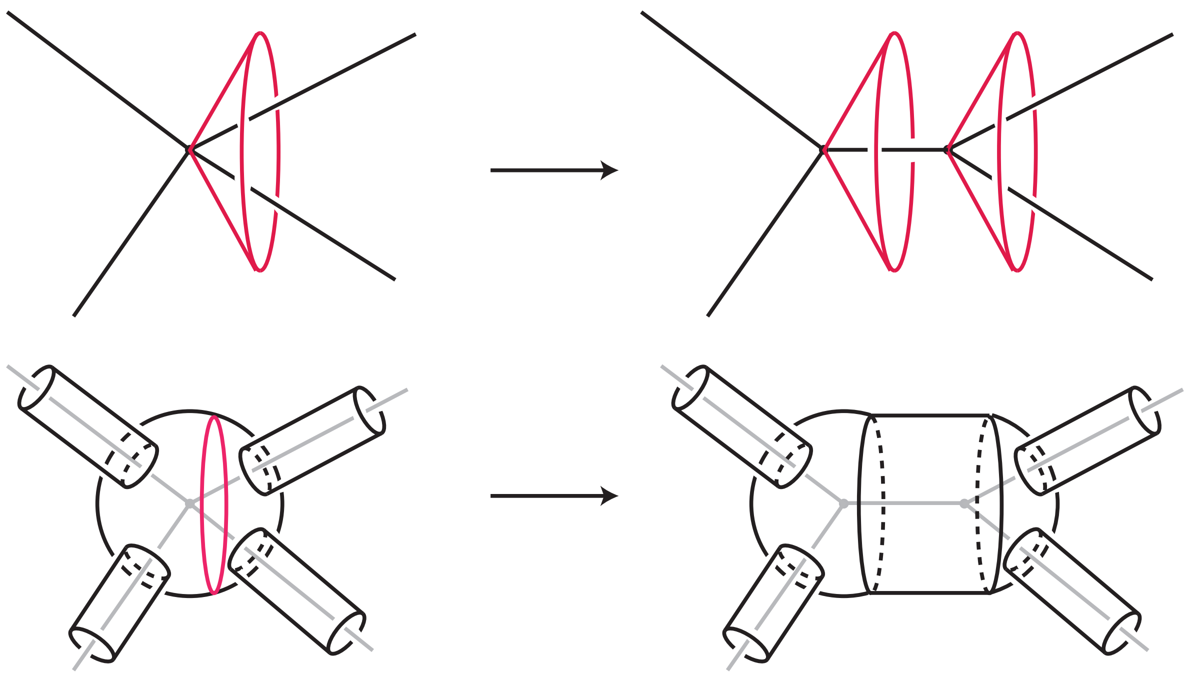
Two different ways to do this:

- Very general: Allow arbitrary motion of the subtree inside a shrinking ball. This is what Galatius did.
- Much more restrictive: Shrinking that can easily be thickened to handlebodies.

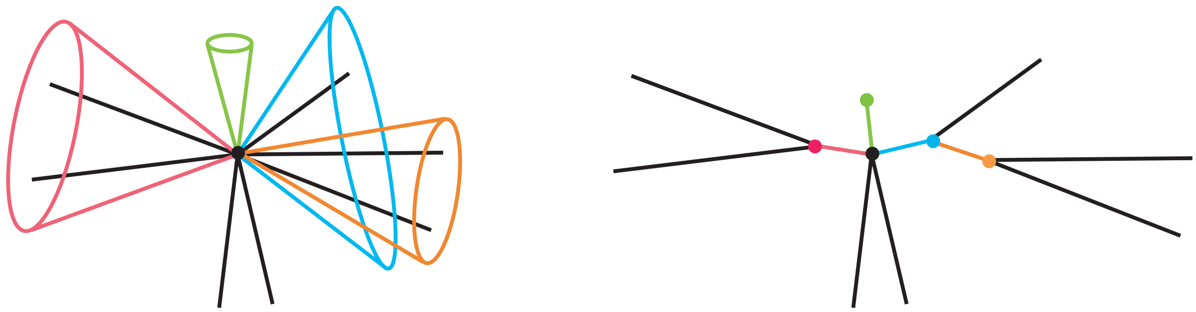
Will use the latter type, called *conical collapsing*.

Easier to describe the inverse operation: *conical expansion*.

<u>Example</u>:



More complicated example:

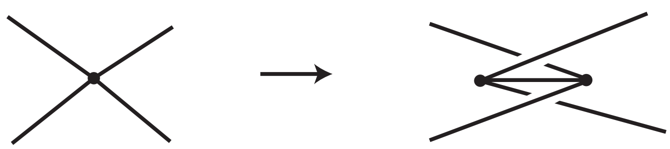


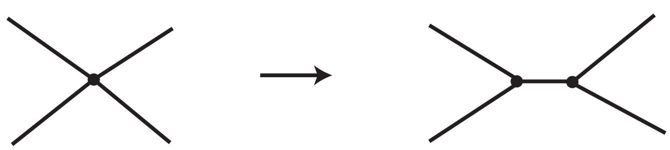
A conical expansion is determined by inserting cones such that:

- the vertices of the cones are at the given vertex of the graph
- cones are disjoint from the graph and from each other, except at the vertex
- cones can be nested.

Then translate the part of the graph inside a cone along the axis of the cone, with the vertex tracing out a new edge.

An example of a nonconical expansion:

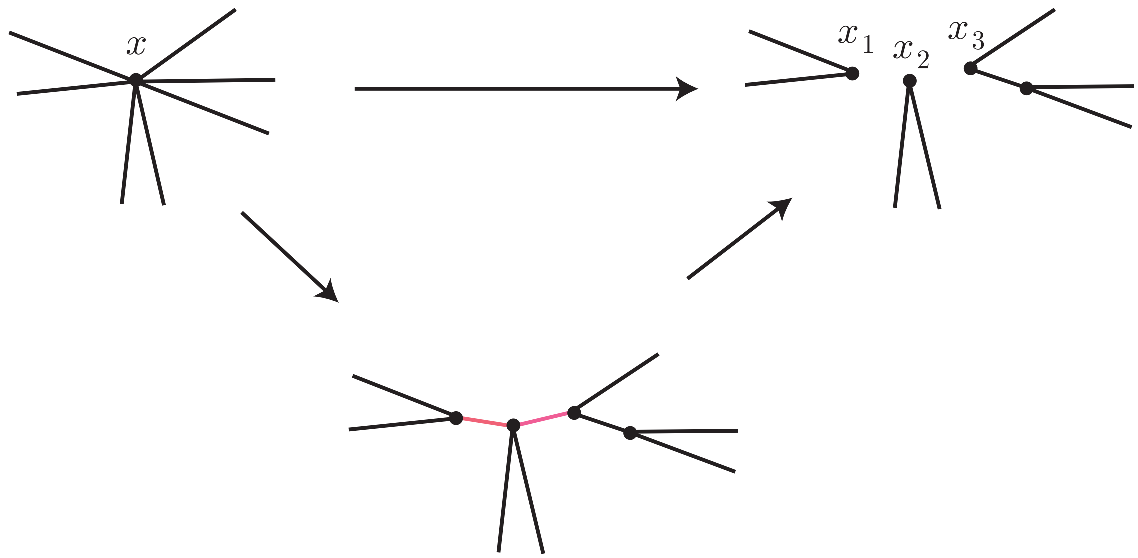




conical version

Attaching a graph to $X$: label some vertices with points in $X$.

Allow these labeled vertices to split into several labeled vertices: Do a conical expansion, delete the edges of a subtree of the new edges, and label its vertices with continuously varying labels in $X$.



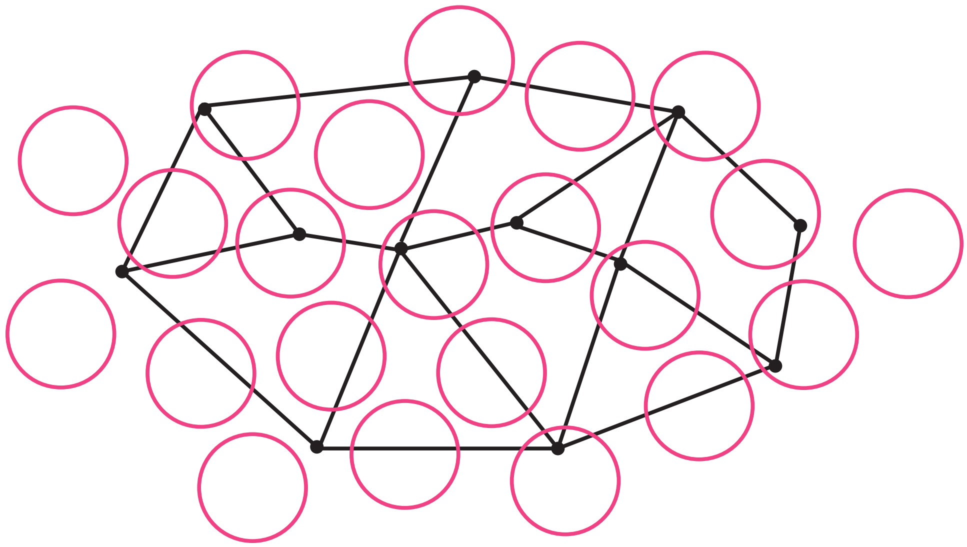

Note: We allow an isolated labeled vertex to be deleted since it denotes attaching nothing to $X$.

Notation:

- $\mathcal{G}^k(X)$ — the space of all such labeled graphs in $\mathbb{R}^k$.
- $\mathcal{G}(X) = \cup_k \mathcal{G}^k(X)$, space of labeled graphs in $\mathbb{R}^\infty$.
- $\mathcal{G}_n(X) \subset \mathcal{G}(X)$, the graphs $\simeq X \vee_n S^1$ (after attaching to $X$).

**Proposition**: $\mathcal{G}_n(X) \simeq BG_n(X)$.

Relate $\mathcal{G}_n(X)$ to $\Omega^\infty S^\infty$ by a *scanning* process. The rough idea: Given a finite graph $K \subset \mathbb{R}^k$, look at it up close by moving a magnifying lens (a jeweler's *loupe*) over all of $\mathbb{R}^k$, recording what appears in the lens.



Equivalent process: fix the lens at the origin, take all possible translations of the graph.

Regarding the lens as a smaller copy of $\mathbb{R}^k$, one sees a graph in $\mathbb{R}^k$ whose edges can extend to infinity. Moving the lens (or the graph), the graph can slide out to infinity and disappear entirely.

Enlarge $\mathcal{G}^k(X)$ to a space $\mathcal{G}^{k,k}(X)$ of such graphs whose edges can extend to infinity. Put a "compact-open" topology on $\mathcal{G}^{k,k}(X)$ allowing parts of graphs to slide to infinity.

For each choice of a lens size we get a scanning map

$$\mathcal{G}^k(X) \longrightarrow \Omega^k \mathcal{G}^{k,k}(X)$$
$$K \subset \mathbb{R}^k \;\longmapsto\; \left(\mathbb{R}^k \cup \{\infty\} \longrightarrow \mathcal{G}^{k,k}(X)\right)$$

This is homotopic to a composition

$$\mathcal{G}^k(X) \simeq \mathcal{G}^{k,0}(X) \to \Omega\mathcal{G}^{k,1}(X) \to \Omega^2\mathcal{G}^{k,2}(X) \to \cdots \to \Omega^k\mathcal{G}^{k,k}(X)$$

where $\mathcal{G}^{k,\ell}(X) \subset \mathcal{G}^{k,k}(X)$ is the subspace of graphs contained in $\mathbb{R}^\ell \times (-1,1)^{k-\ell}$, graphs that can go to infinity in only the first $\ell$ coordinates. Natural map $\mathcal{G}^{k,\ell}(X) \to \Omega\mathcal{G}^{k,\ell+1}(X)$ by translating graphs from $-\infty$ to $+\infty$ in the $(\ell+1)$ st coordinate.

Three steps in the proof:

(1) $\mathcal{G}^{k,\ell}(X) \to \Omega\mathcal{G}^{k,\ell+1}(X)$ is a (weak) homotopy equivalence when $\ell > 0$.

(2) $\lim_n \mathcal{G}_n(X) \to \Omega_0\mathcal{G}^{\infty,1}(X)$ is a homology equivalence.

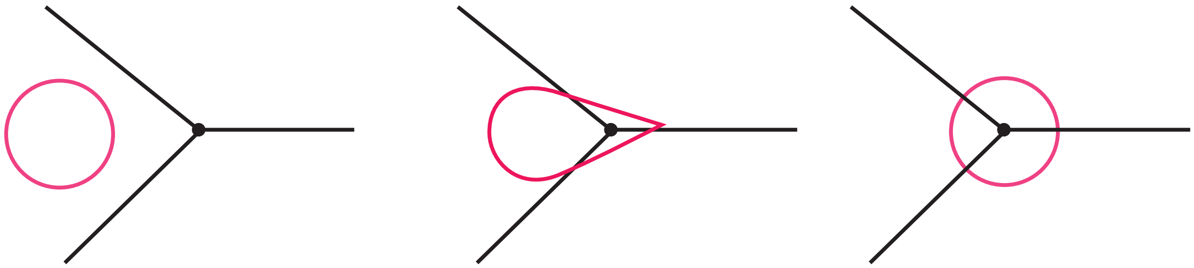
(3) $\mathcal{G}^{k,k}(X) \simeq S^k$, the graphs in $\mathbb{R}^k$ with $\leq 1$ point (unlabeled).

(1) and (2) are proved using classifying spaces of monoids instead of loopspaces, using the Group Completion Theorem for (2).

Combine (1), (2), (3) to get the theorem.

For (3) the idea is to expand a suitably chosen small ball (lens) about the origin to all of $\mathbb{R}^k$, deforming each graph in $\mathcal{G}^{k,k}(X)$ to a tree or the empty graph.

Tricky point: To get continuity as the graph varies, need to choose the ball in a shape to contain only a small piece of the graph that is a tree:



Then shrink this tree:

- Shrink to labeled vertices, if there are any, then delete these labeled vertices. (This is where the dependence on $X$ disappears.)
- Shrink a tree with no labels to a point.

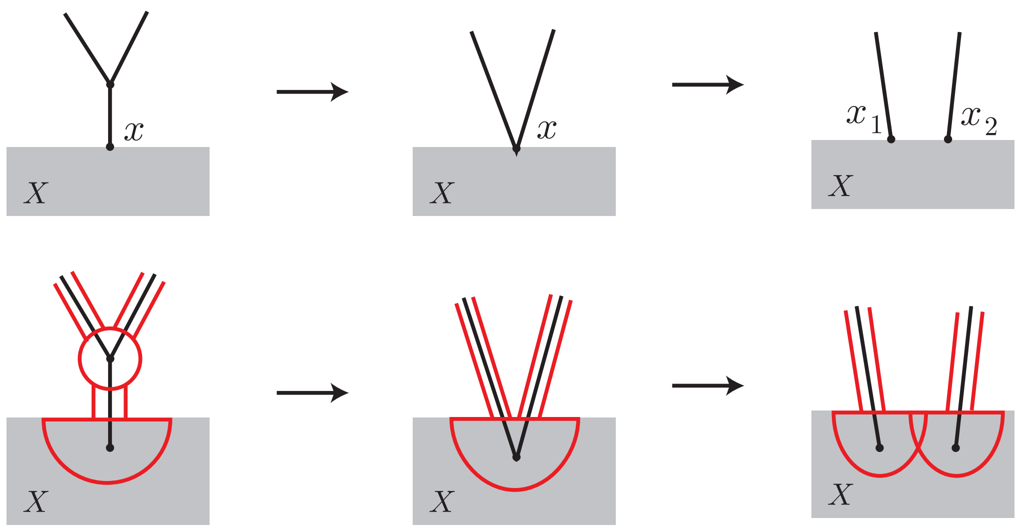
## Handlebodies

Extra data needed to go from a graph $K \subset \mathbb{R}^k$ to a $d$-dimensional (oriented) round handlebody thickening of $K$: a field of (oriented) $d$-planes $P_x \subset \mathbb{R}^k$, $x \in K$, such that $P_x$ contains all tangent lines to edges of $K$ containing $x$.

To attach to a manifold $X^d$ in a submanifold $M \subset \partial X$, label some vertices by points of $M$ (with some tangential data).

Need distinct labels on distinct vertices.

Example:




Get a handlebody space $\mathcal{H}^k(X, M, d)$ analogous to $\mathcal{G}^k(X)$. Points of $\mathcal{H}^k(X, M, d)$ are graphs with extra data of $d$-plane fields and attaching data.

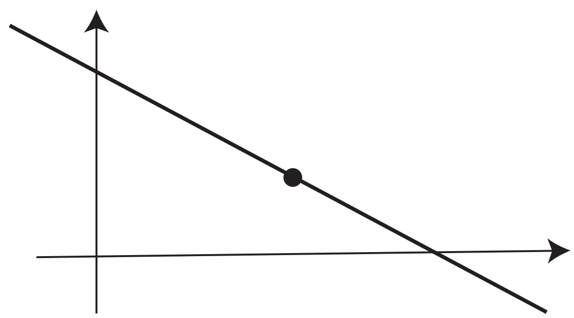
Take $(X, M) = (D^d, D^{d-1})$ for simplicity. Write $\mathcal{H}^k$ for $\mathcal{H}^k(D^d, D^{d-1}, d)$.

Same three steps:

(1) $\mathcal{H}^{k,\ell} \to \Omega\mathcal{H}^{k,\ell+1}$ is a homotopy equivalence when $\ell > 0$.

(2) $\lim_n \mathcal{H}_n \to \Omega_0\mathcal{H}^{\infty,1}$ is a homology equivalence when $d \geq 3$, where $\mathcal{H}_n$ denotes the component of $\mathcal{H}^\infty$ consisting of handlebodies $\simeq \vee_n S^1$.

(3) $\mathcal{H}^{k,k} \simeq$ the graphs in $\mathbb{R}^k$ with $\leq 1$ point (unlabeled), with a $d$-plane at that point. This is just $S^k Gr_+^{k,d}$, the Thom space of the trivial $k$-dimensional bundle over the Grassmannian $Gr^{k,d}$ of oriented $d$-planes in $\mathbb{R}^k$.



Thus we have

$$\lim_n H_i(\mathcal{H}_n) = H_i(\Omega_0^\infty S^\infty BSO(d)_+) \quad \text{for } d \geq 3$$

Remark: $\Omega_0^\infty S^\infty BSO(d)_+$ is the natural analog of the Madsen-Tillmann spectrum for $d$-dimensional manifolds with boundary. Just a coincidence?

**Applications to 3-manifolds.**

$\underline{d = 3}$:

Let $V_n$ be the 3-dimensional handlebody of genus $n$.

**Proposition**: $\mathcal{H}_n \simeq \mathrm{BDiff}(V_n \operatorname{rel} D^2)$ for $D^2 \subset \partial V_n$.

Proof uses special 3-manifold fact: the space of handlebody structures on $V_n$ is contractible.

Thus:

**Theorem**: $H_i(\mathrm{BDiff}(V_n \operatorname{rel} D^2)) \cong H_i(\Omega_0^\infty S^\infty BSO(3)_+)$ for $n >> i$.

Remarks:

- This is the same as the homology of the mapping class group of the handlebody since $\mathrm{Diff}(V_n \operatorname{rel} D^2)$ has contractible components.
- Homology stability known, and the "rel $D^2$" can be dropped.
- $H_*^{stable}(-;\mathbb{Q}) = \mathbb{Q}[x_4, x_8, x_{12}, \cdots]$, half of the MMM classes.

$\underline{d = 4}$:

Let $V_n$ = 4-dimensional handlebody of genus $n$, $\partial V_n = \#_n(S^1 \times S^2)$.

**<u>Proposition</u>**: $\mathcal{H}_n \simeq \mathrm{BDiff}(\#_n(S^1 \times S^2) \operatorname{rel} D^3)$.

**<u>Theorem</u>**: $\lim_n H_i(\mathrm{BDiff}(\#_n(S^1 \times S^2) \operatorname{rel} D^3)) \cong H_i(\Omega_0^\infty S^\infty BSO(4)_+)$.

<u>Remarks</u>:

- This is *not* the same as the stable homology of the mapping class group, which is essentially $\mathrm{Aut}(F_n)$ (with a $(\mathbb{Z}/2)^n$ kernel). The components of $\mathrm{Diff}(\#_n(S^1 \times S^2) \operatorname{rel} D^3)$ are far from contractible.
- Homology stability probably holds.
- Can the "rel $D^3$" be dropped?
- $H_*^{stable}(-;\mathbb{Q})$ is a polynomial algebra on generators corresponding to monomials in $p_1$ and $e$, both in degree 4.

<u>Generalization of the $d = 4$ case</u>: Let $M$ be a compact connected orientable 3-manifold containing $S^1 \times S^2$ as a connected summand. Then

$$\lim_n H_i(\mathrm{BDiff}(\#_n M \operatorname{rel} D^3)) \cong H_i(\Omega_0^\infty S^\infty BSO(4)_+)$$