

Linearization in 3-Dimensional Topology

A. E. Hatcher

The message from Thurston is clear: *geometry dominates topology in dimension three*. Our title refers to one important aspect of this geometry, *linearity* in various forms. We shall consider here linearization of automorphisms of 3-manifolds. Specifically, we ask, does $\text{Diff}(M)$, the group of self-diffeomorphisms of the 3-manifold M (with the C^∞ topology), have the homotopy type of the subgroup of diffeomorphisms which preserve a given “linear” structure on M ? If so, this is strong evidence that the linear structure is really intrinsic to the topology of M .

EXAMPLES (M closed, orientable).

(I) $M = S^3$. The linear diffeomorphisms of S^3 are the isometries, $\text{Isom}(S^3) = O(4)$, the orthogonal group.

The Smale Conjecture. $O(4) \hookrightarrow \text{Diff}(S^3)$ is a homotopy equivalence.

We shall indicate some of the ideas which go into a proof of this below. Previously, Cerf had shown that $\pi_0 \text{Diff}(S^3) \approx \pi_0 O(4)$. He also proved that the Smale Conjecture implies that $\text{Diff}(M) \rightarrow \text{Homeo}(M)$, the natural map to the homeomorphism group, is a weak homotopy equivalence for all 3-manifolds M .

(II) $M = S^3/\Gamma$, $\Gamma \subset \text{SO}(4)$ acting freely on S^3 , the so-called spherical or elliptic 3-manifolds. One expects $\text{Diff}(M) \approx \text{Isom}(M)$, but this is known only for RP^3 .

(III) $M = H^3/\Gamma$, $\Gamma \subset \text{Isom}(H^3)$, hyperbolic 3-manifolds. Again, one expects $\text{Diff}(M) \approx \text{Isom}(M)$, and this is known when M is Haken (=“irreducible, sufficiently large” in the older terminology). By a theorem of Mostow, $\text{Isom}(M) \approx \text{Out}(\pi_1 M)$, the outer automorphism group of $\pi_1 M$ (automorphisms modulo inner automorphisms), which is not only discrete but finite.

(IV) $M = E^3/\Gamma$, $\Gamma \subset \text{Isom}(E^3)$, euclidean or flat 3-manifolds. Here the linear

diffeomorphisms are the affine diffeomorphisms (i.e., affine in the universal cover E^3). Such an M is Haken, so one can show $\text{Diff}(M) \simeq \text{Aff}(M)$. For example, $\text{Diff}(T^3) \simeq \text{GL}(3, \mathbf{Z}) \times T^3$ (semidirect product). This is considerably larger than $\text{Isom}(T^3)$, which is compact.

(V) $M = S^1 \times S^2$. This is best regarded as a bundle $S^2 \rightarrow M \rightarrow S^1$ with linear structure group $O(3)$. Then $\text{Diff}(S^1 \times S^2) \simeq O(2) \times O(3) \times \Omega O(3)$, the bundle automorphisms (Rourke–César de Sá).

(VI) $M = T^2$ -bundle over S^1 with gluing map in $\text{SL}(2, \mathbf{Z})$ having distinct real eigenvalues. Again $\text{Diff}(M)$ has the homotopy type of the bundle automorphism group.

(VII) M Seifert fibered, $S^1 \rightarrow M \rightarrow B$, over a closed surface B . These include the manifolds of I, II, IV, V, but none of those in III or VI. Excluding the manifolds in I, II, IV, and V, the Seifert fiber structure is unique, and $\text{Diff}(M) \simeq \{\text{fiber-preserving diffeomorphisms}\}$ except perhaps when $B = S^2$ and there are only three singular fibers (the non-Haken cases). Incidentally, the linear structure in a Seifert fibering is in the base B , which is naturally a quotient of the spherical, euclidean, or hyperbolic plane by a discrete group G of isometries (perhaps with torsion). For example, M could be the unit tangent bundle of such a B , which has singular fibers arising from the elements of G with fixed points (rotations of finite order).

(VIII) M having a *torus decomposition*, i.e., a splitting of M into submanifolds M_j which are the components of the complement of a finite collection (perhaps empty) of disjointly embedded tori T_i in M , such that

(1) $\pi_1 T_i \rightarrow \pi_1 M$ is injective for each i (to rule out the possibility that T_i bounds a solid torus in M).

(2) Each M_j is either

(a) the interior of a compact Seifert fibered manifold, or

(b) a hyperbolic manifold H^3/Γ of finite volume (having finite volume is almost as good as being compact).

(3) $\{T_i\}$ is minimal, with respect to inclusion, among collections $\{T_i\}$ satisfying (1) and (2).

Small Exception. For the T^2 -bundles in VI above it seems better to choose $\{T_i\} = \emptyset$ rather than a single fiber T^2 , which is what (1)–(3) would yield. (The other T^2 -bundles over S^1 are Seifert-fibered.)

Perhaps the deepest result in 3-manifolds to date is:

THEOREM. *Every Haken manifold has a torus decomposition which is unique up to isotopy.*

This is due to Johannson and (independently) Jaco–Shalen for the Seifert part, and to Thurston for the (much harder) hyperbolic part. As far as is known, all prime 3-manifolds (i.e., indecomposable as a connected sum) have torus decompositions, since all known prime 3-manifolds are either Seifert, hyperbolic, or Haken.

THEOREM. $\text{Diff}(M)$ deformation retracts onto the subgroup of diffeomorphisms leaving $\bigcup_i T_i$ invariant.

If $\{T_i\} \neq \emptyset$, the components of $\text{Diff}(M)$ are contractible. So the content of the preceding theorem is to reduce $\pi_0 \text{Diff}(M)$ essentially to the $\pi_0 \text{Diff}(M_j)$'s.

For example, one can say that $\pi_0 \text{Diff}(M)$ is generated by:

diffeomorphisms which permute the T_i 's and the M_j 's,

Dehn twists along the T_i 's,

isometries of the hyperbolic M_j 's,

fiber-preserving diffeomorphisms of the Seifert M_j 's.

This is reminiscent of Thurston's normal form for diffeomorphisms of surfaces.

(IX) M nonprime. Rourke and César de Sá have largely reduced $\text{Diff}(M)$ in this case to $\text{Diff}(M_j)$ for the prime factors M_j of M , plus the homotopy theory of certain "configuration spaces". This seems to be rather complicated in general.

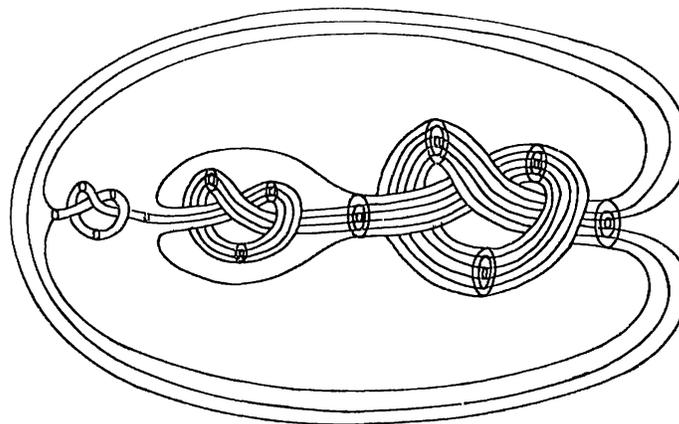
THE SMALE CONJECTURE: $\text{Diff}(S^3) \simeq O(4)$.

There are many statements well-known to be equivalent to this, e.g.,

(1) The space of unknotted smoothly embedded circles in \mathbb{R}^3 deformation retracts onto the subspace of round (i.e., planar, constant curvature) circles.

(2) The space of smoothly embedded 2-spheres in \mathbb{R}^3 deformation retracts onto the subspace of round (i.e., constant curvature) 2-spheres.

Of these, (1) seems hopeless: there appears to be no canonical way of unknotting the unknot. At first glance, (2) seems even harder if one looks at embedded 2-spheres with apparent knots, like the following:

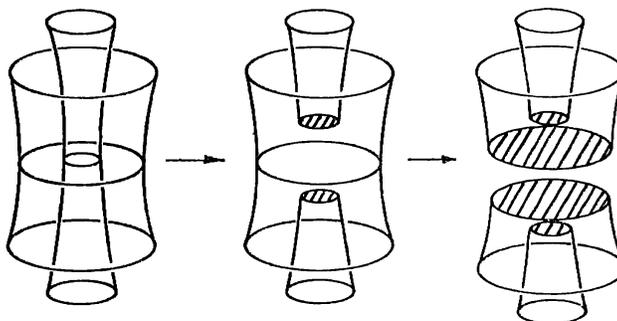


Nonetheless, (2) can be proved, using only elementary (but complicated) differential topology. Intuition suggests there ought, also, to be an analytic proof of (2), based on some physical model for 2-spheres in \mathbb{R}^3 . However, somewhere the topology or geometry of three dimensions will have to enter, since the analog of (2)

either for 3-spheres in \mathbf{R}^4 or for 4-spheres in \mathbf{R}^5 is definitely false (this is directly traceable to the existence of exotic 7-spheres).

What one actually proves is the following technical variant of (2): a smooth family $g_t: S^2 \hookrightarrow \mathbf{R}^3$ parametrized by $t \in S^k$ extends to a smooth family $\bar{g}_t: B^3 \hookrightarrow \mathbf{R}^3$. For $k=0$ this is essentially due to Alexander, and for $k=1$ this is what Cerf showed to calculate $\pi_0 \text{Diff}(S^3) \approx \mathbf{Z}_2$.

The good property of 2-spheres in \mathbf{R}^3 is that they can be sliced into simpler and simpler 2-spheres by surgery on horizontal circles:



(Note this fails for $S^{n-1} \subset \mathbf{R}^n, n \geq 4$.)

The process can be iterated: surger all the circles of intersection of the given 2-sphere with more and more horizontal transverse planes. Eventually a point is reached when further surgeries no longer yield significantly simpler 2-spheres. We call such 2-spheres, somewhat loosely, “indecomposable”.

It is easy to reverse the surgery process, gluing together extensions \bar{g}_t . So the problem becomes to construct \bar{g}_t on the “indecomposables”. This must be done in a canonical way, which works for k -parameter families of “indecomposables”.

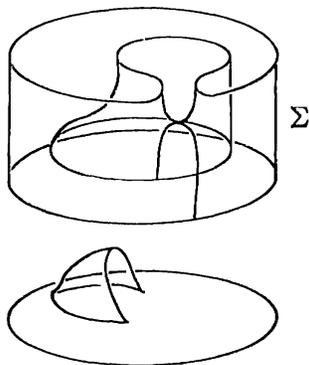
A further problem is that one cannot choose the same horizontal slicing planes for all $t \in S^k$, but only locally in t . That is, one covers S^k by balls B_i , associated to each of which is a finite collection of horizontal planes P_{ij} transverse to $g_t(S^2)$ for $t \in B_i$. One surgers $g_t(S^2)$ using the planes P_{ij} for $t \in B_i$. So on intersections of B_i 's, the “indecomposables” are being further surgered (subdivided), and one must take pains to make the extensions \bar{g}_t on “indecomposables” invariant under such subdivision.

Thus the heart of the problem is understanding the “indecomposables”. For small values of k , one can perturb the family g_t so that the height functions on $g_t(S^2)$ form a generic k -parameter family, and then write down a complete catalog of the types of “indecomposables”. This is how Cerf proceeded for $k=1$. But for general k this approach fails (because smooth singularities are classified only for small codimensions k).

So one must forget height functions, and instead look at “indecomposable” 2-spheres *from the top*. This viewpoint leads to the following basic definition: The *contour* of a 2-sphere $\Sigma \subset \mathbf{R}^3$ is the quotient space of the 3-ball $\bar{\Sigma} \subset \mathbf{R}^3$ bounded

by Σ , obtained by identifying points x and y in $\bar{\Sigma}$ whenever there is a vertical line segment in $\bar{\Sigma}$ joining x and y .

EXAMPLE.



Contour=2-disc with a “flap” or “tongue”

In general, the part of $g_t(S^2)$ between two adjacent slicing planes P_{ij} has the following key property (after a preliminary normalization): Each vertical line in \mathbb{R}^3 meets this part of $g_t(S^2)$ in a connected set (perhaps empty). Using this, one proves that the contour of an “indecomposable” is always a disc with finitely many tongues attached successively, either to the disc or to previously attached tongues. (“Attaching a tongue” means attaching a disc D along a subdisc which meets ∂D nontrivially.)

To get the canonical extensions \bar{g}_t on “indecomposables”, the procedure is: Shrink each tongue in turn down to the arc along which it attaches. This lifts to an isotopy of the “indecomposable”, ending with a 2-sphere whose contour is a disc. For this, \bar{g}_t is easily constructed. Then reversing the isotopy which shrank the tongues, one obtains \bar{g}_t on the original “indecomposable” by isotopy extension (which is canonical). This uses Smale’s theorem $\text{Diff}(S^2) \simeq O(3)$, to make \bar{g}_t canonical.

The hard work comes in making this shrinking-of-contours process mesh with the subdivision (slicing) of “indecomposables” mentioned earlier.

One might well ask if shrinking of contours could be applied to the original $g_t(S^2)$. Unfortunately, one can easily construct examples of 2-spheres in \mathbb{R}^3 whose contours cannot be continuously shrunk, within themselves, to any subdisc. (Such contours are contractible but not collapsible, in the sense of PL topology.) So the slicing process is necessary.

APPLICATION.

THEOREM (C. B. THOMAS). *If the Smale Conjecture is true, then: A 3-manifold M with universal cover S^3 has the homotopy type of one of the spherical manifolds S^3/Γ (for some $\Gamma \subset \text{SO}(4)$ acting freely on S^3 as isometries).*

In particular, $\pi_1 M \approx \Gamma$. To classify such M 's there remains the problem of showing that Γ can act on S^3 only in the standard linear ways. This is known for some Γ 's e.g., Z_2, Z_4, Z_6, Z_8 , generalized quaternion (of order 2^k), binary tetrahedral and octahedral. (See the article of Rubinstein for references.) It is unknown, in particular, for Γ cyclic of odd order (e.g., $Z_3!$), and for the binary icosahedral group.

References

1. J. W. Alexander, *On the subdivision of 3-space by a polyhedron*, Proc. Nat. Acad. Sci. U.S.A **10** (1924), 6—8.
2. J. Cerf, *Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4=0$)*, Lecture Notes in Math., vol. 53, Springer-Verlag, Berlin and New York, 1968.
3. E. César de Sá and C. Rourke, *The homotopy type of homeomorphism groups of 3-manifolds*; Math. Inst. Warwick, 1978 (preprint).
4. A. Hatcher, *Homeomorphisms of sufficiently large P^2 -irreducible 3-manifolds*, Topology **15** (1976), 343—347.
5. N. V. Ivanov, *Diffeomorphism groups of Waldhausen manifolds*, Issledovania po Topologii II, ed. A. A. Ivanov, Izdatelstvo "Nauka" Leningrad, 1976, pp. 172—176.
6. W. H. Jaco and P. B. Shalen, *Seifert fibered spaces in 3-manifolds* (preprint).
7. K. Johannson, *Équivalences d'homotopie des variétés de dimension 3*, C. R. Acad. Sci. Paris Sér. A, **281** (1975), 1009—1010.
8. P. Orlik, *Seifert manifolds*, Lecture Notes in Math., vol. 291, Springer-Verlag, Berlin and New York, 1972.
9. J. H. Rubinstein, *On 3-manifolds that have finite fundamental group and contain Klein bottles* (preprint).
10. S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621—626.
11. W. P. Thurston, *Geometry and topology of 3-manifolds*, notes from a course at Princeton University, 1977—1978.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES
LOS ANGELES, CALIFORNIA 90024, U.S.A.