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BOUNDARY SLOPES FOR MONTESINOS KNOTS

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FOR A KNOT $K \subset S^3$, let $S(K) \subset \mathbb{Q} \cup \{\infty\}$ be the set of slopes of boundary curves of incompressible, ∂-incompressible orientable surfaces in the knot exterior, slopes being normalized in the standard way so that a longitude has slope 0, a meridian slope ∞. These sets $S(K)$ of ∂-slopes are of special interest because of their relation with Dehn surgery and character varieties; see e.g., [2]. The only general results known so far are that $S(K)$ is always finite [5] and contains at least two elements [3], including of course 0 (coming from a minimal genus Seifert surface). Only for special classes of knots has $S(K)$ been determined exactly. For the (p, q) torus knot, $S(K) = \{0, pq\}$. For 2-bridge knots, $S(K)$ is an arbitrarily large set of even integers computable via continued fractions [7]. A non-integer ∂-slope was found first for the $(-2, 3, 7)$ pretzel knot, where by combining results of Culler–Gordon–Luecke–Shalen, Fintushel–Stern, and Oertel it was observed that there must be a fractional ∂-slope between 18 and 19. Other instances of non-integer ∂-slopes have been found subsequently by Takahashi [13].

In this paper we describe an algorithm for computing $S(K)$ for a class of knots which is the "vector sum" of 2-bridge and pretzel knots, the Montesinos knots $K(p_1/q_1, \ldots, p_n/q_n)$ obtained by connecting n rational tangles of slopes $p_1/q_1, \ldots, p_n/q_n$ in a simple cyclic pattern. In this notation 2-bridge knots are the cases $n \leq 2$, while the $(q_1, \ldots, q_n)$ pretzel knot is $K(1/q_1, \ldots, 1/q_n)$. The algorithm is quite effective. Simple cases like the $(-2, 3, 7)$ pretzel, or more generally any (q_1, q_2, q_3) pretzel, can easily be done by hand; for the motivating example $(-2, 3, 7)$ we find $S(K) = \{0, 16, 18\frac{1}{2}, 20\}$. For somewhat more complicated cases, a small computer can do the work rather quickly. Some examples of these computer calculations are given in the last section of the paper. These include the Montesinos knots of ≤ 10 crossings, plus a few other random examples of greater complexity.

Non-integer ∂-slopes occur quite often, starting with the first non-torus, non-alternating knot in the tables, 8_{20}, where $8/3 \in S(K)$. For more complicated Montesinos knots, ∂-slopes occur in such abundance that it seems difficult to find general patterns in the sets $S(K)$. We produce examples showing that all rational numbers occur among the ∂-slopes of Montesinos knots. On the other hand, we show that Montesinos knots having an easily seen alternating projection, namely the ones with all p_i/q_i's of the same sign, have all ∂-slopes even integers. Perhaps this is a general property of alternating knots.

To compute $S(K)$ we in fact determine fairly explicitly all the incompressible surfaces with non-empty, non-meridional boundary in the exterior of $K = K(p_1/q_1, \ldots, p_n/q_n)$. (For meridional boundary it was shown in [11] that $\infty \in S(K)$ if and only if $n \geq 4$, assuming without loss of generality that $|q_i| \geq 2$ for each i.) Incompressible surfaces in rational tangles having been analyzed thoroughly in [7], our main task is understanding how to fit together incompressible surfaces in the separate tangles so as to form a surface in $S^3 - K$ which is still

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incompressible in $S^3 - K$. Partial results in this direction were contained in the second author's thesis [10], and our techniques extend and refine those introduced there. The first step, in §1, is to find a reasonably efficient list of candidate surfaces which includes all incompressible surfaces; these are essentially the surfaces in $S^3 - K$ which are incompressible within each tangle separately and are isotoped to meet the "axis" of K minimally. Then in §2 we analyze compressions which involve interactions between different tangles, to weed out the (relatively few, in general) compressible surfaces on the list. This latter step is fairly delicate, and could probably be applied in other similar contexts, for example to arborescent knots. Fortunately the final algorithm for computing $S(K)$ operates at a more superficial level, involving just elementary algebra.

A thorough understanding of [7] is assumed in much of this paper, and we shall not give specific references when citing [7].

§1. THE CANDIDATE SURFACES

Curve Systems on the 4-Punctured Sphere

In the projective lamination space of a 4-punctured sphere, let PL_e be the 2-disk slice defined by the condition that the weights, or transverse measures, at the four punctures be equal. (See [6] and references cited there for general background on projective lamination spaces; the special case needed here can easily be understood directly, without the general theory, however.) Part of PL_e is the 2-simplex of projective weights (a, b, c) for the train track in Fig. 1.1, where we view S^2 as the one-point compactification of the plane of the page, with the four punctures drawn as heavy dots. A subdivision of this 2-simplex into infinitely many subsimplices is also indicated. Vertices in this subdivision on the right-hand edge (where $a = 0$) correspond to single circles on the 4-punctured sphere, of rational slope c/b. The other vertices correspond to pairs of arcs joining the four punctures, of slope $c/(a+b)$. Points along edges or in 2-simplices of this subdivision having integral ($\equiv$ rational) projective coordinates correspond to curve systems formed from copies of the circles or arc-pairs at the boundary vertices of the edge or 2-simplex.

There is also a mirror-image train track which carries curve systems of negative slope. Its 2-simplex of projective weights meets the former 2-simplex in PL_e along the edge where $c = 0$ and at the c vertex. It will be convenient to regard c as being negative for the second 2-simplex; slopes are then given uniformly by $c/(a + b)$. The rest of PL_e is a disk formed by joining the left-hand edges of the two 2-simplices, which together form a circle in PL_e, to the point of PL_e corresponding to a pair of slope ∞ arcs joining the four punctures. Figure 1.2 shows the corresponding subdivision of PL_e, with slopes indicated.

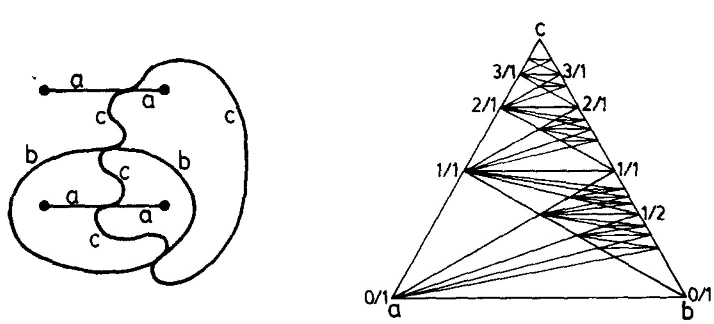


Fig. 1.1.

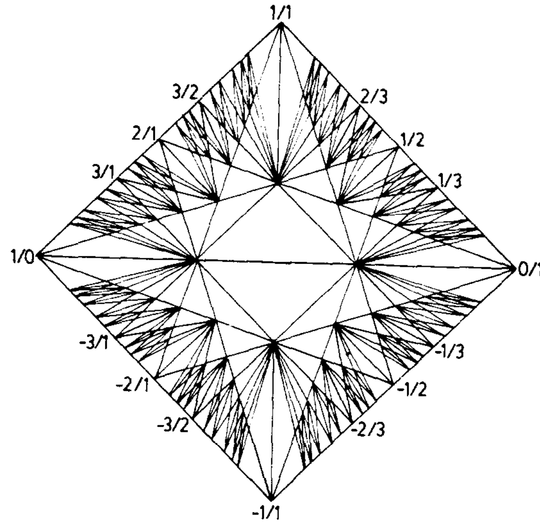


Fig. 1.2.

A more useful picture for our purposes is obtained by slitting PL_e open along the slope ∞ edge. Then the 2-simplex of Fig. 1.1 and its negative-slope counterpart form an infinite strip $\mathscr{S}$, pictured vertically in Fig. 1.3, with vertical coordinate "slope", $c/(a+b)$, and horizontal coordinate $b/(a+b)$. The left border of $\mathscr{S}$ is joined to the slope-∞-arcs vertex. We call this whole picture the diagram $\mathscr{D}$. The slope ∞ edge along which we slit is not a part of $\mathscr{D}$, but adjoining it as a horizontal segment to the left of the slope ∞ vertex of $\mathscr{D}$ defines the augmented diagram $\hat{\mathscr{D}}$. Let $\langle p/q, r/s \rangle$ denote the edge of $\hat{\mathscr{D}}$ whose endpoints correspond to curve systems having slopes p/q and r/s ($p/q = r/s$ is possible), and let $\langle p/q \rangle$ denote the vertex corresponding to slope p/q arcs; we will not need a special notation for the other vertices of $\hat{\mathscr{D}}$.

For an edge $\langle p/q, r/s \rangle$ with $p/q \neq r/s$, let $k/m \langle p/q \rangle + (m-k)/m \langle r/s \rangle$ denote the point of $\mathscr{D}$ corresponding to the curve system consisting of k parallel copies of the pair of arcs of slope p/q joining the four punctures, together with $m-k$ parallel copies of the pair of arcs of slope r/s. This curve system thus has m arcs coming into each puncture. In (a, b, c) coordinates, $k/m \langle p/q \rangle + (m-k)/m \langle r/s \rangle = k(1, q-1, p) + (m-k)\ (1, s-1, r)$. Fixing m and letting k vary, the $m+1$ points $k/m \langle p/q \rangle + (m-k)/m \langle r/s \rangle$ in an edge $\langle p/q, r/s \rangle$ divide that edge into m subsegments. These subsegments in a vertical edge in $\mathscr{S}$ all have equal length, and we may assume this is true also for the edges in $\mathscr{D} - \mathscr{S}$. However, the subsegments of non-vertical edges in $\mathscr{S}$ do not have equal length. For example , along the edge $\langle 0/1, 1/2 \rangle$ the points $k/m \langle 1/2 \rangle + (m-k)/m \langle 0/1 \rangle$ have horizontal coordinates $b/(a+b) = k/(k+m)$ and so are not equally spaced.

The Knot $K = K(p_1/q_1, \ldots, p_n/q_n)$.

Viewing S^3 as the join of two circles A and B, let the circle B be subdivided as an n-sided polygon. Then the join of A with the ith edge of B is a ball B_i. These n balls B_i cover S^3, meeting each other only in their boundary spheres. The circle $A \subset S^3$, called the *axis*, is the singular locus of the union of the ∂B_i's (assuming $n \geq 3$). Identify ∂B_i with the 4-punctured sphere so that: the axis is a standard slope ∞ circle; the circle which is the join of the two points of $B \cap \partial B_i$ with two chosen antipodal points of A is a standard slope 0 circle; and one puncture is in each of the four quadrants of ∂B_i formed by the slope 0 and slope ∞ circles.

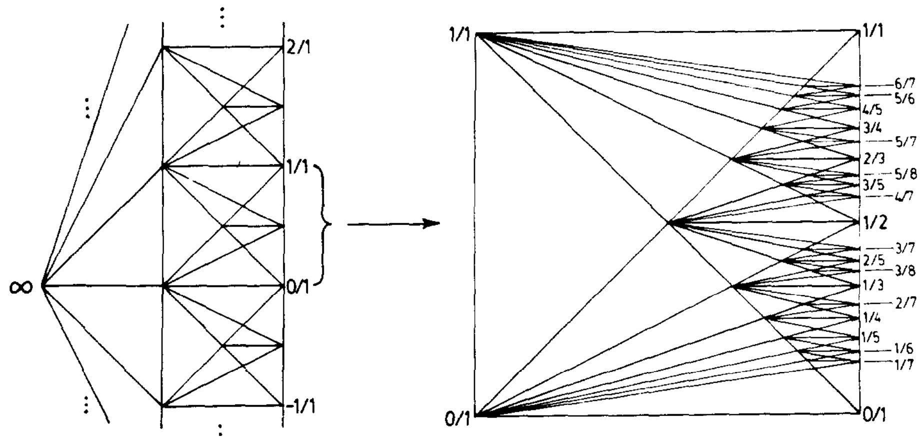


Fig. 1.3. The Diagram $\mathscr{D}$.

Let $\partial B_i \times [0, 1]$ be a collar on ∂B_i inside B_i, with $\partial B_i = \partial B_i \times 1$. Given rational numbers p_i/q_i, $i = 1, \ldots, n$, with $(p_i, q_i) = 1$, the tangle $K_i \subset B_i$ is defined to be the union of the four arcs (puncture) $\times [0, 1] \subset \partial B_i \times [0, 1]$ with the two arcs of slope p_i/q_i in $\partial B_i \times 0$ joining the four punctures. Thus K_i consists of two disjoint embedded arcs in B_i with endpoints on ∂B_i. We may assume the puncture points are chosen so that the endpoints of K_i match up with those of $K_{i\pm 1}$ in the adjacent balls $B_{i\pm 1}$ (subscripts mod n). Then the union of the K_i's is a knot or link $K = K(p_1/q_1, \ldots, p_n/q_n)$ in S^3. It is not hard to see that K is a knot only in two cases: (1) just one q_i is even, and (2) all q_i's are odd and the number of odd p_i's is odd. (The pairing of the four punctures on ∂B_i determined by the two arcs of K_i depends only on the parity of p_i and q_i.)

We shall restrict our attention to the case that K is a knot. Further, we assume $n \geq 3$ since otherwise K is a 2-bridge knot (already studied), and we assume $|q_i| \geq 2$ for each i, since a tangle with $q_i = \pm 1$ can be combined with an adjacent tangle, reducing n.

Addition of Curve Systems

The axis divides each sphere ∂B_i into left and right hemispheres. Suppose we are given two curve systems on ∂B_i and ∂B_{i+1}, corresponding to rational points of $\hat{\mathscr{D}}$. We may assume the two curve systems have been isotoped (rel punctures, always) to minimize the number of intersections with the axis, i.e., neither curve system contains an arc in one hemisphere which can be isotoped (rel endpoints) into the axis. If the two curve systems can be isotoped, preserving the axis, to agree on $\partial B_i \cap \partial B_{i+1}$, then after throwing away this common half of the two curve systems, the two remaining halves form a curve system on $\partial(B_i \cup B_{i+1})$, which we regard as the sum of the two curve systems we started with. When is such an addition possible?

Consider first the strip $\mathscr{S}$ in $\mathscr{D}$, where curve systems have the coordinates (a, b, c). The intersection of such a curve system with either hemisphere consists of $2a$ arcs with one endpoint on the axis and the other at a puncture, and b arcs with both endpoints on the axis; see the train track in Fig 1.1. So the obvious necessary and sufficient condition for being able to add two such curve systems is that they have the same a and b coordinates. The sum system then also has the same a and b coordinates, and its c coordinate is simply the sum of the c coordinates of the two summand systems. (The c coordinate measures "twisting" along the axis, positive or negative.) In terms of the diagram $\mathscr{D}$, if the common value of a is specified, addible curve systems are those lying on a vertical line.

The same conclusions hold also for the rest of $\mathscr{D}$, since this just amounts to allowing a number of slope ∞ arcs in our curve systems. (The addible curve systems must have the same number of slope ∞ arcs.)

Addition of curve systems containing slope ∞ circles is also possible under the obvious necessary condition: same numbers of slope ∞ arcs and slope ∞ circles in the common hemisphere.

The List of Candidates for Incompressible Surfaces

By an *edgepath* in the diagram $\mathscr{D}$ we shall mean a piecewise linear path in the 1-skeleton of $\mathscr{D}$, starting and ending at points which may not be vertices of $\mathscr{D}$.

Let edgepaths γ_i in $\mathscr{D}$ be given, $i = 1, \ldots, n$, with the following properties:

(E1) The starting point of γ_i lies on the edge $\langle p_i/q_i, p_i/q_i \rangle$, and if this starting point is not the vertex $\langle p_i/q_i \rangle$, then the edgepath γ_i is constant.

(E2) γ_i is *minimal*, i.e., it never stops and retraces itself, nor does it ever go along two sides of the same triangle of $\mathscr{D}$ in succession.

(E3) The ending points of the γ_i's are rational points of $\mathscr{D}$ which all lie on one vertical line and whose vertical coordinates add up to zero.

(E4) γ_i proceeds monotonically from right to left, "monotonically" in the weak sense that motion along vertical edges is permitted.

Choose a positive integer m so that the ending point of each γ_i corresponds to a curve system having m arcs coming into each puncture. (The rational endpoint of γ_i corresponds to an integer triple (a_i, b_i, c_i); let m be a common multiple of the a_i's.) To each edgepath γ_i we associate a finite number of surfaces $S_i \subset B_i$ with $\partial S_i \subset K_i \cup \partial B_i$, such that small meridian circles of K_i meet S_i in m points, i.e., *m-sheeted* surfaces. If γ_i is non-constant, S_i is constructed just like the surfaces in [7]. Namely, S_i lies in the collar $\partial B_i \times [0, 1]$, with the projection to $[0, 1]$ a Morse function on S_i all of whose critical points are saddles in $S_i - \partial S_i$. The sequence of transverse intersections $S_i \cap (\partial B_i \times u)$ as u increases is the sequence of curve systems corresponding to points of γ_i having a-coordinate equal to m, that is, points of the form $k/m \langle p/q \rangle + (m-k)/m \langle r/s \rangle$ as described earlier. Going from one such point of γ_i to the next is achieved by a saddle of S_i. Up to level-preserving isotopy there are two choices for each such saddle, as in Fig. 1.4, so we have finitely many m-sheeted surfaces S_i for each non-constant edgepath γ_i.

To a constant γ_i we associate one surface S_i, meeting each level $\partial B_i \times u$ in the curve system corresponding to γ_i with a-coordinate equal to m. The rest of S_i consists of disjoint disks capping off the circles of $S_i \cap (\partial B_i \times 0)$, these disks lying in the ball in B_i bounded by $\partial \mathbf{B}_i \times 0$.

In view of condition (E3), the surfaces S_i fit together to form a surface $S \subset S^3 - K$, called a *candidate surface*.

Fig. 1.4.

In case all the γ_i's end at the vertex $\langle \infty \rangle$, there is also the possibility of augmenting the corresponding surfaces S_i, by the process described in the following paragraph, so as to introduce slope ∞ circles (disjoint from the axis) in the final curve system $S_i \cap \partial B_i$. As we shall see, this augmentation contributes nothing new to the set S(K) of ∂-slopes, but it is necessary for obtaining a complete list of all the incompressible surfaces in $S^3 - K$, in general (though not for $n = 3$; see the Remark later in this subsection). The reader may wish to ignore these augmented surfaces in a first reading of the paper.

Slope ∞ circles are introduced in two ways. First, one can have a saddle joining a slope ∞ arc to itself; there are four essentially different possible positions here, shown in Fig. 1.5(a, b). Within a single B_i we use only one of these four types of saddles. The slope ∞ circle created by such a saddle remains in $\partial B_i \times u$ until $u = 1$, where we position it to lie in the opposite hemisphere from the one where the saddle which created it occurs. And second, we can insert parallel annulus components of S_i, each cutting off from B_i a solid torus neighborhood of the axis. The augmented surfaces S_i so constructed fit together to form a surface $S \subset S^3 - K$ provided the number of slope ∞ circles in the right half of ∂B_i and in the left half of ∂B_{i+1} are the same, for each i. To the augmented S_i there corresponds an augmented edgepath $\hat{\gamma}_i$ which goes out along the edge $\langle \infty, \infty \rangle$ of $\hat{\mathscr{D}}$, as dictated by S_i.

The surfaces S constructed by this augmentation procedure we also call *candidate surfaces*, with the following exceptions. In some cases, an isotopy of S_i (rel ∂S_i) can exchange a saddle of one type in Fig. 1.5(a, b) for a saddle of one of the two types in the opposite hemisphere. The new surface S is not on our list of candidate surfaces because the new saddle creating the slope ∞ circle is in the same hemisphere as the final position of this circle in ∂B_i. In this situation we delete the former surface S from our list of candidate surfaces. As in [7], the only way such an isotopy of S_i producing a saddle in the opposite hemisphere can occur is if two successive saddles of S_i corresponding to segments of $\hat{\gamma}_i$ in the $\langle k, \infty \rangle$ and $\langle \infty, \infty \rangle$ edges of $\hat{\mathscr{D}}$ can be put on the same level, as in Fig. 1.6(a). Then reversing the levels of these two saddles changes both their types, in particular making the new saddle for the $\langle \infty, \infty \rangle$ edge appear in the opposite hemisphere from before. Of the two possible positions of the saddle for the edge $\langle k, \infty \rangle$, only one can be put on the same level with the given saddle for the edge $\langle \infty, \infty \rangle$. But the type of the saddle for the edge $\langle k, \infty \rangle$ may possibly be reversible also, by putting it on the same level with an earlier saddle, then interchanging the levels of these two saddles; see Fig. 1.6(b). By [7], if this saddle reversal is possible, then the successive edges of γ_i involved must lie in triangles of $\mathscr{D}$ sharing a common edge; with this condition on γ_i, a saddle reversal can be realized only for two of the four possible positions for the pair of saddles. If each pair of successive edges of γ_i satisfies this condition, we call γ_i *completely reversible*: all saddles of S_i are reversible in this case, because initial saddles of S_i

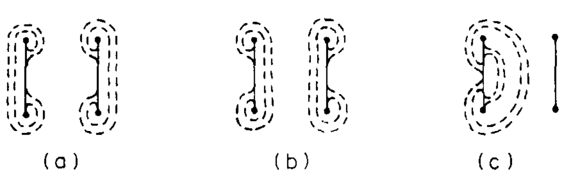


Fig. 1.5.

Fig. 1.6.

can always be reversed, individually (by pushing them across the ball in B_i bounded by $\partial B_i \times 0$) and then reversals can be transmitted along γ_i edge by edge, at will. Thus if γ_i is completely reversible, we do not allow saddles extending $\hat{\gamma}_i$ along the edge $\langle \infty, \infty \rangle$. But otherwise there will be some choice of saddles not permitting the first saddle along $\langle \infty, \infty \rangle$ to be switched to the opposite hemisphere.

This idea of reversing saddle types will play a significant role later, in §2, when we come to decide which candidate surfaces are incompressible.

PROPOSITION 1.1. *Every incompressible, ∂-incompressible surface in S^3-K having non-empty boundary of finite slope is isotopic to one of the candidate surfaces.*

Proof. First isotope S so that its boundary is transverse to all small meridian disks of K, intersecting each of them in m points; all future isotopies of S are to preserve this property. Next isotope S to minimize the number of intersections with the axis. Trivial circles of $S \cap \partial B_i$ (bounding disks in ∂B_i disjoint from the punctures) can be eliminated in the usual way. Trivial arcs of $S \cap \partial B_i$ cannot occur, by ∂-incompressibility. So $S \cap \partial B_i$ is a curve system defining a rational point of $\hat{\mathscr{D}}$. Assume first that this point lies in $\mathscr{D}$ for each i. We apply the analysis of [7] to $S_i = S \cap B_i$ inside B_i, with the minor variant that we now work rel ∂B_i. At non-critical levels, $S_i \cap (\partial B_i \times u)$ defines a point of $\hat{\mathscr{D}}$, and consecutive points lie in a common simplex of $\hat{\mathscr{D}}$, so a finite polygonal path γ_i is traced out, with this sequence of points as its successive vertices. The starting point of γ_i is on the edge $\langle p_i/q_i, p_i/q_i \rangle$. By the argument in [7], γ_i must then stay in the 1-skeleton of $\hat{\mathscr{D}}$ and trace out a minimal edgepath, staying in $\mathscr{D}$ since by assumption it ends in $\mathscr{D}$. Since we have minimized intersections of S with the axis, the ending point of γ_i is its farthest point to the left in $\mathscr{D}$ (otherwise we could push part of S_i out of B_i to reduce the number $2b + 2m$ of intersections with the axis). After eliminating initial motion of γ_i along $\langle p_i/q_i, p_i/q_i \rangle$ as in [7], (E1) is satisfied. Since the S_i's fit together to form S, (E3) holds.

If some $S \cap \partial B_i$ contains slope ∞ circles, a small additional argument not in [7] is needed. The argument for eliminating all non-saddle critical points of S_i comes from [4], and allows also the possibility of ∂B_i-parallel components of S_i disjoint from K, in this case, ∂B_i-parallel annuli of slope ∞. If one of these does not straddle the axis, it can be pushed across ∂B_i to decrease the total number of slope ∞ circles in the ∂B_i's. One shows as in [7] that the edgepath $\hat{\gamma}_i$ in $\hat{\mathscr{D}}$ associated to S_i is minimal. If a slope ∞ circle created by a saddle remains in the same hemisphere as the saddle, this saddle can be pushed across this hemisphere of ∂B_i into the adjacent B_{i+1}, decreasing the total number of slope ∞ circles in the ∂B_i's. Now, of the four types of saddles in Fig. 1.5(a, b), only saddles in one hemisphere of ∂B_i are possible. If both types of saddles in one hemisphere are present, two consecutive saddles of opposite type could be pushed into the same level (Fig. 1.5(c)). Reversing the order of these two saddles then creates a trivial level circle of S_i. By incompressibility this bounds a disk in S, and S can be isotoped to eliminate one of the saddles. □

Remark. When $n = 3$, it is not necessary to include augmented surfaces on the list of candidate surfaces, as the following argument shows. Axis-parallel annuli of S_i in all three B_i's would produce a compressible torus component of S. Having such annuli in two of the three B_i's is clearly impossible in a candidate surface, and an annulus in just one B_i could be pushed across the axis, decreasing the number of slope ∞ circles in the ∂B_i's. So we may assume no S_i has axis-parallel annuli. If a slope ∞ circle of $S \cap \partial B_i$ remains, then pushing it across the axis temporarily increases the number of such circles by one, but the two saddles in the adjacent B_i's which created the original slope ∞ circle are now in the same hemisphere as the new slope ∞ circles, so we can eliminate these two circles by a step in the proof of Proposition 1.1.

Computing ∂-Slopes

We shall show how to compute ∂-slopes of the candidate surfaces in terms of the associated n-tuple of edgepaths γ_i.

All the candidate surfaces look alike near $K \cap (\partial B_i \times 0)$, lying in the collar $\partial B_i \times [0, 1]$. So to measure ∂-slopes it suffices to consider how the surfaces twist around the remaining arcs of K. This twisting can be measured by counting (with signs) how many times the inward normal vectors to ∂S in S pass through slope ∞ as we follow the levels $\partial B_i \times u$ from $u = 0$ to $u = 1$. To do this, it is convenient to view the 4-punctured sphere as the quotient orbifold $\mathbb{R}^2/\Gamma$, where Γ is the group generated by 180° rotations about $\mathbb{Z}^2 \subset \mathbb{R}^2$, the four punctures being $\mathbb{Z}^2/\Gamma$ (two adjacent unit squares form a fundamental domain). Then the straight lines in $\mathbb{R}^2$ of slope p/q passing through the points of $\mathbb{Z}^2$ project down to arcs of slope p/q on the 4-punctured sphere. The twisting we are measuring can be seen perhaps more easily by lifting to the cover $\mathbb{R}^2$.

Passing a saddle involves replacing one pair of opposite sides of a parallelogram by the other pair of opposite sides. If this change increases the slope, we are twisting through slope ∞ at two of the punctures in the clockwise direction (Fig. 1.7). For a decrease in slope, the direction is counterclockwise. We choose the convention that counterclockwise is positive. Note that both choices for the saddle realizing this change of slopes produce twisting of the same sign. Thus the total number $\tau(S)$ of twists is

$$\tau(S) = 2(s_- - s_+)/m$$

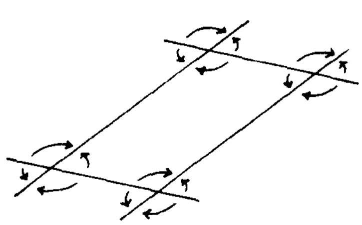


Fig. 1.7.

where s_+ (s_-) is the number of saddles of S which increase (decrease) slope. (Saddles which produce slope ∞ arcs or circles do not contribute to $\tau(S)$.)

A formula more directly in terms of the edgepaths γ_i is

$$\tau(S) = 2(e_- - e_+)$$

where e_+ (e_-) is the number of edges of the γ_i's which increase (decrease) slope, and we allow fractional values for $e_\pm$, corresponding to a final edge of γ_i traversing only a fraction of an edge of $\mathscr{D}$. (The segment from $\langle r/s \rangle$ to $k/m\langle p/q \rangle + (m-k)/m\langle r/s \rangle$ counts as the fraction k/m of an edge).

Finally, the boundary slope of S is $\tau(S) - \tau(S_0)$ for S_0 a Seifert surface on our list of candidate surfaces.

Finding such a Seifert surface is not difficult. We must have $m = 1$, and each γ_i must turn across an even number of triangles at each vertex in order for S_0 to be orientable in B_i. To analyze this condition it is convenient to take the mod 2 values of p and q in all slopes p/q. When we reduce mod 2 in this way, the triangles in $\mathscr{D}$ fold up onto the single triangle

$\langle 1/0, 0/1, 1/1 \rangle$ with vertices $\langle 1/0 \rangle$, $\langle 0/1 \rangle$, and $\langle 1/1 \rangle$. The condition on γ_i is then that its mod 2 reduction can use only one edge of $\langle 1/0, 0/1, 1/1 \rangle$. There are two cases. If one q_i is even, we choose each γ_i to be a minimal edgepath from $\langle p_i/q_i \rangle$ to $\langle \infty \rangle$ whose mod 2 reduction uses only one edge of $\langle 1/0, 0/1, 1/1 \rangle$. If q_i is odd this choice is unique, but for the one even q_i there are two possible choices. Only one of these choices will make the union of all the final saddles (and hence S_0 itself) orientable, namely, the choice which makes the number of odd-integer penultimate slopes of the γ_i's even. This prescription will not work if all the q_i's are odd, since in this case the number of odd p_i's is odd. Instead, we choose γ_i in this case to go only as far as the left-hand border of the strip $\mathscr{S}$ in $\mathscr{D}$, passing only through vertices whose slopes have odd denominator, if necessary extending one γ_i by vertical edges so that the final slopes of the γ_i's add to zero. The corresponding candidate surface is then easily seen to be orientable. (At both points where it meets the axis a normal orientation points in the same direction along the axis.)

Finding Systems (γ_i) ***Satisfying Conditions*** ***(E1 – 4)***

Consider the collection of minimal edgepaths in $\mathscr{S}$ which start at $\langle p_i/q_i \rangle$ and proceed strictly monotonically to the left, ending at a vertex of $\mathscr{S}$. The quotient space of the disjoint union of these edgepaths obtained by identifying common initial segments (possibly just the vertex $\langle p_i/q_i \rangle$ itself) is a tree T_i, naturally immersed in $\mathscr{S}$, whose vertices correspond bijectively with the edgepaths in the given collection (by sending an edgepath to the image of its endpoint in T_i). The tree T_i can be built recursively, by starting with the constant edgepath $\langle p_i/q_i \rangle$ and then, for the recursion, adding an edge for each of the two ways of extending by a leftward-directed edge each edgepath constructed at the previous stage, discarding new edges of T_i which happen to yield non-minimal edgepaths. (Calculations here are quite tame, involving nothing fancier than the Euclidean algorithm to find the continued fraction expansion of p_i/q_i, as in [7]. Non-minimality of edgepaths can easily be detected using the fact that a pair p/q, r/s forms an edge $\langle p/q, r/s \rangle$ of $\mathscr{S}$ iff $ps - qr = \pm 1$.)

In order to have all the "roots" (initial vertices) of the T_i's lying on the same vertical line in $\mathscr{S}$, enlarge each T_i to a tree T_i' by adding an initial edge along $\langle p_i/q_i, p_i/q_i \rangle$ extending as far to the right as the right-most vertex $\langle p_j/q_j \rangle$.

A system (γ_i) satisfying (E1–4) with each γ_i staying in $\mathscr{S}$ and having no vertical segments—briefly, a *type I* system—corresponds to an n-tuple of points $t_i \in T_i'$ lying on a vertical line $u = u_0 \in \mathbb{Q}$, with the property that $\Sigma v(t_i) = 0$, where $(u, v) = (b/a + b, c/a + b)$ are the horizontal and vertical coordinates in $\mathscr{S}$. The function $\Sigma v(t_i)$ is linear (with rational coefficients) in each strip between adjacent vertical lines $u = k/(k+1)$ through vertices of $\mathscr{S}$, $k = 0, 1, \ldots$, so solving $\Sigma v(t_i) = 0$ is straightforward. Solutions may not be isolated, but it is not hard to see that ∂-slopes are constant on an interval of solutions.

Given an n-tuple of vertices $t_i \in T_i$ on the left border of $\mathscr{S}$, it may or may not be possible to extend the corresponding edgepaths $\gamma(t_i)$ by vertical edges of $\mathscr{S}$ to obtain a system (γ_i) satisfying (E1–4). For example, no such extension is possible if all the $\gamma(t_i)$'s end with the edge $\langle 1/2, 1 \rangle$. In general there can be infinitely many extensions of an n-tuple $(\gamma(t_i))$, but they all yield the same ∂-slope. (Note that deciding whether extensions (γ_i) exist is easily a finite process; in fact it suffices to consider the case that $\gamma_i = \gamma(t_i)$ for all but one i.) Systems (γ_i) with vertical edges we call *type II*.

A *type III* system (γ_i) satisfying (E1–4) is obtained by extending to $\langle \infty \rangle$ an n-tuple of edgepaths $\gamma(t_i)$ ending on the left border of $\mathscr{S}$. We could also truncate this extension at a vertical line some fraction of the way between the left border of $\mathscr{S}$ and $\langle \infty \rangle$ if $\Sigma v(t_i) = 0$. As we have seen, the ∂-slopes for such systems (γ_i) extending to the left of $\mathscr{S}$ depend only on the portions $\gamma(t_i)$ in $\mathscr{S}$, so such truncation has no effect on ∂-slopes.

These three types of systems (γ_i) satisfying (E1–4) exhaust all possibilities. It should be clear that the construction can easily be made into an efficient computer algorithm.

Example. Alternating Montesinos Knots. If all p_i/q_i's have the same sign, then the standard alternating projections of the tangles K_i of K fit together to form an alternating projection of K. Since all p_i/q_i's have the same sign, systems (γ_i) generating candidate surfaces cannot stop before reaching the left border of $\mathscr{S}$, by (E3). So the numbers $e_\pm$ must be (or, in the case of type II systems, may be taken to be) integers, and therefore the ∂-slopes in $\mathbf{S}(K)$ are all even integers.

Example: $K(-1/2, 1/3, 1/7)$, *the* $(-2, 3, 7)$ *pretzel knot.* There are just two minimal edgepaths from $\langle \pm 1/q \rangle$ to the left border of $\mathscr{S}$, one moving always upward, the other always downward. So for 3 tangles this give 2^3 possible combinations. We write these as, e.g., $++-$, meaning: the upper edgepaths in the first two tangles and the lower edgepath in the third. A Seifert surface S_0, according to our earlier rules, is $+++$ continued to $\langle \infty \rangle$, with $\tau(S_0) = -18$. Edgepath systems of the types II and III described above then give rise to the ∂-slopes listed in the table below.

	II	III
$+++$	—	0*
$++-$	16	14
$+-+$	8	6
$+--$	20*	20
$-++$	6	4
$-+-$	18	18
$--+$	10	10
$---$	22	24

Except for the two entries marked with an asterisk, all of the associated candidate surfaces turn out to be compressible, as the results in §2 will show. (This high proportion of compressible candidate surfaces is not typical of Montesinos knots, but results from the very special nature of the fractions $-1/2$, $1/3$, and $1/7$.) For edgepath systems of the type I, it is not hard to check that there are just two possibilities:

(a) constant edgepaths in the first two tangles, and in the third tangle the edgepath from $\langle 1/7 \rangle$ to $\langle 1/6 \rangle$. Hence $\tau = -2$ and the ∂-slope is 16.

(b) a constant edgepath in the first tangle, the edgepath from $\langle 1/3 \rangle$ to $1/2\langle 1/3 \rangle + 1/2\langle 1/2 \rangle$ in the second tangle, and the edgepath from $\langle 1/7 \rangle$ to $1/4\langle 1/7 \rangle + 3/4\langle 0/1 \rangle$ in third tangle. (These edgepaths end at the points $(a, b, c) = (4, 6, -5)$, $(4, 6, 4)$, and $(4, 6, 1)$, respectively, since $1/2\langle 1/3 \rangle + 1/2\langle 1/2 \rangle = (1, 2, 1) + (1, 1, 1) = (2, 3, 2) = (4, 6, 4)$ and $1/4\langle 1/7 \rangle + 3/4\langle 0/1 \rangle = (1, 6, 1) + 3(1, 0, 0) = (4, 6, 1)$.) Hence $\tau = 1/2$ and the ∂-slope is $18\ 1/2$.

Both these values are actual ∂-slopes by Proposition 2.1, so the final list of ∂-slopes is 0, 16, $18\ 1/2$, 20.

§2. WHICH CANDIDATE SURFACES ARE COMPRESSIBLE?

Preliminaries on Compressing Disks

Suppose $D \subset S^3 - K$ is a compressing disk for a candidate surface S. We may assume D meets the spheres ∂B_i and the axis transversely. Then the intersection points of D with all the ∂B_i's form a graph $G \subset D$. Interior vertices of G (i.e., vertices not in ∂D) all have valence n since they are the transverse intersections of D with the axis.

The first configuration of interest in G is a cycle bounding a disk $D' \subset D - G$. If B_i is the tangle containing D' and D' does not separate the two arcs of K in B_i, then D' cuts off a ball in $B_i - K$ disjoint from S, and isotoping D' across this ball will either eliminate some intersections of D with the axis or eliminate a component of G. So we may assume D' separates the two arcs of K in B_i. Since D' is disjoint from S_i, S_i can contain no saddles, and the edgepath γ_i is constant. Furthermore, $\partial D'$ is a circle of slope p_i/q_i in ∂B_i, so we may assume $\partial D'$ contains $2q_i \geq 4$ vertices of G (otherwise there would be extra vertices which could be eliminated by isotopy of D). Conversely, if γ_i is constant then we may assume all components of $D \cap B_i$ are such disks D'.

The second type of configuration in G which will be of special interest is a disk component of $D - G$ meeting ∂D in a single arc. But before analyzing this we need to introduce a technical idea which will play a key role in the rest of the paper, the idea of r-saddles. The level curves $S_i \cap (\partial B_i \times u)$, as u decreases through the level of a saddle of S_i, change by surgery along an arc which we call the *cocore* of the saddle. Taking these level curves prior to the surgery to be in standard position, lifting to straight lines in the cover $\mathbb{R}^2$, consider isotopic variations of the cocore within its level $\partial B_i \times u$, keeping endpoints on the level curves. The saddle is called an *r-saddle* if within this isotopy class r is the minimum number of intersections of the cocore with the axis in $\partial B_i \times u$ (i.e., the projection of the axis in $\partial B_i \times 1$ to $\partial B_i \times u$). For example, the configuration in the upper right-hand corner of Fig. 2.14 (near the end of the paper) shows a 4-saddle with cocore the dotted arc from c to d.

The value of r for an r-saddle corresponding to a segment of γ_i in $\mathscr{D}$ is determined by that segment according to the following rules:

—For leftward directed segments in $\mathscr{D}$, the straight line containing the segment meets the right-hand border of $\mathscr{D}$ in a point whose slope has denominator r.

—For vertical and rightward directed segments, $r = 0$.

Rightward-directed segments do not occur in the γ_i's of candidate surfaces, but we will eventually have to consider them nonetheless.

To verify the first rule for a leftward directed segment in $\mathscr{S}$, look in the cover $\mathbb{R}^2$ of the 4-punctured sphere. For a leftward directed edge $\langle p/q, u/v \rangle$ we have $0 < v < q$, the saddle changing two arcs of slope p/q to two arcs of slope u/v. In $\mathbb{R}^2$ these four arcs form a parallelogram spanned by the vectors (q, p) and (v, u). The cocore of the saddle meets the axis minimally when it is isotoped to lie near the diagonal from (v, u) to (q, p), so $r = q - v$. The vectors $(v, u) + k(q - v, p - u)$ for $k = 0, 1, 2, \ldots$ correspond to vertices $\langle [u + k(p - u)]/[v + k(q - v)] \rangle$ in $\mathscr{S}$ on the line containing $\langle p/q, u/v \rangle$, approaching the right-hand border of $\mathscr{S}$, with slopes approaching $(p - u)/(q - v)$. This has denominator $r = q - v$, as claimed in the first rule.

Note that if we took the "dual" saddle, corresponding to a rightward directed segment in the edge $\langle p/q, u/v \rangle$, then the cocore could be chosen to be a vertical arc disjoint from the axis, so $r = 0$ in this case, an instance of the second rule.

For leftward directed segments in $\mathscr{D} - \mathscr{S}$, the first rule gives the value $r = 1$, which is easily checked, as are the remaining special cases of the second rule (vertical segments in $\mathscr{S}$, rightward directed edges in $\mathscr{D} - \mathscr{S}$).

For a candidate surface, a final segment of γ_i in the edge $\langle \infty, \infty \rangle$ has $r = 1$ since we have excluded the possibility that a saddle producing a slope ∞ circle in $S_i \cap \partial B_i$ lies in the same hemisphere as that circle. Thus the cocore of the saddle connects a slope ∞ circle to a slope ∞ arc in the opposite hemisphere, giving the saddle an r-value of 1.

Returning now to the earlier situation of a compressing disk D containing the graph G, let R be an edgemost disk component of $D - G$, meeting ∂D in just one arc. Say $R \subset B_i$ and ∂R contains r interior vertices of G. A special case is that R meets an annulus component of S_i which cuts off a neighborhood of the axis in B_i. Then we may assume R is an essential ∂-compressing disk for this annulus and $r = 1$, for otherwise we could isotope D to eliminate R and simplify G, either eliminating intersections of D with the axis or eliminating a component of G.

If R does not meet such an annulus of S_i, consider the isotopy of S obtained by pushing the arc $R \cap S$ across R to a position near $\partial R - \partial D$. This produces a new final saddle of S with cocore the (isotoped) arc $R \cap S = R \cap \partial D$. This isotopy leaves the edgepath $\hat{\gamma}_i$ unchanged, by [7]. If the new saddle is inessential, not producing the final segment of $\hat{\gamma}_i$, we can isotope D to simplify G as before. So we may assume the new saddle is essential. Its cocore intersects the axis r times, by construction, and we may assume the cocore cannot be isotoped to decrease this number, otherwise we could again isotope D to decrease its intersections with the axis. Thus the new saddle is an r-saddle.

More generally, define the r-value of a component R of $D - G$ lying in a B_i with γ_i non-constant to be the r-value of a saddle realizing the final edge of γ_i. Thus, when R is an edgemost disk this r-value is the number of interior vertices of G on ∂R. An important property to note is that, since the cycle of r-values around each interior vertex of G (including the value "undefined" for the disks $D' \subset B_i$ with γ_i constant) is determined by the γ_i's, this cycle is the same around all interior vertices, up to rotation and reflection.

A Special Case

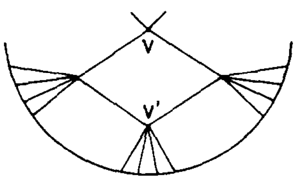
PROPOSITION 2.1. *A candidate surface is incompressible if at least one of its edgepaths γ_i is constant.*

Proof. Let D be a compressing disk for a candidate surface S, as above. We may assume the graph $G \subset D$ is non-empty, otherwise $D \subset B_i$ for some i and D would not be an essential compressing disk since S_i is incompressible in B_i by [7].

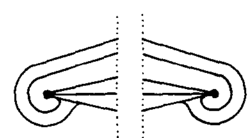
We show first that only one γ_i can be constant. Consider an innermost component U of the union of all the (closed) disks $D' \subset D$ as above, i.e., disk components of $D - G$ disjoint from ∂D. Being innermost, U is simply-connected. Topologically, U is either a disk or a union of disks Δ meeting only at isolated "singular" points, which must be all the vertices of G in U since the cyclic pattern of disks D' is the same around all interior vertices of G. Thus if U has singular points, each disk Δ has more than one singular point on its boundary (since each D' has more than one vertex). This evidently contradicts the fact that U is simply-connected. So U is topologically a disk. It can have no vertices of G in its interior, for if it did, all interior vertices of G would be completely surrounded by disks D' and so the disks D' would cover D entirely, which is impossible since they are disjoint from ∂D. If U consists of more than one D', the common edges between the disks $D' \subset U$ are then arcs with endpoints on ∂U. An edgemost such arc α cuts off a disk D' from U having at least two other vertices besides the endpoints of α, and at these other vertices there is only one disk D', unlike the situation at $\partial\alpha$, a contradiction. So U consists of a single D', and therefore only one γ_i can be constant.

Let C be an innermost component of G, not enclosing any other components, and let $\bar{C}$ be obtained from C by filling in the disks D' with $\partial D' \subset C$. Since C is innermost, $\bar{C}$ is simply-connected. The disks $D' \subset \bar{C}$ are all disjoint, so collapsing them to points turns $\bar{C}$ into a tree T. Vertices of T which are collapsed disks D' have valence at least four since each D' has at least four vertices on its boundary. All other vertices of T are extremal vertices, coming from the points of C on ∂D. Such extremal vertices must exist, so in particular C meets ∂D. Since C was chosen innermost, this means all components C of G meet ∂D, so all components C are innermost.

Now let C be an edgemost component of G, with $G - C$ contained in one component K of $D - C$. By the previous paragraph, $\bar{C}$ is simply-connected, so C must have an extremal cycle $z = \partial D'$ connected to other cycles by at most one edge, say at the vertex v of z. (If C contains only one cycle, there is no v, and references to v in what follows can be ignored.) All other edges leading away from z go to ∂D. Since z has at least four vertices, it has a vertex v' different from v and not adjacent to v. Rechoosing z if necessary, we may assume v' is not contained in ∂K (∂K meets z in at most an edge). Around v' we see, as in Fig. 2.1(a), that the cycle of r-values consists of two 2's separated by $n - 3$ 1's.



(a)



(b)

Fig. 2.1

Next we observe that the final edges of the non-constant γ_i's are alternately slope-increasing and slope-decreasing as i varies. This is because two adjacent edgemost regions R of $D - G$ (say at the vertex v') meet the common hemisphere $\partial B_i \cap \partial B_{i+1}$ in configurations which appear as mirror images when viewed from within B_i and B_{i+1}, see Fig. 2.1(b).

Using our knowledge of which edges of $\mathscr{S}$ have $r = 1$ and $r = 2$, we can see that this alternation of slope-increasing and slope-decreasing edges implies that the sum of the final slopes of the non-constant γ_i's is an integer. Namely, if an even number of γ_i's have final edge with $r = 1$, these final slopes add to an integer, as do the final slopes of the two γ_i's with $r = 2$; and if an odd number of γ_i's have final edge with $r = 1$, then all but one of these final slopes add to an integer and the remaining one adds with the two $r = 2$ final slopes to give an integer.

Hence the one constant γ_i lies on an integer-slope edge. But this is impossible since we assume $|q_i| \geq 2$ for all i. □

Application: Arbitrary Rational Numbers as ∂-Slopes

PROPOSITION 2.2. *For each $p/q \in \mathbb{Q}$, there exists an incompressible, ∂-incompressible orientable surface in the complement of some Montesinos knot, with ∂-slope p/q.*

Proof. To begin, take three tangles with slopes $p_1/q_1 = 2/7$, $p_2/q_2 = 1/(2k+1)$, and $p_3/q_3 = -1/3$, and choose edgepaths γ_i as follows (see Fig. 2.2):

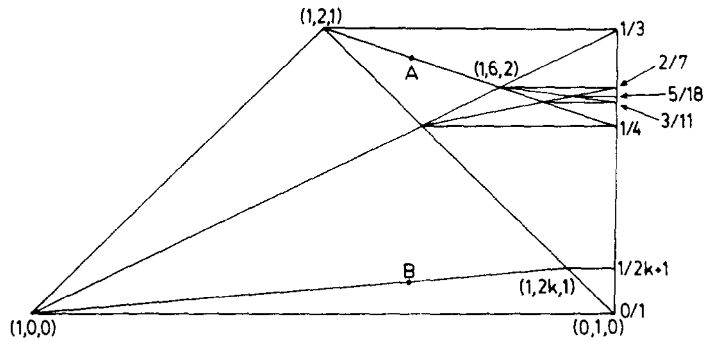


Fig. 2.2

—γ_1 goes linearly from $(1, 6, 2)$ to the point $A = (k-6, 2k, k-3) = \alpha(1, 2, 1) + \beta(1, 6, 2)$ for $\alpha = k-9$, $\beta = 3$.

—γ_2 goes linearly from $(1, 2k, 1)$ to the point $B = (k-6, 2k, 1) = \gamma(1, 0, 0) + \delta(1, 2k, 1)$ for $\gamma = k-7$, $\delta = 1$.

—γ_3 is constant, at the point $C = (k-6, 2k, 2-k)$.

Note that the sum of the slopes of A, B, and C is zero, so we obtain (if $k \geq 9$) a candidate surface S which is incompressible by Proposition 2.1. The twisting number $\tau(S)$ for its boundary is

$$\tau(S) = 2[\gamma/(\gamma+\delta) - \alpha/(\alpha+\beta)] = 4/(k-6).$$

Taking $k = 4q + 6$, we realize the twisting number $\tau(S) = 1/q\,(q \geq 1)$. Repeating this block of three tangles p times, we get an incompressible surface S with $\tau(S) = p/q$. Since all denominators q_i are odd, the $K(p_1/q_1, \ldots, p_n/q_n)$ constructed might not be a knot. To correct this, precede one edgepath γ_i starting at $\langle 2/7 \rangle$ by the edge $\langle 5/18, 2/7 \rangle$, replacing this $p_i/q_i = 2/7$ by $5/18$. This decreases $\tau(S)$ by 2.

To compute $\tau(S_0)$ for a Seifert surface S_0, we use edgepaths with vertices having the following sequences of slopes (see §1 for information on Seifert surfaces):

—$2/7, 1/4, 0/1, 1/0$.
—$1/(2k+1), 1/2k, \ldots, 1/2, 1/1, 1/0$.
—$-1/3, -1/2, -1/1, 1/0$.
—either $5/18, 3/11, 1/4, 1/3, 1/2, 1/1, 1/0$ or $5/18, 2/7, 1/4, 0/1, 1/0$.

Except in the $5/18$ tangle, the twisting for S_0 is then a multiple of 4. In the $5/18$ tangle, the twisting is ± 2. Since $\tau(S) = p/q - 2$, this means the ∂-slope of S is congruent to p/q mod 4.

To change the ∂-slope by a multiple of 4, choose one tangle with $p_i/q_i = 1/(2k+1)$. The given edgepath γ_i we change by preceding it with an even number of edges along either of the two lines in Fig. 2.3 just above or below the horizontal line through the vertex

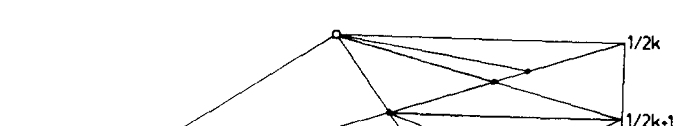


Fig. 2.3.

$\langle 1/(2k+1) \rangle$. This allows us to change $\tau(S)$ by any multiple of ± 4. The Seifert surface S_0 must also be modified. If the new $p_i/q_i > 1/(2k+1)$, the initial edge $\langle 1/(2k+1), 1/2k \rangle$ is replaced by the edge $\langle p_i/q_i, 1/2k \rangle$; if the new $p_i/q_i < 1/(2k+1)$, the initial edge is preceded by the edges $\langle p_i/q_i, 1/(2k+2) \rangle$ and $\langle 1/(2k+2), 1/(2k+1) \rangle$. In both cases $\tau(S_0)$ is unchanged. □

The General Case: Reduction to Saddle Transfers

We return now to the situation at the beginning of §2, where we assume we have a compressing disk D for a candidate surface S. After Proposition 2.1, we may assume no γ_i is constant, and hence the graph G contains no cycles. Let $T \subset G$ be a component tree which is edgemost: $G - T$ lies in one component of $D - T$.

PROPOSITION 2.3. *T has one of the forms shown in Fig.* 2.4.

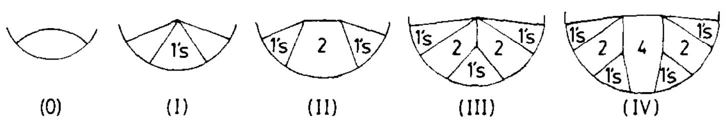


Fig. 2.4.

Proof. We may assume T has at least three interior vertices, since the trees with two or fewer interior vertices are included in Fig. 2.4. The first observation is:

(∗) There must be at least $n - 2$ consecutive regions with $r = 1$ around each interior vertex of T.

To see this, let T' be the subtree of T obtained by deleting all extremal edges, i.e., edges meeting ∂D. At an extremal vertex of T', $n - 1$ extremal edges of T were deleted. These $n - 1$ edges enclose $n - 2$ consecutive regions of $D - T$ with one interior vertex of T. (One of these regions might contain $G - T$, but then look at a different extremal vertex of T'; since we assume T has at least three interior vertices, T' has at least two extremal vertices.)

Consider the arc $A \subset T$ bounding the component of $D - T$ containing $G - T$ (or bounding any component of $D - T$ if $G = T$). If all interior vertices of T are in A, then we have the configuration in Fig. 2.5(a). Here the cycle of r-values at the two left-most interior vertices is not the same, a contradiction. So we may assume T has interior vertices not on A. These must be only one edge away from A, by (∗), as shown in Fig. 2.5(b). It follows that:

—No r's are greater than 4.
—Around each interior vertex there are exactly two r's ≥ 2.
—The region above A has $r = 1$.

If $n \geq 4$, there can be at most two interior vertices on A, by (∗) again, and the only configuration not excluded is the last one in Fig. 2.4. If $n = 3$, one easily checks (e.g., starting

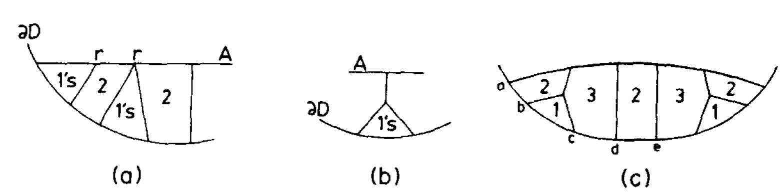


Fig. 2.5.

from one end of A) that there is at most one more configuration compatible with (*) and the fact that the cycle of r-values is the same at all vertices, the configuration shown in Fig. 2.5(c). To rule this out, consider the sequence of segments $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, $\overline{de}$ in T. The position of the first three of these segments in the spheres ∂B_i is shown in Fig. 2.6. But now we cannot continue with the fourth segment $\overline{de}$ (shown as a dashed line) because the beginning of $\overline{de}$ cannot be made to match up with the ending of $\overline{cd}$ in their common hemisphere. The final edge of the γ_i for the middle tangle in Fig. 2.6 determines the position of $\overline{de}$ up to a two-fold ambiguity: the position shown, or the position of $\overline{ab}$. The latter possibility is ruled out by the location of the endpoints of $\overline{de}$. □

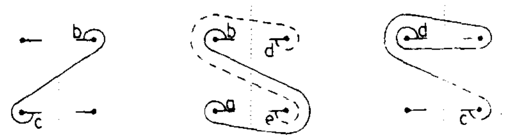


Fig. 2.6.

COROLLARY 2.4. *A candidate surface is incompressible unless the cycle of r-values for the final edges of the γ_i's is of one of the following types*: $(0, r_2, \ldots, r_n)$, $(1, \ldots, 1, r_n)$, *or* $(1, \ldots, 1, 2, r_n)$. □

By proposition 2.3, if a candidate surface S is compressible, there is a compressing disk D containing one of the types of component trees T shown in Fig. 2.4. We shall see that pushing S across the disk D_T cut off from D by T, thereby eliminating T from D, brings about a corresponding type of *transfer* of saddles of S among the various B_i's. The new system (γ_i) need not satisfy the normalization condition (E4), but the crucial question is whether the new γ_i's are still minimal. If they remain minimal after all possible saddle transfers, then in particular after transfers eliminating all T's D would be a compressing disk in B_i for an incompressible surface in B_i, a contradiction.

Systems (γ_i) with Slope ∞ Arcs

Given a system (γ_i) satisfying (E1–4), there are three possibilities which we wish to be able to distinguish:

—(γ_i) is *incompressible*: every candidate surface associated to (γ_i) is incompressible.

—(γ_i) is *compressible*: every candidate surface associated to (γ_i) is compressible.

—(γ_i) is *indeterminate*: there exist both compressible and incompressible candidate surfaces associated to (γ_i).

Here we are disregarding candidate surfaces some of whose augmented edgepaths $\hat{\gamma}_i$ end to the left of $\langle \infty \rangle$ in $\hat{\mathscr{D}}$. These surfaces will be considered separately, later.

PROPOSITION 2.5. *Suppose the γ_i's end at points of $\mathscr{D}$ to the left of $\mathscr{S}$. Then (γ_i) is*:

(a) *incompressible if the integer vertices $\langle k_i \rangle$ of the γ_i's have $|\Sigma_i k_i| \geq 2$*
(b) *compressible if $|\Sigma_i k_i| \leq 1$ and at least $n-2$ γ_i's are completely reversible*
(c) *indeterminate otherwise.*

Proof. All final saddles of a candidate surface S associated to (γ_i) have $r = 1$, so only a type I tree T can occur (Fig. 2.4). Suppose we do have such a tree T, cutting off the disk $D(T)$ from D. Each of the $n-1$ sectors of $D(T)$ lies in one B_i, with the two "radii" in its boundary lying in ∂B_i and the arc of its boundary in ∂D lying on S_i as the cocore of a final 1-saddle of S_i (after isotopy of S_i rel ∂S_i). There are two possible types for each such final saddle, given the final edge of γ_i. Since adjacent sectors of $D(T)$ have a common boundary arc, this means that once the type of the 1-saddle is specified in one sector, this determines the type in all the other sectors.

Pushing S across $D(T)$ has the effect of deleting these final saddles from the $n-1$ S_i's, truncating their γ_i's, while introducing a new 0-saddle in the one remaining S_i, say S_n, extending γ_n by a segment along an edge $\langle \infty, k \rangle$; see Fig. 2.7. (Conventions: On the left side of the "equation" are the duals of the $n-1$ final saddles prior to the transfer. On the right side is the new saddle produced.) Notice that if $\Sigma_i k_i \neq 0$ then the original γ_i's must end at $\langle \infty \rangle$. The new system (γ_i) still satisfies (E3), since we have only isotoped S. So the value of k is determined by the numbers k_i, namely $k = -k_1 - k_2 - \dots - k_{n-1}$, and $|\Sigma_i k_i|$ is the number of triangles at $\langle \infty \rangle$ between the last two edges of the new γ_n. If $|\Sigma_i k_i| \leq 1$, the extended γ_n is not minimal, so S is compressible. Statements (b) and (c) now follow: If at least $n-2$ γ_i's are completely reversible, then we choose $n-1$ of the γ_i's including these $n-2$, and we can reverse final saddles if necessary to assure that the saddle transfer is possible. On the other hand, if fewer than $n-2$ γ_i's are completely reversible, then in any choice of $n-1$ γ_i's there will be at least two not completely reversible, so for some choices of final saddles the saddle transfer will be possible and for other choices it will not.

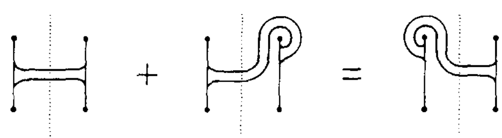


Fig. 2.7.

For statement (a) we must continue further and consider a sequence of saddle transfers eliminating the component trees of G one at a time, edgemost first, checking that the edgepaths γ_i produced by any sequence of saddle transfers remain minimal.

Claim. The net effect of a sequence of saddle transfers on the γ_i's is either to return them to their initial position or to truncate all but one of them while lengthening this last γ_i, by rightward-directed segments lying on two straight lines, one to the left of $\mathscr{S}$ and the other in $\mathscr{S}$.

The new γ_i's will then be minimal provided the extended γ_i is minimal when it passes through the vertex $\langle \infty \rangle$, which will follow from the hypothesis (a) just as for the first transfer, considered above.

The Claim will be verified by induction on the number of transfers. So assume inductively that it holds, and that γ_n, say, has been lengthened. Then since $r_n = 0$ for the extended γ_n, only transfers of types 0, I, and II are possible.

Suppose first that the extended γ_n ends at a point in $\langle \infty, k \rangle$ to the left of $\mathscr{S}$, so only type 0 and I transfers are possible. A type I transfer would simply extend γ_n rightward along $\langle \infty, k \rangle$, truncating the other γ_i's, as before. A type 0 transfer has the opposite effect, truncating γ_n and extending the other γ_i's, thus returning all the γ_i's to an earlier state. This is obvious if the 0-saddle involved in the type 0 transfer is the one just created by a type I transfer, since the 0-transfer then simply cancels the I-transfer. This special case implies the general case since the effect of a saddle transfer depends only on the saddle or saddles being transferred, and here there are just the usual two choices, one a reversal of the other, for a 0-saddle realizing a given rightward-directed segment of the extended γ_n.

Next suppose the inductively extended γ_n ends at the vertex $\langle k \rangle$, on the left border of $\mathscr{S}$. A type 0 transfer now would be just like the 0-transfers considered in the preceding paragraph, so consider a type I transfer. Two examples of such transfers with n odd and n even are shown in Fig. 2.8 (with the same conventions as in Fig. 2.7; an alternative transfer is shown in dashed lines). We see in these examples that the $n-1$ saddles with $r = 1$ alternate between slope-increasing and slope-decreasing as we go from one B_i to the next. In general, we call an r-saddle which increases slope a $+r$-saddle, an r-saddle which decreases slope a $-r$-saddle. This alternation of $+1$'s and -1's holds generally for a type I transfer in $\mathscr{S}$ because the half-saddles in the common hemisphere of ∂B_i and ∂B_{i+1} must be mirror images (the two sectors of $D(T)$ involved have a common edge on this hemisphere), hence the whole saddles must be mirror images modulo Dehn twists along the axis, i.e., vertical translation in $\mathscr{S}$.

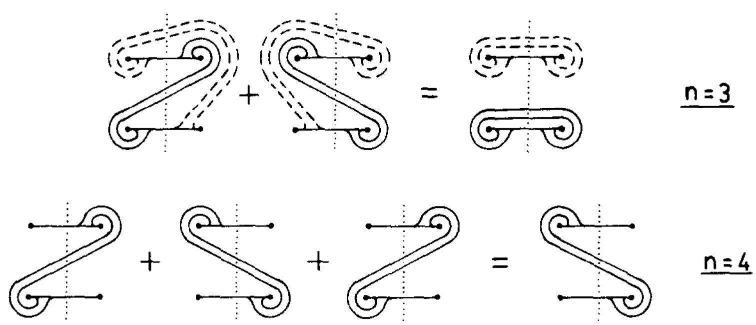


Fig. 2.8.

In the present case, if a type I transfer is made, γ_i for $i < n$ is truncated, while γ_n is lengthened by a segment in $\mathscr{S}$ directed to the right, corresponding to the new saddle created by the transfer. We observe that the new segment of γ_n lies on a line which meets the right border of $\mathscr{S}$ in a point with integer slope. This follows from the fact that the truncated segments of the other γ_i's lay alternately on slope-increasing and slope-decreasing diagonals of the squares in Fig. 1.3, together with the fact that the final slopes of all the new γ_i's still add up to zero.

The case of a type II transfer starting with γ_i's having endpoints on the left border of $\mathscr{S}$ is similar. Typical examples are shown in Fig. 2.9. Again the $n-1$ non-zero r's have alternating signs, and it follows that the new segment of γ_n lies on a line in $\mathscr{S}$ which meets the right border of $\mathscr{S}$ in a point with slope of denominator 2.

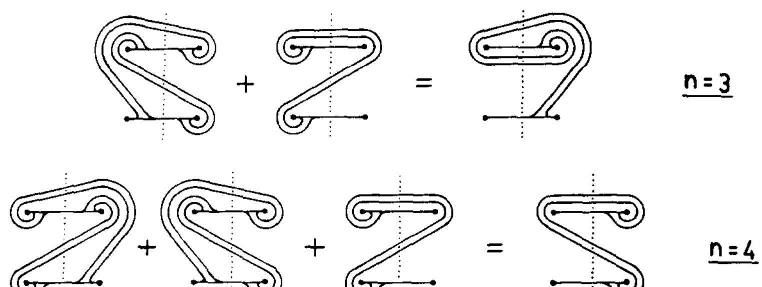


Fig. 2.9.

More generally, consider a sequence of type I and II transfers (with the two types possibly intermingled) starting with the transfers just considered. This truncates final segments of γ_i for $i < n$, segments corresponding to 1-saddles and 2-saddles. Looking at Fig. 1.3, we see that the truncated segments of each γ_i must all lie on one straight line since γ_i was minimal before the truncation. Hence the extension of γ_n in $\mathscr{S}$ must lie on a single line. (It follows that there cannot be transfers of both types I and II.)

Lastly, consider a type 0 transfer involving the 0-saddle corresponding to a final segment of γ_n in $\mathscr{S}$. If this segment is not in an edge $\langle k, k \rangle$ then there are two choices for the 0-saddle, differing by a reversal of saddle type, so, as argued above, the transfer of such a 0-saddle just returns the edgepaths γ_i to an earlier state. On the other hand, for a 0-saddle corresponding to a segment of γ_n in an edge $\langle k, k \rangle$ there are four possible positions: the two shown in the upper right corner of Fig. 2.8, plus their reflection across a horizontal line. This reflection interchanges $+$ 1-saddles and $-$ 1-saddles, so all the 0-saddles at the end of γ_n produced by type I transfers must belong either to the pair shown in Fig. 2.8 or to the opposite pair. (Namely, the original γ_i's for $i < n$ determine whether the 1-saddles in question are $+$ 1-saddles or $-$ 1-saddles.) Conceivably, commuting the first 0-saddle along $\langle k, k \rangle$ with the last 0-saddle along $\langle \infty, k \rangle$ might replace a 0-saddle of the pair in Fig. 2.8 with a 0-saddle of the opposite pair, but Fig. 2.10 shows this cannot happen. Since all the 0-saddles along $\langle k, k \rangle$ belong to a single pair, it follows that type 0 transfers of these saddles return the γ_i's to an earlier state. This establishes the Claim, and hence the Proposition. □

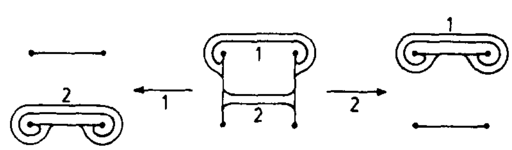


Fig. 2.10.

Systems (γ_i) in $\mathscr{S}$ without Vertical Edges

PROPOSITION 2.6. *Suppose (γ_i) lies in $\mathscr{S}$ and has as its cycle of final r-values $(1, \ldots, 1, r_n)$ with $r_n \geq 1$. Then (γ_i) is incompressible unless the signs of each pair of adjacent r_i's equal to 1 in the cycle are opposite. If these signs do alternate, then:*

(1) *For n odd, (γ_i) is*
 (a) *incompressible if r_n is odd*
 (b) *compressible if r_n is even and at least $n - 1$ γ_i's, including γ_n, are completely reversible*
 (c) *indeterminate otherwise.*

(2) *For n even, (γ_i) is*
 (a) *incompressible unless all signs alternate and the last edge of γ_n lies in the same triangle of $\mathscr{S}$, and has the same ending point, as an edge with $r = 1$*
 (b) *compressible if the conditions in* (a) *are satisfied and at least $n - 2$ γ_i's with final $r_i = 1$ are completely reversible*
 (c) *indeterminate otherwise.*

Proof. Suppose n is odd, and consider a type I transfer. This involves $n - 1$ γ_i's with $r_i = 1$ of alternating sign. Since $n - 1$ is even, the final slopes in these γ_i's add to an integer, so the endpoint of the other γ_i, say γ_n, must be on an edge $\langle k, k \rangle$ with $k \in \mathbb{Z}$, hence at the vertex $\langle k \rangle$. Thus the γ_i's end on the left border of $\mathscr{S}$. The transfer truncates γ_i for $i < n$ and extends γ_n along $\langle k, k \rangle$, so the new γ_n is still minimal.

We must examine the possibility of a 0-transfer using this new segment of γ_n. There are four possible positions for a saddle realizing this segment of γ_n, rather than the usual two for non-horizontal edges of $\mathscr{S}$. Type 0 transfers using two of these four possible 0-saddles would simply cancel I-transfers (the I-transfer just performed, and its "opposite" involving the opposite final 1-saddles for $\gamma_1, \ldots, \gamma_{n-1}$). Can the other two possibilities arise? For convenience, let us redraw the top half of Fig. 2.8, replacing all saddles shown by their duals; see the top half of Fig. 2.11 where we have also change from 1-sheeted to 2-sheeted surfaces. For both type I transfers, the new 0-saddle produced connects the circle to the *top* of the slope 0 arcs. It may be possible to switch to a 0-saddle connecting to the *bottom* of a slope 0 arc, by reversing the new saddle in γ_n with a final saddle of the original γ_n. Two ways this can happen are illustrated in Fig. 2.12. Any other way differs from these only by some number of full Dehn twists along a slope 0 circle, so this switch can happen only if (the original) r_n is even and the corresponding saddle is the right type. (Note that this saddle, labelled "2" in Fig. 2.12, is of the same type in both cases shown.) If this switch occurs, we are in the situation of the lower half of Fig. 2.11, where a 0-transfer produces saddles in the other $n-1$ tangles with visible ∂-compressing disks. So the surface S would be compressible. This somewhat subtle compression is bound to occur if the original γ_n and at least $n-2$ of the remaining original γ_i's are completely reversible, since then we can be sure that both the I-transfer and the switch in type of the new 0-saddle are possible. Otherwise, these moves may not be possible.

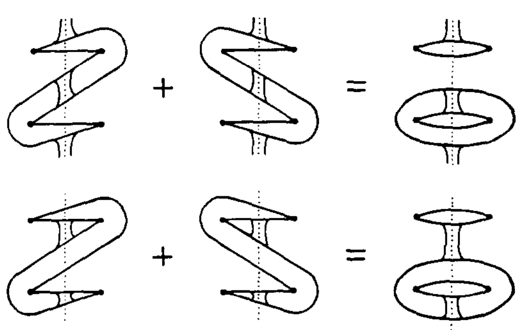


Fig. 2.11.

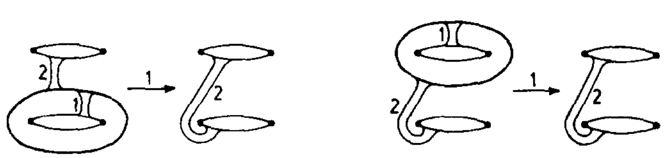


Fig. 2.12.

This example is completely general, as is easily seen. Note that we are assuming that our surface has at least two sheets. This represents no loss of generality since we are considering only orientable surfaces, and taking two parallel copies of a surface doubles the number of sheets without affecting incompressibility. [With single-sheeted surfaces the argument in the preceding paragraph would not work, as one can see by redrawing the lower half of Fig. 2.11 with one sheet. It is possible that these single-sheeted surfaces may be non-orientable and incompressible only in the weaker sense, not π_1-injective.]

Except when $r_n = 2$, when there are initial type II transfers to consider, statement (1) now follows as in Proposition 2.5: γ_n is being lengthened only by rightward-directed edges along a straight line in $\mathscr{S}$, so minimality is preserved.

If $r_n = 2$ and a II-transfer occurs, lengthening γ_1 and truncating $\gamma_2, \ldots, \gamma_n$, say, then $r_2, \ldots, r_n$ must have alternating signs (Fig. 2.9). The extended γ_1 fails to be minimal if and only if r_1 also has opposite sign from r_2. With these alternating signs, we can be sure such a transfer is possible only if $\gamma_2, \ldots, \gamma_n$ are completely reversible, so the statement of (1) does not need to be modified for this case $r_n = 2$. (As in Proposition 2.5 again, subsequent transfers cannot produce new compressions.)

Now assume n is even. For a type I or II transfer to take place, all but one of the signs of the r_i's must alternate, the one whose γ_i is lengthened by the transfer. The lengthened γ_i will be minimal unless the conditions in (2a) hold. With these conditions, a I-transfer must be possible if $n - 2$ γ_i's with $r_i = 1$ are completely reversible, which is also necessary (but not sufficient) to assure that a type II transfer is possible. This gives (2b, c), and (2a) follows since later transfers cannot create any new compressions, as in Proposition 2.5. □

Remark. By examining $\mathscr{S}$ it is easy to see that in Proposition 2.6, condition (2a) can be restated as "incompressible unless all signs alternate and $r_n = d_n + 1$ where d_n is the denominator of the slope of the ending point of γ_n". Similarly, in the following Proposition, condition (3a) can be restated as "incompressible unless $r_n = d_n + 2$".

PROPOSITION 2.7. *Suppose (γ_i) lies in $\mathscr{S}$ and has as its cycle of final r-values $(1, \ldots, 1, 2, r_n)$ with $r_n \geq 2$. Then (γ_i) is incompressible unless each $r_i = 1$ has opposite sign from both its neighbors in the cycle. Assuming this sign condition holds, then:*

(1) *For $r_n = 2$, (γ_i) is*
   (a) *compressible if at least $n - 1$ γ_i's are completely reversible*
   (b) *indeterminate otherwise.*

(2) *For $r_n = 4$, (γ_i) is*
   (a) *compressible if all γ_i are completely reversible*
   (b) *indeterminate otherwise.*

(3) *For $r_n \neq 2, 4$, (γ_i) is*
   (a) *incompressible unless the last edge of γ_n lies in the same triangle of $\mathscr{S}$, and has the same ending point, as an edge with $r = 2$*
   (b) *compressible if the condition in* (a) *holds and $\gamma_1, \ldots, \gamma_{n-1}$ are completely reversible*
   (c) *indeterminate otherwise.*

Proof. Initial transfers can be of types II, III (if $r_n = 2$), or IV (if $r_n = 4$). Type II transfers are treated just as in the earlier propositions and statement (3) follows as before. For $r_n = 2$, a type II transfer produces retracing in γ_n (or, symmetrically, in γ_{n-1}) if the sign condition holds; otherwise, minimality of edgepaths is preserved. So with the sign condition, (γ_i) is compressible if all γ_i's except either γ_n or γ_{n-1} are completely reversible. For $r_n = 4$ a type II transfer yields an extended γ_n which is still minimal, so no compressions result.

Consider next a type III transfer, when $r_n = 2$. Typical examples of such transfers are shown in Fig. 2.13. Here the duals of the shaded saddles are replaced by the unshaded saddle (which could be any one of the 1-saddles). Note that each $r_i = 1$ has opposite sign from its two neighbors in the cycle. It is not hard to see that this necessary condition for a type III transfer is also sufficient in general, provided one has in addition the right types of 2-saddles for $i = n - 1$ and n and both types of 1-saddles for all but one $i < n - 1$. The effect

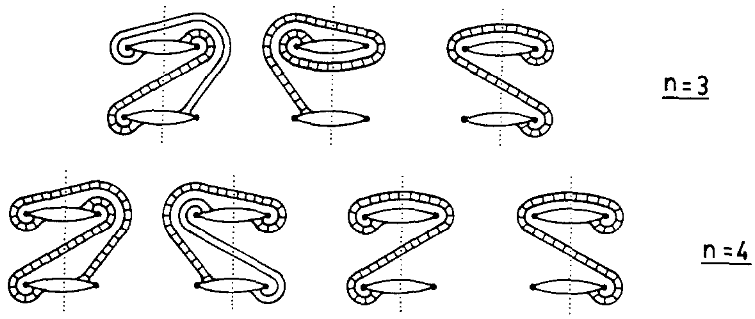


Fig. 2.13.

of a type III transfer is to truncate all the γ_i's and to lengthen one γ_i (corresponding to the one unshaded saddle created) by a rightward directed segment of an edge in $\mathscr{S}$. We may double the number of sheets of $\mathscr{S}$ before performing the transfer, and then the one lengthened γ_i is certain to retrace itself, making $\mathscr{S}$ compressible whenever a type III transfer is possible. This is guaranteed if γ_{n-1}, γ_n, and all but one of the remaining γ_i's are completely reversible. Combining this with the earlier analysis of type II transfers, we obtain statement (1) in the Proposition. (As usual, subsequent transfers can produce no new compressions.)

A type IV transfer, when $r_n = 4$, is more complicated. Here again we must have the alternating sign condition. It follows by considering the location of the edges of $\mathscr{S}$ with $r = 1, 2$, and 4 that the γ_i's must end at the left border of $\mathscr{S}$. A typical example is shown in Fig. 2.14. The sequence of dotted lines $\overline{ab}, \overline{bc}, \ldots, \overline{ef}$ denotes the position of the arc $\partial D(T) \cap \partial D = \partial D(T) \cap \mathrm{S}$. Isotoping S by pushing this arc across $D(T)$ changes the original collection of slope 0 arcs of $\mathrm{S} \cap \partial B_i$ by surgering them along all these dotted arcs $\overline{ab}, \ldots, \overline{ef}$ and also along the arc $\overline{fa}$ in the left-hand configuration. Observe that the arc $\overline{fa}$ creates in this way a ∂-compressing disk (shaded) for the isotoped S. So S is compressible if a type IV transfer is possible. Since we need all the handles shown in Fig. 2.14 in the given positions, we are assured of having a type IV transfer only when all the γ_i's are completely reversible. $\square$

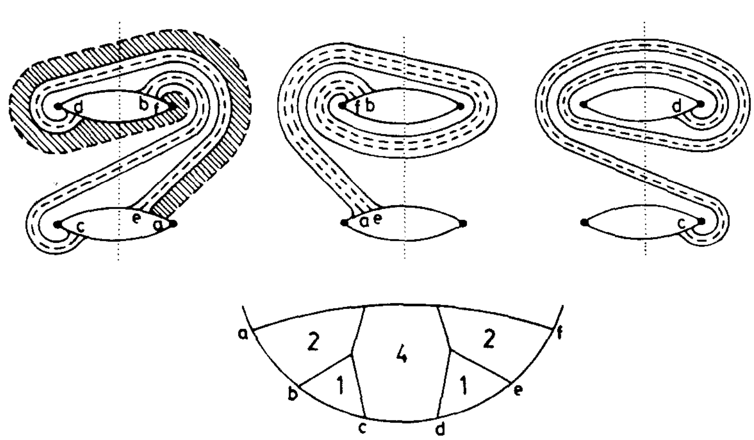


Fig. 2.14

Systems (γ_i) with Vertical Edges

PROPOSITION 2.8. *Suppose (γ_i) has some final $r_i = 0$. Then*:

(a) *(γ_i) is incompressible if in the cycle $(r_1, \dots, r_n)$ there is no sequence $\pm 0, \underline{\mp 2}, \mp 1$ or $\pm 0, \underline{\mp 2}, ?, \underline{\pm 2}, \mp 0$, where the underline denotes a chain (perhaps empty) of repeated symbols and "?" means that one arbitrary r_i may be inserted at this point.*

(b) *(γ_i) is compressible if the cycle $(r_1, \dots, r_n)$ contains one of the units*

$$\text{(A)}\quad \pm\tilde{0}, \underline{\mp\tilde{2}}, \mp 0 \text{ or } \pm\tilde{0}, \underline{\mp\tilde{2}}, \mp 1$$
$$\text{(B)}\quad \pm\tilde{0}, \underline{\mp\tilde{2}}, ?, \underline{\pm\tilde{2}}, \mp\tilde{0}$$

where "$\tilde{\ }$" indicates that the corresponding γ_i must be completely reversible. (Note that $\pm\tilde{0}, \underline{\mp\tilde{2}}, \mp\tilde{0}$ can be viewed as either an A or a B unit.) More generally, (γ_i) is compressible if the cycle $(r_1, \dots, r_n)$ contains a sequence obtained by connecting together a string of A or B units at common $\pm\tilde{0}$'s, thus forming a word $A^{-1}B^kA$, $A^{-1}B^k$, B^kA, or B^k $(k \geq 0)$, then deleting the $\tilde{\ }$'s from these common $\pm\tilde{0}$'s. The first and last terms of such a sequence are allowed to overlap also, but if this overlap term is a $\pm\tilde{0}$, we do not delete its $\tilde{\ }$ unless an adjacent term is ∓ 0.

(c) *(γ_i) is indeterminate otherwise.*

Proof. With some r_i's zero, initial type 0 transfers are possible. The two possible 0-saddles realizing a final vertical segment of γ_i lie in opposite hemispheres (Fig. 2.15). A 0-transfer simply shifts such a saddle across the common hemisphere into an adjacent $B_{i\pm 1}$, where it remains a 0-saddle of the same sign, say $+0$. The adjacent $\gamma_{i\pm 1}$ is thereby lengthened by a vertical segment, and this new $\gamma_{i\pm 1}$ will be non-minimal only if (the original) $r_{i\pm 1}$ is -0 or -1. If $r_{i\pm 1} = -2$, we may be able to reverse the type of the new $+0$-saddle at the end of the extended $\gamma_{i\pm 1}$, obtaining a $+0$-saddle in the opposite hemisphere, then transfer this saddle into $\gamma_{i\pm 2}$, etc. Statement (b) describes the cases when non-minimal edgepaths must be obtainable from sequences of such moves.

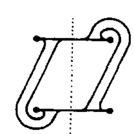


Fig. 2.15

There is also the possibility of a type I or II transfer if only one $r_i = 0$, say $r_n = +0$. A type II transfer would extend γ_n to be still minimal, so we need consider only a type I transfer. For n even, this would extend γ_n by a non-horizontal segment corresponding to the new saddle, which is the dual of a 1-saddle, so γ_n would be non-minimal only if $r_1 = r_{n-1} = -1$, in which case the cycle of r-values contains successive terms $-1, +0, -1$, a situation covered already in (b). The proposition now follows for n even, since later type I and II transfers cannot yield new compressions, as we saw in Proposition 2.5. For n odd, a type I transfer can lead to the subtle ∂-compression in Fig. 2.11 just as it did in Proposition 2.6, since $r_n = +0$ is even. As before, to guarantee that this must happen we must have $r_n = +\tilde{0}$. An adjacent $r_{n\pm 1} = -1$, so this case is also already covered in (b).

Finally, there is the possibility that a sequence of 0-transfers and reversals of 0-saddle type could reduce the number of $r_i = 0$ to one and we could get a compression after a type I transfer, the situation just considered. Reversals of 0-saddle type would involve $r_i = \pm 2$, so these need not concern us here. For 0-transfers, it is not hard to see that only the case n odd can lead to compressions, and this is covered in the final phrase in statement (b). $\square$

Observe that incompressible surfaces whose γ_i's have both upward and downward directed vertical edges cannot occur for $n \leq 3$, by the final condition in (b). But if we do have one incompressible surface whose γ_i's have both upward and downward vertical edges, we can produce infinitely many more by extending these edges arbitrarily far upward and downward (subject to (E3), of course). These surfaces all have the same boundary, but their genus is unbounded. In particular, this can sometimes be done with Seifert surfaces; compare [12].

The following result, together with its analogue with opposite signs, is useful in implementing Proposition 2.8.

PROPOSITION 2.9. *Given a collection* (γ_i) *of minimal edgepaths from* $\langle p_i/q_i \rangle$ *to the left border of* $\mathcal{S}$, *without vertical edges and having final slopes with a positive sum, then these* γ_i*'s can be extended by vertical edges to form a system associated to some incompressible surface except when the following condition* $(*)$ *is satisfied, when no such extension is possible*:

$(*)$ *The cycle of final r-values for the* γ_i*'s contains at least one* $+1$, *and each* $+1$ *in this cycle is separated from the next* $+1$ *by* r_i*'s all but possibly one of which are* $+\tilde{2}$*'s.*

Proof. First we check that, apart from the exceptional cases in $(*)$, we can extend one γ_i by the appropriate number of vertical edges with $r = -0$ to form a non-compressible system. If no r_i is $+1$, we can do this with any γ_i. If some r_i's are $+1$ and there are two consecutive $+1$'s separated by $+\tilde{2}$'s and two r_i's not equal to $+\tilde{2}$, we can choose either of the two corresponding γ_i's to augment with -0 edges.

For the converse we need to see that adding -0 edges to at least one γ_i, and possibly also some $+0$ edges to other γ_i's, always leads to a compressible system in case $(*)$ holds. Think of adding these edges one γ_i at a time, starting with a -0 edge. Clearly, adding the first -0 edge produces a compressible system, and adding further ± 0 edges does not change this situation. $\square$

Exercise. Suppose that (γ_i) is a compressible system ending at the left border of $\mathcal{S}$, without vertical edges. Then any system obtained from (γ_i) by extending some γ_i's by vertical edges is also compressible.

Systems (γ_i) *with Slope* ∞ *Circles*

Recall that slope ∞ circles in curve systems $S \cap \partial B_i$ arise in the *augmentation* process described in §1, when edgepaths γ_i ending at $\langle \infty \rangle$ are extended to augmented edgepaths $\hat{\gamma}_i$ ending on the edge $\langle \infty, \infty \rangle$.

PROPOSITION 2.10. *For* (γ_i) *ending at* $\langle \infty \rangle$, *incompressible augmented surfaces exist if and only if at least four* γ_i*'s are not completely reversible. Hence incompressible augmented surfaces contribute nothing new to the set* $\mathrm{S}(K)$ *of* ∂*-slopes.*

Proof. For an augmented system $(\hat{\gamma}_i)$, we need only consider type (I) transfers. The $n-1$ sectors of $D(T)$ meet S in arcs crossing either duals of final 1-saddles or axis-parallel annuli in S_i's, say for $i < n$. If the remaining S_n contains axis-parallel annuli, then $D(T)$ can be combined with a suitably chosen spanning disk for an axis-parallel annulus to produce a compressing disk for S, which is non-trivial since it intersects a slope ∞ circle on S in one point transversely. If S_n contains no axis-parallel annuli, but at least one saddle producing a slope ∞ circle, then the isotopy of S obtained by pushing across $D(T)$ yields a saddle which eliminates a slope ∞ circle. The new S_n therefore has consecutive saddles producing retracing in its edgepath $\hat{\gamma}_n$, so S is compressible.

The remaining case is that ∂S_n contains no slope ∞ circles. Isotoping S by pushing across $D(T)$ yields a saddle in S_n extending $\hat{\gamma}_n$ along an edge $\langle \infty, k \rangle$. This new S intersects the axis in two points. However, these two points can be cancelled by eliminating an inessential arc in a hemisphere of some ∂S_i where there was originally a slope ∞ circle (Fig. 2.16). The result is now that γ_n must retrace back to $\langle \infty \rangle$ on the edge $\langle \infty, k \rangle$, making S compressible.

Fig. 2.16

Thus S is compressible if a type I transfer involving slope ∞ circles can be made. Suppose γ_i, γ_j, γ_k, and γ_l are not completely reversible, $i < j < k < l$. We can construct an augmented surface for which no I-transfers are possible, as follows. Begin with S_i and S_j whose final saddles, producing slope ∞ arcs, are of a single irreversible type. Augment S_i and S_j by final saddles producing slope ∞ circles in the common hemispheres with S_{i+1} and S_{j-1} as in §1, and chosen so that these saddles cannot be put on the same level with the preceding saddles. (This is possible with candidate surfaces if γ_i and γ_j are not completely reversible.) Connect these two slope ∞ circles by inserting axis-parallel annuli in the intervening S_t's. Then choose the type of the final saddles in S_k and S_l (producing slope ∞ arcs) so that no type I disk $D(T)$ exists in any collection of $n - 1$ S_t's.

On the other hand, for an augmented surface with at most three γ_i's not completely reversible, a type I transfer is always possible. This is obvious if all the augmentation consists of axis-parallel annuli. If there are saddles producing slope ∞ circles, these must occur in at least two γ_i's which are not completely reversible. Then a type I disk $D(T)$ exists in these two B_i's plus $n - 3$ remaining B_i's with completely reversible γ_i's.

If incompressible augmented surfaces exist, then so do incompressible unaugmented surfaces with the same edgepaths to $\langle \infty \rangle$, by Proposition 2.5. Since ∂-slopes are unchanged by augmentation, the rest of Proposition 2.10 follows. □

Note that if one incompressible augmented surface exists, we can produce infinitely many more by simply duplicating the saddles and annuli which produce slope ∞ circles arbitrarily often. These surfaces all have the same boundary but unbounded genus.

§3. EXAMPLES

This section contains the results of some computer calculations of ∂-slopes.

The following table gives the ∂-slopes for Montesinos knots of ten or fewer crossings (including 2-bridge knots).

$3_1 = K(1/3)$: 0, 6	$4_1 = K(2/5)$: −4, 0, 4
$5_1 = K(1/5)$: 0, 10	$5_2 = K(3/7)$: 0, 4, 10
$6_1 = K(4/9)$: −4, 0, 8	$6_2 = K(4/11)$: −4, 0, 2, 8
$6_3 = K(5/13)$: −6, −2, 0, 2, 6	$7_1 = K(1/7)$: 0, 14

$7_2 = K(5/11)$: 0, 4, 14
$7_3 = K(4/13)$: −14, −8, 0
$7_4 = K(4/15)$: −14, −8, 0
$7_5 = K(7/17)$: 0, 4, 6, 10, 14
$7_6 = K(7/19)$: −4, 0, 4, 6, 10
$7_7 = K(8/21)$: −8, −4, 0, 6
$8_1 = K(6/13)$: −4, 0, 12
$8_2 = K(6/17)$: −4, 0, 6, 12
$8_3 = K(4/17)$: −8, 0, 8
$8_4 = K(5/19)$: −8, −2, 0, 8
$8_5 = K(1/3, 1/3, 1/2)$: −4, 0, 2, 8, 10, 12
$8_6 = K(10/23)$: −4, 0, 2, 6, 12
$8_7 = K(9/23)$: −10, −6, −2, 0, 6
$8_8 = K(9/25)$: −10, −6, −4, 0, 2, 6
$8_9 = K(7/25)$: −8, −2, 0, 2, 8
$8_{10} = K(1/3, 2/3, 1/2)$: −6, −2, 0, 6, 8, 10
$8_{11} = K(10/27)$: −4, 0, 6, 12
$8_{12} = K(12/29)$: −8, −4, 0, 4, 8
$8_{13} = K(11/29)$: −10, −6, −4, −2, 0, 6
$8_{14} = K(12/31)$: −4, 0, 4, 6, 8
$8_{15} = K(2/3, 2/3, 1/2)$: −16, −12, −10, −8, −4, −2, 0
$8_{19} = K(1/3, 1/3, -1/2)$: 0, 12 [(4,3) torus knot]
$8_{20} = K(1/3, 2/3, -1/2)$: −10, 0, 8/3
$8_{21} = K(2/3, 2/3, -1/2)$: −12, −6, −2, 0, 1
$9_1 = K(1/9)$: 0, 18
$9_2 = K(7/15)$: 0, 4, 18
$9_3 = K(6/19)$: −18, −12, 0
$9_4 = K(5/21)$: 0, 8, 18
$9_5 = K(6/23)$: −18, −12, −8, 0
$9_6 = K(11/27)$: 0, 4, 10, 14, 18
$9_7 = K(13/29)$: 0, 4, 6, 10, 18
$9_8 = K(11/31)$: −8, −4, 0, 4, 6, 10
$9_9 = K(9/31)$: 0, 6, 8; 14, 18
$9_{10} = K(10/33)$: −18, −12, −6, 0
$9_{11} = K(14/33)$: −14, −10, −4, 0, 4
$9_{12} = K(13/35)$: −4, 0, 6, 8, 14
$9_{13} = K(10/37)$: −18, −14, −12, −8, −6, 0
$9_{14} = K(14/37)$: −12, −8, −4, 0, 6
$9_{15} = K(16/39)$: −14, −10, −8, −4, 0, 4
$9_{16} = K(1/3, 1/3, 3/2)$: 0, 4, 6, 10, 12, 14, 16, 18
$9_{17} = K(14/39)$: −8, −4, −2, 0, 4, 10
$9_{18} = K(17/41)$: 0, 4, 8, 10, 12, 14, 18
$9_{19} = K(16/41)$: −8, −4, 0, 4, 10
$9_{20} = K(15/41)$: −4, 0, 2, 6, 8, 14
$9_{21} = K(18/43)$: −14, −10, −8, −4, 0, 4
$9_{22} = K(3/5, 1/3, 1/2)$: −8, −4, −2, 0, 2, 4, 6, 8, 10
$9_{23} = K(19/45)$: 0, 4, 8, 10, 14, 18
$9_{24} = K(1/3, 2/3, 3/2)$: −10, −6, −4, 0, 2, 4, 6, 8
$9_{25} = K(2/5, 2/3, 1/2)$: −14, −10, −8, −6, −4, −2, 0, 2, 4
$9_{26} = K(18/47)$: −12, −8, −6, −4, 0, 6
$9_{27} = K(19/49)$: −8, −4, −2, 0, 2, 4, 6, 10
$9_{28} = K(2/3, 2/3, 3/2)$: −12, −8, −6, −2, 0, 2, 4, 6
$9_{30} = K(3/5, 2/3, 1/2)$: −10, −6, −4, −2, 0, 2, 4, 6, 8
$9_{31} = K(21/55)$: −6, −2, 0, 2, 6, 12
$9_{35} = K(1/3, 1/3, 1/3)$: −18, −12, −4, 0
$9_{36} = K(2/5, 1/3, 1/2)$: −4, 0, 2, 4, 6, 8, 10, 12, 14
$9_{37} = K(1/3, 2/3, 2/3)$: −10, −4, 0, 4, 8
$9_{42} = K(2/5, 1/3, -1/2)$: −8, 0, 8/3, 6
$9_{43} = K(3/5, 1/3, -1/2)$: −4, 0, 6, 8, 32/3
$9_{44} = K(2/5, 2/3, -1/2)$: −10, −2, 0, 1, 2, 14/3
$9_{45} = K(3/5, 2/3, -1/2)$: −14, −10, −8, −4, −2, 0, 1
$9_{46} = K(1/3, 1/3, -1/3)$: −12, 0, 2
$9_{48} = K(2/3/, 2/3, -1/3)$: −4, 0, 4, 8, 11
$10_1 = K(8/17)$: −4, 0, 16
$10_2 = K(8/23)$: −4, 0, 10, 16
$10_3 = K(6/25)$: −8, 0, 12
$10_4 = K(7/27)$: −12, −6, 0, 8
$10_5 = K(13/33)$: −14, −10, −6, 0, 6
$10_6 = K(16/37)$: −4, 0, 6, 10, 16
$10_7 = K(16/43)$: −4, 0, 6, 10, 16
$10_8 = K(6/29)$: −8, 0, 2, 12
$10_9 = K(11/39)$: −12, −6, −2, 0, 8
$10_{10} = K(17/45)$: −14, −10, −6, −4, 0, 6
$10_{11} = K(13/43)$: −8, −2, 0, 6, 12
$10_{12} = K(17/47)$: −14, −10, −8, −2, 0, 6
$10_{13} = K(22/53)$: −8, −4, 0, 4, 8, 12
$10_{14} = K(22/57)$: −4, 0, 4, 8, 10, 12, 16
$10_{15} = K(19/43)$: −10, −6, 0, 2, 4, 10
$10_{16} = K(14/47)$: −12, −6, −2, 0, 4, 8
$10_{17} = K(9/41)$: −10, −2, 0, 2, 10
$10_{18} = K(23/55)$: −8, −4, 0, 2, 4, 8, 12
$10_{19} = K(14/51)$: −10, −4, −2, 0, 2, 4, 10
$10_{20} = K(16/35)$: −4, 0, 2, 6, 16
$10_{21} = K(16/45)$: −4, 0, 4, 10, 16
$10_{22} = K(13/49)$: −12, −6, 0, 2, 8
$10_{23} = K(23/59)$: −14, −10, −8, −6, −4, 0, 6
$10_{24} = K(24/55)$: −4, 0, 4, 6, 10, 16
$10_{25} = K(24/65)$: −4, 0, 2, 6, 10, 12, 16
$10_{26} = K(17/61)$: −12, −6, −2, 0, 4, 8
$10_{27} = K(27/71)$: −14, −10, −6, −4, 0, 6
$10_{28} = K(19/53)$: −14, −10, −8, −4, −2, 0, 6
$10_{29} = K(26/63)$: −8, −4, −2, 0, 2, 4, 8, 12
$10_{30} = K(26/67)$: −4, 0, 4, 8, 10, 12, 16
$10_{31} = K(25/57)$: −10, −6, −4, 0, 2, 4, 10
$10_{32} = K(29/69)$: −8, −4, −2, 0, 2, 4, 6, 8, 12
$10_{33} = K(18/65)$: −10, −4, −2, 0, 2, 4, 10
$10_{34} = K(13/37)$: −14, −10, −4, 0, 2, 6
$10_{35} = K(20/49)$: −12, −8, −4, 0, 4, 8
$10_{36} = K(20/51)$: −4, 0, 4, 8, 10, 16
$10_{37} = K(23/53)$: −10, −6, −4, 0, 4, 6, 10
$10_{38} = K(25/59)$: −4, 0, 4, 6, 8, 10, 12, 16
$10_{39} = K(22/61)$: −4, 0, 2, 4, 6, 8, 10, 12, 16
$10_{40} = K(29/75)$: −14, −10, −6, −4, −2, 0, 2, 6
$10_{41} = K(26/71)$: −8, −4, 0, 2, 4, 6, 8, 12
$10_{42} = K(31/81)$: −10, −6, −4, −2, 0, 2, 4, 6, 10
$10_{43} = K(27/73)$: −10, −6, −4, 0, 4, 6, 10
$10_{44} = K(30/79)$: −8, −4, 0, 2, 4, 6, 12
$10_{45} = K(34/89)$: −10, −6, −4, −2, 0, 2, 4, 6, 10
$10_{46} = K(1/5, 1/3, 1/2)$: −4, 0, 2, 6, 8, 12, 14, 16
$10_{47} = K(1/5, 2/3, 1/2)$: −6, −2, 0, 4, 6, 10, 12, 14
$10_{48} = K(4/5, 1/3, 1/2)$: −10, −6, −4, −2, 0, 2, 6, 8, 10
$10_{49} = K(4/5, 2/3, 1/2)$: −20, −16, −14, −12, −10, −8, −4, −2, 0
$10_{50} = K(3/7, 1/3, 1/2)$: −4, 0, 2, 6, 8, 10, 12, 14, 16
$10_{51} = K(3/7, 2/3, 1/2)$: −6, −2, 0, 4, 6, 8, 10, 12, 14

$10_{52} = K(4/7, 1/3, 1/2)$: $-10, -6, -4, 0, 2, 6, 8, 10$

$10_{53} = K(4/7, 2/3, 1/2)$: $-20, -16, -14, -12, -10, -8, -4, -2, 0$

$10_{54} = K(2/7, 1/3, 1/2)$: $-10, -6, -4, 0, 2, 4, 6, 8, 10$

$10_{55} = K(2/7, 2/3, 1/2)$: $-20, -16, -14, -10, -8, -6, -4, -2, 0$

$10_{56} = K(5/7, 1/3, 1/2)$: $-4, 0, 2, 4, 6, 8, 12, 14, 16$

$10_{57} = K(5/7, 2/3, 1/2)$: $-6, -2, 0, 2, 4, 6, 10, 12, 14$

$10_{58} = K(2/5, 2/5, 1/2)$: $-12, -8, -4, -2, 0, 2, 4, 6, 8$

$10_{59} = K(2/5, 3/5, 1/2)$: $-8, -4, 0, 2, 4, 6, 8, 10, 12$

$10_{60} = K(3/5, 3/5, 1/2)$: $-12, -8, -4, 0, 2, 4, 6, 8$

$10_{61} = K(1/4, 1/3, 1/3)$: $-8, -2, 0, 6, 8, 12$

$10_{62} = K(1/4, 1/3, 2/3)$: $-6, 0, 2, 8, 10, 14$

$10_{63} = K(1/4, 2/3, 2/3)$: $-20, -14, -12, -8, -6, -4, 0$

$10_{64} = K(3/4, 1/3, 1/3)$: $-8, -2, 0, 2, 6, 8, 12$

$10_{65} = K(3/4, 1/3, 2/3)$: $-6, 0, 2, 4, 8, 10, 14$

$10_{66} = K(3/4, 2/3, 2/3)$: $-20, -14, -12, -10, -8, -6, -4, 0$

$10_{67} = K(2/5, 1/3, 2/3)$: $-16, -12, -10, -8, -4, -2, 0, 4$

$10_{68} = K(3/5, 1/3, 1/3)$: $-14, -10, -8, -6, -4, 0, 2, 4, 6$

$10_{69} = K(3/5, 2/3, 2/3)$: $-6, -2, 0, 2, 4, 6, 8, 10, 12, 14$

$10_{70} = K(2/5, 1/3, 3/2)$: $-8, -4, -2, 0, 2, 4, 6, 8, 10, 12$

$10_{71} = K(2/5, 2/3, 3/2)$: $-10, -6, -4, -2, 0, 2, 4, 6, 8, 10$

$10_{72} = K(3/5, 1/3, 3/2)$: $-4, 0, 2, 4, 6, 8, 10, 12, 14, 16$

$10_{73} = K(3/5, 2/3, 3/2)$: $-14, -10, -8, -6, -4, -2, 0, 2, 4, 6$

$10_{74} = K(1/3, 1/3, 5/3)$: $-16, -10, -4, 0, 4$

$10_{75} = K(2/3, 2/3, 5/3)$: $-8, -2, 0, 4, 8, 12$

$10_{76} = K(1/3, 1/3, 5/2)$: $-4, 0, 2, 6, 8, 12, 14, 16$

$10_{77} = K(1/3, 2/3, 5/2)$: $-6, -2, 0, 4, 6, 10, 12, 14$

$10_{78} = K(2/3, 2/3, 5/2)$: $-16, -12, -10, -6, -4, 0, 2, 4$

$10_{124} = K(1/5, 1/3, -1/2)$: $0, 15$ [(5, 3) torus knot]

$10_{125} = K(1/5, 2/3, -1/2)$: $-10, 0, 4, 32/5$

$10_{126} = K(4/5, 1/3, -1/2)$: $-14, -8, -6, -4, 0, 8/3$

$10_{127} = K(4/5, 2/3, -1/2)$: $-16, -10, -8, -6, -2, 0, 1$

$10_{128} = K(3/7, 1/3, -1/2)$: $0, 8, 32/3, 16$

$10_{129} = K(3/7, 2/3, -1/2)$: $-10, -2, 0, 1, 4, 20$

$10_{130} = K(4/7, 1/3, -1/2)$: $-14, -8, -4, 0, 8/3$

$10_{131} = K(4/7, 2/3, -1/2)$: $-16, -10, -6, -2, 0, 1$

$10_{132} = K(2/7, 1/3, -1/2)$: $-14, -2, 0, 3/2, 2$

$10_{133} = K(2/7, 2/3, -1/2)$: $-16, -6, -10/3, -2, 0, 1/2$

$10_{134} = K(5/7, 1/3, -1/2)$: $0, 4, 6, 8, 10, 14, 50/3$

$10_{135} = K(5/7, 2/3, -1/2)$: $-10, -6, -4, -2, 0, 4, 6, 7$

$10_{136} = K(2/5, 2/5, -1/2)$: $-8, -4, 0, 1, 14/3, 8$

$10_{137} = K(2/5, 3/5, -1/2)$: $-12, -8, -4, 0, 1, 14/3$

$10_{138} = K(3/5, 3/5, -1/2)$: $-8, -4, -2, 0, 4, 8, 9, 12$

$10_{139} = K(1/4, 1/3, -2/3)$: $0, 12, 13, 14, 18$

$10_{140} = K(1/4, 1/3, -1/3)$: $-14, 0, 8/5$

$10_{141} = K(1/4, 2/3, -1/3)$: $-12, -2, 0, 2, 9/2$

$10_{142} = K(3/4, 1/3, -2/3)$: $0, 8, 12, 16$

$10_{143} = K(3/4, 1/3, -1/3)$: $-14, -6, -2, 0, 8/3$

$10_{144} = K(3/4, 2/3, -1/3)$: $-12, -8, -4, -2, 0, 2, 5$

$10_{145} = K(2/5, 1/3, -2/3)$: $-18, -6, -4, 0$

$10_{146} = K(2/5, 2/3, -1/3)$: $-10, -4, 0, 2, 3, 4, 20/3$

$10_{147} = K(3/5, 1/3, -1/3)$: $-8, -4, 0, 4, 6, 26/3$

The program also computes the Euler characteristic and number of boundary components of the "simplest" surface with given edgepath system (γ_i). (Orientability is not quite so easily determined, so we have not tried to include this in the program.) One finds that the only genus zero incompressible surfaces for the above knots are, as expected, the incompressible annuli for the torus knots 3_1, 5_1, 7_1, 8_{19}, 9_1, and 10_{124}. Genus one incompressible surfaces occur in a number of cases. Specifically, for the non-2-bridge Montesinos knots the following ∂-slopes are realized by genus one surfaces:

8_5:	12	8_{20}:	0 (non-Seifert surface)			9_{35}:	0 (Seifert surface)
9_{42}:	6	9_{46}:	2, 0 (Seifert surface)			10_{46}:	16
10_{61}:	12	10_{125}:	4	10_{126}:	-4	10_{139}:	12, 13
10_{140}:	0	10_{142}:	12	10_{145}:	$-4, -6$		

Here are a few slightly more complicated examples of ∂-slope calculations:

$K(2/5, 3/7, -1/3, -5/8)$: $-14, -10, -8, -6, -23/6, -4, -2, -8/23, 0, 2, 4, 6, 8, 10, 12, 14, 16, 20, \infty$

$K(2/3, 1/3, -3/5, -3/4, 3/7)$: $-18, -14, -12, -10, -8, -6, -4, -2, -4/3, 0, 1/2, 80/51, 2, 24/7, 38/11, 4, 16/3, 6, 8, 10, 12, 14, 16, 20, \infty$

$K(1/3, 1/3, -1/3, -2/5, 1/5, -3/4, 2/3)$: $-16, -12, -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 58/5, 12, 40/3, 122/9, 124/9, 14, 102/7, 190/13, 76/5, 168/11, 142/9, 16, 434/27, 146/9, 18, 20, 22, 24, 26, 28, 30, 34, \infty$

K($-15/32$, $3/11$, $7/41$): -34, -30, -28, -26, -24, -22, -20, -18, -16, -14, -12, -10, -8, -6, $-74/15$, -4, $-26/7$, -2, $-13/8$, $-16/17$, $-10/11$, 0, 2, 44/19, 40/13, 34/11, 74/21, 4, 6, 86/11, 8, 148/17, 10, 83/8, 152/13, 12, 127/10, 216/17, 14, 272/19, 16, 167/10, 18, 20, 22, 24

K($11/53$, $17/43$, $-13/21$): -36, -32, -28, -24, $-47/2$, -22, $-62/3$, -20, $-39/2$, -18, $-390/23$, $-50/3$, -16, $-44/3$, $-594/41$, $-72/5$, -14, $-68/5$, -12, -10, $-48/5$, -8, $-13/2$, -6, $-28/5$, $-26/5$, $-14/3$, $-22/5$, -4, $-5/2$, -2, $-6/5$, $-2/3$, $-2/5$, 0, 2, 3, 7/2, 4, 24/5, 6, 15/2, 8, 44/5, 10, 23/2, 12, 38/3, 64/5, 14, 16, 50/3, 18, 20, 22, 24

K($1/3$, $3/5$, $-3/4$, $-2/7$, $3/11$, $-5/13$): -14, -10, -8, -6, -4, -2, 0, 2, 4, 664/117, 6, 20/3, 38/5, 8, 62/7, 9, 19/2, 776/81, 48/5, 260/27, 10, 32/3, 98/9, 58/5, 82/7, 12, 110/9, 112/9, 90/7, 13, 27/2, 122/9, 68/5, 96/7, 14, 72/5, 44/3, 134/9, 15, 46/3, 108/7, 78/5, 110/7, 16, 146/9, 148/9, 118/7, 17, 52/3, 35/2, 88/5, 230/13, 124/7, 18, 92/5, 170/9, 96/5, 58/3, 136/7, 138/7, 20, 106/5, 64/3, 43/2, 282/13, 152/7, 22, 116/5, 24, 276/11, 26, 28, 30, 32, 34, 36, 38, 42, ∞

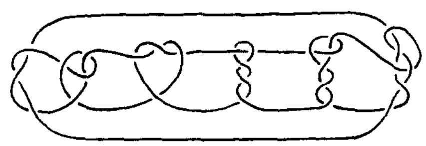


Fig. 3.1. K($1/3$, $3/5$, $-3/4$, $-2/7$, $3/11$, $-5/13$).

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