Stable Homology of Spaces of Graphs

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Goal: An analog of the Madsen-Weiss theorem for certain 3-manifolds.

Theorem: Let M be a compact connected orientable 3-manifold containing $S^1 \times S^2$ as a connected summand. Then

$$\lim_n H_i(\mathrm{BDiff}(\#_n M \operatorname{rel} D^3)) = H_i(\Omega_0^\infty S^\infty BSO(4)_+)$$

- Different Thom spectrum from the one in the Madsen-Weiss theorem. Ebert showed that the Madsen-Tillmann spectrum doesn't give the right answer for odd-dimensional manifolds.
- Components of $\mathrm{Diff}(\#_n M \operatorname{rel} D^3)$ are not contractible, so no corollary about mapping class groups.
- Homology stability hasn't yet been proved for these 3-manifolds.

Starting point: Galatius' theorem, the analog of the Madsen-Weiss theorem for $\mathrm{Aut}(F_n)$:

$$\lim_n B\mathrm{Aut}(F_n)^+ \simeq \Omega_0^\infty S^\infty, \quad \text{one component of } \Omega^\infty S^\infty$$

Equivalently, the inclusion of the symmetric group $\Sigma_n \hookrightarrow \mathrm{Aut}(F_n)$ induces isomorphisms $H_i(\Sigma_n) \cong H_i(\mathrm{Aut}(F_n))$ for $n >> i$.

Will talk about two extensions:

(I) Relative version, for "relative graphs": attach 0-cells and 1-cells to a fixed base space X.

(II) Handlebody version, where "handlebody" means a d-dimensional thickening of a graph, for fixed $d \geq 3$.

Relative Graphs

Motivation: connection with algebraic K-theory.

$$K(\mathbb{Z}) = BGL_\infty(\mathbb{Z})^+ = \lim_n B\mathrm{Aut}(\mathbb{Z}^n)^+ \qquad (\times \mathbb{Z})$$

"Nonabelian algebraic K-theory": replace $\mathbb{Z}^n$ by F_n.

Galatius' theorem computes this to be $\Omega^\infty S^\infty$.

Waldhausen's $A(*)$:

$$A(*) = \lim_{n,k} B\mathrm{HomEq}(\vee_n S^k, *)^+ \qquad (\times \mathbb{Z})$$

- when $k = 0$, $\mathrm{HomEq}(\vee_n S^0, *) = \Sigma_n$
- when $k = 1$, $\mathrm{HomEq}(\vee_n S^1, *) \simeq \mathrm{Aut}(F_n)$

$B\mathrm{HomEq}(\vee_n S^1)$ classifies fibrations with fibers $\simeq \vee_n S^1$.

More geometric: Fibrations with fibers actual finite graphs $\simeq \vee_n S^1$.

Such graphs form a category G_n with morphisms generated by graph isomorphisms and collapsing subtrees (so-called *simple* maps).

BG_n classifies these more geometric fibrations.

Culler-Vogtmann: $BG_n \simeq B\mathrm{HomEq}(\vee_n S^1)$

<u>Remark</u>: This fails for fibrations with fibers higher-dimensional finite complexes. The difference is measured by $\mathrm{Wh}^{PL}(fibers)$.

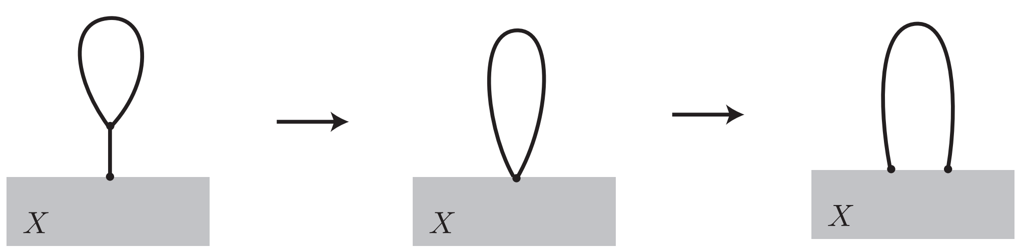
More generally,

$$A(X) = \lim_{n,k} B\mathrm{HomEq}(X \vee_n S^k \,\mathrm{rel}\, X)^+ \qquad (\times K_0(\mathbb{Z}[\pi_1 X]))$$

Again take $k = 1$. More geometric version: Graphs on X — attach a finite graph to X by identifying some of its vertices with points in X.

Category $G_n(X)$, graphs on X that are $\simeq X \vee_n S^1 \,\mathrm{rel}\, X$, morphisms as before. (Assume X path-connected for simplicity.)

Topological category: attachments to X can vary continuously.



Conjecture: $BG_n(X) \simeq B\text{HomEq}(X \vee_n S^1 \,\text{rel}\, X)$.

True when $X = point$ via the basepointed version of Culler-Vogtmann.

Theorem: $\lim_n BG_n(X)^+ \simeq \Omega_0^\infty S^\infty$.

Independent of X — surprise!

When $X = point$ this is Galatius' theorem, and the proof follows his general plan and later improvements by Galatius and Randal-Williams.

Sketch of proof:

Start with a nice geometric model for $BG_n(X)$, a space of graphs in $\mathbb{R}^\infty$ with data on attaching to X.

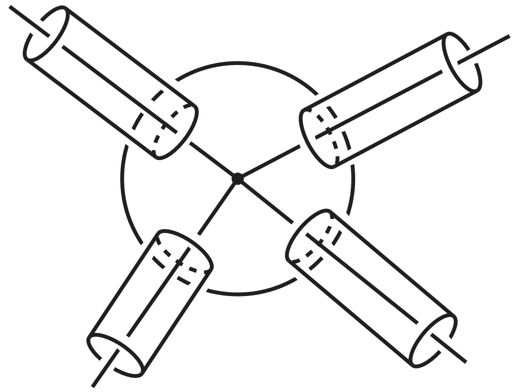
Edges of graphs are smooth curves. Allow variation of graphs by smooth isotopy, but also want to allow subtrees to shrink continuously to points.

Two different ways to do this:

- Very general: Allow arbitrary motion of the subtree inside a shrinking ball. This is what Galatius did.
- Much more restrictive: Shrinking that can easily be thickened to handlebodies.

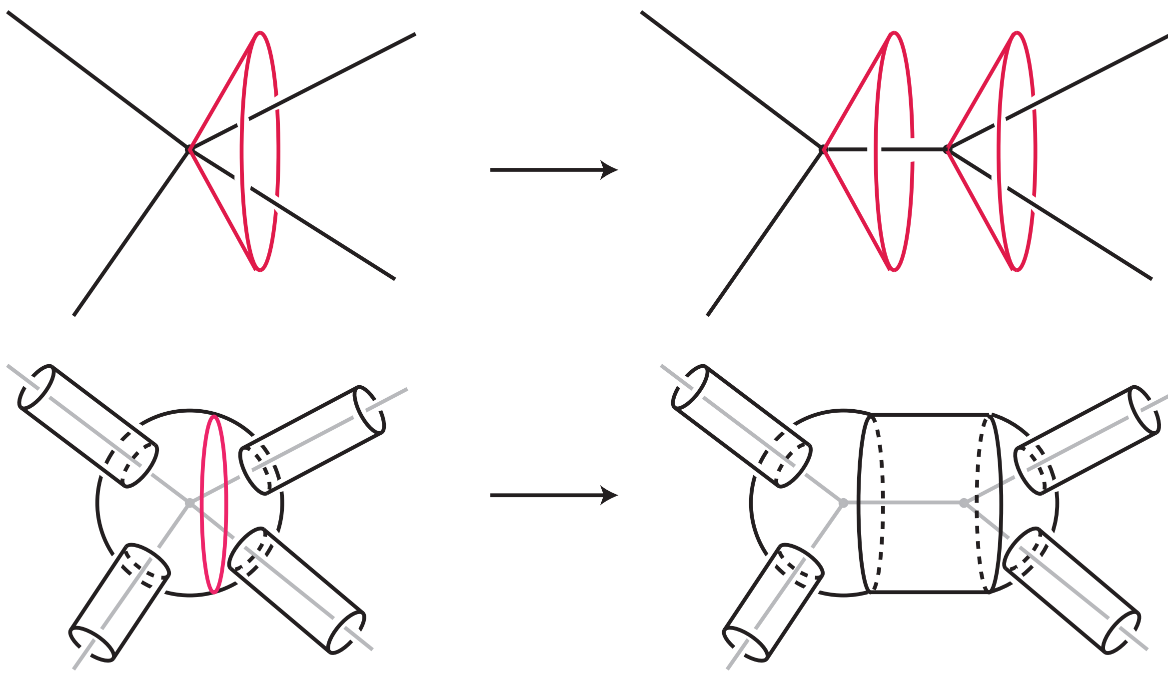
Will use the latter type, called *conical collapsing*. Inverse operation: *conical expansion*.

Round Handlebodies: Thicken vertices to 0-handles which are round balls, truncated along disjoint disks where the 1-handles attach. Thicken the edges to 1-handles with round cross-sectional disks.



Assume edges are linear near vertices.

Example of a conical expansion:

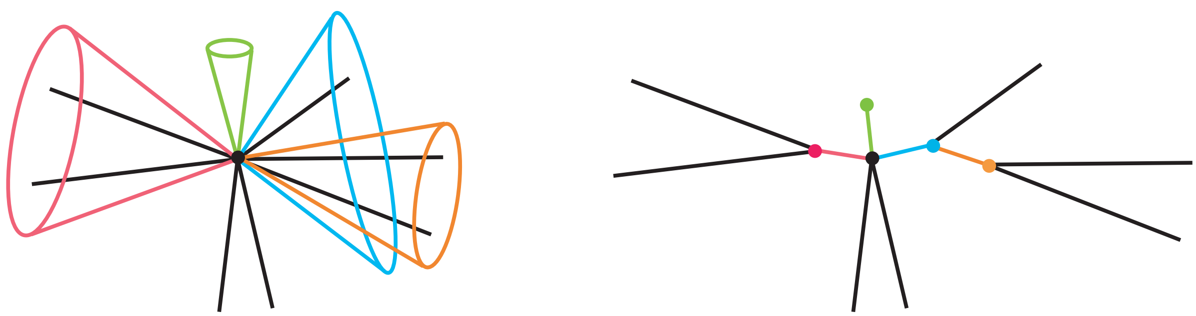


A conical expansion is determined by inserting cones such that:

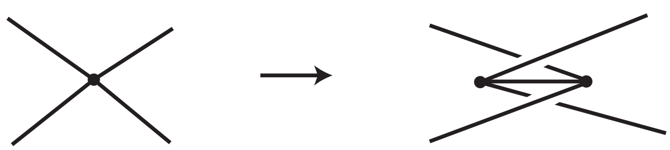
- the vertices of the cones are at the given vertex of the graph
- cones are disjoint from the graph and from each other, except at the vertex
- cones can be nested.

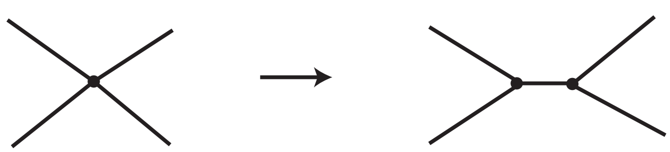
Then translate the part of the graph inside a cone along the axis of the cone, with the vertex tracing out a new edge.

More complicated example:



An example of a nonconical expansion:

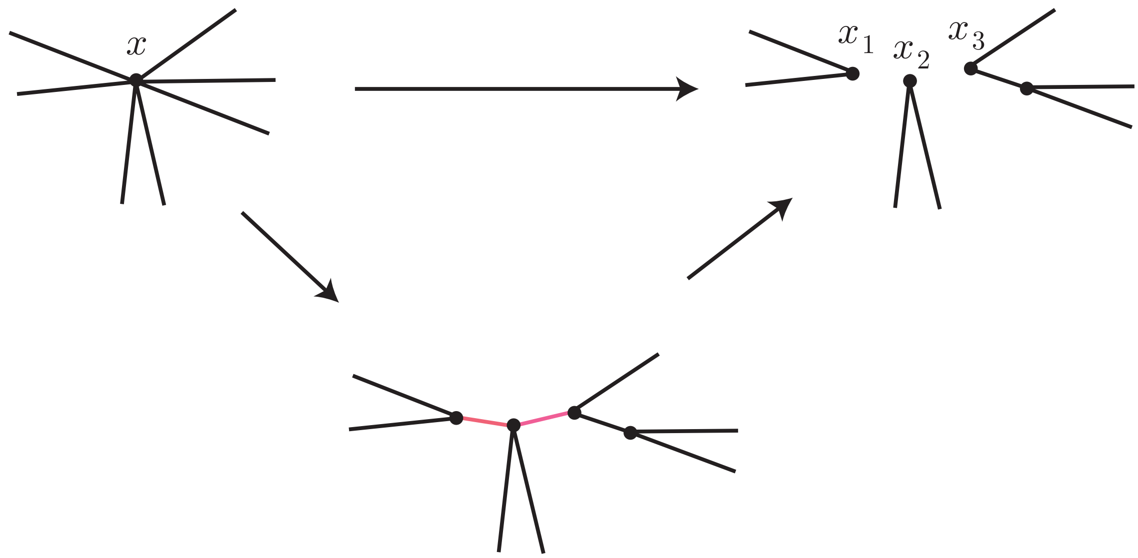




conical version

Attachment to X: label some vertices with points in X.

Allow these labeled vertices to split into several labeled vertices: Do a conical expansion, delete the edges of a subtree and label its vertices with continuously varying labels in X.



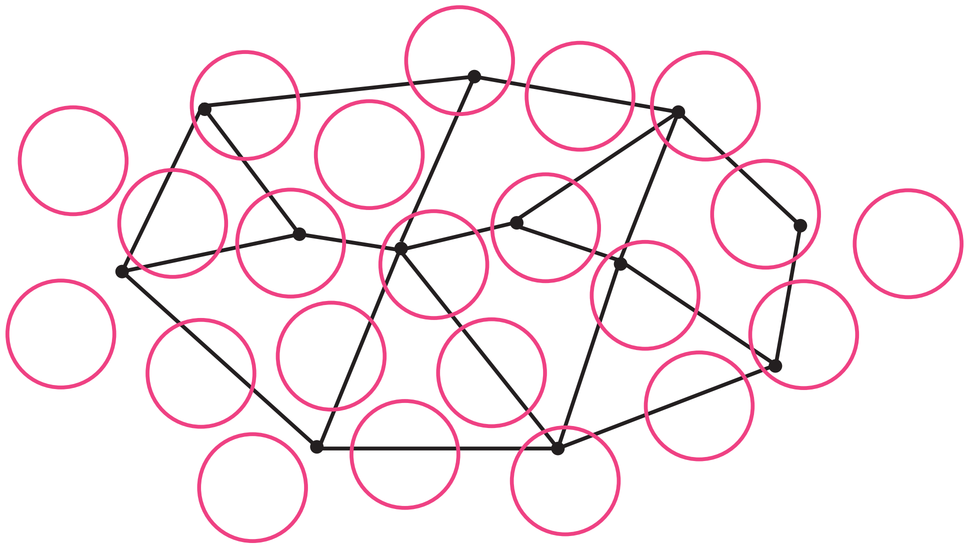
An isolated labeled vertex can be deleted since it denotes attaching nothing to X.

Notation:

- $\mathcal{G}^k(X)$ — the space of all such labeled graphs in $\mathbb{R}^k$.
- $\mathcal{G}(X) = \cup_k \mathcal{G}^k(X)$
- $\mathcal{G}_n(X) \subset \mathcal{G}(X)$, the graphs $\simeq X \vee_n S^1$ (after attaching to X).

<u>**Proposition**</u>: $\mathcal{G}_n(X) \simeq BG_n(X)$.

Relate $\mathcal{G}_n(X)$ to $\Omega^\infty S^\infty$ by a *scanning* process. The rough idea: Given a finite graph $K \subset \mathbb{R}^k$, look at it up close by moving a magnifying lens (a jeweler's *loupe*) over all of $\mathbb{R}^k$, recording what appears in the lens.



Regarding the lens as a smaller copy of $\mathbb{R}^k$, one sees a graph in $\mathbb{R}^k$ whose edges can extend to infinity. Moving the lens, the graph can slide out to infinity and disappear entirely.

Enlarge $\mathcal{G}^k(X)$ to a space $\mathcal{G}^{k,k}(X)$ of such graphs whose edges can extend to infinity. Put a "compact-open" topology on $\mathcal{G}^{k,k}(X)$ allowing parts of graphs to slide to infinity.

For each choice of a lens size we get a scanning map

$$\mathcal{G}^k(X) \to \Omega^k \mathcal{G}^{k,k}(X)$$
$$K \subset \mathbb{R}^k \mapsto \left(\mathbb{R}^k \cup \{\infty\} \to \mathcal{G}^{k,k}(X)\right)$$

This is homotopic to a composition

$$\mathcal{G}^k(X) \to \Omega \mathcal{G}^{k,1} \to \Omega^2 \mathcal{G}^{k,2} \to \cdots \to \Omega^k \mathcal{G}^{k,k}$$

where $\mathcal{G}^{k,\ell}(X) \subset \mathcal{G}^{k,k}(X)$ is the subspace of graphs contained in

$\mathbb{R}^\ell \times (-1,1)^{k-\ell}$, graphs that can go to infinity in only the first ℓ coordinates. Natural map $\mathcal{G}^{k,\ell}(X) \longrightarrow \Omega\mathcal{G}^{k,\ell+1}(X)$ by translating graphs from $-\infty$ to $+\infty$ in the $(\ell+1)$st coordinate.

Three steps:

(1) $\mathcal{G}^{k,\ell}(X) \longrightarrow \Omega\mathcal{G}^{k,\ell+1}(X)$ is a (weak) homotopy equivalence when $\ell > 0$.

(2) $\lim_n \mathcal{G}_n(X) \longrightarrow \Omega_0\mathcal{G}^{\infty,1}(X)$ is a homology equivalence.

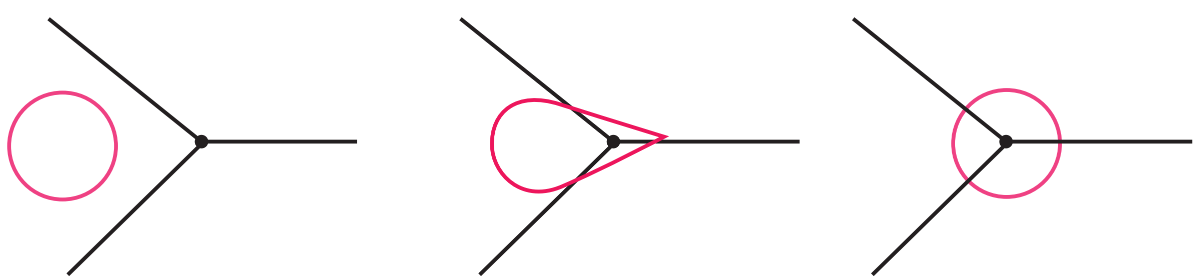
(3) $\mathcal{G}^{k,k}(X) \simeq S^k$, the graphs in $\mathbb{R}^k$ with ≤ 1 point (unlabeled).

Thus

$$\begin{aligned}\lim_n \mathcal{G}_n(X) &\sim_{H_*} \Omega_0\mathcal{G}^{\infty,1} \\ &= \lim_k \Omega_0\mathcal{G}^{k,1}(X) \\ &\simeq \lim_k \Omega_0^k\mathcal{G}^{k,k}(X) \simeq \lim_k \Omega_0^k S^k = \Omega_0^\infty S^\infty\end{aligned}$$

(1) and (2) are proved using classifying spaces of monoids instead of loopspaces, using the group completion theorem for (2).

For (3) the idea is to expand a suitably chosen small ball (lens) about the origin to all of $\mathbb{R}^k$. The ball is chosen in a shape to contain only a small piece of the graph that is a (relative) tree:



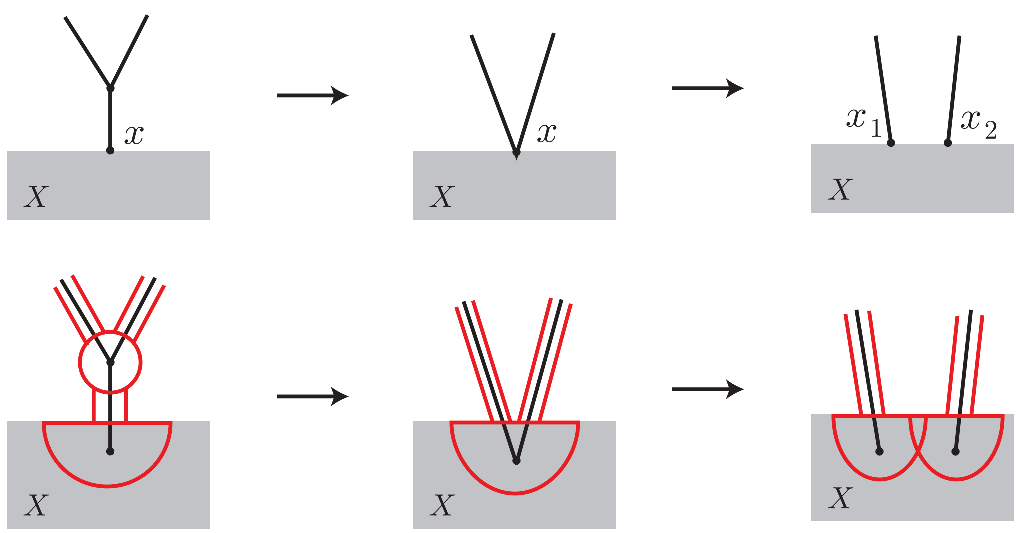
Then shrink this tree:

- Shrink to labeled vertices, if there are any, then delete these labeled vertices.
- Shrink a tree with no labels to a point.

Handlebodies

Extra data needed to go from a graph $K \subset \mathbb{R}^k$ to a d-dimensional oriented handlebody thickening of K: a field of oriented d-planes $P_x \subset \mathbb{R}^k$, $x \in K$, such that P_x contains all tangent lines to edges of K containing x.

To attach to a manifold X^d in a submanifold $M \subset \partial X$, label some vertices by points of M (with some tangential data). Need distinct labels on distinct vertices.



Get a handlebody space $\mathcal{H}^k(X, M, d)$ analogous to $\mathcal{G}^k(X)$.

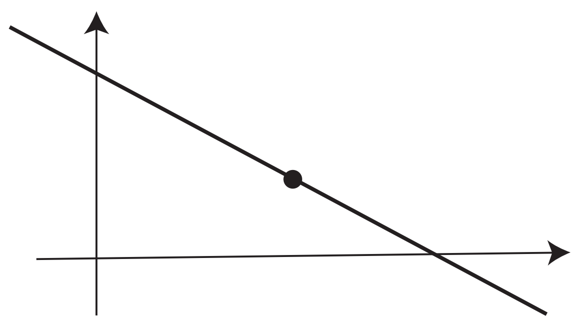
Take $(X, M) = (D^d, D^{d-1})$ for simplicity. Write $\mathcal{H}^k$ for $H^k(D^d, D^{d-1}, d)$.

Same three steps:

(1) $\mathcal{H}^{k,\ell} \longrightarrow \Omega\mathcal{H}^{k,\ell+1}$ is a homotopy equivalence when $\ell > 0$.

(2) $\lim_n \mathcal{H}_n \longrightarrow \Omega_0\mathcal{H}^{\infty,1}$ is a homology equivalence when $d \geq 3$, where $\mathcal{H}_n$ denotes the component of $\mathcal{H}^\infty$ consisting of handlebodies $\simeq \vee_n S^1$.

(3) $\mathcal{H}^{k,k} \simeq$ the graphs in $\mathbb{R}^k$ with ≤ 1 point (unlabeled), with a d-plane at that point. This is just $S^k Gr_+^{k,d}$, the Thom space of the trivial k-dimensional bundle over the Grassmannian $Gr^{k,d}$ of oriented d-planes in $\mathbb{R}^k$.



Thus we have

$$\lim_n H_i(\mathcal{H}_n) = H_i(\Omega_0^\infty S^\infty BSO(d)_+)$$

Applications to 3-manifolds:

- $d = 3$: $\mathcal{H}_n \simeq \mathrm{BDiff}(V_n \operatorname{rel} D^2)$ for V_n a 3-dimensional handlebody of genus n. Thus for $n >> i$ we have

$$H_i(\mathrm{BDiff}(V_n \operatorname{rel} D^2)) = H_i(\Omega_0^\infty S^\infty BSO(3)_+)$$

  This is the same as the homology of the mapping class group of the handlebody.

- $d = 4$: $\mathcal{H}_n \simeq \mathrm{BDiff}(\#_n(S^1 \times S^2) \operatorname{rel} D^3)$, where $\#_n(S^1 \times S^2)$ is the boundary of a 4-dimensional handlebody $\simeq \vee_n S^1$. Thus

$$\lim_n H_i(\mathrm{BDiff}(\#_n(S^1 \times S^2) \operatorname{rel} D^3)) = H_i(\Omega_0^\infty S^\infty BSO(4)_+)$$

Generalization of the $d = 4$ case: Let M be a compact connected orientable 3-manifold containing $S^1 \times S^2$ as a connected summand. Then

$$\lim_n H_i(\mathrm{BDiff}(\#_n M \operatorname{rel} D^3)) = H_i(\Omega_0^\infty S^\infty BSO(4)_+)$$