Triangulations of Surfaces

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This is a slightly revised and expanded version of a paper published in 1991 with a very similar title.

This paper uses an elementary surgery technique to give a simple topological proof of a theorem of Harer which says that the simplicial complex having as its top-dimensional simplices the isotopy classes of triangulations of a compact surface with a fixed set of vertices is contractible, except in a few special cases when it is homeomorphic to a sphere. (Triangulations here are allowed to have triangles with coinciding vertices or edges.) The proof yields mild generalizations of Harer's theorem, allowing more general vertex sets, as well as extending to a larger complex whose simplices correspond to curve systems consisting of circles as well as arcs. As a corollary we deduce the well-known classical fact that any two isotopy classes of triangulations of a compact surface with a fixed set of vertices are related by a finite sequence of elementary moves in which only one edge changes at a time.

Let S be a compact connected surface, possibly with boundary. For our fixed set of vertices we choose any nonempty finite subset $V = \{v_1, \cdots, v_n\}$ of S. By an essential arc in (S, V) we mean an arc α embedded in S so as to meet $\partial S \cup V$ only in the endpoints of α, which lie in V and are permitted to coincide, with the condition that α does not separate S into two components, one of which is a disk meeting V only in the endpoints of α. By an essential circle in (S, V) we mean an embedded circle in S disjoint from $\partial S \cup V$ which does not separate S into two components, one of which is either a disk meeting V in at most one point or an annulus disjoint from V.

A collection $\{\alpha_0, \cdots, \alpha_k\}$ of essential circles and arcs in S which are disjoint except perhaps for the endpoints of the arcs, and such that no two α_i's are ambient isotopic fixing V, we call a curve system. The ambient isotopy classes (rel V) of curve systems $\{\alpha_0, \cdots, \alpha_k\}$ form the k-simplices $[\alpha_0, \cdots, \alpha_k]$ of a simplicial complex $C = C(S, V)$, the faces of $[\alpha_0, \cdots, \alpha_k]$ being obtained by passing to subcollections of $\{\alpha_0, \cdots, \alpha_k\}$. Let $\mathcal{A} = \mathcal{A}(S, V)$ be the subcomplex consisting of simplices $[\alpha_0, \cdots, \alpha_k]$ with each α_i an arc. We will exclude without further mention the trivial cases when $\mathcal{A}(S, V)$ or $C(S, V)$ is empty.

Theorem. (a) *The complex $\mathcal{A}(S,V)$ is contractible except when S is a disk with V contained in ∂S or an annulus with V contained in one component of ∂S. In these exceptional cases $\mathcal{A}(S,V)$ is homeomorphic to a sphere.*
(b) *If V contains at least one point in the interior of S, then $C(S,V)$ is contractible.*

When V is contained in ∂S, the complex $\mathcal{A}(S,V)$ is the complex denoted $AZ(\Delta)$ in [H1], whose contractibility is the principal technical result (Theorem 1.5) in that paper, proved using the heavier machinery of Thurston's theory of projective lamination spaces. An intermediate generalization is Theorem 1.1(a) in [H2]. See also [B-E]. The exceptional situation when $\mathcal{A}(S,V)$ is a sphere is Theorem 1.1(b) of [H2].

Another special case of interest is when $\partial S = \varnothing$. If we then form the compact surface S' by puncturing S at the points of V, then $C(S,V)$ can be identified with the subset of the projective lamination space of S' consisting of laminations with all leaves compact. Namely, to a point of a simplex $[\alpha_0,\cdots,\alpha_k]$ in $C(S,V)$ with barycentric coordinates $(t_0,\cdots,t_k)$, associate the measured lamination $\alpha_0\cup\cdots\cup\alpha_k$ with t_i the measure of α_i. (The topology on $C(S,V)$ as a simplicial complex is generally finer than the subspace topology on this subset of the projective lamination space, however.)

In case V meets each component of ∂S, the maximal simplices of $\mathcal{A}(S,V)$ correspond to collections of arcs decomposing S into triangles with the fixed vertex set V, "triangulations" in the generalized sense that vertices of a triangle can coincide, as can a pair of edges, and two triangles can intersect in more than a vertex or edge. In the general case that some components of ∂S are disjoint from V, then maximal simplices of $\mathcal{A}(S,V)$ decompose S also into punctured monogons, the punctures being these components of ∂S. For simplicity we will call these more general decompositions "triangulations" of (S,V) also. Figure 1 shows two elementary moves in which triangulations are modified, nonisotopically, by changing one edge in a quadrilateral or punctured digon.

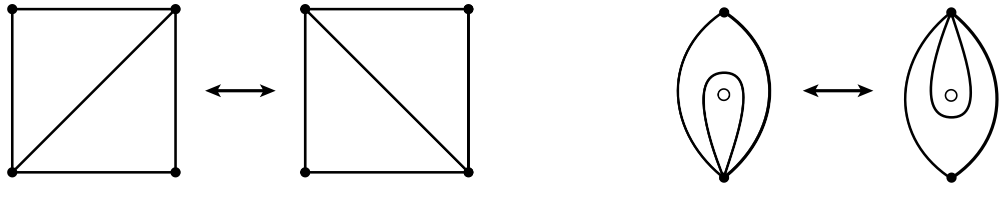


Figure 1

Corollary. *Any two isotopy classes of triangulations of (S, V) are related by a finite sequence of elementary moves.*

When V meets each component of ∂S a much stronger result can be found in [H2], where a contractible cell complex is constructed whose vertices are the triangulations of (S, V) and whose edges are the elementary moves between triangulations. The 2-cells of this complex give certain basic relations among elementary moves, etc.

Proof of the Corollary: Triangulations of (S, V) correspond to maximal simplices of $\mathcal{A}$, which all have the same dimension by Euler characteristic considerations. Codimension-one faces of these simplices are obtained by deleting one edge from a triangulation. In the two cases of elementary moves, the codimension one face belongs to exactly two maximal simplices. The only other possibility is that the deleted edge occurs twice in the same triangle, and the codimension one face belongs to only one maximal simplex. The Corollary thus translates into the statement that any two maximal simplices in $\mathcal{A}$ can be joined by a path passing only through (open) codimension-one faces, except in the cases when $\mathcal{A}$ has dimension zero and more than one point; using the Theorem one sees that this happens only when (S, V) is a quadrilateral or a punctured digon, exactly the two cases shown in Figure 1, when $\mathcal{A}$ consists of two points related by an elementary move.

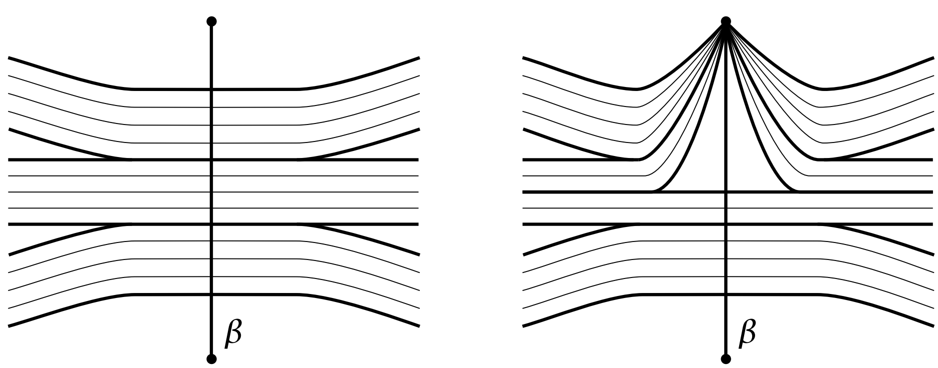
The basic fact we need is that $\mathcal{A}$ is connected whenever it has dimension greater than zero. This follows from the theorem. By this fact, any two maximal simplices of $\mathcal{A}$ can be joined by some polygonal path in $\mathcal{A}$, and it remains to push this path off all simplices of codimension greater than one. This is done by induction on the minimum dimension of the simplices encountered along the path, and will be possible provided the links of simplices of codimension greater than one are connected. But the link of a simplex $[\alpha_0, \cdots, \alpha_k]$ can be identified with $\mathcal{A}(S', V')$ where (S', V') is (S, V) cut open along the arcs α_i. The surface S' might not be connected, but this is not a problem since $\mathcal{A}(S, V)$ can be defined just as well for disconnected surfaces, where it is simply the join of the complexes $\mathcal{A}$ for the separate components. Since $[\alpha_0, \cdots, \alpha_k]$ has codimension at least two, $\mathcal{A}(S', V')$ has dimension at least one, hence is connected. $\square$

Another elementary proof of this corollary can be found in [M], p. 36.

Proof of the Theorem: For part (a), first assume that V has at most one point in each component of ∂S. Let β be an essential arc in (S,V). We will construct a deformation retraction (in fact a flow which is linear on simplices) of $\mathcal{A}$ onto the star of the vertex $[\beta]$. This star consists of the simplices $[\alpha_0,\cdots,\alpha_k]$ in $\mathcal{A}$ coming from curve systems $\alpha_0\cup\cdots\cup\alpha_k$ disjoint from β. (One α_i can be isotopic to β.) Since stars are always contractible, part (a) of the theorem will follow in this special case.

Let $[\alpha_0,\cdots,\alpha_k]$ be a simplex of $\mathcal{A}$. We may assume $\alpha_0\cup\cdots\cup\alpha_k$ has been isotoped to intersect β minimally, so there is no disk $D\subset S$ with ∂D consisting of an arc in β and an arc in an α_i, $\partial D\neq\alpha_i\cup\beta$, and with the interior of D disjoint from V. We will use the well-known fact that any two such minimal positions for $\alpha_0\cup\cdots\cup\alpha_k$ are isotopic through a family of minimum position curve systems. (This is not hard to prove: Take an isotopy $F:\coprod\alpha_i\times I\to S$ between minimum position curve systems, perturb F to be transverse to β, then gradually simplify $F^{-1}(\beta)$, cancelling critical points of the projection $F^{-1}(\beta)\to I$.)

Let $P=\sum t_i\alpha_i$ be a point in the simplex $[\alpha_0,\cdots,\alpha_k]$, expressed in terms of barycentric coordinates t_i with $\sum t_i=1$. We can think of P as a weighted sum of the curves α_i, and we can interpret this weighted sum geometrically by replacing each α_i with a family of nearby parallel curves of total thickness t_i (with the endpoints of arcs within a family pinched together at V). For convenience, pinch all the families of parallel curves in P into a single family where they cross β, with total thickness θ, as shown in Figure 2(a).



(a) (b)

Figure 2

Now for $t\in[0,1]$ let P_t be obtained from P by first cutting part of the way through the thickness θ band crossing β, starting from a chosen end of β and cutting in to a thickness $t\theta$, then redirecting the two ends of the resulting cut band to the given end of β, as indicated in Figure 2(b). The curves of P_t will all be essential, except for ∂-parallel arcs produced as in Figure 3. These we simply discard from P_t; note that not all of P_t is discarded since we assume V has at most one point in each component of ∂S.

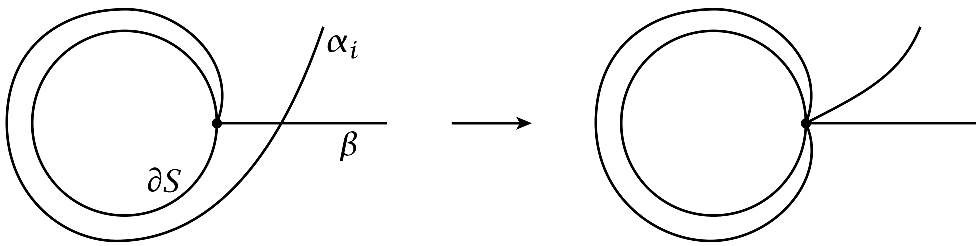


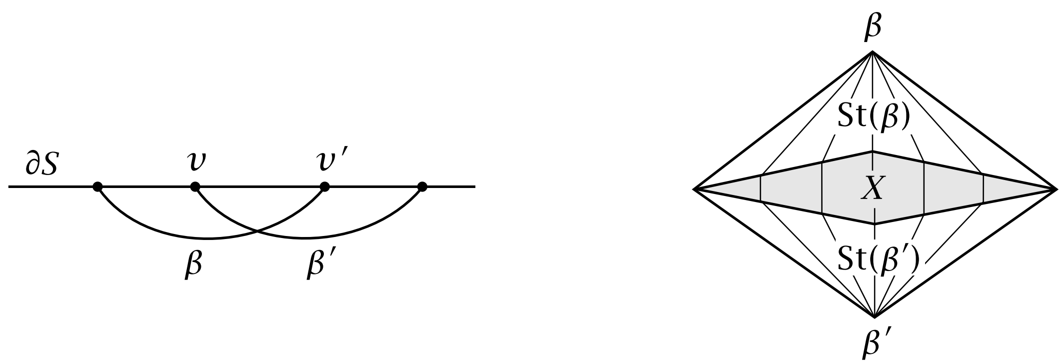
Figure 3

In terms of the curve system $c_1 = \alpha_0 \cup \cdots \cup \alpha_k$ this procedure can be described as follows. Let $x_1, \cdots, x_m$ be the points of intersection of β with c_1, ordered as they are ordered along β starting at the chosen end of β. Sliding the intersection point x_1 to the end of β converts c_1 to a new curve system c_2 (after discarding arcs that are parallel or ∂-parallel) meeting β in $x_2, \cdots, x_m$. The union $c_1 \cup c_2$ is a curve system defining a simplex σ_1 containing $[\alpha_0, \cdots, \alpha_k]$ as a face. Initial segments of the paths P_t move linearly along fibers of a linear projection of σ_1 onto the face corresponding to c_2, stopping when they reach this face. Now repeat the process, sliding x_2 to the end of β to convert c_2 to c_3, etc., until all the x_i's have been eliminated. We see from this that the paths P_t starting at points of $[\alpha_0, \cdots, \alpha_k]$ flow linearly across a finite sequence of simplices $\sigma_1, \cdots, \sigma_m$. So the flow is continuous on $[\alpha_0, \cdots, \alpha_k]$. Furthermore, it is clear that this flow restricted to a face of $[\alpha_0, \cdots, \alpha_k]$ is the flow associated to that face. Hence the flow is continuous on all of $\mathcal{A}$ (it is the identity on the star of $[\beta]$), defining a deformation retraction of $\mathcal{A}$ onto the star of $[\beta]$. This finishes the proof of (a) in the special case that V has at most one vertex in each component of ∂V.

For the general case of (a) we will use the following result, whose proof does not involve the surgery technique:

Proposition. *If V' is obtained from V by adding a new vertex in a component of ∂S that already contains at least one vertex of V, then $\mathcal{A}(S,V')$ is homeomorphic to the suspension of $\mathcal{A}(S,V)$.*

Proof: Let v' be the new vertex, adjacent to a vertex v of V in a component of ∂V. Let β be the ∂-parallel arc in S joining v' to the next vertex of V' on the other side of v (which might be v' again), and let β' be defined similarly by interchanging the roles of v and v'; see Figure 4(a). Let X be the subcomplex of $\mathcal{A}(S,V')$ consisting of arc systems not containing β or β'. We can obtain $\mathcal{A}(S,V')$ from X by attaching the stars $\mathrm{St}(\beta)$ and $\mathrm{St}(\beta')$ of the vertices β and β' along the links of these vertices, these links being subcomplexes of X.



(a) (b)

Figure 4

The suspension $\Sigma\mathcal{A}(S,V)$ can be parametrized as the quotient of $\mathcal{A}(S,V)\times[-1,1]$ with each of $\mathcal{A}(S,V)\times\{-1\}$ and $\mathcal{A}(S,V)\times\{1\}$ collapsed to points. There is an embedding $f:X\to\Sigma\mathcal{A}(S,V)$ sending a point $x\in X$, viewed as a weighted arc system, to the point $f(x)=(\overline{x},(\theta(x)-\theta'(x))/2)$ where $\overline{x}$ is obtained from x by collapsing the segment of ∂S joining v and v' to a point, and $\theta(x)$ and $\theta'(x)$ are the total weights of the arcs of x at v and v', respectively. These weights lie in the interval $[0,1]$ so $(\theta(x)-\theta'(x))/2$ is in $[-1/2,1/2]$. For fixed $\overline{x}$ the values of $(\theta(x)-\theta'(x))/2$ range over a symmetric interval about 0 whose length equals $\theta(x)+\theta'(x)$. This varies continuously with x and can be 0 if V' contains other vertices besides v and v'. The map $X\to\mathcal{A}(S,V)$, $x\mapsto\overline{x}$, is surjective, so the complement of $f(X)$ in $\Sigma\mathcal{A}(S,V)$ consists of the open cones on the links of β and β', so the closures of these two open cones are the stars of β and β'. Thus we have decomposed both $\mathcal{A}(S,V')$ and $\Sigma\mathcal{A}(S,V)$ into the same three pieces, intersecting in the same way, so they are homeomorphic, in fact piecewise linearly homeomorphic. □

The general case of (a) now follows by an inductive procedure, the inductive step being given by the Proposition. In most cases the induction starts with one of the special cases considered earlier when V has at most one point in each component of ∂S, so

$\mathcal{A}(S,V)$ is contractible. The exceptional cases where $\mathcal{A}(S,V)$ is a sphere arise from starting the induction with the two cases that $\mathcal{A}(S,V)$ is the sphere S^0, shown in Figure 1, the cases that S is a disk and V consists of four points in its boundary, or S is an annulus and V consists of two points in one of its boundary components.

For (b) the same technique used to prove the special case of (a) can be applied provided that we choose for β an essential arc with both endpoints at a point of V in the interior of S. □

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