Chapter 3
Eilenberg-Moore Spectral Sequences

There are two Eilenberg-Moore spectral sequences that we shall consider, one for homology and the other for cohomology. In contrast with the situation for the Serre spectral sequence, for the Eilenberg-Moore spectral sequences the homology and cohomology versions arise in two different topological settings, although the two settings are in a sense dual. Both versions share the same underlying algebra, however, involving Tor functors.

The first occurrence of a Tor functor in algebraic topology is in the universal coefficient theorem. Here one has a group $\mathrm{Tor}(A,B)$ associated to abelian groups A and B which measures the common torsion of A and B. The formal definition of $\mathrm{Tor}(A,B)$ in terms of tensor products and free resolutions extends naturally from the context of abelian groups to that of modules over an arbitrary ring, and the result is a sequence of functors $\mathrm{Tor}_n^R(A,B)$ for modules A and B over a ring R. In case R is a principal ideal domain such as $\mathbb{Z}$ the groups $\mathrm{Tor}_n^R(A,B)$ happen to be zero for $n>1$, and $\mathrm{Tor}_1^R(A,B)$ is the $\mathrm{Tor}(A,B)$ in the universal coefficient theorem. This same Tor functor appears also in the general form of the Künneth formula for the homology groups of a product $X\times Y$. The Eilenberg-Moore spectral sequences can be regarded as generalizations of the Künneth formula to fancier kinds of products where extra structure is involved. The rings R that arise need not be principal ideal domains, so the Tor_n groups can be nonzero for large n.

For the case of homology the E^2 page of the spectral sequence consists of groups $E^2_{p,q} = \mathrm{Tor}_{p,q}^{H_*(G)}(H_*(X),H_*(Y))$, where the index p has the same meaning as the subscript in Tor_n and the second index q arises from the fact that the various homology groups involved are graded, so $\mathrm{Tor}_p = \bigoplus_q \mathrm{Tor}_{p,q}$. In order for the notation $\mathrm{Tor}_{p,q}^{H_*(G)}(H_*(X),H_*(Y))$ to make sense $H_*(G)$ must be a ring, and the simplest situation when this is the case is when G is a topological group and homology is taken with coefficients in a commutative ring, so the product in G induces, via the cross product in homology, a product in $H_*(G)$, the Pontryagin product. We also need $H_*(X)$ and $H_*(Y)$ to be modules over $H_*(G)$, and the most natural way for this structure to arise is if G acts on X and Y, the actions being given by maps $G\times X\to X$ and $G\times Y\to Y$

inducing the module structures on homology. These are the ingredients needed in order for the terms in the E^2 page to be defined, and then with a few additional hypotheses of a more technical nature (namely that the coefficient ring is a field and the action of G on Y is free, defining a principal bundle $Y \to Y/G$) the spectral sequence exists and converges to $H_*(X \times_G Y)$, where $X \times_G Y$ is $X \times Y$ with the diagonal action of G factored out. One can think of $X \times_G Y$ as the topological analog of the tensor product of modules. Thus the spectral sequence measures whether the homology of a 'tensor product of spaces' is the tensor product of the homology of the spaces.

For cohomology with coefficients in a commutative ring we always have a ring structure coming from cup product, so we can replace the topological group G by any space B. In order for $H^*(X)$ and $H^*(Y)$ to be modules over $H^*(B)$ it suffices to specify maps $X \to B$ and $Y \to B$. Converting one of these maps into a fibration, we can use the other map to construct a pullback square with fourth space Z, and then, again with some technical hypotheses, there is an Eilenberg-Moore spectral sequence having $E_2^{p,q} = \mathrm{Tor}_{p,q}^{H^*(B)}(H^*(X), H^*(Y))$ and converging to $H^*(Z)$.

When X is a point the two spectral sequences specialize in the following ways:

- For a principal bundle $G \to Y \to Y/G$ one has a spectral sequence converging to $H_*(Y/G;k)$ with $E^2_{p,q} = \mathrm{Tor}_{p,q}^{H_*(G)}(k, H_*(Y;k))$, for k a field.
- For a fibration $F \to Y \to B$ with B simply-connected one has a spectral sequence converging to $H^*(F;k)$ with $E_2^{p,q} = \mathrm{Tor}_{p,q}^{H^*(B)}(k, H^*(Y;k))$, with k again a field.

In some situations these spectral sequences can be more effective than the Serre spectral sequence. If one has a fibration and one is trying to compute homology or cohomology of the base or fiber from the homology or cohomology of the other two spaces, then in the Serre spectral sequence one has to argue backward from E^∞ to E^2, whereas here one is going forward, which is usually easier. In some important cases where the differentials in the Serre spectral sequence are fairly complicated the differentials in the Eilenberg-Moore spectral sequence are all trivial, and one has only the problem of computing the Tor groups in E^2. This is generally easier than computing differentials.

The original derivations of these spectral sequences by Eilenberg and Moore were fairly algebraic, but here we shall follow (not too closely) a more topological route first described in [Smith 1970] and [Hodgkin 1975].

3.1 The Homology Spectral Sequence

If G is a topological group, its homology $H_*(G;k)$ with coefficients in a commutative ring k has a ring structure with multiplication the Pontryagin product, which is the composition of cross product with the map induced by the group multiplication:

$$H_*(G;k)\times H_*(G;k) \xrightarrow{\times} H_*(G\times G;k) \longrightarrow H_*(G;k)$$

Similarly, if G acts on a space X, the map $G\times X \to X$ defining the action gives the homology $H_*(X;k)$ the structure of a module over $H_*(G;k)$, via the composition

$$H_*(G;k)\times H_*(X;k) \xrightarrow{\times} H_*(G\times X;k) \longrightarrow H_*(X;k)$$

If we take k to be a field, the Künneth formula gives an isomorphism $H_*(X\times Y;k) \approx H_*(X;k)\otimes_k H_*(Y;k)$, and we may ask whether there is an analog of this formula that takes the module structure over $H_*(G;k)$ into account, so that $\otimes_k$ is replaced by $\otimes_{H_*(G;k)}$, when actions of G on X and Y are given. We might expect that $X\times Y$ would have to be replaced by some quotient space of itself taking the actions into account since $H_*(X;k)\otimes_{H_*(G;k)} H_*(Y;k)$ is a quotient of $H_*(X;k)\otimes_k H_*(Y;k)$.

Since the ring $H_*(G;k)$ need not be commutative, even in the graded sense, we need to pay attention to the distinction between left and right modules. This matters in the definition of $A\otimes_R B$, where in the case that R is noncommutative, A must be a right R-module and B a left R-module, and we obtain $A\otimes_R B$ from $A\otimes_{\mathbb{Z}} B$ by imposing the additional relations $ar\otimes b = a\otimes rb$. Topologically, we should then consider a right action $X\times G \to X$ and a left action $G\times Y \to Y$. If we start with a left action on X we can easily convert it into a right action via the formula $xg = g^{-1}x$, and conversely a right action can be made into a left action, so there is no intrinsic distinction between left and right actions.

The topological analog of $A\otimes_R B$ is the quotient space $X\times_G Y$ of $X\times Y$ under the identifications $(xg,y) \sim (x,gy)$. This definition leads naturally to the following question:

- Is $H_*(X\times_G Y;k)$ isomorphic to $H_*(X;k)\otimes_{H_*(G;k)} H_*(Y;k)$? Or if they are not isomorphic, how are they related?

Consider for example the important special case that Y is a point, so $X\times_G Y$ is just the orbit space X/G. Then we are asking whether $H_*(X/G;k)$ is $H_*(X;k)\otimes_{H_*(G;k)} k$, which is $H_*(X;k)$ with the action of $H_*(G;k)$ factored out. It is easy to find instances where this is not the case, however. A simple one is $\mathbb{C}\mathrm{P}^n$, regarded as the orbit space of an action of $G = S^1$ on $X = S^{2n+1}$. Here $H_*(\mathbb{C}\mathrm{P}^n;k)$ is quite a bit larger than $H_*(S^{2n+1};k)\otimes_{H_*(S^1;k)} k$, which is just $H_*(S^{2n+1};k)$ since the action of $H_*(S^1;k)$ cannot produce any nontrivial identifications, for dimension reasons.

The isomorphism $H_*(X\times_G Y;k) \approx H_*(X;k)\otimes_{H_*(G;k)} H_*(Y;k)$ does sometimes hold. A fairly trivial case is when X is a product $Z\times G$ with G acting just on the second factor, $(z,g)h = (z,gh)$. Then $X\times_G Y$ is homeomorphic to $Z\times Y$ via the map

$(z,g,y) \mapsto (z,gy)$ with inverse $(z,y) \mapsto (z,1,y)$. In this case the isomorphism $H_*(X \times_G Y;k) \approx H_*(X;k) \otimes_{H_*(G;k)} H_*(Y;k)$ becomes

$$\big(H_*(Z;k) \otimes_k H_*(G;k)\big) \otimes_{H_*(G;k)} H_*(Y;k) \approx H_*(Z;k) \otimes_k H_*(Y;k)$$

which is a special case of the algebraic isomorphism $(A \otimes_k R) \otimes_R B \approx A \otimes_k B$. This special case will play a role in the construction of the spectral sequence. One can in fact view the spectral sequence as an algebraic machine for going from this rather uninteresting special case to the general case.

Constructing the Spectral Sequence

To save words, let us call a space with an action by G a ***G*-space**. A ***G*-map** between G-spaces is a map f that preserves the action, so $f(xg) = f(x)g$ for right actions and similarly for left actions.

It will be convenient to have basepoints for all the spaces we consider, and to have all maps preserve basepoints. To be consistent, this would require that elements of G act by basepoint-preserving maps, in other words basepoints are fixed by the group actions. This excludes many interesting actions, but there is an easy way around this problem. Given a space X with a G-action, let X_+ be the disjoint union of X with a new basepoint x_0, and extend the action to fix x_0, so $x_0 g = x_0$ for all $g \in G$. This trick makes it possible to assume all actions fix basepoints. It also allows us to use reduced homology since $\widetilde{H}_*(X_+;k) \approx H_*(X;k)$. So in what follows we assume all maps and all actions preserve the basepoint.

In basepointed situations it is often best to replace the product $X \times Y$ by the smash product $X \wedge Y$, the quotient of $X \times Y$ with $\{x_0\} \times Y \cup X \times \{y_0\}$ collapsed to a point, the basepoint in $X \wedge Y$. Notice that $X_+ \wedge Y_+ = (X \times Y)_+$. For actions fixing the basepoint the quotient $X \wedge_G Y$ is defined, and $X_+ \wedge_G Y_+ = (X \times_G Y)_+$. So we will be working with $X \wedge_G Y$ rather than $X \times_G G$.

Recall the definition of $\mathrm{Tor}_n^R(A,B)$. One chooses a resolution

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

of A by free right R-modules and then tensors this over R with B, dropping the final term $A \otimes_R B$, to get a chain complex

$$\cdots \longrightarrow F_1 \otimes_R B \longrightarrow F_0 \otimes_R B \longrightarrow 0$$

whose n^{th} homology group is $\mathrm{Tor}_n^R(A,B)$. If R is a graded ring and A and B are graded modules over R, as will be the case in our application, then a free resolution of A can be chosen in the category of graded modules, with maps preserving grading. Tensoring with B stays within the graded category, so there is an induced grading of $\mathrm{Tor}_n^R(A,B)$ as a direct sum of its q^{th} grading subgroups $\mathrm{Tor}_{n,q}^R(A,B)$.

The ideal topological realization of this algebraic construction would require a sequence of G-spaces and G-maps $\cdots \to K_1 \to K_0 \to X$ such that applying the functor

$H_*(-;k)$ gave a free resolution of $H_*(X;k)$ as a module over $H_*(G;k)$. To start the inductive construction of such a sequence we would want a G-space K_0 with a G-map $f_0\colon K_0 \to X$ such that f_0 induces a surjection on homology and $H_*(K_0;k)$ is a free $H_*(G;k)$-module. Algebraically, the simplest way to construct a free R-module F_0 and a surjective R-module homomorphism $F_0 \to A$ is to take F_0 to be a direct sum of copies of R, one for each element of A. One can regard this direct sum as a family of copies of R parametrized by A. The topological analog of this is to choose K_0 to be the product $X \times G$, a family of copies of G parametrized by X. For the map $f_0\colon K_0 \to X$ we choose the action map $(x,g) \mapsto xg$. This will be a G-map if we take the action of G to be trivial on the X factor, so $(x,g)h = (x,gh)$. This action does not fix the basepoint, but we can correct this problem by taking K_0 to be the quotient of $X \times G$ with $\{x_0\} \times G$ collapsed to a point. For this new K_0 there is an induced quotient map $f_0\colon K_0 \to X$ since the action of G on X fixes x_0.

If the coefficient ring k is a field the Künneth formula gives an isomorphism $\widetilde{H}_*(K_0;k) \approx \widetilde{H}_*(X;k) \otimes_k H_*(G;k)$. From this we see that $H_*(K_0;k)$ is free as a module over $H_*(G;k)$ since G acts trivially on the factor $\widetilde{H}_*(X;k)$. The map f_0 induces a surjection on homology since it is a retraction with respect to the inclusion $X \hookrightarrow K_0$, $x \mapsto (x,1)$. Another way of seeing that f_{0*} is surjective is to identify it with the map $\widetilde{H}_*(X;k) \otimes_k H_*(G;k) \to \widetilde{H}_*(X;k)$ induced by the action, and this map is surjective since the identity element of G gives an identity element of $H_0(G;k)$.

Thus if we let X_1 be the mapping cone of f_0 we have short exact sequences

$$0 \longrightarrow \widetilde{H}_*(X_1;k) \xrightarrow{\partial} \widetilde{H}_*(K_0;k) \xrightarrow{f_{0*}} \widetilde{H}_*(X;k) \longrightarrow 0$$

For basepoint reasons we should take the reduced mapping cone, the quotient of the ordinary mapping cone with the cone on the basepoint collapsed to a point. The actions of G on X and K_0 extend naturally to an action on the mapping cone since it is the mapping cone of a G-map. For future reference let us note the following:

$(*)$ If $\widetilde{H}_i(X;k) = 0$ for $i < n$ then the same is true for K_0, and $\widetilde{H}_i(X_1;k) = 0$ for $i < n+1$.

The first statement holds by the isomorphism $\widetilde{H}_*(K_0;k) \approx \widetilde{H}_*(X;k) \otimes_k H_*(G;k)$, and the second follows from the short exact sequence displayed above.

Now we iterate the construction to produce a diagram

$$\begin{tikzcd} & K_0 \arrow[d] & K_1 \arrow[d] & K_2 \arrow[d] & \\ X = & X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & \cdots \end{tikzcd}$$

with associated short exact sequences

$$0 \longrightarrow \widetilde{H}_*(X_{p+1};k) \longrightarrow \widetilde{H}_*(K_p;k) \longrightarrow \widetilde{H}_*(X_p;k) \longrightarrow 0$$

These can be spliced together as in the following diagram to produce a resolution of $\widetilde{H}_*(X;k)$ by free $H_*(G;k)$-modules:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \widetilde{H}_*(K_2) \arrow[rr] \arrow[dr] & & \widetilde{H}_*(K_1) \arrow[rr] \arrow[dr] & & \widetilde{H}_*(K_0) \arrow[r] & \widetilde{H}_*(X) \arrow[r] & 0 \\
\cdots & & \widetilde{H}_*(X_2) \arrow[ur] \arrow[dr] & & \widetilde{H}_*(X_1) \arrow[ur] \arrow[dr] & & & \\
\cdots & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

The next step is to apply $\wedge_G Y$. Since the map $K_p \to X_p$ is a G-map with mapping cone X_{p+1}, there is an induced map $K_p \wedge_G Y \to X_p \wedge_G Y$ and its mapping cone is $X_{p+1} \wedge_G Y$. The associated long exact sequences of reduced homology may no longer split since the inclusions $X_p \hookrightarrow K_p$, $x \mapsto (x,1)$, are not G-maps, but we can assemble all these long exact sequences into a staircase diagram:

$$\begin{tikzcd}[column sep=small]
 & & \arrow[d] & & \arrow[d] & \\
\arrow[r] & \widetilde{H}_n(K_p \wedge_G Y) \arrow[r] & \widetilde{H}_n(X_p \wedge_G Y) \arrow[r] \arrow[d] & \widetilde{H}_{n-1}(K_{p-1} \wedge_G Y) \arrow[r] & \widetilde{H}_{n-1}(X_{p-1} \wedge_G Y) \arrow[r] \arrow[d] & {} \\
\arrow[r] & \widetilde{H}_n(K_{p+1} \wedge_G Y) \arrow[r] & \widetilde{H}_n(X_{p+1} \wedge_G Y) \arrow[r] \arrow[d] & \widetilde{H}_{n-1}(K_p \wedge_G Y) \arrow[r] & \widetilde{H}_{n-1}(X_p \wedge_G Y) \arrow[r] \arrow[d] & {} \\
 & & {} & & {} &
\end{tikzcd}$$

Thus we have a spectral sequence.

Let us set $E^1_{p,q} = \widetilde{H}_{p+q}(K_p \wedge_G Y; k)$. We will show in a moment that $E^1_{p,q} = 0$ for $q < 0$, so the spectral sequence lives in the first quadrant. From the staircase diagram we see that the differentials have the form $d_r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ just as in the Serre spectral sequence.

The E^1 page consists of the chain complexes

$$\cdots \widetilde{H}_{q+2}(K_2 \wedge_G Y; k) \longrightarrow \widetilde{H}_{q+1}(K_1 \wedge_G Y; k) \longrightarrow \widetilde{H}_q(K_0 \wedge_G Y; k) \longrightarrow 0$$

Recall that $K_p = (X_p \times G)/(\{x_p\} \times G)$ with the action $(x,g)h = (x,gh)$. By an earlier observation we have $H_*((X_p \times G) \times_G Y; k) \approx H_*(X_p \times G; k) \otimes_{H_*(G;k)} H_*(Y;k)$. The space $(X_p \times G) \times_G Y$ retracts via G-maps onto its G-subspaces $(\{x_p\} \times G) \times_G Y$ and $(X \times G) \times_G \{y_0\}$, and collapsing these subspaces produces $K_p \wedge_G Y$. It follows that $\widetilde{H}_*(K_p \wedge_G Y; k) \approx \widetilde{H}_*(K_p; k) \otimes_{H_*(G;k)} \widetilde{H}_*(Y;k)$. In particular, the assertion $(*)$ implies inductively that $\widetilde{H}_i(K_p; k) = 0$ for $i < p$, so the same holds for $K_p \wedge_G Y$, proving that $E^1_{p,q} = 0$ for $q < 0$.

Under the isomorphism $\widetilde{H}_*(K_p \wedge_G Y; k) \approx \widetilde{H}_*(K_p; k) \otimes_{H_*(G;k)} \widetilde{H}_*(Y;k)$ the differential d_1, which is the composition of two horizontal maps in the staircase diagram, corresponds to $f_{p*} \otimes \mathbb{1}$ where f_p is the composition $K_p \to X_p \to \Sigma K_{p-1}$, the second map being part of the mapping cone sequence $K_{p-1} \to X_{p-1} \to X_p \to \Sigma K_{p-1}$. By the definition of $\mathrm{Tor}_{p,q}$ this says that $E^2_{p,q} = \mathrm{Tor}^{H_*(G;k)}_{p,q}(\widetilde{H}_*(X;k), \widetilde{H}_*(Y;k))$.

In order to prove that the spectral sequence converges to $\widetilde{H}_*(X \wedge_G Y; k)$ we need to impose some restrictions on the action of G on Y. We shall assume that Y has the form Y_+ for a G-space Y on which G acts freely in such a way that the projection $\pi : Y \to Y/G$ is a principal G-bundle. This means that each point of Y/G has a neighborhood U for which there is a G-homeomorphism $\pi^{-1}(U) \to G \times U$ where the latter space is a G-space via the action $g(h,y)=(gh,y)$. This hypothesis guarantees that the projection $X\times_G Y\to Y/G$ induced by $X\times Y\to Y$, $(x,y)\mapsto y$, is a fiber bundle with fiber X, since $X\times_G(G\times U)$ is just $X\times U$, by an argument given earlier in a slightly different context.

Theorem 3.1. *Suppose X is a right G-space and Y is a left G-space such that the projection $Y\to Y/G$ is a principal bundle. Then there is a first-quadrant spectral sequence with $E^2_{p,q}=\mathrm{Tor}^{H_*(G;k)}_{p,q}(H_*(X;k),H_*(Y;k))$ converging to $H_*(X\times_G Y;k)$.*

The convergence statement means that the groups $E^\infty_{p,q}$ for $p+q=n$ form the successive quotients in a filtration of $H_n(X\times_G Y;k)$.

Proof: We take the preceding spectral sequence for the G-spaces X_+ and Y_+. The E^2 terms have already been identified, so it remains only to check convergence. At the top of each A column of the staircase diagram, the columns with the arrows, we have the groups $H_*(X\times_G Y;k)$, so by Proposition 1.2 it will suffice to show that all the terms sufficiently far down each A column are zero, that is, $\widetilde{H}_n(X_p\wedge_G Y_+;k)=0$ for sufficiently large p.

Since $Y\to Y/G$ is a principal G-bundle, the projection $X_p\times_G Y\to Y/G$ is a bundle with fiber X_p. Since the action of G on X_p fixes the basepoint x_p, this bundle has a section $\{x_p\}\times Y/G$ and $X_p\wedge_G Y_+$ is $X_p\times_G Y$ with this section collapsed to a point. So it will be enough to show that $H_i(X_p\times_G Y,\{x_p\}\times Y/G;k)=0$ for $i<p$. The quickest way to see this is to use the relative Serre spectral sequence for this pair of fiber bundles, with local coefficients if Y/G is not simply-connected, together with the earlier fact that $H_i(X_p,\{x_p\};k)=0$ for $i<p$.

Alternatively, for an argument not using the Serre spectral sequence we can start with the following two more elementary facts, which together imply inductively that K_p and X_p are $(p-1)$-connected:

- The mapping cone of a retraction of n-connected spaces is $(n+1)$-connected.
- If Z is n-connected then so is $(Z\times W)/(\{z_0\}\times W)$ for any space W, assuming that the point $z_0\in Z$ is a deformation retract of some neighborhood.

Since (X_p,x_p) is $(p-1)$-connected, so is the pair $(X_p\times_G Y,\{x_p\}\times Y/G)$, from the homotopy lifting property. Thus the relative homology groups for this pair vanish below dimension p, and this says that $\widetilde{H}_n(X_p\wedge_G Y_+;k)=0$ for $n<p$. $\square$

3.2 The Cohomology Spectral Sequence

The situation we are interested in here is that the cohomology $H^*(X;k)$ of a space X is a module over the cohomology ring $H^*(B;k)$ of another space B by means of a map $f:X\to B$, which allows us to define $rx = f^*(r)\smile x$ for $r\in H^*(B;k)$ and $x\in H^*(X;k)$. We shall take B to be fixed and consider different choices for X, each choice having a specified map to B. Of particular interest is a pullback diagram involving a pair of spaces mapping to B, a commutative square as shown at the right, where Z is the subspace of $X\times Y$ consisting of pairs (x,y) mapping to the same point in B. Eventually we will be assuming one or both of the maps $X\to B$ and $Y\to B$ is a fibration, so Z is the pullback fibration, but for the moment we do not need any assumptions about fibrations.

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array}$$

The pullback can be regarded as a product of the two maps to B in a categorical sense, since it has the property that if we have a commutative square with the pullback Z replaced by some other space W, then there is a unique map $W\to Z$ making the enlarged diagram at the right commute. From this point of view, what we are looking for is a Künneth-type formula for the cohomology of the 'product' Z in terms of the cohomology of X and Y, regarded as modules over the cohomology of B. When B is a point the pullback Z is just the usual product $X\times Y$. We can expect things to be quite a bit more complicated for a general space B, and the Künneth formula that we will obtain will be in the form of a spectral sequence rather than the simpler form of the classical Künneth formula.

$$\begin{tikzcd} W \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[r] \arrow[d] & Y \arrow[d] \\ & X \arrow[r] & B \end{tikzcd}$$

Theorem 3.2. *Given a pair of maps $X\to B$ and $Y\to B$, the latter being a fibration, then there is a spectral sequence with $E_2^{p,q} = \mathrm{Tor}_{p,q}^{H^*(B;k)}(H^*(X;k),H^*(Y;k))$ converging to $H^*(Z;k)$ if B is simply-connected and the cohomology groups of X, Y, and B are finitely generated over k in each dimension.*

The finite generation hypothesis is needed since we will be using the Künneth formula repeatedly, and this needs finiteness assumptions in the case of cohomology, unlike homology.

The derivation of this spectral sequence will be formally rather similar to what we did for the spectral sequence in the previous section, once the proper categorical framework is established. Instead of considering arbitrary maps $X\to B$ we will consider only maps that are retractions onto a subspace $B\subset X$. This may seem too restrictive at first glance, but it actually includes the case of an arbitrary map $f:X\to B$ by enlarging X to $X_B = X\amalg B$ with the retraction $r:X_B\to B$ that equals f on X and the identity on B. When B is a point this amounts to enlarging X to X_+ by adding a disjoint basepoint. Thus X_B is X with a disjoint *basespace* adjoined. In the situation we will be considering of retractions $r:X\to B$ we can similarly regard B as a base*space* for X instead of just a base*point*.

To formalize, we will be working in the category $\mathcal{C}_B$ whose objects are retractions $r:X\to B$ and whose morphisms are commutative triangles as at the right. The category $\mathcal{C}_B$ has quotients: Given a pair (X,A) in $\mathcal{C}_B$, with the retraction $X\to B$ restricting to the retraction $A\to B$, we can form the quotient space of X obtained by identifying points of A with their images under the retraction to B. This idea allows us to construct the (reduced) mapping cone of a map $f:X\to Y$ in $\mathcal{C}_B$. First form the ordinary mapping cylinder of f and collapse its subspace $B\times I$ to B, then collapse the copy of X at the source end of the mapping cylinder to B via the retraction $X\to B$. The retractions of X and Y to B induce a retraction of the resulting mapping cone to B, so we stay within $\mathcal{C}_B$.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & B & \end{tikzcd}$$

The pullback of two retractions $r_X:X\to B$ and $r_Y:Y\to B$ in $\mathcal{C}_B$ serves as their product, as we observed earlier, and we shall use the notation $X\times_B Y$ for this product, to emphasize the analogy with the object $X\times_G Y$ in the previous section. The product $X\times_B Y$ lies in $\mathcal{C}_B$ since the retractions r_X and r_Y induce a well-defined retraction of $X\times_B Y$ to B sending (x,y) to $r_X(x)=r_Y(y)$.

We can also define a smash product $X\wedge_B Y$ in $\mathcal{C}_B$ as the quotient space of $X\times_B Y$ obtained by collapsing $X\times_B B = X$ to B via r_X and $B\times_B Y = Y$ to B via r_Y. For the operation of adjoining disjoint basespaces we have $X_B\wedge_B Y_B = (X\times_B Y)_B$.

Since $H^*(X_B,B)\approx H^*(X)$ we will frequently be working with cohomology relative to the basespace B in what follows. This can be thought of as the analog of reduced cohomology for the category $\mathcal{C}_B$. For a pair (X,A) in $\mathcal{C}_B$ with quotient X/A in $\mathcal{C}_B$ obtained by collapsing A to B via the retraction there is a long exact sequence

$$\cdots \longrightarrow H^n(X/A,B)\longrightarrow H^n(X,B)\longrightarrow H^n(A,B)\longrightarrow\cdots$$

assuming that A is a deformation retract of a neighborhood in X so that excision can be applied. Given also a space Y in $\mathcal{C}_B$ it is easy to check from the definitions that $(X\wedge_B Y)/(A\wedge_B Y)=(X/A)\wedge_B Y$ so there is also an exact sequence

$$\cdots \longrightarrow H^n(X/A\wedge_B Y,B)\longrightarrow H^n(X\wedge_B Y,B)\longrightarrow H^n(A\wedge_B Y,B)\longrightarrow\cdots$$

We will be using this in the case that X is a mapping cylinder with A its source end, so that X/A is the mapping cone.

Proof of 3.2: The first step will be to construct a commutative diagram

$$\begin{tikzcd} X = X_0 \arrow[r] & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[r] & \cdots \\ & & X_1 \arrow[ur] & & X_2 \arrow[ur] & \cdots & \end{tikzcd}$$

such that applying $H^*(-,B;k)$ to the horizontal row gives a resolution of $H^*(X,B;k)$ by free $H^*(B;k)$-modules. Then we will apply $\wedge_B Y$ to the diagram and again take $H^*(-,B;k)$ to get a staircase diagram, which will give the spectral sequence we want.

Let $K_0=(X/B)\times B$, viewed as an object in $\mathcal{C}_B$ by including B in $(X/B)\times B$ as the subspace $(B/B)\times B$ and taking the projection $(X/B)\times B\to B$ as the retraction. Then

$H^*(K_0;k) \approx H^*(X/B;k) \otimes_k H^*(B;k)$ and hence $H^*(K_0,B) \approx H^*(X,B) \otimes_k H^*(B;k)$. This is a free right $H^*(B;k)$-module since the module structure is given by $(a \otimes b)c = a \otimes bc$, the retraction $K_0 \to B$ being projection onto the second factor. There is a natural map $f : X \to K_0$, $f(x) = (x, r(x))$, which is a morphism in $\mathcal{C}_B$. This induces a surjection $f^* : H^*(K_0, B; k) \to H^*(X, B; k)$ since the composition $X/B \to K_0/B \to X/B$ of the maps induced by f and the projection onto the first factor is the identity map. (Note that these quotient maps are not maps in $\mathcal{C}_B$.) Another way to see that f^* is a surjection is to identify $H^*(K, B; k)$ with $H^*(X,B) \otimes_k H^*(B;k)$, and then f_* can be viewed as the map $H^*(X,B) \otimes_k H^*(B;k) \to H^*(X,B;k)$ defining the module structure on $H^*(X,B;k)$. This map is obviously onto since there is an identity element in $H^*(B;k)$.

Let X_1 be the mapping cone of f in the category $\mathcal{C}_B$. We will eventually need the following statement about vanishing of cohomology:

($*$) If $H^i(X,B;k) = 0$ for $i < n$, then this is true also for (K_0,B), and $H^i(X_1,B;k) = 0$ for $i < n+1$ if B is path-connected.

The first half of this assertion is an immediate consequence of the isomorphism $H^*(K_0,B;k) \approx H^*(X,B) \otimes_k H^*(B;k)$ while the second half is evident from the exact sequence

$$0 \longrightarrow H^n(X_1,B;k) \longrightarrow H^n(K_0,B;k) \xrightarrow{f^*} H^n(X,B;k) \longrightarrow 0$$

since f^* is an isomorphism in this dimension if $H^0(B;k) \approx k$.

Iteration of the construction of K_0 and X_1 from X now produces the diagram displayed at the beginning of the proof. The long exact sequences of cohomology $H^*(-,B;k)$ break up into short exact sequences that splice together to give a free resolution

$$\begin{tikzcd}
\cdots \arrow[r] & H^*(K_2,B) \arrow[rr] \arrow[dr] & & H^*(K_1,B) \arrow[rr] \arrow[dr] & & H^*(K_0,B) \arrow[r] & H^*(X,B) \arrow[r] & 0 \\
\cdots & & H^*(X_2,B) \arrow[ur] \arrow[dr] & & H^*(X_1,B) \arrow[ur] \arrow[dr] & & & \\
\cdots & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

After applying $\wedge_B Y$ we obtain long exact sequences of cohomology $H^*(-,B;k)$ that may no longer split into short exact sequences, but do form a staircase diagram

$$\begin{tikzcd}
 & & {} \arrow[d] & & {} \arrow[d] & \\
{} \arrow[r] & H^n(K_p \wedge_B Y, B) \arrow[r] & H^n(X_p \wedge_B Y, B) \arrow[r] \arrow[d] & H^n(K_{p-1} \wedge_B Y, B) \arrow[r] & H^n(X_{p-1} \wedge_B Y, B) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^{n+1}(K_{p+1} \wedge_B Y, B) \arrow[r] & H^{n+1}(X_{p+1} \wedge_B Y, B) \arrow[r] \arrow[d] & H^{n+1}(K_p \wedge_B Y, B) \arrow[r] & H^{n+1}(X_p \wedge_B Y, B) \arrow[r] \arrow[d] & {} \\
 & & {} & & {} &
\end{tikzcd}$$

hence we get a spectral sequence.

To recognize the E_2 terms as Tor groups we argue as follows. The pullback $X \times_B Y$ will be a product $Z \times B$ if X is a product $Z \times B$, with projection onto the second factor as the retraction. Thus in this case we have isomorphisms

$$\begin{aligned} H^*(X\times_B Y;k) &\approx H^*(Z;k)\otimes_k H^*(Y) \\ &\approx H^*(Z;k)\otimes_k [H^*(B;k)\otimes_{H^*(B;k)} H^*(Y;k)] \\ &\approx [H^*(Z;k)\otimes_k H^*(B;k)]\otimes_{H^*(B;k)} H^*(Y;k) \approx H^*(X;k)\otimes_{H^*(B;k)} H^*(Y;k) \\ &\approx [H^*(X,B;k)\oplus H^*(B;k)]\otimes_{H^*(B;k)} [H^*(Y,B;k)\oplus H^*(B;k)] \end{aligned}$$

This last tensor product can be expanded out as the sum of four terms, and after cancelling three of these we obtain

$$H^*(X\wedge_B Y,B;k) \approx H^*(X,B;k)\otimes_{H^*(B;k)} H^*(Y,B;k)$$

In particular this applies to the products $K_p = (X_p/B)\times B$, so the groups in the E_1 page are obtained from the groups in the free resolutions by tensoring over $H^*(B;k)$ with $H^*(Y,B;k)$. The differentials d_1 are obviously obtained by tensoring the boundary maps in the resolutions with the identity map on the $H^*(Y,B;k)$ factor, so the E_2 page consists of $\mathrm{Tor}_{*,*}^{H^*(B;k)}(H^*(X,B;k),H^*(Y,B;k))$ groups.

To make the indexing precise, we set $E_1^{p,q} = H^{p+q}(K_p\wedge_B Y,B;k)$. The nonzero terms in the E_1 page then all lie in the first quadrant. In the staircase diagram we replace n by $p+q$, so q is constant on each column of the diagram. In the E_1 page the differential d_1 maps $E_1^{p,q}$ to $E_1^{p-1,q+1}$, diagonally upward to the left, so the diagonals with $p+q$ constant form chain complexes with homology groups $E_2^{p,q} = \mathrm{Tor}_{p,q}^{H^*(B;k)}(H^*(X,B;k),H^*(Y,B;k))$. Fixing p and letting q vary, the direct sum of the terms in the p^{th} column of the E_2 page is $\mathrm{Tor}_p^{H^*(B;k)}(H^*(X,B;k),H^*(Y,B;k))$.

The differential d_r in the E_r page maps $E_r^{p,q}$ to $E_r^{p-r,q+1}$, going r units to the left but only one unit upward. This means that it is no longer automatically true that the sequence of groups $E_r^{p,q}$ for fixed p and q and increasing r stabilizes at some finite stage, as the differentials mapping to $E_r^{p,q}$ could perhaps be nonzero for infinitely many values of r. However, this does not actually happen since all the terms $E_1^{p,q}$ are finite-dimensional vector spaces over k, hence this is also true for $E_r^{p,q}$, and each nonzero differential starting or ending at a given term $E_r^{p,q}$ reduces its dimension by at least one so this cannot happen infinitely often.

At the top of the q^{th} A column of the staircase diagram we have the group $H^q(X\wedge_B Y,B;k)$. This is filtered by the kernels of the compositions of the vertical maps downward from this group, with successive quotients the entries in the q^{th} row of the E_∞ page. For the general convergence results at the beginning of Chapter 1 to be applicable we need the terms in the q^{th} A column of the staircase diagram to be zero sufficiently far down this column. We claim that this will happen in the situation of the theorem where we assume that B is simply-connected. As a preliminary step to seeing why this is true, recall that $H^*(K_p,B;k) \approx H^*(X_p,B;k)\otimes H_*(B)$ and $H^*(X_{p+1},B;k)$ is the kernel of the module structure map $H^*(X_p,B;k)\otimes H_*(B)\to H^*(X_p,B;k)$, so if $\widetilde{H}^*(B;k)$ vanishes below dimension 2 we see that $H^*(X_{p+1},B;k)$ will vanish in two more dimensions than $H^*(X_p, B; k)$. By induction it follows that both $H^i(X_p, B; k)$ and $H^i(K_p, B; k)$ are zero for $i < 2p$. (In particular, in the E_1 page this means that $E_1^{p,q} = 0$ for $p > q$, which gives a stronger reason for the terms $E_r^{p,q}$ to stabilize as r goes to infinity.)

Now we can prove the claim about the A columns. In the situation of the theorem we take Y to be of the form $Y_B = Y \amalg B$ for a fibration $Y \to B$ with B simply-connected. Then $X_p \wedge_B Y_B$ is $X_p \times_B Y$ with the subspace $B \times_B Y$ collapsed to B, so $H^*(X_p \wedge_B Y_B, B; k)$ is $H^*(X_p \times_B Y, B \times_B Y; k)$. Thus we are looking at the cohomology of the pullback of the fibration $Y \to B$ over X_p and B. With B simply-connected we have seen that $H^i(X_p, B; k) = 0$ for $i < 2p$ so by the Serre spectral sequence the relative cohomology of the pair $(X_p \times_B Y, B \times_B Y)$ vanishes in the same dimensions. The simple-connectivity assumption guarantees that the action of π_1 of the base on the cohomology of the fiber is trivial for the fibration $Y \to B$ and hence also for the pullback. Thus we have $H^i(X_p \times_B Y, B \times_B Y; k) = 0$ for $i < 2p$, which implies that each A column of the staircase diagram consists of zeroes from some point downward, as we claimed.

There is also a more elementary argument for this that does not use the Serre spectral sequence. One proves inductively that the pairs (X_p, B) and (K_p, B) are $(2p-1)$-connected if B is simply-connected. Since the cohomology vanishes in this range with coefficients in any field it suffices to show that X_p and K_p are simply-connected when $p > 0$, and this can be done by a van Kampen argument after modifying the construction by attaching cones to subspaces rather than collapsing them to a point. Once one knows that (X_p, B) is $(2p-1)$-connected, the homotopy lifting property then implies that $(X_p \times_B Y, B \times_B Y)$ is also $(2p-1)$-connected. □

References

L. HODGKIN, The equivariant Künneth theorem in K-theory, *Springer Lecture Notes* 496 (1975), 1–101.

L. SMITH, *Lectures on the Eilenberg-Moore Spectral Sequence*, *Springer Lecture Notes* 134 (1970).