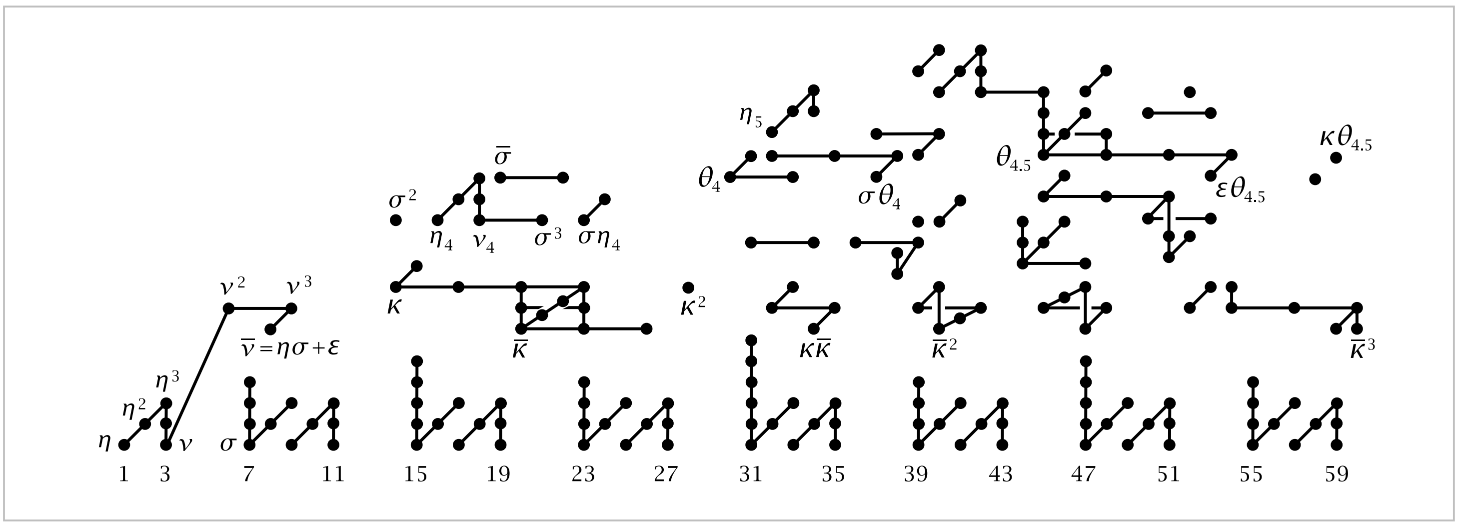
Pictures of Stable Homotopy Groups of Spheres

It is a classical theorem of Freudenthal (ca. 1940) that the homotopy group $\pi_{n+i}(S^n)$ is independent of n for n sufficiently large, namely $n > i+1$. This stable value is often denoted π_i^s. For $i = 0$ this group is infinite cyclic, while for $i > 0$ it is a finite abelian group (Serre 1950), hence is the direct sum of its p-primary subgroups, the elements of order a power of a prime p. Computing these groups π_i^s has been quite difficult, with the difficulty increasing as i increases.

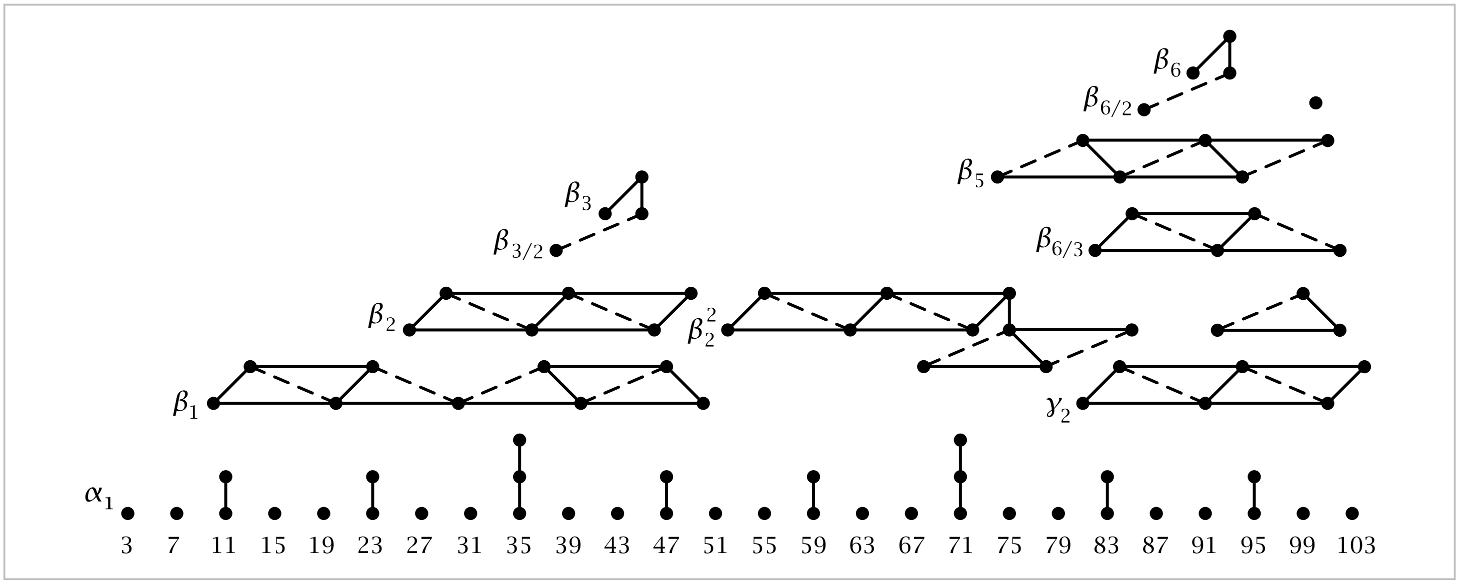
The charts below show the extent of the complete calculations to date for the p-primary parts of π_i^s for $p = 2,\ 3$, and 5. First we show the 2-primary part of π_i^s for $i \le 60$:



The numbers across the bottom represent the dimension i. A vertical string of k connected dots in the ith column corresponds to a direct summand of π_i^s which is cyclic of order 2^k. For example for $i = 47$ we have five cyclic summands of orders 32, 4, 2, 2, and 2. The diagonal line segments moving one unit to the right represent composition with the (suspended) Hopf map $\eta \in \pi_1^s$, while the segments moving three units to the right represent composition with the Hopf element $\nu \in \pi_3^s$. There are other nontrivial compositions but these are not shown, to avoid complicating the diagram too much. The labels on some of the elements are the standard names that have been given to these elements.

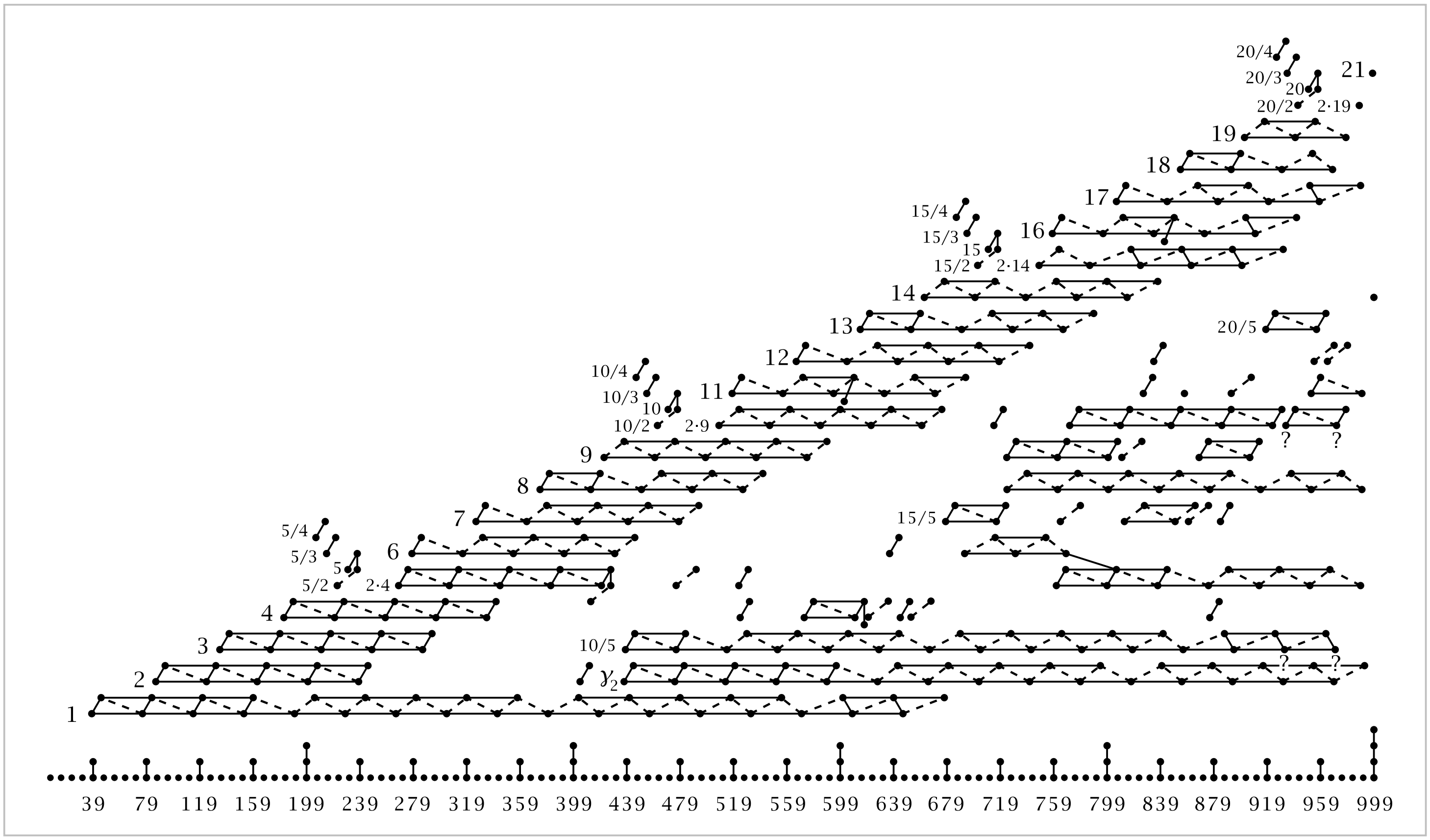
The regular pattern across the bottom is known to extend out to infinity on the right, with the order of the cyclic summand in dimension $4n-1$ being 2^{m+1} where 2^m is the highest power of 2 dividing $4n$. All the elements in this regular sequence are detected by K-theory, as Adams showed, and the elements in dimensions congruent to 1, 3, and 7 mod 8, along with the elements in dimensions congruent to 0 and one of the two summands in dimensions congruent to 1, make up the image of the J homomorphism.

Here is the corresponding picture for the 3-primary part of π_i^s:



The "triadic ruler" across the bottom again extends to infinity to the right, and represents the 3-primary part of the image of the J homomorphism. The solid nonvertical line segments represent composition with the elements α_1 and β_1, while the dashed lines denote Toda brackets $[\alpha_1, \alpha_1, -]$.

The 5-primary groups exhibit greater regularity and have been computed even farther, as shown in the picture on the next page.



The image of the J homomorphism again forms an infinite regular pattern across the bottom. In the rest of the diagram a label i (integer or rational) denotes the element called β_i.

The chart for $p = 2$ was drawn from calculations due to many people over the years, most recently Dan Isaksen and Zhouli Xu who have corrected a few errors in earlier calculations. The charts for $p = 3$ and $p = 5$ are drawn from tables in Ravenel's "green book" *Complex Cobordism and Stable Homotopy Groups of Spheres.*