AN INTRODUCTION TO KNOT THEORY

SEMINAR 2, FALL 2018

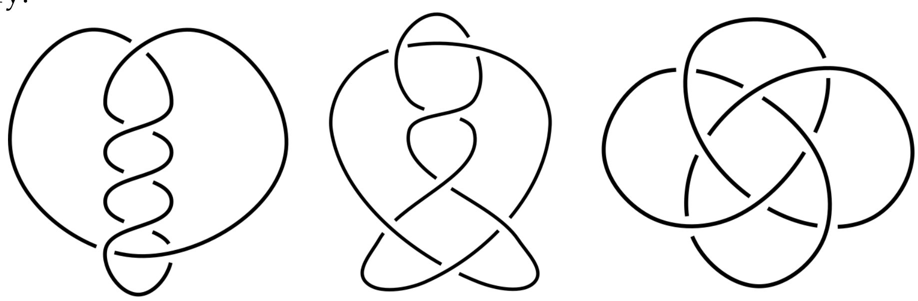
These notes were written for a two-month seminar for high school seniors I taught at Ithaca High School in Fall 2018. I would like to thank Severin Drix and Mary Ann Huntley for giving me the opportunity to teach this course.

Herein, I largely (and closely) follow Colin Adams' excellent 'The Knot book' [Adams, 2004] and Inga Johnson's wonderful 'An Interactive Introduction to Knot Theory' [Johnson, 2017]. Any mistakes introduced in these notes are my own.

The majority of the pictures were drawn using Inkscape or tikz, with a few exceptions taken from [Adams, 2004].

FIGURE 1. Cornell Robotic Construction Laboratory's "Log Knot"

Our aim for the next few weeks is to study knot theory: a field of mathematics that is over 100 years old (though by mathematical standards this means it's relatively young!). It all started around the 1880s when chemists thought that the universe was made out of a substance called the "ether", and Lord Kelvin thought that different types of matter were caused by different knots in the ether. This meant that physicists, such as Peter Guthrie Tait, tried to create a table of the elements corresponding to a classification of all knots. This theory of the ether was thoroughly debunked by 1890, so that chemists all moved on, but by that point the question of understanding knots had proven to be extremely interesting from mathematicians' point of view. Quite fascinatingly, about a century later in the 1980s, biochemists noticed that DNA molecules are knotted, and that these knots can have an effect on the properties of the molecules, so knots are back in vogue in biology and chemistry!



Hydrogen? Radium? Potassium?

Physically, what is a knot? We have a length of string that we tangle or knot up somehow, and then we connect the ends. From now on, we're not allowed to cut the string in any way, we can only move the string around while it is still looped. Let's compare this to a more mathematical definition of a knot:

Definition 0.1. A **knot** is a simple closed curve in space.
Equivalently,

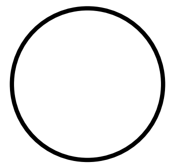
Definition 1*: A **knot** is a continuous map

$$\gamma : [a, b] \to \mathbb{R}^3$$

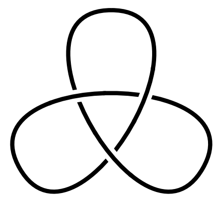
such that $\gamma(a) = \gamma(b)$ and such that γ is injective on the open interval (a, b).

What does this mean in terms of our physical interpretation?
A *curve* is a one-dimensional object, like the real line $\mathbb{R}$, a segment $[a, b]$ or a circle (it is infinitely thin, unlike our physical knot, but this won't really matter for now).
A curve is *closed* if its ends are connected.
A curve is *simple* if it does not intersect itself (except at the ends, as we require): the curve cannot cut through itself. This is the same as saying that the map γ is injective: for any x_1 and x_2 in (a, b), if $\gamma(x_1) = \gamma(x_2)$, then $x_1 = x_2$. Physically, we're saying that a strand can't be pulled through another strand, which is true since we're mathematicians, not magicians.
By continuous, we mean that we can't cut the string in any way.
By $\mathbb{R}^3$, we simply mean three-dimensional space, the one we live in.

Example 0.2. Perhaps our most important example of a knot is the most trivial one: the unknot. For this knot, we don't twist or knot up our length of string at all before connecting the ends:



Example 0.3. Interestingly, the next simplest knot (we will explain and prove this later!) is the trefoil:



Generally, since we write on two-dimensional paper, we will always present our knots as if we have taken a photograph of them, as in our examples above. Note that if we simply tried to project our knot down into two dimensions (for example, by shining a light at our physical knot and looking at the shadow it creates on the wall behind) is not quite enough information to remember our knot. Why not?

Definition 0.4. A two-dimensional picture of a knot, in which we take note of the strand that goes over and the strand that goes under at each crossing is called a **knot projection**.

One of the interesting things about knot theory is that the big problems in the field, namely open problems that are well-known, are generally quite easy to state, but difficult to solve. For example:

Question 0.5. When are two knots the same?

Take any two knots that we make as described above. Are these the same knot? Namely, can I manipulate the strands of one of them so that it ends up looking exactly the same as the second one? Perhaps you've tied the knots in very similar ways, and with little trouble you can make them look the same. But what if you can't? Can you be sure that you just haven't spent enough time trying? What if you gave it two weeks? Three weeks? A year? (10 years? 20? ...?!) It turns out that this is a surprisingly difficult question to answer in general.

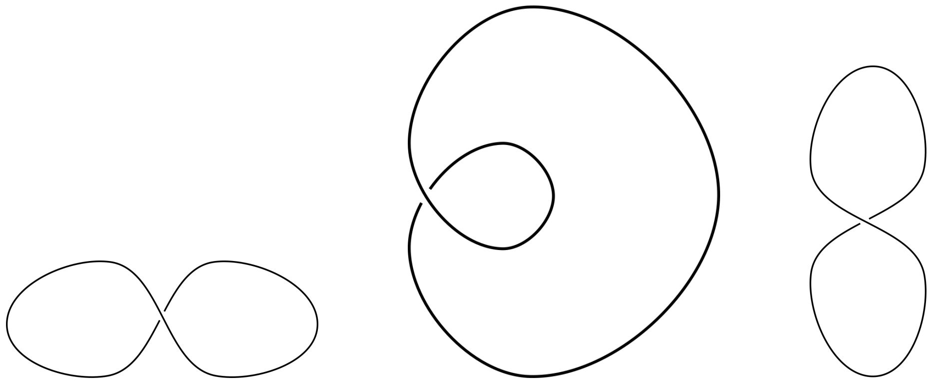
A related question that we'll be asking ourselves today is:

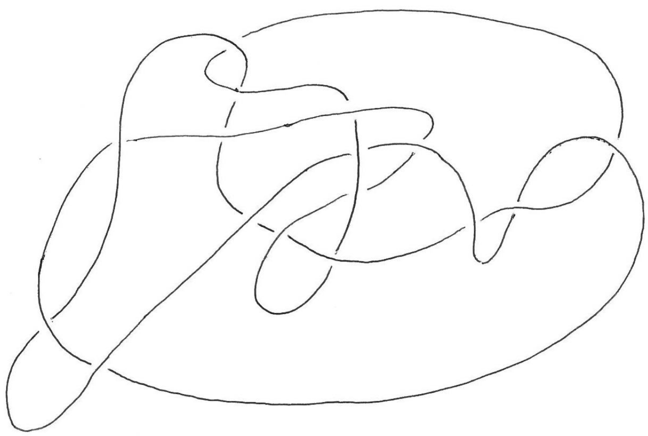
Question 0.6. Given a knot, for example, the trefoil, can we untangle the knot into the unknot?

Another way to ask this question in terms of our knot projections is:

Question 0.7. Given a knot projection, is this just a complicated projection of the unknot?

For example, consider the following knot projections:



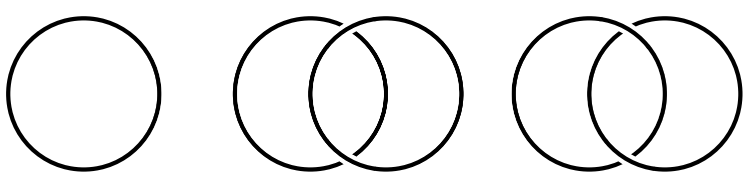


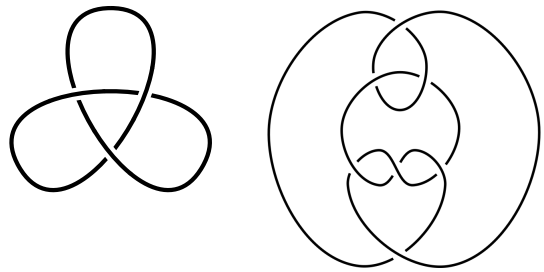
Exercise 0.8. Show that all of the projections above are projections of the unknot.

On the plus side, in 1961 Wolfgang Haken showed that the question of whether a given projection is the projection of the unknot is decidable: namely he gave a procedure telling us how to check. There are several algorithms for unknotting knots, but the complexity of the algorithms (are there algorithms that can be implemented in polynomial time?) is still an open question!

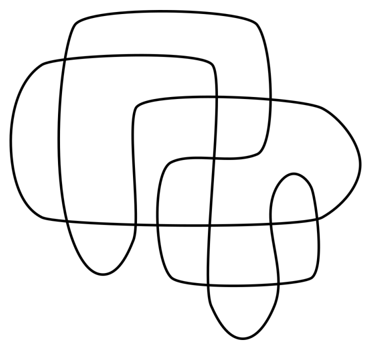
Now we want to start thinking about these big questions in some specific examples, to lead ourselves to some methods mathematicians have devised when pondering these exact questions.

Exercise 0.9. Consider the following knot projections. Which, if any, are projections of the same knot? If you are convinced that the are not the same, how can you *prove* that they are not the same? What are some of the intrinsic characteristics of these knots which allow you to say that they are not the same?





Exercise 0.10. Suppose that we simply projected our knot in three dimensions down into two dimensions using its shadow as described above. We would get a picture something like this:



Show that by choosing over- and under-crossings wisely, you can always turn such a two-dimensional picture into a projection of the unknot. Equivalently, show that given any knot projection, you can obtain a projection of the unknot by changing some number of crossings from over to under or vice versa.

Exercise 0.11. Based on our characterisation of the knots above, how could we start thinking about classifying knots? How can we determine the complexity of a knot? How could we start making a table of the simplest knots? What do we mean by simple here?

1. The Reidemeister Theorem

Let's recall our definition of a knot:

Definition 1.1. A **knot** is a simple closed curve in three-dimensional space.

Remember: we are not allowed to cut the string or pull any strands through other strands.

We also had a way of keeping track of our knots in two dimensions:

Definition 1.2. A **knot projection** is a two-dimensional picture of a knot with the added information of the **crossings** of the knot.

Remember: there are **LOTS** of possible projections for the same knot!

Example 1.3.

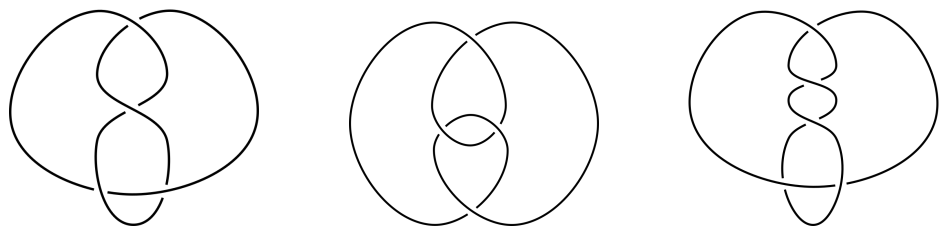


FIGURE 2. These are all projections of the **figure-eight knot**

Exercise 1.4.
Show that all of the following projections are projections of the same knot (i.e. show that they are **equivalent**).

Last time, we also noted that we could have projections that had different **components**, i.e. different closed loops. These are technically not knots, we call them **links**:

Definition 1.5. A **link** is a finite union of pairwise nonintersecting knots. Each knot in the link is called a **component** of the link.

How could we make a link physically? We would take a few strings and tangle them up before connecting up the pairs of ends to end up with several knots tangled together. By *finite union*, we just mean that there should be only be a finite number of components, and by *pairwise nonintersecting* we mean that the strands of any two knots should not pass through each other (you should think of what this means in terms of our physical knots).

Example 1.6.

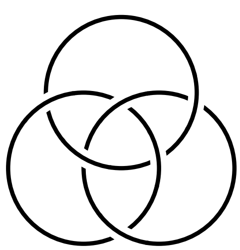


FIGURE 3. Borromean rings

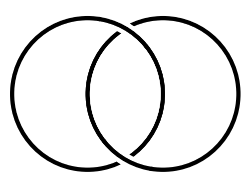


FIGURE 4. the Hopf link

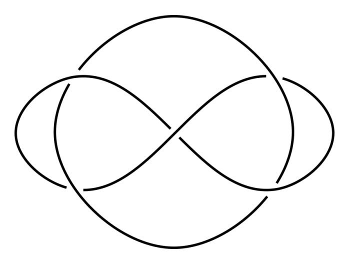


FIGURE 5. The Whitehead link

Exercise 1.7. Compute the number of components of the links in the example above. How many components does the trefoil have? How many components does a knot have?

Example 1.8. If a link can be rearranged to be a disjoint, unlinked union of n unknots, we say that it is the **unlink of** n **components**.

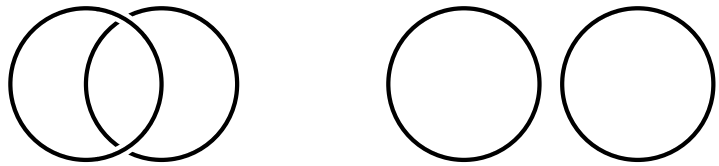


FIGURE 6. Two projections of the unlink of two components

The number of components of a link is a simple way of telling apart links. For example, we can quickly tell apart the Hopf link and the trefoil by counting their components. It is clear that no matter how we manipulate the strands of the link, if we don't cut the string in any way, we can't change the number of components of the link. Another way of putting this is that any two projections of the same link will still have the same number of components.

We call such a quantity or property that doesn't depend on the chosen projection of a link a **link invariant**. These invariants are very useful: they are much less complicated than links themselves, and if we find that a particular invariant is different for two projections, then they must be projections of different knots. Therefore, these invariants make it much easier to tell knots apart.

Our discussion above tells us that the number of components is a link invariant. However, unfortunately, this quite a weak invariant: we can't use it to tell any knots apart, for example. Our aim will be to find some more knot invariants. To do this, we should first think about what it means for two projections to be projections of the same knot. It means that we can rearrange our strands to make the knots looks exactly the same. Looking at our projections, we have different ways of seeing this rearrangement of the physical knot. It's very important to remember that these rearrangements **should not change** the physical knot: though the projections change, the knot stays exactly the same. Now, it might seem like a monumental task to check if some property is a knot invariant: there are so many projections of the same knot! It turns out that there are only a small number of rearrangements to consider.

The rearrangements that are allowed are called **ambient isotopies**. You should imagine that your link is made out of a very stretchy rubber material: you can move the strands around and stretch them or shrink them as much as you like, as long as you don't pull the strands through each other or pull a knot so tight that it disappears. That is the *isotopy* part. *Ambient* means that we are considering this rearrangement in the space the knot lives in.

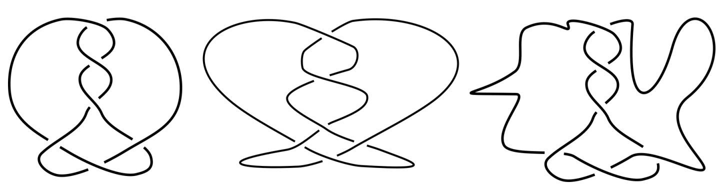
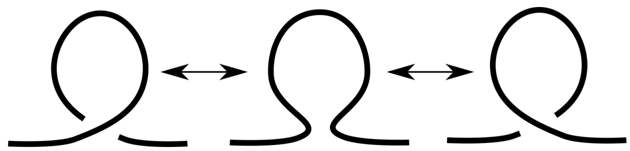


FIGURE 7. Ambient isotopies

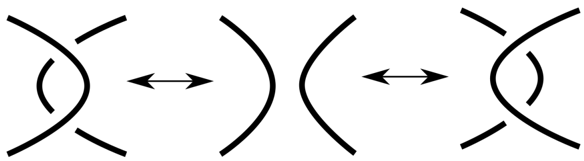
There are other special ambient isotopies that tell us ways that we can change our crossings, known as **Reidemeister moves**. They come in three types:

Definition 1.9. The **Reidemeister moves** are:

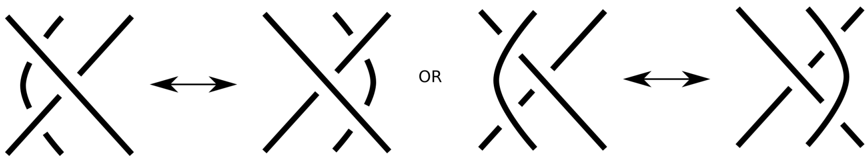
R1



R2



R3

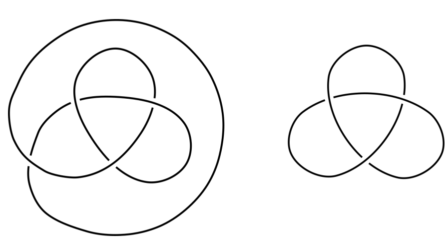


Note! These are **local** moves: away from the small piece of the projection that is changed by one of these Reidemeister moves, we do not change anything else about the knot. It seems clear that these Reidemeister moves modify the knot projection without changing the knot itself, but perhaps more surprisingly these are the only (crossing-changing) moves that you need to get from any projection of a knot to any other!

Theorem 1.10 (**Reidemeister, 1926).** Given any two projections of the same knot, there exists a (finite) sequence of Reidemeister moves and ambient isotopies taking us from one projection to the next.
The proof is a little technical and long for this class, so we will skip it here but you should at least convince yourself that these moves will not affect our knot. What is this powerful theorem saying? If we want to answer the global question of when two knot projections come from the same knot, we only really need to consider what is happening at each crossing, i.e. using local information.

It might seem like we're done now, we just need to check if there's a sequence of Reidemeister moves to get from one projection to another. The problem is that not only might it be difficult to find such a sequence (you might have to introduce hundreds if not thousands of new crossings in the intermediate projections!), it's also not clear when there *isn't* such a sequence.

Exercise 1.11. What is a sequence of Reidemeister moves taking us from the first projection of the trefoil to our more common one?



2. Our first link invariants

Now we're going to start computing some more link invariants: quantities or properties that don't depend on our choice of projection, that are intrinsic to the knot or link itself. Why are these link invariants so useful? Our big question in knot theory is figuring out how to tell knots apart (or be able to say that they are the same), but this can be really difficult! Given that there are so many different projections of a given knot, and so many ways to tangle it up, it could take us years to tell if a given projection is the same as another. These knot and link invariants, as we'll see, are much easier to compute - remember our first example of a link invariant, the number of components - so we can much more quickly say that certain projections are different.

Be careful though, since these invariants are simpler than the knots themselves, they will not be **complete invariants**, namely, we won't be able to tell apart all knots and links using them: recall for example that there are many links with the same number of components!

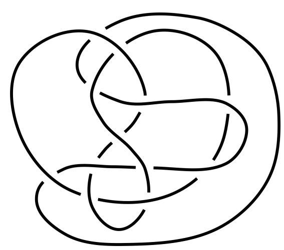
2.1. The linking number.

Our next example of a link invariant is the **linking number**. This will be a way of measuring how "linked up" or "twisted up" two components of a link are. Let's first get an idea of what this twisting up of components means.

Definition 2.1. A link is **splittable** if the components of the link can be deformed so that they lie on different sides of a plane in three-dimensional space.

This means exactly what you think: a link is splittable if you can move around the strands until the components can be separated. What is an example of a splittable link that we have already seen?

Exercise 2.2. Show that the following link is splittable. What are the components?



On the other hand, the Hopf link is *not* splittable. The components pass through each other so that we can't separate them by a plane. The linking number will help us quantify this linking up of the two components, counting how many times a given component of the link twists around another component. First we need to define the **orientation** of a knot:

Definition 2.3. An **orientation** on a knot is a choice of direction to travel around the knot. This is denoted by placing directed arrows consistently along a projection of the knot. A knot with a chosen orientation is called an **oriented knot**.

Let M and N be two components in a link, and choose an orientation on each of them. This means that the crossings between strands of M and N will come in two types:

Let p be the number of $+1$ crossings between M and N and let n be the number of -1 crossings between M and N. **NOTE:** we do not consider self-crossings of the components here! Then we define

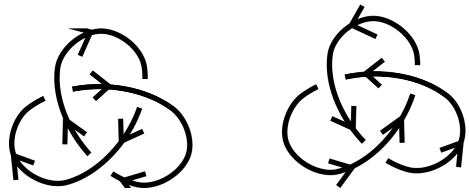


FIGURE 8. Two orientations for the trefoil

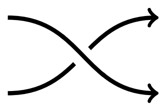


FIGURE 9. +1

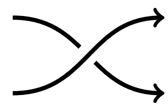
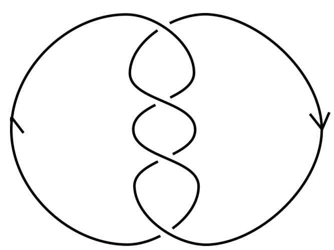


FIGURE 10. -1

Definition 2.4. The **linking number** $l(M, N)$ of M and N is

$$l(M, N) = \frac{p - n}{2}$$

Exercise 2.5.
(1) Compute the linking number of the Hopf link (after choosing an orientation on each of its components).
(2) Compute the linking number of the following oriented link:



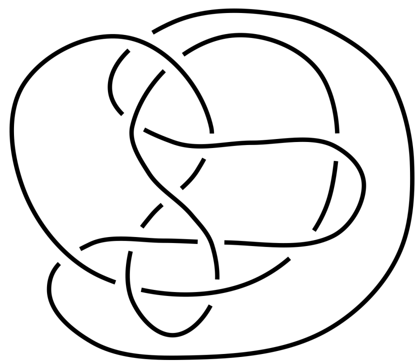
(3) Explain how to make a link with two components whose linking number is any integer k of your choosing.
(4) Compute the linking number of the link above after changing the orientation of one of the components. Do the same after switching the orientations of both components.
(5) How does the linking number depend on your choice of orientation? Is there a different quantity that you could compute that does not depend on the orientation?

Our aim now is to show that the linking number does not depend on the choice of projection of our link, namely that it is a link invariant. The way that we will show this is by showing that the linking number is not affected by the Reidemeister moves. Since any two projections are connected by a sequence of Reidemeister moves, this tells us that any two projections of the same link will have the same linking number.

Theorem 2.6. The linking number is unchanged by Reidemeister moves and is therefore a link invariant.

Proof. Check the Reidemeister moves. ☐

Exercise 2.7. (1) Show that a splittable link always has linking number 0.
(2) After choosing orientations as appropriate, compute the linking number of the following link:

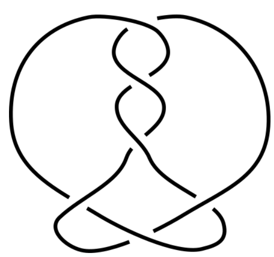


(3) Is it true that any link with linking number 0 is splittable? If yes, explain why. If not, give a counter-example.

2.2. **Tricolourability.** At last, we are going to see an invariant that tells us that we have knots that aren't the unknot!

Definition 2.8. An **arc** in a projections of a link is a piece of the link that goes from one undercrossing to another undercrossing with only overcrossings in between

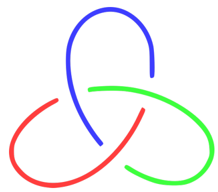
Example 2.9. Find all of the arcs in the following projection:



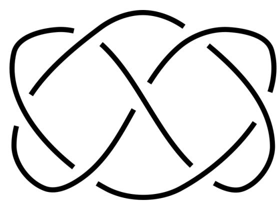
Definition 2.10. A projection of a knot or link is **tricolourable** if every arc of the knot can be coloured one of three different colours such that:

(1) At each crossing, either arcs of all three colours meet, or only arcs of a single colour meet.
(2) At least two colours are used.

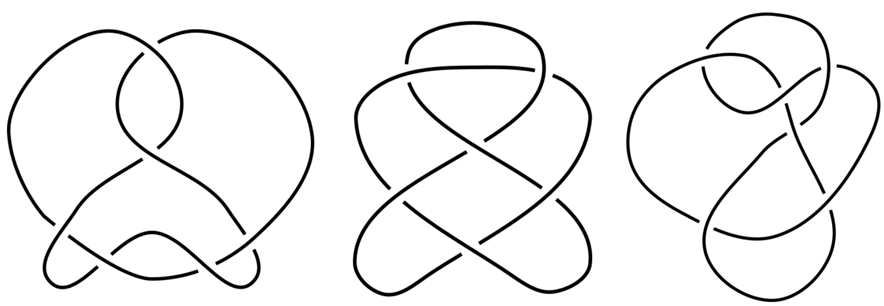
Example 2.11. The trefoil is tricolourable:



Exercise 2.12. (1) Show that the unknot is not tricolourable.
(2) Show that the following knot projection is tricolourable:



(3) Are the following projections tricolourable?



It turns out that tricolourability is also a knot invariant:

Theorem 2.13. Tricolourability is independent of the Reidemeister moves and is therefore a link invariant. We say that a link whose projections are tricolourable is a **tricolourable link**.

Proof. This is an exercise! □

Since tricolourability is a knot invariant and we have a projection for which the trefoil is tricolourable and a projection for which the unknot is not, this tells us that the trefoil is not the unknot! (Huzzah!) Unfortunately, this is again not a very strong invariant: a knot is either tricolourable or it is not, so we cannot tell two tricolourable knots apart.

Exercise 2.14. Show that the following projection of the figure eight knot is not tricolourable. What can we conclude about the unknot, the trefoil and the figure eight knot? [*Hint: How many arcs does the projection have? How many crossings are there? Is it possible to have three arcs with the same colour? Is it possible to have two arcs that have the same colour?*]

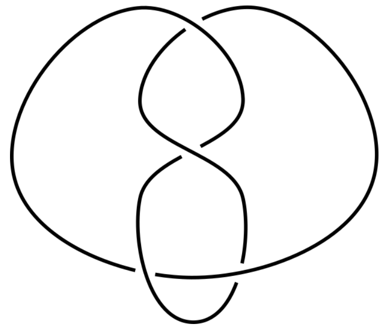
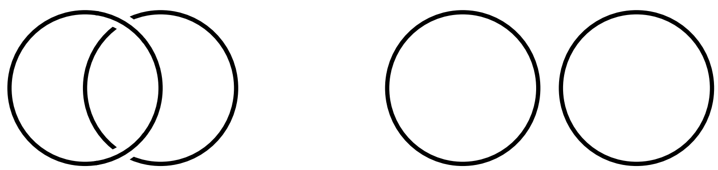


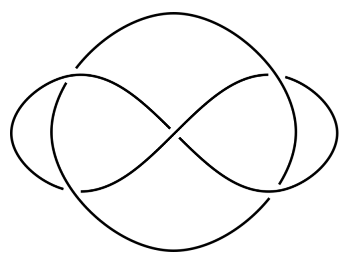
FIGURE 11. The figure eight knot is not tricolourable

We can also consider the tricolourability of links, where our definition of arcs remains the same as for knots.

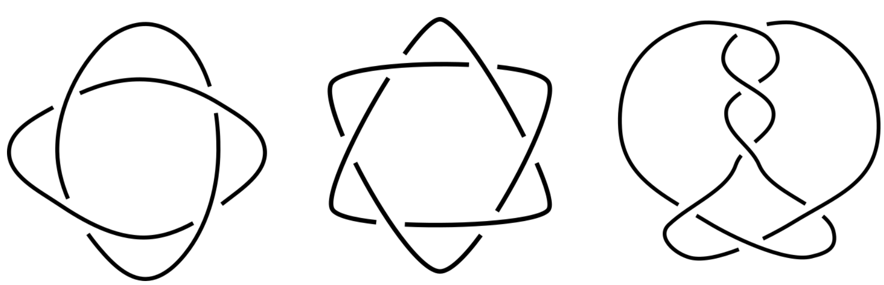
Exercise 2.15. Is the unlink of two components tricolourable? How about the unlink of n components? If possible, find a tricolouration of the following projections of the two-component unlink:



Exercise 2.16. In class we showed that the Whitehead link, shown below, has linking number 0 (if you don't remember this fact, check it now!), so that we could not distinguish it from the two-component unlink. Is the Whitehead link tricolourable? What does this tell you about the Whitehead link?



Exercise 2.17. Determine whether the following links are tricolourable or not:



3. Examples and classification

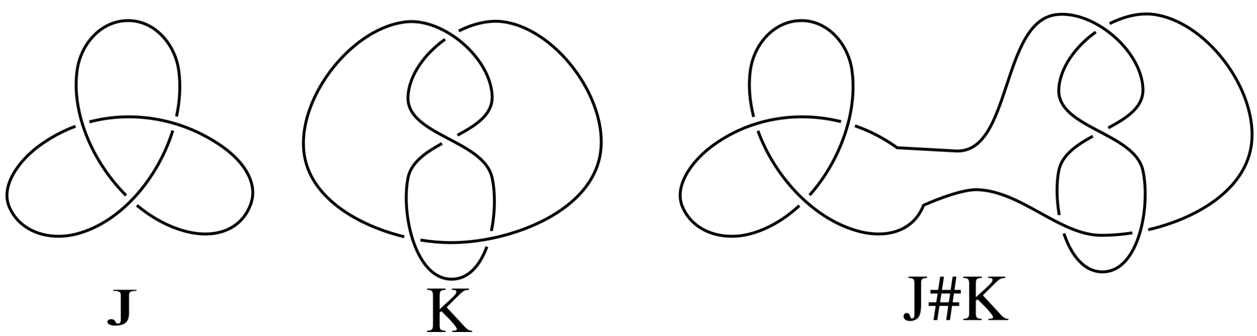
Now that we know that not all knots are just the same, let's look at some more examples of knots. Furthermore, knowing that we have different knots, we can wonder how many of them there are, and if we can begin to classify them. Could we use our knot invariants as an aid in this classification endeavour?

3.1. **Some more knots.**

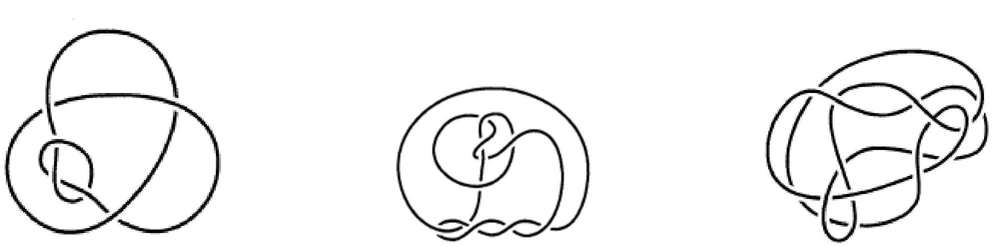
3.1.1. *Prime and Composite knots.* Interestingly, knots have a similar structure to the natural numbers. Recall that any (positive) integer can be written as a product of prime numbers (and the unit 1). We call a number that is a product of more than one prime a **composite** number. Similarly, we have a concept of **prime** and **composite** knots. First, we want to understand the version of putting together, or **composing** two knots.

Definition 3.1. Let J and K be knots. The **composition** or **(connected) sum** of J and K, denoted J#K, is constructed as follows: make a cut in each of the knots and join the loose edges pairwise so that no new crossings are introduced. If a knot is the composition of two knots then is called a **composite knot**. If a knot cannot be drawn as a connect sum of two knots, it is called a **prime knot**.

An example of the connected sum of two knots is shown below.



Exercise 3.2. Show that the following knots are composites:



You can think of prime knots in the same way you think of prime numbers, these are the indecomposable knots, so if we want to begin tabulating knots, we only need to keep track of the prime knots, since the others are "built" from these. In this analogy, you can see the unknot as the "unit" 1, as the next exercise shows:

Exercise 3.3. Show that for any knot J, the composition $J\#U$ of J with the unknot U is isotopic to J itself. (You should think of this as analogous to the equality $n \times 1 = n$ for any number n.)

One question you could ask at this point is about our simplest knot, namely the unknot. Is the unknot a composition of knots? Or more generally, is it possible to put together two complicated knots and get a "simpler" knot? As you would expect from our integer analogy, this is not possible, and we can prove a slightly weaker version of this fact using our knot invariants: tricolourability.

Exercise 3.4. Show that the composition of any knot J with a tricolourable knot K is tricolourable. Conclude that the unknot is not the composition of any number of trefoils.

You might wonder if the composition of knots is well-defined - that is, if you connect the knots at different points, will you obtain different composite knots?

Exercise 3.5. How many knots is it possible to obtain taking the composition $J\#K$? *Hint: consider the pictures below: orient each of your knots and consider how your choice of join affects the induced orientation (or not) on the composition. If two joins lead to an orientation on the composite, show that their outcomes are isotopic. What about if the composition does not have a consistent orientation?*

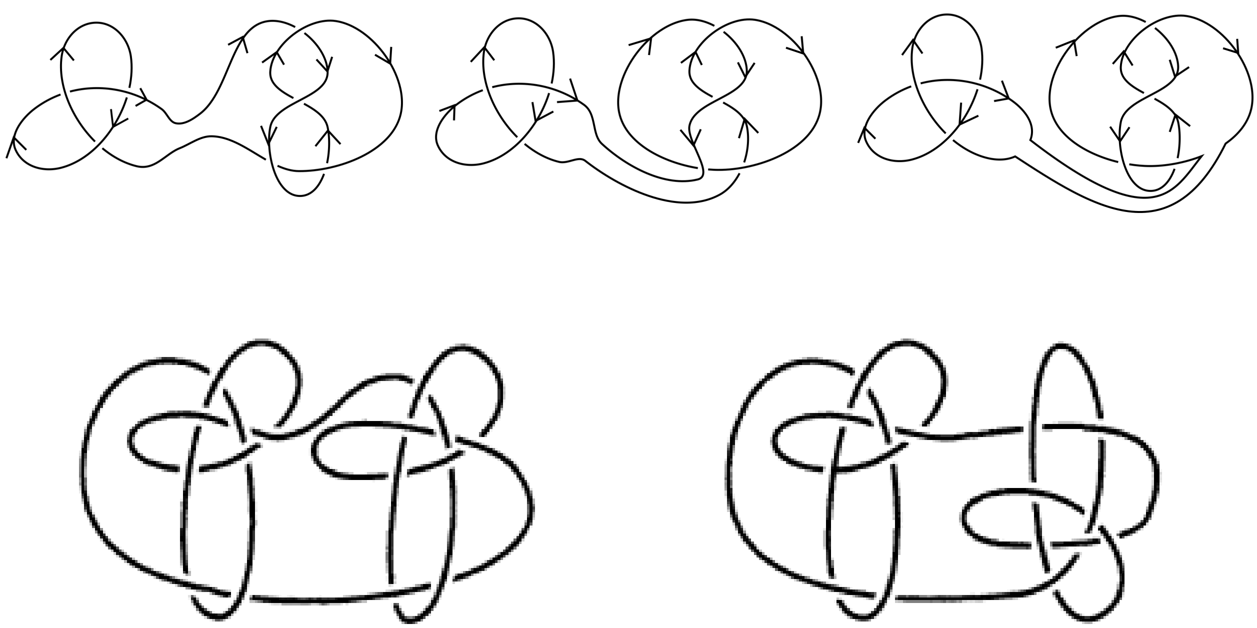
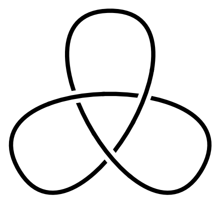


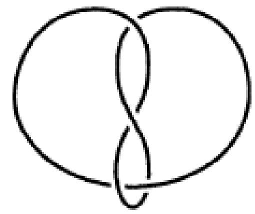
FIGURE 12. The composition(s) of the knot 8_17 with itself

3.1.2. *Alternating knots.*

Definition 3.6. An **alternating knot** is a knot with a projection that has crossings that alternate between over- and undercrossings as you traverse the knot in a fixed direction. An **alternating link** is a link with a projection that has crossings that alternate between over- and undercrossings as you traverse any component of the link in a fixed direction.

Exercise 3.7. Show that the trefoil and the figure-eight knot projections shown below are alternating





Careful! We **do not** require that **every** projection of our knot or link has this alternating property, only that it has one such projection.

Exercise 3.8.
(1) Modify the projections above to make projections of the trefoil and the figure-eight knot that are not alternating.
(2) More generally, do any of the Reidemeister moves break the alternating property of a link projection? What would it mean if all of the Reidemeister moves preserved this alternating property?

Definition 3.9. We (this notation is *not* standard) say that a projection of a knot is a **forgetful knot projection** if we forget whether each crossing is an overcrossing or an undercrossing. We call such crossings **precrossings**.

Note: this amounts to saying that a forgetful knot projection really is a projection of the knot! (Even though we don't call it a knot projection...)

Example 3.10.

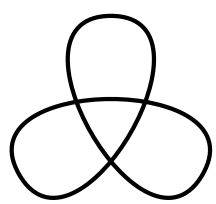
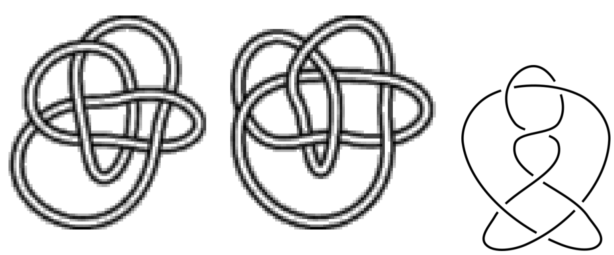
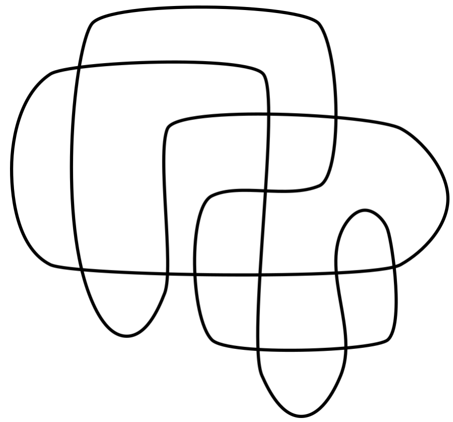


FIGURE 13. A forgetful projection of the trefoil

Exercise 3.11. Which of the following projections are alternating?



Exercise 3.12. Show that by choosing whether each precrossing is an under- or overcrossing, you can turn the following forgetful projection into an alternating projection.



Exercise 3.13. Our aim here is to show that you can choose over- and undercrossings for **any** forgetful projection so that the resulting knot projection is alternating.

(1) First, determine a method for choosing over- and under-crossings on a forgetful projection so that you should end up with an alternating projection (think about how you did this in the previous exercise). The difficult part is to show that this method always works!
(2) Suppose you have an arbitrary forgetful projection of a knot. Choose an orientation on it. Choose a precrossing and label it with 1. Follow the strand around using your choice of orientation until your next crossing, and label this with a 2. Continue this way until you arrive back at your first crossing for the second time. Each crossing should have two labels. **Claim:** Each crossing is labeled with an even number and an odd number. Confirm this claim in the example above.
(3) Our aim now is to prove the claim above. Consider any precrossing in the projection. Is it possible to have a precrossing where your strand loops back on itself without crossing any other strands? What does this look like? Is it possible to have exactly one other (pre)crossing before the strand loops back to your original (pre)crossing? Why or why not? How about 2, 3 or 4 crossings? What happens in general? What does this tell you about your labels at each crossing?
(4) How can you relate this fact that each precrossing has an odd and an even label to over- and undercrossings?
(5) Conclude that your method for choosing crossing types works in general.
(6) Conclude that given any knot projection, you can turn it into alternating projection by changing a finite number of crossings from over- to undercrossings or vice versa (careful! This will almost certainly change the knot!)? In a projection with n crossings, what is the largest number of crossings you should need to change?
(7) **Bonus!** Show that you can always choose over- and undercrossings in a forgetful projection in such a way that the resulting knot projection is a projection of the unknot.

3.1.3. *Pretzel links.*

Definition 3.14. Let p, q and r be integers. A **3-strand pretzel link** $P_{p,q,r}$ is constructed as follows: take three pairs of string segments and arrange them vertically. Twist the bottom ends of the first pair p times (counterclockwise if $p > 0$ and clockwise if $p < 0$). Twist the bottom ends of the second pair q times and the bottom end of the third pair r times. After twisting the pairs, connect the ends of the three pairs as shown below

Example 3.15.

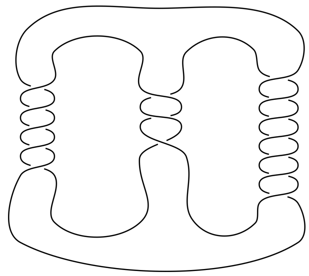
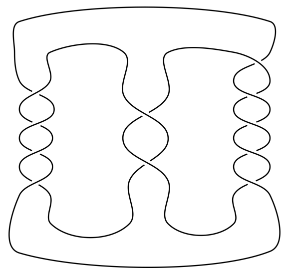


FIGURE 14. The pretzel link $P_{5,-3,7}$

Exercise 3.16.

(1) Which pretzel link is shown below?



(2) Draw the pretzel link $P_{-3,1,0}$.

Exercise 3.17.

(1) How many components do our three examples of pretzel links have? Are any of them knots?
(2) What is the largest number of components a 3-strand pretzel link can have?
(3) For which values of p, q and r is the pretzel link $P_{p,q,r}$ a knot?
(4) More generally, how does the number of components of $P_{p,q,r}$ depend on the choice of p, q and r?
(5) **Bonus!** Answer these same questions for 2-strand and 4-strand pretzel links.
(6) **Bonus bonus?** What about for n-strand pretzel links??

Exercise 3.18.

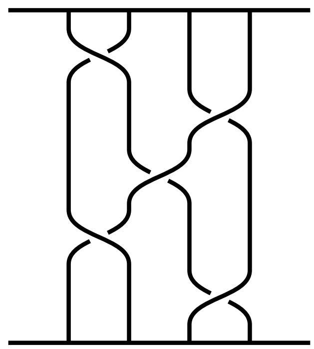
(1) Show that for positive integers p and r, the link $P_{p,-1,r}$ is alternating.
(2) For what choices of p, q, r is $P_{p,q,r}$ alternating?

3.1.4. *Braids.* At first it might seem like this topic is a little unrelated: there are no knots in sight! It turns out that braids are a nice way to describe knots, and are incredibly interesting in their own right!

Definition 3.19. A **braid** is a set of n strings, all attached to a horizontal bar at the top and at the bottom, in such a way that the strings can tangle around each other, but must never turn back up: each string travels from the top bar to the bottom bar.

Note: another way to say this is that each string hits any horizontal plane between the bars exactly once.

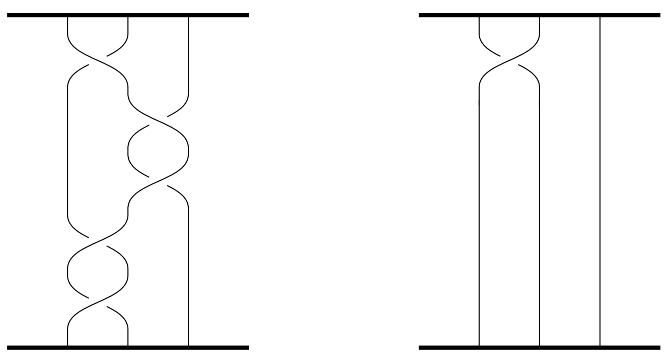
Example 3.20. A braid:



Exercise 3.21. Can a string in a braid have a self-loop? Why or why not?

Definition 3.22. We say that two braids are **equivalent** if we can rearrange the strings in the two braids to look the same without pasising strings through one another or themselves, while keeping the bars fixed and without cutting or detaching the strings.

Example 3.23. The following two braids are equivalent:



So, what does all this have to do with knots and links?

Definition 3.24. A **braid closure** is the knot or link obtained by pulling around the bottom bar and gluing it to the top bar so that the strands match up (then erasing the bar).

Therefore any braid defines a knot or link. Very interestingly, the converse is also true: namely any knot is the closure of a braid. This is a theorem of Alexander.

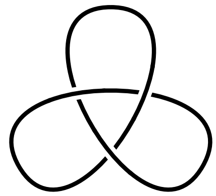
4. Tabulating and classifying knots

4.1. **Crossing number.** This invariant is a good way for determining the complexity of a knot, and will help us create a table of all known distinct knots.

Definition 4.1. The **crossing number** of a knot K, denoted $c(K)$, is the least number of crossings that occur in any projection of the knot.

Example 4.2. Since the unknot has a (well-known) projection with no crossings, the crossing number of the unknot U is 0: $C(U) = 0$.

Example 4.3. The trefoil T has a well-known projection with 3 crossings. I claim that this is the smallest number of crossings that a projection of the trefoil can have.

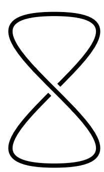


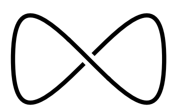
As with almost anything in knot theory, this is quite a difficult quantity to pin down, since it requires us finding some sort of minimal projection, in which the number of crossings is minimal. If we find a projection of K with n crossings, then all we can say is that the crossing number is at most n: $c(K) \leq n$. One way to show that it is indeed n would be to show that the knot is distinct from any knot with fewer crossings (again, this can be **very** difficult in general). Let's do this for the trefoil.

Claim 4.4. The trefoil has crossing number 3.

Proof. We have seen that the unknot is the only knot with crossing number zero.

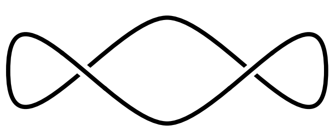
What knots can we get with crossing number one? There are only two possibilities (up to rotation and planar isotopies):

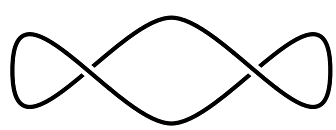




These are both just projections of the unknot (we have Reidemeister 1 moves), so in fact the crossing numbers of these knots is simply 0: there are no knots with crossing number 1!

How about for crossing number two? We can check case by case by drawing two crossings and then connecting them up in all the possible ways. (This is an exercise!) The possibilities, up to rotation and isotopies are



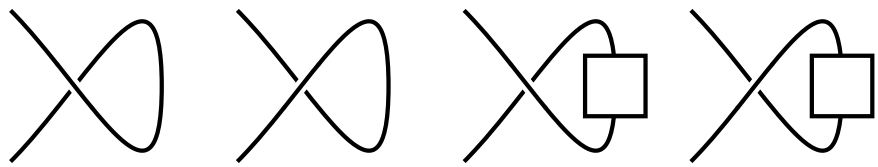


and links (which we disregard here since we are only interested in knots). Again, these are just projections of the unknot. Therefore there are no knots with crossing number 2.

From our usual projection, we have seen that the trefoil has crossing number at most 3. We have also seen that the trefoil is not trivial (it is tricolourable while the unknot is not!), so it cannot have crossing number 0. Therefore the crossing number of the trefoil is indeed 3. (*Aside:* this confirms my claim made in the very beginning that the trefoil is the next simplest knot after the unknot!) ☐

As I mentioned earlier, this process of showing that a knot's projection produces the crossing number is generally very difficult, but in some cases it is possible to tell whether a projection is minimal (i.e. produces the crossing number).

Definition 4.5. A projection of a knot is **reduced** if it does not have any crossings of the following form, called **easily removed crossings**:



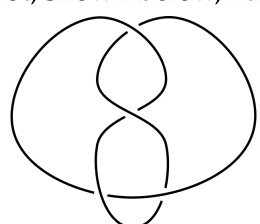
where the squares denote some piece of the knot.

This tells us that an easily removed crossing is some Reidemeister 1-type move that does not affect some part of the knot.

Theorem 4.6 (Kauffman, Murasugi, Thistlethwaite)**.** An alternating knot in a reduced alternating projection of n crossings has crossing number n.

Thus, in this special case for alternating knots, we can determine the crossing number more easily. Again, this is unfortunately more difficult in general.

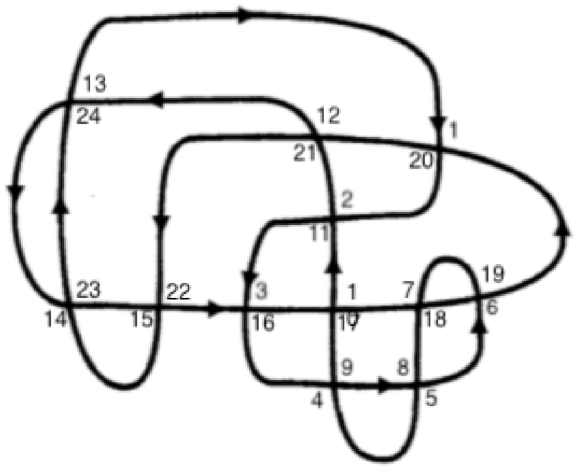
Exercise 4.7. Show that the figure 8 knot, shown below, has crossing number 4.



Exercise 4.8. Show that the following knot has crossing number 7:

4.2. **Dowker notation.** It turns out that the labelling process we used to prove that any forgetful projection could be turned into an alternating projection is another way to tabulate knots, known as Dowker notation. Remember, given an alternating projection, we had numbers on our projection at each crossing: one odd and one even. This of this as a **pairing** between the odd and even numbers from 1 to $2n$ where n is the number of crossings.

Consider our example of a forgetful projection from earlier, labelled as in the exercise:



Writing the odd numbers in order, we get the following pairing:

1	3	5	7	9	11	13	15	17	19	21	23
20	16	8	18	4	2	24	22	10	6	12	14

So we only really need to remember the sequence of even numbers, remembering that the odd numbers are in order:

20 16 8 18 4 2 24 22 10 6 12 14

Interestingly, we can go back the other way. Try to think of a systematic way to draw a knot from the one-line notation above!

4.3. **Tangles and Conway notation.** We introduce here a notation of John H. Conway that proved very useful in tabulating knots (Conway used his notation to list all prime knots with up to 11 crossings BY HAND, and missed only four of them!).

I have notes for this that I will add at some point... 25/07/19

5. Polynomial invariants

Polynomial invariants of knots and links are extremely powerful invariants that are also interesting in their own right. The first of these polynomial invariants was discovered by Alexander in the 1920s, and is known as the **Alexander polynomial**. Around 50 years later, John Conway determined a method for computing this polynomial, known as **skein relations**, which is what we'll use here.

In the 1980s, Vaughan Jones, from New Zealand, discovered another polynomial invariant now called (you guessed it!) the Jones polynomial. It was born from his work on Operator Algebras, an entirely unrelated area of mathematics, that happened to have a very similar relation that occurs in knot theory.

While there are other polynomial invariants that were discovered after Jones' work, most notably the HOMFLY-PT[1] polynomial, named for its discoverers[2]. We will focus on the Jones polynomial.

6. The bracket polynomial and the Jones polynomial

Let's begin by thinking about how we could define a polynomial invariant of knots and links. We'll follow some ideas of Louis Kauffman here. It turns out that what we will define is a **Laurent polynomial** rather than a regular polynomial, but don't let this worry you. The coefficients we will come up with will eventually be related to We will start by defining the **bracket polynomial**, where the bracket polynomial of a knot K is denoted $\langle K \rangle$ We will have some basic requirements:

First, we want the polynomial of the unknot to be 1:

Rule 1: $\left\langle \bigcirc \right\rangle = 1.$

Next let's start thinking about crossings. This is where the **skein relation** comes into play: we want to build our bracket polynomial from polynomials with fewer crossings, so we **resolve** the crossing, and write the bracket of a crossing as a linear combination of the brackets of the two possible resolutions:

Rule 2: $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + B \left\langle)(\right\rangle,$
where A and B are considered to be some variables.

Note that I could consider the opposite crossing: $\left\langle \times \right\rangle$, but really, this is the same as rotating my crossing 90 degrees to the right, so I would get a similar equation:

$$\left\langle \times \right\rangle = B \left\langle \asymp \right\rangle + A \left\langle)(\right\rangle .$$

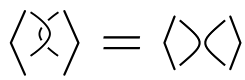
Turning my head to the right, this is the same equation as before, so I only really need to remember the first equation from rule 2.

Finally, we can imagine that by resolving one of these crossings, we obtain a link with multiple components, so we should have a rule explaining what to do once we have a link with a trivial component:

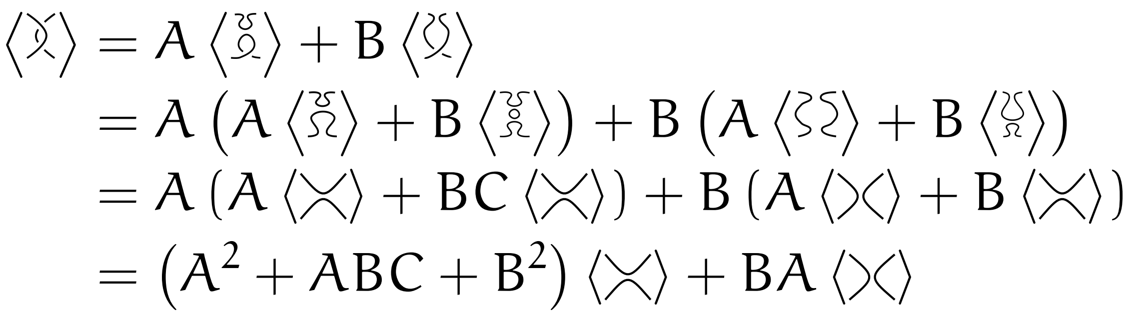
[1]Prof. Allen Knutson thinks this should instead be written "FLYPMOTH"
[2]Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, Przytycki and Traczyk

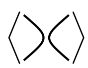
Rule 3: $\left\langle L \cup \bigcirc \right\rangle = C\langle L\rangle$,
where L is a link and C is a variable.

Now, if we want this bracket polynomial to be a link invariant, it must (invariably) be invariant under the Reidemeister moves! Let us begin with the second Reidemeister move. Remember, what we're trying to do here is impose the bracket relation



Resolving the two crossings, we find:

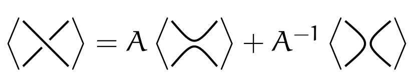


If we want this to be equal to the bracket , we need $BA = 1$, so that $B = A^{-1}$. Similarly, we need the coefficient in front of the  term to be zero, so using the above, we have

$$A^2 + AA^{-1}C + A^{-2} = 0,$$

so that $C = -(A^2 + A^{-2})$. Our updated rules are therefore:

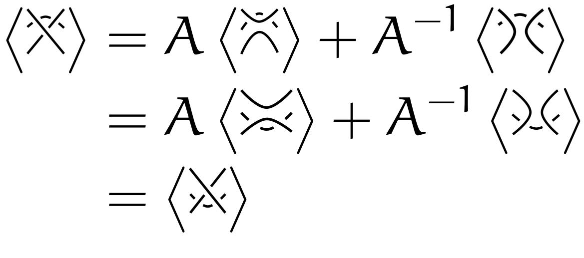
Rule 1: $\left\langle \bigcirc \right\rangle = 1.$

Rule 2: 

Rule 3: $\left\langle L \cup \bigcirc \right\rangle = -(A^2 + A^{-2})\langle L\rangle$

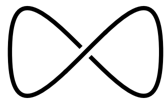
Exercise 6.1. Let L be a link consisting of n disjoint unknots. Compute its bracket polynomial.

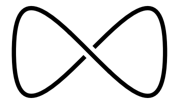
Let's check the third Reidemeister move, remember that we want to show that the bracket polynomial is invariant under this move:



This shows us that since we have forced the bracket polynomial to satisfy the second Reidemeister move (and ambient isotopy, namely in this case a rotation of the entire diagram by 180°), it now automatically is invariant under the third move, and we don't have to tweak the bracket polynomial for this move.

Exercise 6.2. Compute the bracket polynomial for the following links. Do these computations worry you? If yes, why?





Finally, let's see what happens when we apply an R1 move:

$$\begin{aligned} \langle\ \ \rangle &= A\langle\ \ \rangle + A^{-1}\langle\ \ \rangle \\ &= A(-A^2 - A^{-2})\langle\ \ \rangle + A^{-1}\langle\ \ \rangle \\ &= -A^3\langle\ \ \rangle \end{aligned}$$

$$\begin{aligned} \langle\ \ \rangle &= A\langle\ \ \rangle + A^{-1}\langle\ \ \rangle \\ &= A\langle\ \ \rangle + A^{-1}(-A^2 - A^{-2})\langle\ \ \rangle \\ &= -A^{-3}\langle\ \ \rangle \end{aligned}$$

Uh-oh... The bracket polynomial needed to be invariant under an R1 move. But to do this, we would need to define $A = -1$, which is a bit pointless - we would just have numbers and no variables, and we wouldn't get any fun polynomial invariants this way. So what should we do? We should fix our bracket polynomial so that it remains invariant under the Reidemeister 1 move, without losing any of the freedom of our variable. To do this, let's remember a quantity that we computed for links a few weeks ago: the linking number. This time, we are going to compute a modified version of this linking number so that we count all crossings in a diagram, not just those between two different components. Recall, first we orient our knot or link, so that we have positive and negative crossings:

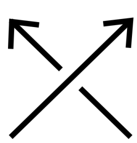


FIGURE 15. +1 crossing

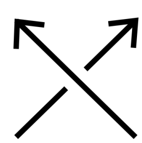


FIGURE 16. -1 crossing

Definition 6.3. Let L be a projection of an oriented link. Let p be the number of positive crossings in L and let n be the number of negative crossings in L. The **writhe** $w(L)$ of the link projection L is

$$w(L) = p - n$$

Example 6.4. The writhe of the following oriented knot K is 1:

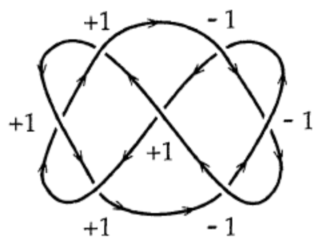


FIGURE 17. $w(K) = 4 - 3 = 1$

Exercise 6.5. Show that the writhe of a link diagram is invariant under Reidemeister 2 and 3 moves.

Exercise 6.6. What is the effect of a Reidemeister 1 move on the writhe of a link diagram?

Exercise 6.7. What is the effect of the choice of orientation on the writhe?

So, now we have that both the writhe and the bracket polynomial are affected only by the Reidemeister 1 moves, so we can use their effects to cancel each other out. Let us define the **beta-polynomial** (beta for better) of a link as follows:

$$\beta(L) = (-A^3)^{-w(L)}\langle L\rangle$$

Since both the writhe and the bracket polynomial are invariant under R2 and R3, the beta-polynomial is also. Now let's check what happens when we apply a Reidemeister 1 move. Let L' be the diagram of our link with the twist in it, and let L be the diagram of our link without the twist:

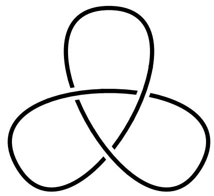
$$\begin{aligned}
\beta(L') &= (-A^3)^{-w(L')}\langle L'\rangle \\
&= (-A^3)^{-(w(L)+1)}\langle L'\rangle \\
&= (-A^3)^{-(w(L)+1)}((-A)^3\langle L\rangle) \\
&= (-A^3)^{-w(L)}\langle L\rangle \\
&= \beta(L)
\end{aligned}$$

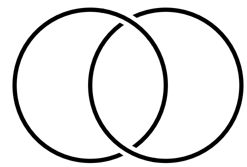
This tells us that our beta-polynomial is unaffected by R1! Therefore:

Theorem 6.8. The beta polynomial is an invariant of knots and links!

As we wanted.

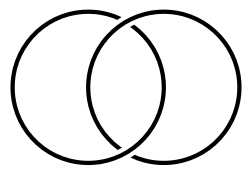
Exercise 6.9. Show that the beta-polynomial for the following links are $A^{-4} + A^{-12} - A^{-16}$ and $A^{-2} + A^{-10}$ respectively.





What effect does your choice of orientation have?

Exercise 6.10. Compute the beta-polynomial for the following link. What do you expect from the previous exercise?



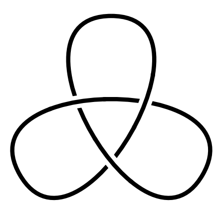
We can finally define the Jones polynomial!

Definition 6.11. The **Jones polynomial** is obtained from the beta-polynomial by replacing each A by $t^{-1/4}$.

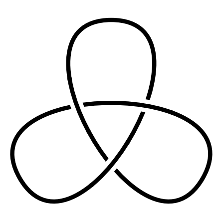
Exercise 6.12. Compute the Jones polynomial of the trefoil and the figure eight knot (*Hint: Show that the beta-polynomial of the figure-eight knot is* $A^8 - A^4 + 1 - A^{-4} + A^{-8}$).

7. A small lie, resolved

So far, I have been lying to you a little. I consistently refer to the trefoil as, well... THE trefoil. But it turns out this is not quite true. Consider the trefoil that I have been consistently drawing on the board:



And now consider what happens when I reverse all the crossings (namely, turn every over-crossing into an under-crossing and vice versa):



We say that the two trefoils are **mirror images** of each other. Now, I would not be lying to you in calling the trefoil the trefoil if these two mirror images were equivalent to each other, namely if I could come up with some ambient isotopy between them. This would mean that the mirror image projections would just be two different projections of the same knot. We say that a knot that is equivalent to its mirror image **amphichiral**.

An example of an amphichiral knot is the figure-eight knot, as shown by the sequence of Reidemeister moves below:

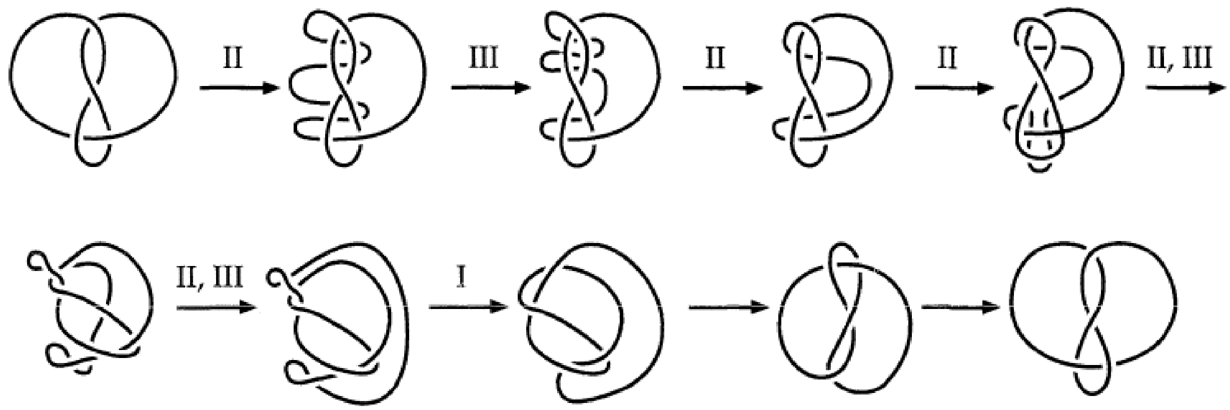


FIGURE 18. The figure eight knot is amphichiral

Exercise 7.1. Using a piece of string, show that the figure-eight knot is amphichiral.

By telling you that I have been lying to you this whole time, I am telling you that the trefoil is **not** amphichiral (we say that it is a **chiral knot**). We will prove this over the next few exercises, using the power of our shiny new polynomial invariants. To make notation simpler, we write the following: for a link L, the beta-polynomial for L "in the variable A" as $\beta_L(A)$.

Exercise 7.2. Let K be a knot and let K^* be its mirror image. Show that the beta-polynomial of K^* is just the beta-polynomial of K with each A replaced by A^{-1}. In our new notation, this amounts to showing: $\beta_{K^*}(A) = \beta_K(A^{-1})$. *Hint: Show this for the bracket polynomial first!*

Exercise 7.3. Suppose now that K is amphichiral. What does this tell you about $\beta_K(A)$ and $\beta_K(A^{-1})$?

This tells us that the polynomial of an amphichiral knot must be **palindromic** (just like my name!), namely, the coefficients must be the same backwards or forwards, where we consider **all** the coefficients, including the zeros. Hence the coefficients should be symmetric about 0.

Exercise 7.4. Check that the beta-polynomial of the figure-eight knot is indeed palindromic.

Exercise 7.5. Check that the beta-polynomial of the trefoil is not palindromic and conclude. Confirm your conclusion by computing the beta-polynomial of the mirrored trefoil. What does this computation tell you?

8. Tying up some loose ends

Recall a few classes ago that we used the following theorem about alternating knots, first conjectured by the physicist Peter Guthrie Tait in the late 19th century:

Theorem 8.1. If K is a knot in a reduced, alternating projection with n crossings, then the crossing number of K is n.

namely, this projection determines the crossing number of the knot and we don't need to keep looking for knots with fewer crossings. This conjecture remained unproven until the mid 80s, when several topologists proved the theorem independently. We now have the techniques to prove the theorem ourselves! For now, we will prove part of this result, namely:

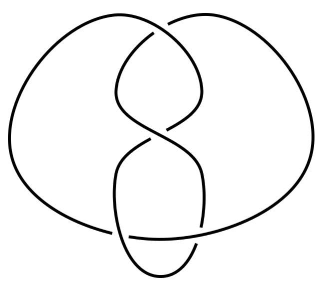
Theorem 8.2. Let K_1 and K_2 be any two reduced alternating knot projections of an alternating knot with n_1 and n_2 crossings respectively. Then $n_1 = n_2$.

We will rely on the second rule for the bracket polynomial, namely:

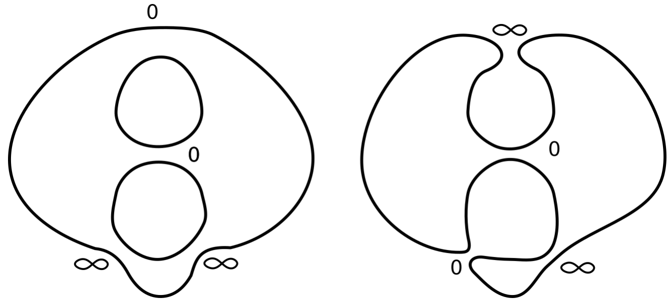
allowing us to compute the bracket polynomial for an n-crossing projection in terms of the polynomials for $n-1$-crossing projections.

Using our tangle notation, let us call the resolution with coefficient A the 0-resolution, and the resolution with the coefficient A^{-1} the ∞-resolution. Given a knot projection K, let us call a complete resolution of K, namely, a choice of 0- or ∞-resolution for each crossing of K a **state** of K. We call a **component** of a state any one of the unknots making up the state.

Example 8.3. Consider our usual projection of the figure-eight knot:



Two states for this projection are:



Exercise 8.4. Given a knot projection with n crossings, how many different states does the projection have?

Definition 8.5. Given a knot projection K and a state S of K such that S has i 0-resolutions and j ∞-resolutions, denote by $\mathrm{sgn}(S)$ (pronounced "total sign"), the monomial

$$\mathrm{sgn}(S) = A^i (A^{-1})^j$$

Exercise 8.6. What are the total signs of the example states for the figure-eight knot projection above?

Definition 8.7. The number of components of a state S of a knot projection is called the **size** of the state S, denoted $|S|$.

Exercise 8.8. What are the sizes of the example states for the figure-eight knot projection above?

I claim that we have the following useful formula for computing the bracket polynomial of a knot projection:

Claim 8.9. Given a knot projection with n crossings, the bracket polynomial for K is given by:

$$\langle K \rangle = \sum_S \mathrm{sgn}(S)(-A^2 - A^{-2})^{|S|-1}$$

Our aim now is to prove this claim.

Exercise 8.10.
(1) Use the formula in the claim to compute the bracket polynomial for the trefoil.
(2) Use the formula in the claim to compute the bracket polynomial for the figure-eight knot.
(3) Compare the polynomials you obtain here to the ones you computed previously using the definition.

Exercise 8.11. Using induction on the number of crossings in the projection, prove the claim.

Definition 8.12. Let K be a knot projection. The **span** of K, denoted $\mathrm{span}(K)$, is the maximum degree of the bracket polynomial of K minus the minimum degree of the bracket polynomial of K.

Exercise 8.13. Compute the span of the trefoil (both of them) and the figure-eight knot.

In a pleasing turn of events, despite the fact that the bracket polynomial is not a knot invariant, the span is:

Theorem 8.14. The span is a knot invariant.

Exercise 8.15. Prove the theorem. What is the only thing you need to check and why?

We'll come back to the span at the very end of our proof of the Tait conjecture. The next element that we'll need for the proof is the notion of a checkerboard colouring:

Definition 8.16. A **checkerboard colouring** of a knot or link projection K is a colouring by two colours of regions (including the "outside" region) of the forgetful projection such that each region is adjacent only to regions of opposite colours.

Note that there are two possible checkerboard colourings of any knot or link projection, where the two colours are simply inverted. From now on I call the colours "shaded" and "unshaded".

Example 8.17.

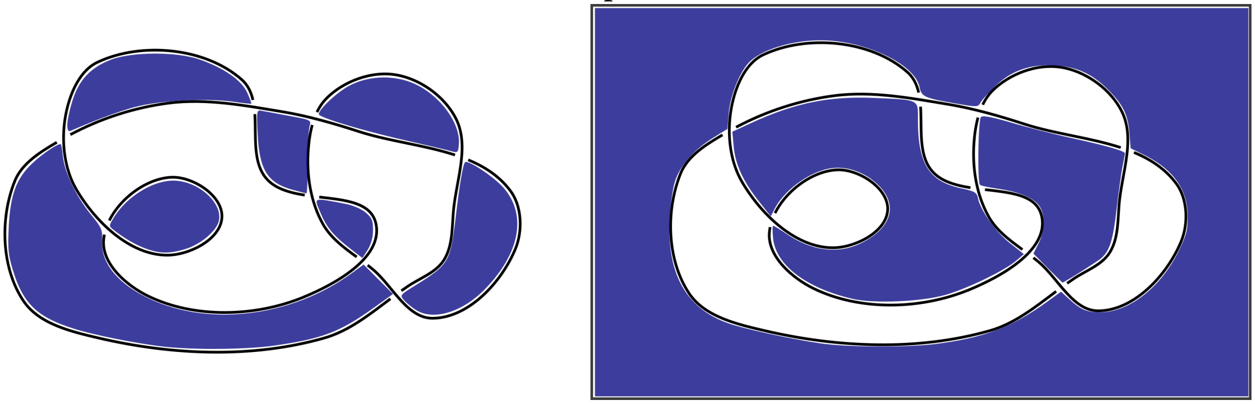


FIGURE 19. Two possible checkerboard colourings of a link projection

Now we choose the checkerboard colouring such that at a given crossing c, the 0-resolution connects the shaded regions around c.

Exercise 8.18. Show that in an alternating diagram, if you choose the checkerboard colouring above, the 0-resolution will connect the shaded regions at **every** crossing.

Example 8.19.

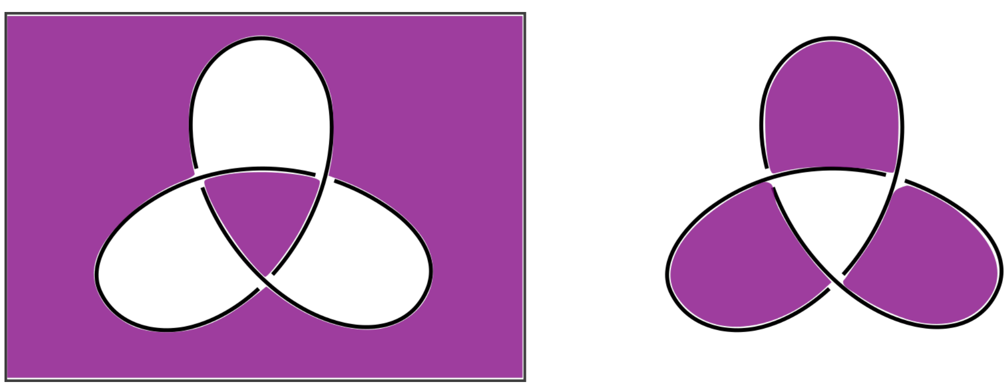


FIGURE 20. Checkerboard colourings of the left- and right-handed trefoils such that 0-resolutions connect the shaded regions

Lemma 8.20. Suppose that K is a reduced alternating knot projection with n crossings. Suppose that K is checkerboard coloured as described above. Then the highest and lowest degree terms in $\langle K \rangle$ are given by:

$$\text{max degree} = (-1)^{W-1} A^{n+2W-2}$$
$$\text{min degree} = (-1)^{B-1} A^{-n-2B+2}$$

where B is the number of shaded regions in the checkerboard colouring, and W is the number of unshaded regions.

Exercise 8.21. Determine the correct checkerboard colouring for the trefoil in our usual projection, and compute the contribution to the bracket polynomial of the state with only 0-resolutions. Check that this is the highest term in the bracket polynomial computed previously.

Exercise 8.22. Show that more generally, given the correct choice of checkerboard colouring of a reduced alternating knot diagram K, the maximum degree in the bracket polynomial of K is max degree $= (-1)^{W-1}A^{n+2W-2}$.

Exercise 8.23. Let K be as a bove. Let S_0 be the state with all 0-resolutions. Let S' be a state with exactly one ∞-resolution and the rest all 0-resolutions. Use the fact that K is **reduced** to show that the maximum degree of the contribution to the bracket polynomial of S' is strictly less than the maximum degree of the contribution of S_0. (*Hint: what part of the contribution of a single state is affected by the choice of resolution at the crossings?*)

Exercise 8.24. Put all the pieces together to show that the maximum degree term in $\langle K \rangle$ is indeed max degree $= (-1)^{W-1}A^{n+2W-2}$. Using the state with all ∞-resolutions, prove the analogous result for the minimum degree and complete the proof of the lemma.

From the lemma, we can now compute the span of the projection of a reduced, alternating knot projection K:

$$\text{span}(K) = \text{max degree } - \text{min degree} = n + 2W - 2 - (-n - 2B + 2) = 2n + 2(B + W) - 4$$

The last thing we need is an extremely useful and neat topological invariant, discovered in the mid-1700s by the great Leonhard Euler:

Definition 8.25. Let X be a surface, considered as a topological space (so we can bend it and stretch or compress it, but we can't cut it). Suppose that X consists of V vertices, E edges and F faces. The **Euler characteristic** of X, denoted $\chi(X)$, is defined by the formula:

$$\chi(X) = V - E + F$$

Note that these vertices, faces and edges must satisfy certain conditions, for example they must have common boundaries and cannot overlap. The really amazing thing is that this is a topological invariant: we can choose different edges and faces and vertices, or we can deform our topological space as much as we like, but we will still end up with the same Euler characteristic!

The example that we'll use is that of a sphere: the Euler characteristic of the sphere is 2: we can see this in a couple of different examples of the sphere (remember, we are in the realm of topology here!).

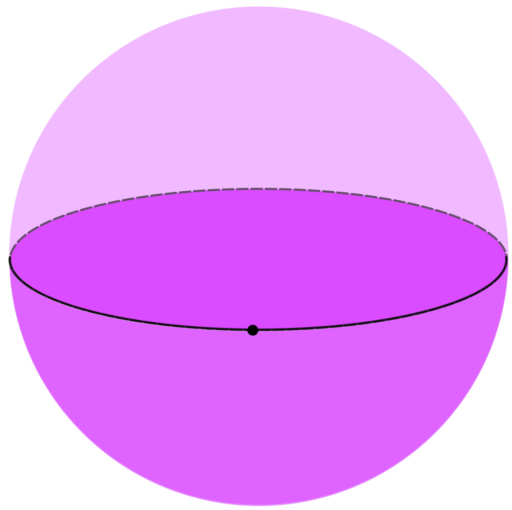


FIGURE 21. The sphere with one vertex, one edge and two faces

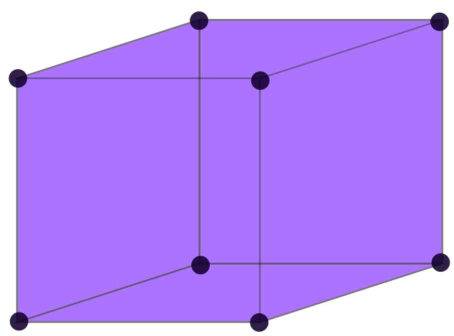


FIGURE 22. The sphere (topologically) with 8 vertices, 12 edges and 6 faces

In each case, we can compute the Euler characteristic of the sphere S. In case one:

$$\chi(S) = 1 - 1 + 2 = 2$$

and in case two:

$$\chi(S) = 8 - 12 + 6 = 2$$

as expected.

How does this relate to knots? For now, we will see our knots as living on the surface of a sphere - to do this we need to make it two-dimensional, so we will consider the forgetful projection of the knot instead. Then this decomposes our sphere into vertices, edges and faces!

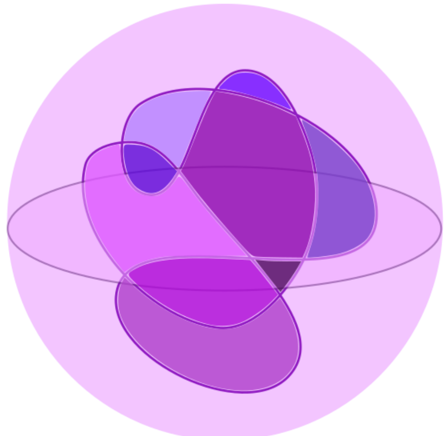


FIGURE 23. A forgetful projection of a knot on the sphere

We see in this example that the sphere has 6 vertices, 12 edges and 8 faces. So we get the Euler characteristic of the sphere again.

Exercise 8.26. Show that when you consider this decomposition of the sphere using a forgetful projection of a reduced, alternating knot projection K with n crossings, the number of vertices is the number of crossings n, the number of edges is $2n$ and the number of faces is $B+W$, where B is the number of shaded regions in a checkerboard colouring of K and W is the number of unshaded regions in the checkerboard colouring.

Using this exercise, we have that

$$\begin{aligned} 2 &= \chi(S) \\ &= V - E + F \\ &= n - 2n + B + W \\ &= B + W - n \end{aligned}$$

Therefore, in a reduced, alternating knot projection, we have $B + W = 2 + n$, and hence, the span of a reduced alternating knot projection K is

$$\text{span}(K) = 2n + 2(B + W) - 4 = 2n + 2(2 + n) - 4 = 4n$$

We have that the span of K is $4n$ and we have already seen that the span is a knot invariant. Therefore, any reduced, alternating knot projection of an alternating has the same number of crossings!

The final exercise (!) of this class (!) is to show that reduced alternating projections have fewer crossings than other projections.

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[Adams, 2004] Adams, C. (2004). *The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots*. American Mathematical Society.

[Johnson, 2017] Johnson, I. (2017). *An Interactive Introduction to Knot Theory*. Dover Publications, Inc.