Current algebras, categorification and annular Khovanov homology

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Chapter 1

Preliminaries

1.1 Introduction

Lie algebras and quantum groups have been widely studied in the context of representation theory, while invariants of knots and links are objects of significant importance in topology. There exist connections between these objects that can be of interest for both of these fields. One example of such a connection is Reshetikhin and Turaev's construction of link invariants generalising the Jones polynomial from representations of quantum groups [RT90]. The Reshetikhin-Turaev invariants can be enriched by categorification: replacing representations of the quantum group by categories equipped with a quantum group action and replacing polynomial invariants by homological link invariants. Our objective in this thesis is to describe several of the categorifications which are important in both representation theory and topology and to study some of the new structure that appears in the categorified world.

On the topological side, one of the triumphs of categorification is Khovanov homology, a categorification of the Jones polynomial to a homological link invariant first introduced by Mikhail Khovanov in the late 1990s [Kho00]. Khovanov homology is strictly stronger than its decategorified counterpart [BN02] and can detect the unknot [KM11]. Khovanov has also made a significant contribution to the categorification of representations of quantum groups [HK01] [HK06], and moreover of quantum groups themselves [KL09].

The added algebraic structure of homological link invariants can furthermore can make these invariants interesting objects from the perspective of representation theory in their own right. Another aim of this thesis is to demonstrate this relationship in the case of a homological invariant of annular links, which arises as representations of a particular Lie algebra, known as a current algebra.

Chapter 1 introduces notation, definitions and basic theorems and properties that will be used in later chapters. In particular, we give an overview of the representation theory of semisimple Lie algebras and quantum groups, following [FH91], [Hum72] and [Lus93]. This representation theory is used extensively in the remaining chapters. The main result is a classification of the irreducible finite-dimensional representations of the Lie algebra $\mathfrak{sl}_2$, and a detailed proof of the complete reducibility of its finite-dimensional representations.

In chapter 2, we study particular types of current algebras and their representations, such as the polynomial current algebras examined in [CG07], and representation algebras $\mathfrak{g}(V)$. We give a description of the representations of particular examples of these algebras that arise in our study of annular Khovanov homology using quiver representations. We first give a proof of a theorem of Loupias [Lou72] on quiver representations of the Lie algebra $\mathfrak{sl}_2(V_1)$. We then state and prove an analogous result for the current algebra $\mathfrak{sl}_2^-(V_2)$, which reappears in chapter 4 in our discussion of annular Khovanov homology.

In chapter 3 we examine the work of Khovanov and Huerfano in [HK01] and [HK06], presenting their work on categorifications of representations of quantum groups. This chapter is largely independent of the preceding chapters, however some of the algebraic objects studied here are encountered in earlier sections. For example, the zigzag algebra plays a significant role in both the representation theory of current algebras in chapter 2 as well as the construction of a categorification of the adjoint representation in this chapter.

The principal objective of chapter 4 is to relate knot homology to the representation theory studied in previous chapters, giving a representation theoretic presentation of annular Khovanov homology, a homology theory of knots and links in the solid torus defined by Asaeda, Przytycki and Sikora in [APS04]. In particular, the main theorem 4.4.1 of this chapter defines an explicit action of the current algebra $\mathfrak{sl}_2^-(V_2)$ on annular Khovanov homology. We give a complete and independent proof of this theorem, originally due to Grigsby-Licata-Wehrli [GLW]. Of particular interest here is the relationship between the current algebra action and Lee's deformation of Khovanov homology, as seen in [Lee05].

1.2 Notation

Calligraphic scripts denote categories ($\mathcal{C}$, $\mathcal{D}$) and functors ($\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$).

Cursive scripts $\mathscr{A}$, $\mathscr{B}$ denote finite-dimensional k-algebras, for algebraically closed field k of characteristic zero.

The letters V and W denote representations or vector spaces. Vector spaces are finite-dimensional and over the field $\mathbb{C}$ unless otherwise specified.

Letters M and N denote modules over some algebra $\mathscr{A}$.

$\mathfrak{g}$ denotes a Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra.

Greek letters $\alpha, \beta, \lambda, \mu$ denote weights of a representation of a Lie algebra.

Categories are written in bold. For example, **Vect** is the category of finite-dimensional vector spaces and linear maps, and $\mathscr{A}$-**Mod** is the category of modules over an algebra $\mathscr{A}$.

A subscript $_g$ before a category indicates that the category has graded objects, for example $_g$**Vect** is the category of graded vector spaces and grading-preserving linear maps.

1.3 Representation theory

Let $\mathscr{A}$ be a finite-dimensional k-algebra, for algebraically closed field k of characteristic zero. Let (ρ, V) denote a finite-dimensional representation of $\mathscr{A}$ where $\rho : \mathscr{A} \to \mathrm{End}(V)$ is an algebra homomorphism, though we will generally omit the homomorphism ρ in our description of representations. An algebra $\mathscr{A}$ is *semisimple* if all its representations split into a direct sum of irreducible representations.

The following theorems are used without further comment throughout this thesis:

Lemma 1.3.1 (Schur)**.** *Let $\mathscr{A}$ be an algebra over some field k, that is not necessarily algebraically closed. Let V and W be representations of $\mathscr{A}$ and $\phi : V \to W$ a non-zero map of representations of $\mathscr{A}$. Then*

(i) If V is irreducible then ϕ is injective.

(ii) If W is irreducible then ϕ is surjective.

Corollary 1.3.2 (Schur's lemma for algebraically closed fields)**.** *Let $\mathscr{A}$ be a k-algebra for algebraically closed k and let V be a finite-dimensional irreducible representation $\mathscr{A}$, with $\phi \in End_{\mathscr{A}}(V)$, then ϕ is a scalar operator: $\phi = \lambda \cdot id_V$ for some $\lambda \in k$.*

Theorem 1.3.3 (Krull-Schmidt)**.** *Any finite-dimensional representation of a finite-dimensional algebra $\mathscr{A}$ can be decomposed uniquely (up to isomorphism and reordering of summands) into a direct sum of indecomposable representations.*

Definition Let M be a finitely generated $\mathscr{A}$-module. Then a *Jordan-Hölder series* (or *composition series*) for M is a chain

$$0 = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_{n-1} \subset M_n = M$$

of submodules of M such that M_{i+1}/M_i is a simple module for all $i < n$.

Theorem 1.3.4 (Jordan-Hölder)**.** *Let M be a finitely-generated $\mathscr{A}$-module with Jordan-Hölder series $0 = M_0 \subset M_1 \subset M_2 \subset \ldots M_{n-1} \ldots M_n = M$ and $0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_{k-1} \subset N_k = M$. Then $n = k$ and there exists some permutation $\sigma \in S_{n-1}$ such that $N_{i+1}/N_i \cong M_{\sigma(i)+1}/M_{\sigma(i)}$ for all $0 \leq i < n$.*

1.4 Quivers and their representations

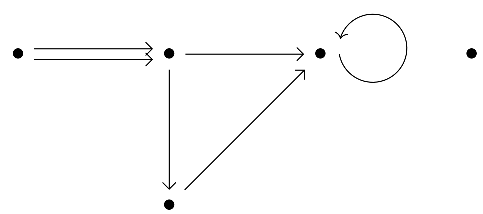
Quivers are particularly simple objects and their representations are very visual, making them ideal for studying other categories of representations.

Definition A *quiver* Q is a directed graph, consisting of a set of vertices V_i, $i \in I$ and a set of directed edges, denoted by arrows, between them a_j, $j \in J$, where I and J are (not necessarily finite) index sets.

Given an arrow a_j from V_i to V_k, we call V_i the *source* and V_k the *target* of a_j. A *path* in Q is a finite word in the a_j, $\omega = a_{j_k} a_{j_{k-1}} \dots a_{j_1}$, $j_n \in J$ such that the target of a_{j_n} is the source of $a_{j_{n+1}}$ for each $n \in \{1, \dots, k-1\}$.

Remark. *There is no restriction on the edges, namely we may have multiple edges between two vertices, and loops from a vertex to itself are also allowed. We also do not require the graph to be connected.*

Example. *An example of a quiver is*



Definition A *quiver representation* of a quiver Q is a pair (M, ϕ), where $M = \{M_i \mid i \in I\}$ is a set of finite-dimensional vector spaces, one for each vertex of Q, and

$$\phi = \{\phi_j : M_i \to M_k \mid i, k \in I, j \in J\}$$

is a set of linear maps between the M_i, one for each edge in Q.

A quiver representation can be considered as a representation of an associative algebra:

Definition The *path algebra* A_Q of a quiver Q is an associative algebra with basis given by the oriented paths of Q, including trivial paths v_i, $i \in I$ corresponding to the vertices of Q. Multiplication is concatenation of paths, with the path ab consisting of the path b followed by the path a. If the ending vertex of a path b is not the same as the starting vertex of a path a, then we define the product ab to be zero.

Theorem 1.4.1. *The category of representations of a quiver Q is equivalent to the category of representations of the path algebra A_Q.*

1.5 Algebras

The following algebras will be studied in later chapters.

1.5.1 Hopf algebras

Many of the algebras of interest to us here, such as the universal enveloping algebra of a Lie algebra, and quantum groups are examples of Hopf algebras. Hopf algebras are bialgebras. This structure is of particular use when considering tensor products of representations: given a Hopf algebra $\mathscr{H}$ and two representations V and W of $\mathscr{H}$, one can regard the tensor product $V \otimes W$ as a representation of $\mathscr{H}$ itself, not just as a representation of $\mathscr{H} \otimes \mathscr{H}$. This is due to a comultiplication map $\Delta : \mathscr{H} \otimes \mathscr{H} \otimes \mathscr{H}$. We introduce some of this structure here.

Let $\mathscr{A}$ and $\mathscr{C}$ be k-linear spaces for some field k.

Definition The triple $(\mathscr{A}, \mu, \iota)$, where the multiplication map $\mu : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ and unit $\iota : k \to \mathscr{A}$ are k-module homomorphisms, is a k-*algebra* if the following diagrams commute:

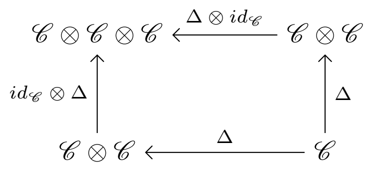
$$\begin{array}{ccc} \mathscr{A}\otimes\mathscr{A}\otimes\mathscr{A} & \xrightarrow{\mathrm{id}_{\mathscr{A}}\otimes\mu} & \mathscr{A}\otimes\mathscr{A} \\ {\scriptstyle \mu\otimes \mathrm{id}_{\mathscr{A}}}\downarrow & & \downarrow{\scriptstyle\mu} \\ \mathscr{A}\otimes\mathscr{A} & \xrightarrow{\mu} & \mathscr{A}\end{array}$$

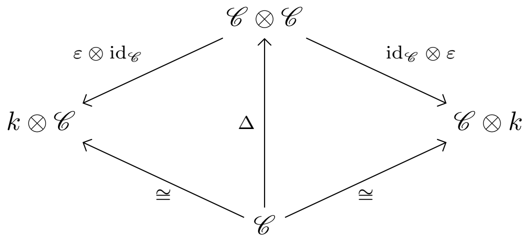
$$\begin{array}{ccccc} & & \mathscr{A}\otimes\mathscr{A} & & \\ & \nearrow^{\iota\otimes\mathrm{id}_{\mathscr{A}}} & & \nwarrow^{\mathrm{id}_{\mathscr{A}}\otimes\iota} & \\ k\otimes\mathscr{A} & & \downarrow{\scriptstyle\mu} & & \mathscr{A}\otimes k \\ & \searrow_{\cong} & & \swarrow_{\cong} & \\ & & \mathscr{A} & & \end{array}$$

where the isomorphisms $k \otimes \mathscr{A} \to \mathscr{A}$ and $\mathscr{A} \otimes k \to \mathscr{A}$ are given by left scalar multiplication. The first diagram shows that multiplication under μ is associative, and the second diagram illustrates the *unit law.*

We can also define the dual of a k-algebra:

Definition The triple $(\mathscr{C}, \Delta, \varepsilon)$ where $\Delta : \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$, called the comultiplication map, and $\varepsilon : \mathscr{C} \to k$, called the counit, are k-module homomorphisms, is called a *coalgebra* if the following diagrams commute:





where the first diagram shows that the comultiplication map Δ is coassociative and the second demonstrates the *counit law.*

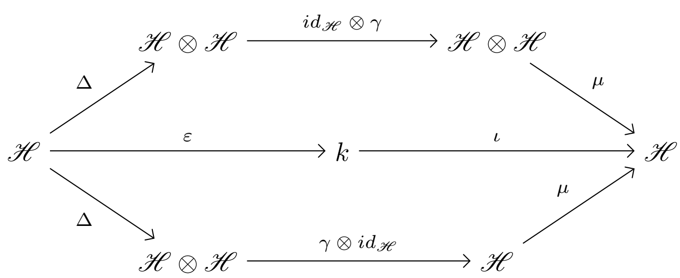
We may also consider an object that combines the structure of an algebra and a coalgebra:

Definition A *bialgebra* over k is a 5-tuple $(\mathscr{B}, \mu, \iota, \Delta, \varepsilon)$ such that

1. $(\mathscr{B}, \mu, \iota)$ is an algebra
2. $(\mathscr{B}, \Delta, \varepsilon)$ is a coalgebra
3. Δ and ε are k-algebra homomorphisms.

We are now able to define a Hopf algebra:

Definition A *Hopf algebra* is a 6-tuple $(\mathscr{H}, \mu, \iota, \Delta, \varepsilon, \gamma)$ such that $(\mathscr{H}, \mu, \iota, \Delta, \varepsilon)$ is a bialgebra and $\gamma : \mathscr{H} \to \mathscr{H}$ is a k-module homomorphism called the antipode such that the following diagram commutes:



1.5.2 Superalgebras

Definition A *superalgebra* over a field k (here, k will always be the field $\mathbb{C}$ of complex numbers) is a $\mathbb{Z}_2$-graded algebra, namely an algebra $\mathcal{A}$ that has a decomposition

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$$

together with a bilinear operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ that preserves the grading as follows: $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j}$ where we consider i, j and $i + j$ as elements of $\mathbb{Z}_2$.

We will call an element a in $\mathcal{A}$ *even* if $a \in \mathcal{A}_0$ or *odd* if $a \in \mathcal{A}_1$. We further define the *parity* $p(a)$ of a homogeneous element a of $\mathcal{A}$ as

$$p(a) = \begin{cases} 0 & \text{if } a \in \mathcal{A}_0 \\ 1 & \text{if } a \in \mathcal{A}_1 \end{cases}$$

In the case of Lie algebras:

Definition A *Lie superalgebra* is a Lie algebra $\mathfrak{g}$ with a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that the Lie bracket satisfies the following condition:

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$$

where we again consider i, j and $i + j$ as elements of $\mathbb{Z}_2$.

Any element $x \in \mathfrak{g}$ decomposes uniquely into homogeneous parts $x = x_0 + x_1$ where x_0 is even and x_1 is odd.

The Lie bracket now satisfies some slightly modified relations:

$$[x, y] = -(-1)^{p(x)\cdot p(y)}[y, x]$$

for homogeneous elements x and y, so that if either x or y is even, we have our usual Lie bracket condition. The Jacobi identity for homogeneous elements x, y and z in $\mathfrak{g}$ becomes:

$$[[x, y], z] + (-1)^{p(x)(p(y)+p(z))}[[y, z], x] + (-1)^{(p(x)+p(y))p(z)}[[z, x], y] = 0.$$

The $\mathbb{Z}_2$-grading becomes apparent when considering the universal enveloping algebra of a Lie superalgebra $\mathfrak{g}$: $U(\mathfrak{g})$ is the associative algebra generated by the elements of $\mathfrak{g}$, modulo the relation $[x, y] = xy - (-1)^{p(x)p(y)}yx$ for all homogeneous elements x and y in $\mathfrak{g}$. Hence, if both x and y are odd, then the Lie bracket of x and y in $U(\mathfrak{g})$ is given by $[x, y] = xy + yx$.

1.6 Homomorphism spaces in graded categories

Given a category $\mathcal{C}$ with $\mathbb{Z}$-graded objects, for example the category of graded vector spaces, morphisms are defined to be grading-preserving, so that any $f : M \to N$ satisfies $f(M_n) \subset N_n$

for all integers n. Hence, the homomorphism space $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is a vector space consisting of all grading-preserving morphisms. Another homomorphism space of interest is the graded homomorphism space $\mathrm{HOM}_{\mathcal{C}}(M, N) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(M\{k\}, N)$. This associates a graded vector space to pairs of objects in $\mathcal{C}$, and will often introduce interesting ties between objects and their categorifications.

1.7 The Grothendieck group of a category

In chapters 4 and 5 we will be discussing a process called decategorification: passing from an object to one with less structure. In given contexts, decategorification has a specific meaning. If the original object is a category then decategorification consists of taking the Grothendieck group of the category. This is defined as follows:

Definition Let $\mathcal{C}$ be an abelian category. Then the *Grothendieck group* $K(\mathcal{C})$ of the category $\mathcal{C}$ is the abelian group generated by isomorphism classes $[M]$ of objects M in $\mathcal{C}$ modulo the relation $[M] = [M_1] + [M_2]$ if there exists an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

There is a similar definition for additive categories:

Definition Let $\mathcal{C}$ be an additive category. Then the *split Grothendieck group* $K_0(\mathcal{C})$ is the abelian group generated by isomorphism classes $[M]$ of objects M in $\mathcal{C}$ modulo the relation $[M_1 \oplus M_2] = [M_1] + [M_2]$ for all objects M_1 and M_2 in $\mathcal{C}$.

Suppose further that objects in the category $\mathcal{C}$ are $\mathbb{Z}$-graded, so that $M = \oplus_{n\in\mathbb{Z}} M_n$. Then the shift functor $\{k\}$ shifts the grading of objects in $\mathcal{C}$ up by some integer k: $M_n\{k\} = M_{n-k}$. This grading gives the Grothendieck group of $\mathcal{C}$ the structure of a $\mathbb{Z}[q, q^{-1}]$-module by defining $[M\{k\}] = q^k[M]$ for any object M in $\mathcal{C}$ and any integer k.

1.8 Semisimple Lie algebras and quantum groups

The language and classical results in Lie theory are used extensively in later chapters. For this reason, we introduce the main results in the representation theory of semisimple Lie algebras and quantum groups. Proofs of the following are well-known, and can in particular be found in [FH91] and [Hum72]. The exception to this general overview is the section on the Lie algebra $\mathfrak{sl}_2\mathbb{C}$, for which we give a complete description of irreducible finite-dimensional representations and give a detailed proof of the complete reducibility of finite-dimensional representations.

1.9 Introductory example: $\mathfrak{sl}_2$

The representation theory of the Lie algebra $\mathfrak{sl}_2$ have a very simple description but is also an archetype for the representation theory of a much larger class of Lie algebras. Our aim here is to study the finite dimensional representations of the Lie algebra $\mathfrak{sl}_2$ over the complex numbers. In so doing, we give a complete proof of the following two theorems:

Theorem 1.9.1. *Up to isomorphism, there exists a unique finite-dimensional irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ with highest weight n for every nonnegative integer n.*

Theorem 1.9.2. *The finite dimensional representations of $\mathfrak{sl}_2$ are completely reducible.*

1.9.1 The Lie algebra $\mathfrak{sl}_2\mathbb{C}$

As a set, $\mathfrak{sl}_2\mathbb{C}$ is the set of traceless matrices with complex entries:

$$\mathfrak{sl}_2\mathbb{C} = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \middle| \; a, b, c \in \mathbb{C} \right\}$$

Considering $\mathfrak{sl}_2$ as a Lie algebra, we use the fact that $\mathfrak{sl}_2$ is an associative algebra with multiplication given by matrix multiplication, so that the Lie bracket is given by

$$[x, y] = xy - yx \text{ for all } x, y \in \mathfrak{sl}_2$$

As a complex vector space, $\mathfrak{sl}_2$ is three-dimensional, with standard basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and relations

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h$$

Thus a representation of $\mathfrak{sl}_2$ is a vector space V together with linear operators E, F and H acting on V such that

$$HE - EH = 2E \qquad HF - FH = -2F \qquad EF - FE = H$$

1.9.2 Irreducible representations

The operator H plays a critical role in the classification of irreducible representations of $\mathfrak{sl}_2$.

Action of $\mathfrak{sl}_2$ on generalised eigenspaces of H

Let V be a finite-dimensional representation of $\mathfrak{sl}_2$ over $\mathbb{C}$. Then H has at least one eigenvalue on V since $\mathbb{C}$ is algebraically closed, and there is a basis for V in which H is upper-triangular.

Furthermore, H has finitely many eigenvalues since V is finite-dimensional. Consequently, let λ be the eigenvalue of H with largest real part, called the highest weight of the representation. More generally a *weight* is an eigenvalue of H, and this terminology is extended to eigenspaces and eigenvectors.

Let $\overline{V}(\lambda)$ be the generalised weight space of H associated to the highest weight λ, and let $v \in \overline{V}(\lambda)$, so that there exists some $k \in \mathbb{N}$ such that $(H - \lambda I)^k v = 0$. I claim that the action of E on this weight space is trivial. First, consider the action of E on any generalised weight space. Let μ be a weight on V, and w some generalised weight vector associated to μ. Then there exists some $n \in \mathbb{N}$ such that $(H - \mu I)^n w = 0$. Consider $(H - \mu I)^n E$. From the relation $HE - EH = 2E$, we have $H^k E = E(H + 2)^k$ for all $k \in \mathbb{N}$. This is seen by induction on k. Then, from a binomial expansion

$$(H - \mu I)^n E = E(H + 2I - \mu I)^n = E(H - (\mu - 2)I)^n$$

From this, we see that

$$(H - (\mu + 2)I)^n Ew = E(H + 2I - (\mu + 2)I)^n w = E(H - \mu I)^n w = 0$$

Thus, Ew is again a generalised eigenvector for H, with weight $\mu + 2$. Returning to $v \in \overline{V}(\lambda)$, Ev is a generalised eigenvector for H with weight $\lambda + 2$. But λ was assumed to have the largest real part, so the eigenspace associated to $\lambda + 2$ must be trivial and $E|_{\overline{V}(\lambda)} = 0$.

Similarly, taking μ as above and following the same process, the relation $HF - FH = -2F$ leads to

$$(H - (\mu - 2)I)^n Fw = 0$$

so that Fw is also a generalised eigenvector for H, with associated eigenvalue $\mu - 2$.

Consider the sequence of vectors in V of the form $F^j v$ for $j \in \mathbb{N} \cup \{0\}$ and $v \in \overline{V}(\lambda)$. Each of the $F^j v$ is a generalised eigenvector for H, with associated weight $\lambda - 2j$, which are clearly all distinct. Thus the $F^j v$ form a linearly independent set in V, and by finite-dimensionality of V there must exist some $M \in \mathbb{N}$ such that $F^M v = 0$. Let N be the smallest integer such that $F^N v = 0$. Since $F^j V = F^{j-N} F^N v$ with $j - N > 0$ for all $j > N$, $F^j v = 0$ for all $j > N$.

This process shows that that V contains a submodule

$$W^\lambda = \bigcup_\mu V_\mu$$

where $\mu = \lambda - 2k$ for $k = \{0, 1, \ldots, N - 1\}$ and V_μ the generalised eigenspace for H associated to μ. For any such weight μ and any $w \in V_\mu$, $Ew \in V_{\mu+2} \subset W$, or $Ew = 0$ for $\mu = \lambda$, and $Fw \in V_{\mu-2} \subset W$, or $Fw = 0$ if $\mu = \lambda - 2(N - 1)$. Thus E and F shift generalised eigenvectors up or down to successive eigenspaces in W.

Any finite-dimensional representation will contain a submodule of the form W^λ, we shift our focus to the highest weight λ and show that λ is not only real, but integral and equal to $N - 1$.

Claim 1.9.3. $EF^j = F^j E + j(H + (j - 1)I)F^{j-1}$

Proof. The case $j = 1$ holds from the relations in $\mathfrak{sl}_2$. Applying the relation $HF - FH = -2F$ successively, $F^j H = (H + 2j)F^j$ for $j \geq 1$. Using this relation,

$$\begin{aligned} EF^{j+1} &= EF^j F \\ &= (F^j E + j(H + (j-1)I)F^{j-1})F \\ &= F^j(FE + H) + j(H + (j-1)I)F^j \\ &= F^{j+1}E + (H + 2j + jH + j(j-1)I)F^j \\ &= F^{j+1}E + (j+1)(H + jI)F^j \end{aligned}$$

By induction on j, the claim holds on $\mathbb{N}$. □

Applying this to V, from the fact that E acts trivially on the highest weight space V_λ,

$$EF^j v = j(H + (j-1)I)F^{j-1}v$$

Letting $j = N$,

$$0 = EF^N v = N(H + (N-1)I)F^{N-1}v$$

By assumption, $F^{N-1}v \neq 0$ since N is the smallest integer such that $F^N v = 0$ and $N \in \mathbb{N}$, so $(H + (N-1)I)F^{N-1}v = 0$, and $F^{N-1}v$ is an eigenvector for H, with eigenvalue $-(N-1)$. However, $F^{N-1}v$ is a (generalised) eigenvector for H with eigenvalue $\lambda - 2(N-1)$. Therefore $\lambda - 2(N-1) = -(N-1)$, so $\lambda = N - 1$ as claimed.

Thus the highest weight of V is a nonnegative integer, and since V is arbitrary, this holds for all finite-dimensional representations of $\mathfrak{sl}_2\mathbb{C}$. We now show existence and uniqueness of irreducible representations with highest weight N for all nonnegative integers N. This is equivalent to the statement that there is exactly one irreducible representation of dimension $N + 1$ for every nonnegative integer N.

Existence

Let V be a vector space of dimension $\lambda + 1$ for λ a nonnegative integer. Then I claim that we can define an action of $\mathfrak{sl}_2\mathbb{C}$ on V such that V is irreducible, with highest weight λ..

Let $\{v_0, v_1, \ldots, v_\lambda\}$ be a basis for V. Define the action of F on V by setting $v_i = F^i(v_0)$ and $F^k(v_i) = 0$ if $i+k > \lambda$. Then we define $H(v_i) = (\lambda - 2i)v_i$. Then by definition the highest weight of V is λ. From the previous calculation of EF^k, define $E(v_i) = i(\lambda - i + 1)v_{i-1}$, with $E(v_0) = 0$. It remains to show that the commutation relations hold to prove that V equipped with this action of E, F and H is an $\mathfrak{sl}_2(\mathbb{C})$ representation. First we check that $HF - FH = -2F$.

$$\begin{aligned} HF(v_i) - FH(v_i) &= HF^{i+1}(v_0) - (\lambda - 2i)F(v_i) \\ &= H(v_{i+1}) - (\lambda - 2i)(v_{i+1}) \\ &= (\lambda - 2(i+1) - \lambda + 2i)v_{i+1} \\ &= -2F(v_i). \end{aligned}$$

For $[H, E]$,

$$\begin{aligned} HE(v_i) - EH(v_i) &= i(\lambda - i + 1)H(v_{i-1}) - (\lambda - 2i)E(v_i) \\ &= i(\lambda - i + 1)(\lambda - 2(i-1))v_{i-1} - (\lambda - 2i)i(\lambda - i + 1)v_{i-1} \\ &= 2i(\lambda - i + 1)v_{i-1} \\ &= 2E(v_i) \end{aligned}$$

Finally, for $[E, F]$,

$$\begin{aligned} EF(v_i) - FE(v_i) &= E(v_{i+1}) - i(\lambda - i + 1)F(v_{i-1}) \\ &= (i+1)(\lambda - i)v_i - i(\lambda - i + 1)v_i \\ &= (\lambda - 2i)v_i \\ &= H(v_i) \end{aligned}$$

This construction turns V into an $\mathfrak{sl}_2(\mathbb{C})$ module, and it remains to show that it is irreducible.

Suppose there is some non-zero submodule W contained in V. Then W must contain some non-zero vector w of the form $w = \sum_{i=0}^{\lambda} a_i v_i$ where the a_i are complex constants. Since W is an $\mathfrak{sl}_2(\mathbb{C})$ module, it must contain all possible images of w under the operators E, F and H. If i is the smallest integer such that a_i is non-zero, apply F $\lambda - i$ times, so that $v_\lambda = \frac{1}{a_i}F^{\lambda - i}(w)$ is an element of W. Successively applying E, all the basis vectors of E must be elements of W, so that $W = V$. Hence V contains no proper submodules.

Therefore, for every positive integer N there exists an irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ with highest weight $N - 1$ and dimension N.

Uniqueness

Define $V_\lambda = \text{Span}\{v, F(v), F^2(v), \ldots, F^\lambda(v)\}$ to be the representation constructed in the previous section with highest weight λ, generated by a highest weight vector v. Let W be an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight $k \in \mathbb{N}$. This is possible since we showed in section 1.1 that all finite dimensional representations have highest weights in $\mathbb{N}$. Thus there exists some non-zero vector w in W such that $H(w) = kw$.

I claim that W is isomorphic to to V_k. Define the homomorphism $\phi : V \to W$ by $\phi(v) = w$. By the requirement that ϕ commute with the action of $\mathfrak{sl}_2$, $\phi(F^k(v)) = F^k(w)$. Then ϕ is a non-zero homomorphism of irreducible representations, so by Schur's lemma, ϕ is an isomorphism.

In section **??**, we show that this classification of irreducible representations in fact determines all finite-dimensional representations of $\mathfrak{sl}_2\mathbb{C}$, since all finite-dimensional representations decompose into a direct sum of irreducibles.

1.9.3 Notation

In our classification of the irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ we called highest weight the largest eigenvalue of H. To make this concept more general for later use, we make the following definition:

Definition Let V be a finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$. A *highest weight vector* $v \in V$ is an eigenvector for H that is sent to zero under the action of E. The eigenvalue of H associated to v is called *highest weight* of the representation.

In the following section, we will see that such a highest weight always exists, and is unique for an irreducible representation.

1.9.4 Complete reducibility of representations

A representation V can in fact be decomposed into eigenspaces of H, not just generalised eigenspaces. We first show that H is diagonalisable on its highest weight space. Recall some notation from the previous section: V is a finite dimensional representation of $\mathfrak{sl}_2\mathbb{C}$, λ is the highest weight of V (based on our original definition of highest weight) with associated generalised eigenspace $\overline{V}(\lambda)$ and V_μ is the unique irreducible representation with highest weight μ.

Diagonalisability of H on $\overline{V}(\lambda)$

Claim 1.9.4. *Let v be some vector in $\overline{V}(\lambda)$. Then for each $k \in \mathbb{N}$, there exists a polynomial P_k of degree k, such that*

$$E^kF^kv = P_k(H)v$$

where

$$P_k(x) = k!\prod_{j=0}^{k-1}(x-j)$$

Proof. From the $\mathfrak{sl}_2$ relations and the fact that $Ev = 0$, the claim holds for $k = 1$. A simple induction argument shows that $HF^j = F^j(H - 2jI)$. Recall that

$$EF^k = F^kE + k(H + (k-1)I)F^{k-1}$$

$$\begin{aligned} E^{k+1}F^{k+1}v &= E^k(EF^k + 1)v \\ &= E^k(F^{k+1}Ev + (k+1)(H + kI)F^kv) \\ &= (k+1)E^kF^k(H + kI - 2kI)v \\ &= (k+1)k!\prod_{j=0}^{k-1}(H - jI)(H - kI) \end{aligned}$$

By induction, the claim holds for all $k \in \mathbb{N}$. □

Taking N large enough that $F^N v = 0$ for all $v \in \overline{V}(\lambda)$ (this is possible by the previous section),

$$0 = E^N F^N v = P_N(H)v = N! \prod_{j=0}^{N-1} (H-j)v$$

This holds for any $v \in \overline{V}(\lambda)$ so that $P_N(H)$ is identically zero on $\overline{V}(\lambda)$. Thus the minimal polynomial of H must divide $P_N(H)$. Since $P_N(H)$ splits into a product of linear factors, the minimal polynomial of H also splits on $\overline{V}(\lambda)$ and consequently H is diagonalisable on $\overline{V}(\lambda)$.

1.9.5 The Casimir operator

Definition The *Casimir operator* acting on a representation V of $\mathfrak{sl}_2\mathbb{C}$ is given by $C = EF + FE + \frac{1}{2}H^2$ where E, F and H are the standard operators in a representation of $\mathfrak{sl}_2\mathbb{C}$.

Lemma 1.9.5. *The Casimir operator has the following properties:*

(i) C commutes with the operators E, F and H.

(ii) On V_λ, C acts as $\frac{\lambda(\lambda+2)}{2} Id$ on V_λ.

Proof. (i)

$$\begin{aligned}
CE &= EFE + FE^2 + \tfrac{1}{2}H^2E \\
&= E(EF-H) + (EF-H)E + \tfrac{1}{2}H(EH+2E) \\
&= E(EF) - EH + E(FE) + \tfrac{1}{2}(EH+2E)H \\
&= EC \\
CH &= EFH + FEH + \tfrac{1}{2}H^3 \\
&= E(HF+2F) + F(HE-2E) + H(\tfrac{1}{2}H^2) \\
&= (HE-2E)F + 2H + (HF+2F)E + H(\tfrac{1}{2}H^2) \\
&= HC
\end{aligned}$$

The calculation for CF is nearly identical to that of CE.

ii) Let $\{v, F(v), \ldots, F^\lambda(v)\}$ be a basis for V_λ, where v is a highest weight vector in V_λ. Then any element of V_λ is of the form $w = \sum_{i=0}^{\lambda} a_i F^i(v)$ for complex constants a_i. Then

$$\begin{aligned}
C(w) &= \textstyle\sum_{i=0}^{\lambda} a_i F^i C(v) \\
&= \textstyle\sum_{i=0}^{\lambda} a_i F^i (EF(v) + FE(v) + \tfrac{1}{2}H^2(v)) \\
&= \textstyle\sum_{i=0}^{\lambda} a_i F^i ((FE(v) + H(v)) + 0 + \tfrac{1}{2}\lambda^2 \cdot v) \\
&= \textstyle\sum_{i=0}^{\lambda} a_i F^i \tfrac{\lambda(\lambda+2)}{2} v \\
&= \tfrac{\lambda(\lambda+2)}{2} w
\end{aligned}$$

□

Note that since C commutes with the action of $\mathfrak{sl}_2\mathbb{C}$, it is a homomorphism of representations from V to itself. Since V is irreducible, C must act as a scalar operator by Schur's lemma for algebraically closed fields.

Proof of complete reducibility

To derive a contradiction, suppose that W is a reducible, indecomposable representation of smallest dimension. The Casimir operator is an intertwining map on W so that the decomposition of W into a direct sum of C-eigenspaces becomes a decomposition of W into subrepresentations since each C-eigenspace is invariant under E, F and H. Thus, C can have only one eigenvalue on W denoted μ.

The representation W is reducible and finite-dimensional, so W must contain a submodule of the form V_λ where λ is the highest weight of W. As seen in the previous lemma, the operator C acts on V_λ as $\frac{\lambda(\lambda+2)}{2}Id$. Hence, $\mu = \frac{\lambda(\lambda+2)}{2}$.

Furthermore, W/V_λ is also a representation of $\mathfrak{sl}_2\mathbb{C}$ and C again must act on W/V_λ with only one eigenvalue, namely $\frac{\lambda(\lambda+2)}{2}$. There are two possibilities:

If W/V_λ is irreducible, then by uniqueness of the eigenvalue of C, we must have $W/V_\lambda \cong V_\lambda$.

If W/V_λ is reducible, we have that $\dim(W/V_\lambda) < \dim W$, and by our assumption on the minimality of W, W/V_λ must be decomposable, and furthermore the summands must be irreducible. Again by the fact that C has only one eigenvalue on the quotient space, all the summands must just be copies of V_λ. Therefore there exists some positive integer n such that $W/V_\lambda \cong nV_\lambda$. Thus the first case is simply a special case of the second, with $n = 1$.

Let $\overline{W}(\lambda) \subset W$ be the eigenspace associated to the highest weight λ. For each V_λ, the eigenspace associated to λ is one-dimensional by irreducibility of V_λ, and H is diagonalisable on $V(\lambda)$ for any finite-dimensional representation V. Hence H is diagonalisable on $\overline{W}(\lambda)$ and on its quotient by V_λ. From the decomposition of W/V_λ, $(W/V_\lambda)(\lambda)$ is n-dimensional, and since H is diagonalisable on $W(\lambda)$, the eigenspace $W(\lambda)$ splits into $n+1$ copies of $V_\lambda(\lambda)$, and is therefore $(n+1)$-dimensional.

Let $\{v_1, v_2, \ldots, v_{n+1}\}$ be a basis for $W(\lambda)$. Then the set $\{F^j(v_i)\}$, such that $1 \leq i \leq n+1,\ 0 \leq j \leq \lambda$ is linearly independent. We have shown that for fixed i, the $F^j(v_i)$ are linearly independent, so it remains to show that

Claim 1.9.6. *For fixed j the set $\{F^j(v_i)\}$, $1 \leq i \leq n+1$ is a linearly independent set.*

Proof. $\{v_1, \ldots, v_{n+1}\}$ is a basis for $W(\lambda)$ so the claim holds for $j = 0$.

Now let $j \geq 1$. Suppose we have $\sum_{i=1}^{n+1} c_i F^j(v_i) = 0$. Then

$$\begin{aligned} 0 &= E(\textstyle\sum_{i=1}^{n+1} c_i F^j(v_i)) \\ &= \textstyle\sum_{i=1}^{n+1} c_i EF^j(v_i) \\ &= \textstyle\sum_{i=1}^{n+1} c_i j(\lambda - j + 1)F^{j-1}(v_i) \end{aligned}$$

since $E(v_i) = 0$ for all $1 \leq i \leq n+1$, given that the v_i are in $W(\lambda)$.

For each $j = 1, \ldots, \lambda$, $j(\lambda - j + 1)$ is non-zero, so $\sum_{i=1}^{n+1} c_i F^{j-1}(v_i) = 0$, and by the inductive step, $c_1 = c_2 = \ldots = c_{n+1} = 0$.

□

Thus, for fixed i and fixed j respectively, the set $\{F^j(v_i)\}$ is linearly independent, so for $1 \leq i \leq n+1$ and $0 \leq j \leq \lambda$ the $F^j(v_i)$ are linearly independent. Furthermore, they span W since the highest weight vectors completely determine the space W. Therefore, $\{F^j(v_i)\}$ forms a basis of W.

Define $W_i = \mathrm{Span}\{v_i, F(v_i), \dots, F^\lambda(v_i)\}$ for each $i = 1, \dots, n+1$. Then each W_i forms an irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ and is thus a subrepresentation of W. But then $W = \oplus_{i=1}^{n+1} W_i$, which contradicts the assumption that W is indecomposable. Therefore, all indecomposable representations are irreducible and the representations of $\mathfrak{sl}_2\mathbb{C}$ are completely reducible.

1.10 Semisimple Lie algebras

We use the example of $\mathfrak{sl}_2$ to give a classification of the finite-dimensional irreducible representations of semisimple complex Lie algebras and introduce the terminology of Lie theory, as seen in [FH91], which will be used extensively in the following chapters.

Let $\mathfrak{g}$ be a Lie algebra.

Definition 1. The *derived series* of $\mathfrak{g}$ is a descending chain of subalgebras $\{\mathscr{D}^k\mathfrak{g}]\}$ of $\mathfrak{g}$ defined inductively by:

$$\mathscr{D}^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \qquad \mathscr{D}^k\mathfrak{g} = [\mathscr{D}^{k-1}\mathfrak{g}, \mathscr{D}^{k-1}\mathfrak{g}] \text{ for } k \geq 1$$

2. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of $\mathfrak{g}$ is an *ideal* if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, and all $Y \in \mathfrak{g}$.

3. $\mathfrak{g}$ is *semisimple* if for all ideals $\mathfrak{h} \subset \mathfrak{g}$, $\mathscr{D}^k\mathfrak{h} = 0$ for some k implies that $\mathfrak{h}$ is the zero ideal in $\mathfrak{g}$.

The most significant property of semisimple Lie algebras that will be used here is the complete reducibility of their finite-dimensional representations:

Proposition 1.10.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra and let V be a finite-dimensional representation with $W \subset V$ a $\mathfrak{g}$-submodule. Then there exists a submodule $W' \subset V$ such that $V = W \oplus W'$ as representations of $\mathfrak{g}$.*

Equivalently, any finite-dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ decomposes into a direct sum of irreducible representations of $\mathfrak{g}$.

For the remainder of this section, let $\mathfrak{g}$ be a semisimple Lie algebra. Consider the adjoint representation, wherein $\mathfrak{g}$ acts on itself under the following map:

$$\mathrm{ad}(x)(y) = [x, y]$$

for all x and y in $\mathfrak{g}$. From here, we hope to find an analogue of the action of the element H in $\mathfrak{sl}_2$. It will no longer be possible to find a single such element, but the generalisation of H for a semisimple Lie algebra is given by the following definition:

Definition A *Cartan subalgebra* $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{g}$ that acts diagonally under the adjoint representation.

More generally, a Cartan subalgebra acts diagonally on any representation of $\mathfrak{g}$.

Example. *In the case of $\mathfrak{sl}_n$, a choice of Cartan subalgebra is the subalgebra of diagonal matrices. Letting $H_i = E_{i,i}$ be the elementary matrix with a 1 in the i-th position on the diagonal and zeros elsewhere. This Cartan subalgebra for $\mathfrak{sl}_n$ is given by:*

$$\mathfrak{h} = \{a_1H_1 + a_2H_2 + \ldots + a_nH_n \mid a_i \in \mathbb{C},\ a_1 + \ldots a_n = 0\}$$

It is necessary to generalise the notion of eigenvectors and eigenvalues to subalgebras $\mathfrak{h}$, rather than just for single elements. An *eigenvector* for $\mathfrak{h}$ acting on a vector space V is an element v of V that is an eigenvector for each $H \in \mathfrak{h}$. Therefore an eigenvalue for $\mathfrak{h}$ is a linear functional $\alpha : \mathfrak{h} \to \mathbb{C}$ such that there exists a non-zero eigenvector $v \in V$ satisfying

$$H \cdot v = \alpha(H) \cdot v$$

for every $H \in \mathfrak{h}$. In the case of the adjoint representation, eigenvectors $x \in \mathfrak{g}$ satisfy $\mathrm{ad}(H)(x) = \alpha(H) \cdot x$.

Given a choice of Cartan subalgebra $\mathfrak{h}$, let $\mathfrak{h}$ act on $\mathfrak{g}$ via the adjoint representation. Since the Cartan subalgebra acts diagonally, $\mathfrak{g}$ can be decomposed into a direct sum of weight spaces $\mathfrak{g}_\alpha$ consisting of eigenvectors with respect to the eigenvalue $\alpha \in \mathfrak{h}^*$.

Definition The set of *roots* $R \subset \mathfrak{h}^*$ of a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ is the set of non-zero weights of the adjoint representation. The corresponding weight spaces $\mathfrak{g}_\alpha$ are called root spaces.

We have the following properties for roots and root spaces:

Proposition 1.10.2.

1. *The adjoint action of $\mathfrak{g}_\alpha$ sends the root space $\mathfrak{g}_\beta$ to the root space $\mathfrak{g}_{\alpha+\beta}$.*
2. *Each root space $\mathfrak{g}_\alpha$ is one-dimensional.*
3. *R is symmetric in the sense that if $\alpha \in R$ then $-\alpha \in R$.*
4. *R generates a lattice $\Lambda_R \subset \mathfrak{h}^*$, of rank equal to the dimension of the Cartan subalgebra.*

These properties extend to general finite dimensional representations V of $\mathfrak{g}$, namely

Proposition 1.10.3. *Let V be a representation of $\mathfrak{g}$. Then*

1. *V decomposes as a direct sum $V = \bigoplus_\alpha V_\alpha$, for finitely many elements $\alpha \in \mathfrak{h}^*$. The dimension of V_α is called the multiplicity of the weight α in V.*

2. *For any root β, $g_\beta : V_\alpha \to V_{\alpha+\beta}$.*

3. *The weights of an irreducible representation are congruent modulo the root lattice Λ_R.*

The roots of $\mathfrak{g}$ allow us to distinguish copies of $\mathfrak{sl}_2$ contained in $\mathfrak{g}$:

Proposition 1.10.4. *For every root $\alpha \in R$ there exists a subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Each of the weight spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are one-dimensional and the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a one-dimensional subspace of $\mathfrak{h}$, so that $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C}$.*

Thus to each pair $(\alpha, -\alpha)$ of roots of opposite sign, there is a unique element H_α in the commutator $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ with eigenvalues 2 and -2 on the weight spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ respectively. Eigenvalues of the H_α are integral and symmetric about the origin in $\mathbb{Z}$.

To express this symmetry in terms of the totality of the weights of $\mathfrak{g}$, we define a subgroup of the group of isometries of the root system R.

Definition The *Weyl group* $\mathscr{W}$ of a Lie algebra $\mathfrak{g}$ is the group generated by the involutions W_α, $\alpha \in R$ acting on $\mathfrak{h}^*$ by reflecting across the hyperplanes

$$\Omega_\alpha = \{\beta \in \mathfrak{h}^* \mid \langle H_\alpha, \beta \rangle = 0\}$$

perpendicular to the line spanned by α, so that

$$W_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$$

Proposition 1.10.5. *The weights of a representation of $\mathfrak{g}$ are invariant under the action of the Weyl group.*

Definition An *ordering* of the roots of a semisimple Lie algebra $\mathfrak{g}$ is a choice of linear functional l on the root lattice that decomposes the set of roots into two disjoint subsets of equal size $R = R^+ \cup R^-$ where $R^+ = \{\alpha \in R \mid l(\alpha) > 0\}$ is called the set of *positive roots* and $R^- = \{\alpha \in R \mid l(\alpha) < 0\}$ is called the set of *negative roots*.

Definition Let V be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$. A *highest weight vector* for V is a vector $v \in V$ such that v is an eigenvector for the Cartan sub algebra $\mathfrak{h}$ and in the kernel of the action of $\mathfrak{g}_\beta$ for all $\beta \in R^+$.

Theorem 1.10.6. *Any finite dimensional representation of a complex semisimple Lie algebra contains a highest weight vector, which generates an irreducible subrepresentation under successive applications of roots spaces $\mathfrak{g}_\beta$ for $\beta \in R^-$. For an irreducible representation, the highest weight vector is unique up to scalars.*

Definition The eigenvalue α associated to a highest weight vector of a finite dimensional representation is the *highest weight* of the representation.

A positive root is *simple* if it cannot be written as a sum of two positive roots. We similarly define simple negative roots.

Definition The *Weyl chamber* $\mathcal{W}$ of a representation is the locus in the real span of the roots satisfying the inequality $\alpha(H_\beta)$ for all $\beta \in R^+$. This is the closure of a connected component of $\mathfrak{h}^*$ contained in the complement of the union of the hyperplanes Ω_β.

We say that α is a *dominant weight* of $\mathfrak{g}$ if $\alpha \in \mathcal{W} \cap \Lambda_W$, where Λ_W is the weight lattice. Denote by Λ_W^α the subset of Λ_W congruent to α modulo the root lattice Λ_R and by $Conv_W^\alpha$ the convex hull of the points conjugate to α under the action of the Weyl group in Λ_W.

The following theorem classifies all irreducible representations of a semisimple Lie algebra $\mathfrak{g}$:

Theorem 1.10.7 (Existence and uniqueness theorem)**.** *The isomorphism classes of irreducible representations of $\mathfrak{g}$ are parametrised by the dominant weights α of $\mathfrak{g}$. The weights of the representation consist of the elements $\beta \in \Lambda_W^\alpha \cap Conv_W^\alpha$, all occurring with multiplicity one.*

Definition The *fundamental weights* of a representation are the elements $\omega_i \in \mathfrak{h}^*$ such that $\omega_i(H_{\alpha_j}) = \delta_{i,j}$ where $\alpha_1, \dots, \alpha_n$ are the simple roots of the Lie algebra.

In the example of the Lie algebra $\mathfrak{sl}_n$, the Cartan subalgebra is the set of diagonal matrices with zero trace:

$$\mathfrak{h} = \{\mathrm{diag}[a_1\ a_2\ \dots\ a_n] \mid \sum_{i=1}^{n} a_i = 0\}$$

so that $\mathfrak{h}^* = \mathbb{C}\{L_1, L_2, \dots, L_n\}/(\sum L_i = 0)$. In this case, the fundamental weights are $\omega_i = L_1 + \dots + L_i$ for $i = 1, \dots, n-1$.

Proposition 1.10.8. *Any highest weight of a representation can be uniquely expressed as a nonnegative integral linear combination of fundamental weights.*

Definition Let $\mathfrak{g}$ be a Lie algebra. The *Killing form* on $\mathfrak{g}$ is a symmetric bilinear form $K(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $K(x, y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$.

We will often want to consider the tensor product of representations of a Lie algebra as a representation of the Lie algebra itself, particularly in the context of link homology. The ability to consider such tensor products is due to the following property of the universal enveloping algebra:

Proposition 1.10.9. *Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. Then $\mathcal{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.*

Proof. We begin by defining the comultiplication and counit on $\mathcal{U}(\mathfrak{g})$: let $\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ be defined by $\Delta(x) = x \otimes id + id \otimes x$ for all $x \in \mathcal{U}(\mathfrak{g})$ and let $\varepsilon : \mathcal{U}(\mathfrak{g}) \to k$ be defined by $\varepsilon(x) = 0$ for all $x \in \mathcal{U}(\mathfrak{g})$ (we will generally take $k = \mathbb{C}$ here). The antipode $\gamma : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is defined by $\gamma(x) = -x$. These maps are algebra homomorphisms by the universal mapping property of $\mathcal{U}(\mathfrak{g})$ and clearly satisfy coassociativity, and the counit and antipode laws. By *cocommutative*, we mean that $\sigma \circ \Delta = \Delta$, where $\sigma : \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ swaps the factors in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$: $\sigma(x \otimes y) = (y \otimes x)$. Thus cocommutativity follows directly from the definition of Δ. □

1.10.1 Dynkin diagrams

Given a root system R, in particular the root system of a Lie algebra, one can construct a *Dynkin diagram* by drawing nodes • for each simple root in R and joining two nodes by a number of edges depending on the angle between them. Of particular interest in this thesis, specifically in the categorification of the adjoint representation of quantum groups encountered in section **??** are the *simply-laced* Dynkin diagrams: those graphs with at most one edge connecting any pair of nodes. In this case, all roots are of the same length. The simply-laced diagrams are all of the following form:

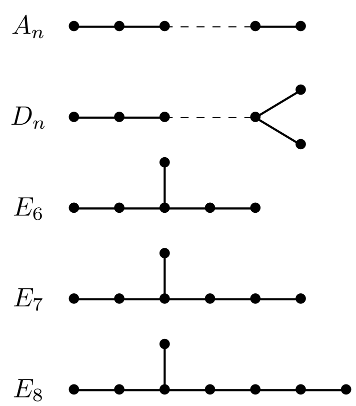


Figure 1.1: The simply-laced Dynkin diagrams

The simply-laced Dynkin diagrams play an important classification role in several areas of mathematics, as demonstrated by a remarkable theorem of Pierre Gabriel [Gab72].

Definition A quiver is *of finite type* if it has finitely many isomorphism classes of indecomposable modules.

Theorem 1.10.10 (Gabriel). *A connected quiver is of finite type if and only if its underlying undirected graph is a simply-laced Dynkin diagram as seen in figure 1.10.1.*

1.11 Quantum groups

Quantum groups, for our purposes, are quantum deformations of semisimple Lie algebras and have the structure of Hopf algebras. In particular, we can define a comultiplication map on a quantum group, so that tensor products of representations are themselves representations of the quantum group. These quantum groups have proven to be very useful in low-dimensional topology and knot theory. For example, representations of quantum groups have been used to determine Reshetikhin-Turaev invariants of tangles. In the case where the original Lie algebra was $\mathfrak{sl}_2$, the Reshetikhin-Turaev invariant of a link is the Jones polynomial. Quantum groups are also interesting in the field of categorification, where the existence of a quantum deformation corresponds to the existence of a grading on the lifted structure.

We discuss here the definition of a quantum group and in particular the quantum group $U_q(\mathfrak{sl}_2)$, and some of the properties of these groups, as well as their representations. Proofs of the theorems and properties here can be found in [Hum72], [Jan96] and [Lus93].

1.11.1 Definitions

For the remainder of this chapter, let $\mathfrak{g}$ be a complex semisimple Lie algebra, with a set of roots R, a set of simple roots Π and Killing form denoted $(\ ,\ )$. We use the following notation: let q be an element of $\mathbb{Q}\backslash\{-1,0,1\}$ and for $\alpha \in \Pi$, let $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$. For $a \in \mathbb{Z}$, $n \in \mathbb{N}$, define

$$[a]_\alpha = \frac{q_\alpha^a - q_\alpha^{-a}}{q_\alpha - q_\alpha^{-1}}$$

$$[n]_\alpha! := [1]_\alpha[2]_\alpha \dots [n]_\alpha \text{ and } [0]_\alpha! = 1$$

$$\begin{bmatrix} a \\ n \end{bmatrix}_\alpha = \frac{[a]_\alpha!}{[n]_\alpha![a-n]_\alpha!} \text{ for } a \geq n$$

and for any $\alpha, \beta \in \Pi$, let $\langle \alpha, \beta \rangle = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$

Definition The *quantum group* of $\mathfrak{g}$ (also called the quantum enveloping algebra of $\mathfrak{g}$ [Jan96]), is the $\mathbb{Q}$-algebra with generators $E_\alpha,\ F_\alpha,\ K_\alpha$ and K_α^{-1} for all $\alpha \in \Pi$ and relations

$$K_\alpha K_\alpha^{-1} = 1 = K_\alpha K_\alpha^{-1} \tag{1.1}$$

$$K_\alpha K_\beta = K_\beta K_\alpha \tag{1.2}$$

$$K_\alpha E_\beta = q^{(\alpha,\beta)} E_\beta K_\alpha \tag{1.3}$$

$$K_\alpha F_\beta = q^{-(\alpha,\beta)} F_\beta K_\alpha \tag{1.4}$$

$$E_\alpha F_\beta - F_\beta E_\alpha = \delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \tag{1.5}$$

$$\sum_{s=0}^{1-\langle\alpha,\beta\rangle} (-1)^s \begin{bmatrix} 1-\langle\alpha,\beta\rangle \\ s \end{bmatrix}_\alpha E_\alpha^{1-\langle\alpha,\beta\rangle-s} E_\beta E_\alpha^s = 0 \tag{1.6}$$

$$\sum_{s=0}^{1-\langle\alpha,\beta\rangle} (-1)^s \begin{bmatrix} 1-\langle\alpha,\beta\rangle \\ s \end{bmatrix}_\alpha F_\alpha^{1-\langle\alpha,\beta\rangle-s} F_\beta F_\alpha^s = 0 \tag{1.7}$$

for all $\alpha, \beta \in \Pi$.

Definition We say that $\mathfrak{g}$ is *simply-laced* if

1. $(\alpha, \alpha) = 2$ for all $\alpha \in \Pi$, and
2. $(\alpha, \beta) \in \{0, -1\}$ for all $\alpha \neq \beta \in \Pi$.

In the case of a simply-laced Lie algebra $\mathfrak{g}$, $\langle \alpha, \beta \rangle = 0$ or 1 for all $\alpha, \beta \in \Pi$. Hence in the relation 1.5, $q_\alpha = q$ and the last two relations 1.6 and 1.7 simplify to the following for a simply-laced Lie algebra:

$$E_\alpha E_\beta = E_\beta E_\alpha \text{ if } (\alpha, \beta) = 0 \tag{1.8}$$

$$F_\alpha F_\beta = F_\beta F_\alpha \text{ if } (\alpha, \beta) = 0 \tag{1.9}$$

$$E_\alpha^2 E_\beta - (q + q^{-1}) E_\alpha E_\beta E_\alpha + E_\beta E_\alpha^2 = 0 \text{ if } (\alpha, \beta) = -1 \tag{1.10}$$

$$F_\alpha^2 F_\beta - (q + q^{-1}) F_\alpha F_\beta F_\alpha + F_\beta F_\alpha^2 = 0 \text{ if } (\alpha, \beta) = -1 \tag{1.11}$$

The quantum group $U_q(\mathfrak{g})$ has the following useful structure:

Theorem 1.11.1. *There is a unique Hopf algebra structure* (Δ, ε, S) *such that for all* $\alpha \in \Pi$*:*

$$\begin{array}{lll} \Delta(E_\alpha) = E_\alpha \otimes 1 + K_\alpha \otimes E_\alpha & \varepsilon(E_\alpha) = 0 & S(E_\alpha) = -K_\alpha^{-1} E_\alpha \\ \Delta(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha & \varepsilon(F_\alpha) = 0 & S(F_\alpha) = -F_\alpha K_\alpha \\ \Delta(E_\alpha) = K_\alpha \otimes K_\alpha & \varepsilon(K_\alpha) = 1 & S(K_\alpha) = -K_\alpha^{-1} \end{array}$$

$U_q(\mathfrak{g})$ is equipped with the following involutions:

There is a unique $\mathbb{Q}(q)$-algebra automorphism $\omega : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$\omega(E_\alpha) = F_\alpha \qquad \omega(F_\alpha) = E_\alpha \qquad \omega(K_\alpha) = K_\alpha^{-1} \qquad \omega(f(q)x) = f(q)\omega(x)$$

There is a unique $\mathbb{Q}$-algebra automorphism $\psi : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$\psi(E_\alpha) = E_\alpha \qquad \psi(F_\alpha) = F_\alpha \qquad \psi(K_\alpha) = K_\alpha^{-1} \psi(f(q)x) \qquad = f(q^{-1})\psi(x)$$

for all $\alpha \in \Pi$ and all $f \in \mathbb{Q}(q)$.

There is an antiautomorphism $\tau : U_q(\mathfrak{g}) \to U_q^{op}(\mathfrak{g})$ defined by

$$\begin{gathered} \tau(xy) = \tau(y)\tau(x) \text{ for all } x, y \in U_q(\mathfrak{g}) \\ \tau(E_\alpha) = qF_\alpha K_\alpha^{-1}, \ \tau(F_\alpha) = qE_\alpha K_\alpha, \ \tau(K_\alpha) = K_\alpha^{-1} \\ \tau(f(q)x) = f(q^{-1})\tau(x) \text{ for all } f \in \mathbb{Q}(q) \text{ and } x \in U_q(\mathfrak{g}) \end{gathered}$$

For any $\alpha, \beta \in \Pi$ and $a, b \in \mathbb{N}$, we can also consider the *divided powers* of elements $U_q(\mathfrak{g})$

$$E_\alpha^{(a)} := \frac{E_\alpha^a}{[a]_\alpha!} \qquad\qquad F_\beta^{(b)} := \frac{F_\beta^b}{[b]_\beta!}$$

Products of these elements span a $\mathbb{Z}[q, q^{-1}]$-submodule of $U_q(\mathfrak{g})$.

We make a brief digression to the particular case $U_q(\mathfrak{sl}_2)$, which is just as illustrative as the original $\mathfrak{sl}_2$ case was for general semisimple Lie algebras.

1.11.2 Example: $U_q(\mathfrak{sl}_2)$

In the case $\mathfrak{g} = \mathfrak{sl}_2$, $U := U_q(\mathfrak{g})$ is a $\mathbb{Q}(q)$-algebra with generators E, F, K and K^{-1}, and relations

$$\begin{aligned} KK^{-1} &= 1 = K^{-1}K \\ KE &= q^2 EK \\ FK &= q^{-2} FK \\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}} \end{aligned}$$

When studying representations of U, we proceed in similar manner to the $\mathfrak{sl}_2$ case, considering instead eigenspaces of K. We restrict ourselves to finite-dimensional representations V that admit a weight space decomposition: $V = \oplus_{n \in \mathbb{Z}} V_n$ where each V_n is an eigenspace of K, and take $q \neq \pm 1$. We have the following theorem for finite-dimensional irreducible representations of U, which is very similar to the corresponding theorem for $\mathfrak{sl}_2$:

Theorem 1.11.2. *For each nonnegative integer n there is an irreducible representation of U L_+ with basis $v_0, v_1, \ldots, v_n$ and an irreducible representation L_- with basis $w_0, w_1, \ldots, w_n$ such that for all $0 \leq i \leq n$*

$$Kv_i = q^{n-2i} v_i \qquad\qquad Kw_i = -q^{n-2i} w_i$$

$$Fv_i = \begin{cases} v_{i+1} & \text{if } i < n \\ 0 & \text{if } i = 0 \end{cases} \qquad\qquad Fw_i = \begin{cases} w_{i+1} & \text{if } i < n \\ 0 & \text{if } i = 0 \end{cases}$$

$$Ev_i = \begin{cases} 0 & \text{if } i = 0 \\ [i][n+1-i]v_{i-1} & \text{if } i > 0 \end{cases} \qquad\qquad Ew_i = \begin{cases} 0 & \text{if } i = 0 \\ -[i][n+1-i]w_{i-1} & \text{if } i > 0 \end{cases}$$

and every irreducible representation of U of dimension $n+1$ is isomorphic to L_+ or L_-.

1.11.3 Representations of $U_q(\mathfrak{g})$

Let $U := U_q(\mathfrak{g})$ and let Λ be the weight lattice of $\mathfrak{g}$ and Φ be the root lattice of $\mathfrak{g}$. We restrict ourselves to representations V that admit a weight decomposition, which means here that

$$V = \bigoplus_{\lambda \in \Lambda} V_\lambda \qquad\qquad \text{where } V_\lambda = \{v \in V \mid K_\mu = q^{(\lambda, \mu)} v \text{ for all } \mu \in \Phi\}$$

As in the $U_q(\mathfrak{sl}_2)$ case, the irreducible representations of $U_q(\mathfrak{g})$ are very similar to the irreducible representations of $\mathfrak{g}$ (particularly when restricted to the representations above).

Lemma 1.11.3. *Let V be a finite-dimensional representation of U with $V \neq 0$ Then*

1. *There exist $\lambda \in \Lambda$ and $v \in V_\lambda$, $v \neq 0$ such that $E_\alpha v = 0$ for all $\alpha \in \Pi$ and*
2. *letting λ and v be as in 1, λ is a dominant weight and $F_\beta^{\langle\lambda,\beta\rangle+1} v = 0$ for all $\beta \in \Pi$.*

For $\lambda \in \Lambda$ define the left ideal $J_\lambda = \sum_{\alpha\in\Pi} UE_\alpha + \sum_{\alpha\in\Pi} U(K_\alpha - q^{(\lambda,\alpha)})$. Then let $M(\lambda) := U/J_\lambda$. This is a representation of U generated by $v_\lambda := 1 + J_\lambda$ such that

$$E_\alpha v_\lambda = 0 \qquad \text{and} \qquad K_\alpha v_\lambda = q^{(\lambda,\alpha)} v_\lambda \qquad \text{for all } \alpha \in \Pi.$$

Call $M(\lambda)$ the *Verma module* (or universal highest weight module) of highest weight λ. It is universal in the following sense:

If V is any representation of U with $v \in V_\lambda$ such that $E_\alpha v = 0$ for all $\alpha \in \Pi$, then there is a unique U-module homomorphism $\phi : M(\lambda) \to V$ with $v_\lambda = v$. Then we can characterise all finite-dimensional irreducible representations of U:

Theorem 1.11.4.
1. *Let $\lambda \in \Lambda$. Then the Verma module $M(\lambda)$ has a unique irreducible quotient representation $L(\lambda)$.*
2. *For each dominant weight $\lambda \in \Lambda$, the irreducible representation $L(\lambda)$ is finite-dimensional. Furthermore, all finite-dimensional irreducible representations of U are isomorphic to exactly one $L(\lambda)$ with $\lambda \in \Lambda$ dominant.*

Chapter 2

Current algebras

Annular Khovanov homology has rich algebraic structure: its homology groups are modules of a particular algebra, called $\mathfrak{sl}_2^-(V)$ here. This is a particular example of a truncated current Lie algebra. We introduce some of these modified Lie algebras and study their representations through the use of quivers.

2.1 Polynomial current algebras

The algebras introduced in this chapter are all built from finite-dimensional semisimple complex Lie algebras, denoted $\mathfrak{g}$. These modified Lie algebras are no longer semisimple and hence have more complex representations.

Definition A polynomial current algebra associated to a complex semisimple Lie algebra $\mathfrak{g}$ is a Lie algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes \mathbb{C}[t]$ with Lie bracket

$$[x \otimes t^k, y \otimes t^l]_{\mathfrak{g}[t]} = [x, y]_{\mathfrak{g}} \otimes t^{k+l}.$$

Current algebras have a natural $\mathbb{Z}_{\geq 0}$-grading given by the powers of the variable t. The pairing $[\ ,\ ]_{\mathfrak{g}[t]}$ satisfies the Jacobi identity and the alternating property since $[\ ,\ ]_{\mathfrak{g}}$ is a Lie bracket.

For what follows, fix the finite-dimensional semisimple complex Lie algebra $\mathfrak{g}$ and choose some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with n the rank of $\mathfrak{h}$. Denote by $R \subset \mathfrak{h}^*$ the set of of roots with respect to $\mathfrak{h}$, P^+ the set of positive integral weights (weights in the dominant Weyl chamber with respect to a choice of ordering) and let θ be the highest root. Let $\mathfrak{g}$**-mod** be the category of finite-dimensional $\mathfrak{g}$-modules, with $\mathfrak{g}$-module homomorphisms as the morphisms in $\mathfrak{g}$**-mod**. Then we have seen that the isomorphism classes of irreducible (simple) representations of $\mathfrak{g}$ are parametrised by the elements of P^+. Given $\lambda \in P^+$, let $V(\lambda)$ be the irreducible representation of highest weight λ, generated by a highest weight vector v_λ. Given a representation V of $\mathfrak{g}$ in

$\mathfrak{g}$-**mod**, decompose V into a direct sum of weight spaces:

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$$

so that $h \cdot v = \mu(h)v$ for all $v \in V_\mu$ and $h \in \mathfrak{h}$.

Let $U(\mathfrak{g}[t])$ be the universal enveloping algebra of $\mathfrak{g}[t]$ inheriting the grading from $\mathfrak{g}[t]$. This is a unital associative algebra generated by the elements in $\mathfrak{g}[t]$, modulo the relation

$$(x \otimes t^i)(y \otimes t^j) - (y \otimes t^j)(x \otimes t^i) - [x \otimes t^i, y \otimes t^j] = 0$$

for all $x, y \in \mathfrak{g}$, and all $i, j \in \mathbb{N}$. The enveloping algebra $U(\mathfrak{g}[t])$ is generated by elements in $\mathfrak{g}$ and $\mathfrak{g} \otimes t$ [CG07]. Then $U(\mathfrak{g}[t])$ is a Hopf algebra by proposition 1.10.9, with comultiplication $\Delta : x \mapsto x \otimes 1 + 1 \otimes x$ for all $x \in \mathfrak{g}[t]$ and multiplication map m. Note that the comultiplication map preserves the grading in $U(\mathfrak{g}[t])$. From now on, we write xt^k in the place of $x \otimes t^k$ for elements in $\mathfrak{g}[t]$. Let $\mathfrak{g}[t]_+$ be the Lie ideal $\mathfrak{g} \otimes t\mathbb{C}[t]$. Then $\mathfrak{g}[t]_+$ is also a graded Lie algebra, with grading given by the powers of t, so that the homogeneous components of $\mathfrak{g}[t]_+$ of nonpositive degree are zero.

Let $\mathcal{G}$ be the category of graded $\mathfrak{g}[t]$-modules with finite-dimensional homogeneous subspaces and grading-preserving $\mathfrak{g}[t]$-module homomorphisms. Then any representation V of $\mathfrak{g}[t]$ in Ob$\mathcal{G}$ can be decomposed as a direct sum over $\mathbb{Z}_+$ of subspaces $V[r]$: $V = \oplus_{r \in \mathbb{Z}_+} V[r]$ such that $(xt^k)V[r] \subset V[r+k]$ for all $x \in \mathfrak{g}$, and all $r, k \in \mathbb{Z}_+$. Thus, each $V[r]$ is a finite-dimensional representation of $\mathfrak{g}$. For a morphism $f \in \mathrm{Hom}_{\mathfrak{g}[t]}(V, W)$, denote by $f[r]$ the restriction of f to $V[r]$. In particular, $f[r] \in \mathrm{Hom}_{\mathfrak{g}}(V[r], W[r])$, since f is grading-preserving.

Let $\mathcal{B}$ be the covariant functor from $\mathfrak{g}$-**mod** to $\mathcal{G}$ defined by

$$\mathcal{B}(V)[0] = V \text{ and } \mathcal{B}(V)[r] = 0 \text{ for all } r > 0 \text{ and any } V \in \mathrm{Ob}\mathfrak{g}\textbf{-mod}.$$

The $\mathfrak{g}[t]$ action on $\mathcal{B}(V)$ is given by $(xt^k) \cdot v = \delta_{k,0} x \cdot v$ for all $x \in \mathfrak{g}, k \in \mathbb{Z}_+, v \in V$. Similarly, for any morphism $f \in \mathfrak{g}$-**mod**, we define $\mathcal{B}(f)[0] = f$ and $\mathcal{B}(f)[r] = 0$ for all $r > 0$, so that $\mathrm{Hom}_{\mathcal{G}}(\mathcal{B}(V), \mathcal{B}(W)) = \mathrm{Hom}_{\mathfrak{g}}(V, W)$. In words, the functor $\mathcal{B}$ sends a representation of $\mathfrak{g}$ to a representation of $\mathfrak{g}[t]$ concentrated in degree zero.

Define $\{j\} : \mathcal{G} \to \mathcal{G}$ to be the shift functor by $j \in \mathbb{Z}_+$:

$$(V)[k]\{j\} = V[k+j] \text{ for all } k \in \mathbb{Z}_+$$

The irreducible representations of $\mathfrak{g}[t]$ can be understood from the classification of irreducible representations of $\mathfrak{g}$:

Proposition 2.1.1. *Define* $V(\lambda, j) := (\mathcal{B}(V(\lambda)))\{j\}$ *for* $\lambda \in P^+$ *and* $j \in \mathbb{Z}_+$. *Then for each pair* $(\lambda, j) \in P^+ \times \mathbb{Z}_+$, $V(\lambda, j)$ *is up to isomorphism the unique irreducible representation of* $\mathfrak{g}[t]$ *in* $Ob(\mathfrak{G})$ *and*

$$Hom_{\mathcal{G}}(V(\lambda, j), V(\mu, k)) = 0 \text{ if } (\lambda, j) \neq (\mu, k)$$
$$Hom_{\mathcal{G}}(V(\lambda, j), V(\lambda, j)) \cong \mathbb{C}.$$

Moreover, if $V \in Ob\mathcal{G}$ is concentrated in degree n for some $n \in \mathbb{Z}$, that is, $V = V[n]$, then V is completely reducible.

Proof. By definition, $V(\lambda, j) = (\mathcal{B}(V(\lambda)))\{j\}$, which is the irreducible representation $V(\lambda)$ of $\mathfrak{g}$ considered as a representation of $\mathfrak{g}[t]$ with xt^k acting trivially for all $k > 0$ and all $x \in \mathfrak{g}$. Furthermore, $V(\lambda, j)$ is concentrated in degree j, so that $V(\lambda, j) = V(\lambda)[j]$. Then $V(\lambda, j)$ is an irreducible representation of $\mathfrak{g}[t]$ since $V(\lambda)$ is an irreducible representation of $\mathfrak{g}$, and any subrepresentation of $V(\lambda, j)$ would also have trivial action of xt^k for $k \neq 0$. Since the $V(\lambda)$ are non-isomorphic as $\mathfrak{g}$-modules, $V(\lambda, j)$ and $V(\mu, k)$ cannot be isomorphic as representations of $\mathfrak{g}[t]$ if $\lambda \neq \mu$. Furthermore, any morphism in $\mathcal{G}$ preserves the grading, so $V(\lambda, j)$ and $V(\lambda, k)$ cannot be isomorphic for $j \neq k$.

I claim that any irreducible representation of $\mathfrak{g}[t]$ must be concentrated in a single degree. Suppose that V is a representation of $\mathfrak{g}[t]$ such that $V[m]$ and $V[m']$ are non-zero for some $m < m' \in \mathbb{Z}_+$. The action of $\mathfrak{g}[t]$ always preserves or raises the degree, so $\oplus_{j>m} V[j]$ is a proper, non-trivial subrepresentation of V so V is reducible.

Now suppose V is an irreducible representation of $\mathfrak{g}[t]$, so that there exists some $m \in \mathbb{Z}_+$ with $V = V[m]$. Then by definition of objects in $\mathcal{G}$, V is finite-dimensional. Furthermore, xt^k must act by zero for all $k > 0$, since $(xt^k)V[m] \subset V[m + k] = 0$. Then V is a representation of $\mathfrak{g}$ and must be irreducible as a $\mathfrak{g}$-module to be irreducible as a $\mathfrak{g}[t]$-module. Therefore V is isomorphic to $V(\lambda)$ as a $\mathfrak{g}$-module for some $\lambda \in P+$, and hence is isomorphic to $V(\lambda, m)$ as a $\mathfrak{g}[t]$-module.

The statements about the morphism spaces follow from Schur's lemma for algebraically-closed fields and the restriction to grading-preserving $\mathfrak{g}[t]$-module homomorphisms in $\mathcal{G}$ (namely, the only maps between elements concentrated in distinct degrees must be trivial).

Finally, suppose $V = V[n]$ for some $n \in \mathbb{Z}_+$. Then as before, xt^k must act trivially for all $k > 0$ and V is a $\mathfrak{g}$-module, so by semisimplicity of $\mathfrak{g}$, V is completely reducible as a $\mathfrak{g}$-module, and hence as a $\mathfrak{g}[t]$-module. □

Since $U(\mathfrak{g}[t])$ is a Hopf algebra, the tensor product $V \otimes W$ of representations of $\mathfrak{g}[t]$ also carries the structure of a representation of $\mathfrak{g}[t]$ via the comultiplication map. Let V, W be representations of $\mathfrak{g}[t]$, $k \in \mathbb{Z}_+$ and define

$$(V \otimes W)[k] = \bigoplus_{i \in \mathbb{Z}_+} V[i] \otimes W[k - i]$$

with $W[j] = 0$ for $j < 0$.

Lemma 2.1.2. (i) $V \otimes W = \bigoplus_{k \in \mathbb{Z}_+} (V \otimes W)[k]$

(ii) *For all $j \in \mathbb{Z}_+$, $xt^j \cdot ((V \otimes W)[k]) \subset (V \otimes W)[k + j]$*

The lemma in particular shows that $\mathcal{G}$ is a tensor category.

Proof. (i) This follows from the distributivity of the tensor product over the direct sum.

(ii) Using the comultiplication map,

$$(xt^j)((V \otimes W)[k]) = \bigoplus_{i \in \mathbb{Z}_+} (xt^j)V[i] \otimes W[k-i] + V[i] \otimes (xt^j)W[k-i].$$

Then $(xt^j)V[i] \otimes W[k-i] \subset V[i+j] \otimes W[k-i] = V[i+j] \otimes W[k+j-(i+j)]$ and $V[i] \otimes (xt^j)W[k-i] \subset V[i] \otimes W[k+j-i]$, which are both clearly subsets of

$$(V \otimes W)[k+j] = \bigoplus_{i \in \mathbb{Z}_+} V[i] \otimes W[k+r-i].$$

□

2.2 Takiff Lie algebras

Another class of Lie algebras consists of truncated polynomial current algebras, first studied by Takiff [Tak71].

Definition Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. A *Takiff Lie algebra* is the truncated current algebra $\mathfrak{g}_t := \mathfrak{g} \otimes \mathbb{C}[t]/t^2$.

More generally, a *generalised Takiff Lie algebra* is the algebra $\mathfrak{g} \otimes \mathbb{C}[t]/t^n$ for some integer $n \geq 2$.

This is a graded Lie algebra with grading given by powers of t and Lie bracket given by $[x \otimes 1, y \otimes 1]_{\mathfrak{g}_t} = [x, y]_{\mathfrak{g}} \otimes 1$, $[x \otimes 1, y \otimes t]_{\mathfrak{g}_t} = [x, y]_{\mathfrak{g}} \otimes t = [x \otimes t, y \otimes 1]$ and $[x \otimes t, y \otimes t] = 0$ for all $x, y, \in \mathfrak{g}$. We again omit the tensor product symbol. If the Lie algebra $\mathfrak{g}$ has dimension n as a vector space over $\mathbb{C}$ and basis $\{x_1, x_2, \ldots, x_n\}$, then $\mathfrak{g}_t$ has dimension $2n$ and basis $\{x_1, \ldots, x_n, x_1t, \ldots, x_nt\}$.

We classify the irreducible and projective indecomposable representations of $\mathfrak{g}_t$. Let $\mathcal{G}_t$ be the category whose objects are finite-dimensional $\mathfrak{g}_t$-modules and whose morphisms are $\mathfrak{g}_t$-module homomorphisms.

Let P^+ be the set of dominant weights of $\mathfrak{g}$ and let $V(\lambda)$ denote the irreducible representation of $\mathfrak{g}$ with highest weight λ. Then:

Proposition 2.2.1.

1. *The isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{g}_t$ in $Ob\mathcal{G}_t$ are parametrised by $\lambda \in P^+$, with $\mathfrak{g}_t$-action given by $xt^k \cdot v = \delta_{k,0} x \cdot v$ for all $x \in \mathfrak{g}$ and all $v \in V_\lambda$.*

2. *Up to isomorphism, there is a unique finite-dimensional projective indecomposable $\mathfrak{g}_t$-representation $V_t(\lambda) := V(\lambda) \otimes \mathbb{C}[t]/t^2$ for each $\lambda \in P^+$, and any finite-dimensional projective representations are isomorphic to $V_t(\lambda)$ for some λ.*

Proof. 1. Let V_λ be a finite-dimensional $\mathfrak{g}_t$-module with $\mathfrak{g}_t$-action as defined in the proposition statement, and such that $V_\lambda \cong V(\lambda)$ as representations of $\mathfrak{g}$ for some $\lambda \in P^+$. Then V_λ is irreducible as a representation of $\mathfrak{g}_t$: xt^k would act trivially on any $\mathfrak{g}_t$-subrepresentation of V_λ for $k > 0$. Thus, a subrepresentation of V_λ as a $\mathfrak{g}_t$-module is a $\mathfrak{g}$-submodule of $V(\lambda)$, and must therefore be trivial.

Suppose that V is an irreducible representation of $\mathfrak{g}_t$. Then xt^k must act trivially for $k > 0$. If not, then consider the subspace W of V spanned by $(xt) \cdot v$ for all $v \in V$. Then by assumption W is a proper, non-trivial subspace of V. It is proper because if every element of v could be expressed as $xt \cdot w$ for some $x \in \mathfrak{g}$ and some $w \in V$, then $yt(v) = (yt)(xt)w = 0$ for all v, contradicting the assumption. But then $\mathfrak{g}(W) \subset W$, and $xt \cdot W = 0$ for all $x \in \mathfrak{g}$, and W is a proper subrepresentation of V, contradicting the irreducibility of V. Thus V is an irreducible representation of $\mathfrak{g}$, and is isomorphic to $V(\lambda)$ for some $\lambda \in P^+$.

2. We omit the proof of 2.

□

2.3 Representation algebras

We consider one final definition of a current algebra. The current algebra appearing in annular Khovanov homology in chapter 4 is an example of this type of current algebra.

Definition Let $\mathfrak{g}$ be a Lie algebra as above and let V be some finite-dimensional representation of $\mathfrak{g}$. Then the *representation algebra* associated to the pair $\mathfrak{g}(V)$ is the vector space $\mathfrak{g}(V) := \mathfrak{g} \oplus V$ with Lie bracket

$$[(x, v), (y, w)]_{\mathfrak{g}(V)} = ([x, y]_{\mathfrak{g}}, x \cdot v - y \cdot w) \tag{2.1}$$

for all $x, y \in \mathfrak{g}$ and all $v, w \in V$.

A routine calculation shows that this bracket satisfies the Jacobi identity, and is alternating, so this is well-defined Lie algebra. There is also a $\mathbb{Z}_2$-graded version of the representation algebra (a definition of a Lie superalgebra is found in section 1.5.2):

Definition Let $\mathfrak{g}$ and V be as above. Then the *representation superalgebra* is the Lie superalgebra $\mathfrak{g}^-(V)$ with underlying vector space $\mathfrak{g} \oplus V$ and Lie bracket as in equation 2.1. The $\mathbb{Z}_2$-grading is defined to be: $\mathfrak{g}^-(V)[0] = \mathfrak{g}$ and $\mathfrak{g}(V)^-[1] = V$.

While from the definition it may seem as though $\mathfrak{g}(V)$ and $\mathfrak{g}^-(V)$ are indistinguishable, the $\mathbb{Z}_2$ structure becomes apparent when passing to the universal enveloping algebras of $\mathfrak{g}(V)$ and $\mathfrak{g}^-(V)$, or when studying their representations. For example, in $U(\mathfrak{g}(V))$ the equality $[(0, v), (0, v)] = 0$ is trivial, whilst in $U(\mathfrak{g}^-(V))$ this equality is equivalent to $v^2 = 0$ for all

$v \in V$, which is not necessarily true in $U(\mathfrak{g}(V))$. Thus, for any $v \in V$, v is a nilpotent linear operator acting on any representation W of $\mathfrak{g}^-(V)$ is nilpotent, but this does not have to hold on a representation of $\mathfrak{g}(V)$.

2.3.1 Examples

The simplest example of a representation algebra is the Lie algebra $\mathfrak{sl}_2(V_1)$, where V_1 is the standard representation of $\mathfrak{sl}_2$. This is a five-dimensional Lie algebra with basis $\{e, f, h, v_1, v_{-1}\}$ and (non-trivial) relations:

1. $\mathfrak{sl}_2$ relations as above
2. $[e, v_1] = [f, v_{-1}] = 0$
3. $[e, v_{-1}] = v_1$
4. $[f, v_1] = v_{-1}$
5. $[h, v_1] = v_1$, $[h, v_{-1}] - v_{-1}$
6. $[v_1, v_{-1}] = 0$

We can also consider the dual of this representation algebra, the Lie superalgebra $\mathfrak{sl}_2^-(V_1)$, with the same definition as above, but with a $\mathbb{Z}_2$-grading, so that we have the addition non-trivial relations $[v_1, v_1] = 0$ and $[v_{-1}, v_{-1}] = 0$.

While these examples have a simple presentation and are in some sense the smallest non-trivial extensions of $\mathfrak{sl}_2$ by one of its representations, we show in section 2.4.2 that representations of $\mathfrak{sl}_2(V_1)$ and $\mathfrak{sl}_2^{-1}(V_1)$ are already much more complex structure than the representations of $\mathfrak{sl}_2$ itself.

Main example: $\mathfrak{sl}_2^-(V_2)$

A representation superalgebra that is of particular interest in the context of annular Khovanov homology is the algebra $\mathfrak{sl}_2^-(V_2)$, where we recall that V_2 is the three-dimensional irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ with highest weight 2. The superalgebra $\mathfrak{sl}_2^-(V_2)$ is a six-dimensional complex vector space, with basis $\{e, f, h, v_{-2}, v_0, v_2\}$ and relations:

1. $\mathfrak{sl}_2$ relations: $[h, e] = 2e, [h, f] = -2f$ and $[e, f] = h$
2. $[e, v_2] = [f, v_{-2}] = [h, v_0] = 0$
3. $[e, v_0] = -2v_2$
4. $[f, v_0] = 2v_{-2}$
5. $[e, v_{-2}] = v_0$

6. $[f, v_2] = -v_0$
7. $[h, v_2] = 2v_2, \; [h, v_{-2}] = -2v_{-2}$
8. $[v_i, v_j] = 0$ for $i, j \in \{-2, 0, 2\}$

Note that the subscripts on basis elements v_{-2}, v_0 and v_2 indicate the weight spaces that the basis elements generate. The relations are directly computed from the definition of the Lie bracket on $\mathfrak{sl}_2^-(V_2)$, and constants arise from the choice of basis vectors v_i.

2.4 Quiver representations

As noted previously, current algebras $\mathfrak{g}(V)$ are not semisimple, so the relations between representations are nontrivial. A way to visualise this added complexity and classify representations is to consider quiver representations, as introduced in section 1.4. Here we demonstrate this method in some simple examples of current algebras and contrast these to the semisimple Lie algebra case.

2.4.1 Semisimple Lie algebras

Semisimple Lie algebras have the property that their finite dimensional representations are completely reducible, from a theorem of Weyl [Wey68]. For this reason, it suffices to understand the irreducible representations of a semisimple Lie algebra, and by Schur's lemma [**?**], there can be no non-trivial maps between non-isomorphic irreducible representations. In the case of $\mathfrak{sl}_2$, where there is exactly one irreducible representation for each highest weight, or equivalently one of each dimension, this means that there can be no non-trivial maps between different V_k. Consequently, the quiver representing the category of finite-dimensional $\mathfrak{sl}_2$-modules is particularly simple, as demonstrated in the following result, which is a restatement in the language of quivers of the classification of the irreducible finite-dimensional representations and complete reducibility of finite-dimensional representations of $\mathfrak{sl}_2$, as seen in section 1.9.

Proposition 2.4.1. *The category of finite-dimensional representations of* $\mathfrak{sl}_2$ *is equivalent to the category of representations of the following quiver* Q*:*

Note that **Rep**-Q is equivalent to **Vect**, since there are no arrows between the vertices of Q, namely a finite-dimensional representation of Q consists of a terminating sequence of vector spaces.

Proof. We begin by defining a functor $\mathcal{F}$ from **Rep**-$\mathfrak{sl}_2$ to **Rep**-Q. Let M be a representation of $\mathfrak{sl}_2$. Then M decomposes into a direct sum of irreducible representations:

$$M = \bigoplus_{i \geq 0} V_i^{n_i}$$

where n_i is the number of copies of the irreducible representation V_i. Let E_i be the highest weight space of $V_i^{n_i}$, namely

$$E_i = \{e_i \in V_i \mid e \cdot e_i = 0 \text{ and } h \cdot e_i = ie_i\}.$$

Since the weight spaces of an irreducible representations are one-dimensional, E_i has dimension n_i. By Schur's lemma, there are no non-trivial maps between the $V_i^{n_i}$, so in particular no maps between the highest weight spaces. Then the functor $\mathcal{F}$ takes a representation M to the representation of Q associating the vector space E_i to the ith vertex in Q.

Conversely, let W_i be the vector spaces corresponding to vertices v_i in the quiver Q. Then define a representation V of $\mathfrak{sl}_2$ by setting each W_i to be a highest weight space of weight i in V. Since V decomposes into a direct sum of irreducible representations, each generated by a single highest weight vector, the definition of the highest weight spaces in V completely determines V as a $\mathfrak{sl}_2$ representation. The number of copies of a given irreducible representation of highest weight i is determined by the dimension of W_i. These constructions are clearly inverse to each other. □

Proposition 2.4.2. *More generally, the category of finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$ is equivalent to the category of finite-dimensional representations of a quiver Q with vertex index set $I = P^+$, where P^+ is the set of dominant weights of $\mathfrak{g}$, and no edges.*

Proof. The proof clearly extends from the $\mathfrak{sl}_2$ case. Any finite-dimensional representation V of $\mathfrak{g}$ decomposes into a direct sum of irreducible representations, each isomorphic to $V(\lambda)$ for some $\lambda \in P^+$, where $V(\lambda)$ is an irreducible representation of highest weight λ. The highest weight spaces for each λ determines the representation V and define a representation of Q.

Conversely, a representation of Q consists of vector spaces V_λ for each $\lambda \in P^+$, and we define a representation of $\mathfrak{g}$ with highest weight spaces V_λ. □

2.4.2 Loupias' results

We prove the following theorem of Loupias [**?**, Lou72]emonstrating the complex structure of $\mathfrak{sl}_2(V_1)$-**rep** through the use of quiver representations.

Theorem 2.4.3 (Loupias)**.** *The category of finite-dimensional representations of $\mathfrak{sl}_2(V_1)$ is equivalent to the category of finite-dimensional representations of the quiver:*

$$\overset{0}{\bullet} \underset{\beta_0}{\overset{\alpha_0}{\rightleftarrows}} \overset{1}{\bullet} \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \overset{2}{\bullet} \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \overset{3}{\bullet} \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \overset{4}{\bullet} \qquad \cdots$$

with relations

$$\beta_0\ \alpha_0 = 0$$
$$\beta_{i+1}\ \alpha_{i+1} = \alpha_i\ \beta_i \textit{ for all } i \in \mathbb{N}$$

Proof. We demonstrate how to construct a representation of the quiver from a representation of $\mathfrak{sl}_2(V_1)$: let W be a representation of $\mathfrak{sl}_2(V_1)$. Then in particular W is a representation of $\mathfrak{sl}_2$ and we can decompose W into a direct sum of irreducible representations $W = \bigoplus_{i\in\mathbb{N}} W_i^{\oplus n_i}$. Let B_i be the highest weight space for each of the W_i, so that $B_i = \{b_i \in W_i \mid e \cdot b_i = 0 \text{ and } h \cdot b_i = ib_i\}$. The dimension of weight spaces in an irreducible representation is one, so each B_i has dimension n_i. To begin constructing a representation of the quiver, assign the highest weight space B_i to each vertex i. The v_i will determine the linear maps assigned to arrows in the quiver. Define

$$\begin{aligned} \alpha_i : B_i &\to B_{i+1} & \beta_i : B_{i+1} &\to B_i \\ b_i &\mapsto v_1 \cdot b_i & b_{i+1} &\mapsto (i+2)v_{-1} \cdot b_{i+1} - fv_1 \cdot b_{i+1} \end{aligned}$$

for all $i \in \mathbb{N}$. The definition of β_i consists of applying v_{-1} to a vector in B_{i+1} then projecting onto B_i, since $v_{-1} \cdot b_{i+1}$ is not necessarily a highest weight vector in W_i. Then the claim is that these α_i and β_i satisfy the quiver relations. We first check that α_i and β_i are indeed maps on the B_i, using the relations defined in section 2.3.1 and noting that the Lie bracket on the algebra becomes the commutator in $End(W)$.

Let $b_i \in B_i$. Then $ev_1 \cdot b_i = v_1 e \cdot b_i = 0$ since $[e, v_1] = ev_1 - v_1 e = 0$ and b_i is a highest weight vector, so $e \cdot b_i = 0$. Also, $hv_1 \cdot b_i = v_1 h \cdot b_i + v_1 \cdot b_i = (i+1)v_1 \cdot b_i$. From these two equalities, it can be seen that $v_1 \cdot b_i \in B_{i+1}$. Similarly,

$$\begin{aligned} e \cdot \beta_i(b_{i+1}) &= e \cdot [(i+2)v_{-1} \cdot b_{i+1} - fv_1 \cdot b_{i+1}] \\ &= (i+2)v_{-1}e \cdot b_{i+1} + (i+2)v_1 \cdot b_{i+1} - fev_1 \cdot b_{i+1} - hv_1 \cdot b_{i+1} \\ &= (i+2)v_1 \cdot b_{i+1} - (i+2)v_1 \cdot b_{i+1} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} h \cdot \beta_i(b_{i+1}) &= h \cdot [(i+2)v_{-1} \cdot b_{i+1} - fv_1 \cdot b_{i+1}] \\ &= (i+2)(i+1)v_{-1} \cdot b_{i+1} - (i+2)v_{-1} \cdot b_{i+1} - fhv_1 \cdot b_{i+1} + 2fv_1 \cdot b_{i+1} \\ &= i[(i+2)v_{-1} \cdot b_{i+1} - fv_1 \cdot b_{i+1}] \end{aligned}$$

so $\beta(b_{i+1}) \in B_i$ for all $b_{i+1} \in B_{i+1}$.

For the quiver relations, since v_1 and v_{-1} commute, $fv_1^2 = v_1^2 f + 2v_{-1}v_1$:

$$\begin{aligned} \beta_0\alpha_0(b_0) &= \beta_0(v_1 \cdot b_0) \\ &= 2v_{-1}v_1 \cdot b_1 - fv_1^2 \cdot b_0 \\ &= 2v_{-1}v_1 \cdot b_1 - v_1^2 f \cdot b_0 - 2v_{-1}v_1 \\ &= 0 \end{aligned}$$

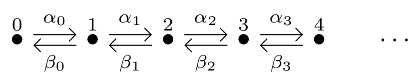
since $b_0 \in B_0$, where B_0 is a direct sum of copies of the trivial representation.

$$\begin{aligned}
\beta_{i+1}\alpha_{i+1}(b_{i+1}) &= \beta_{i+1}(v_1 \cdot b_{i+1}) \\
&= (i+3)v_{-1}v_1 \cdot b_{i+1} - fv_1^2 \cdot b_{i+1} \\
&= (i+3)v_{-1}v_1 \cdot b_{i+1} - v_1^2 f \cdot b_{i+1} - 2v_{-1}v_1 \cdot b_{i+1} \\
&= (i+1)v_{-1}v_1 \cdot b_{i+1} - v_1^f \cdot b_{i+1} \\
\alpha_i\beta_i(b_{i+1}) &= \alpha_i[(i+2)v_{-1} \cdot b_{i+1} - fv_1 \cdot b_{i+1}] \\
&= (i+2)v_1v_{-1} \cdot b_{i+1} - v_1 f v_1 \cdot b_{i+1} \\
&= (i+2)v_1v_{-1} \cdot b_{i+1} - v_1^2 f \cdot b_{i+1} - v_1v_{-1} \cdot b_{i+1} \\
&= \beta_{i+!}\alpha_{i+1}(b_{i+1}).
\end{aligned}$$

This is a representation of the quiver constructed from the representation of $\mathfrak{sl}_2(V_1)$. The inverse process consists of defining each vector space V_i assigned to the vertices i in the quiver to be the highest weight space of the direct sum of irreducible representations W_i of highest weight i. This determines the $\mathfrak{sl}_2$-action on the representation $V = \bigoplus_{i \in \mathbb{N}} W_i$, with $V_i \subset W_i$ is the highest weight space of W_i. Then define the action of the v_i by $v_1 \cdot b_i = \alpha_i(b_i)$ and $v_{-1} \cdot b_{i+1} = \frac{1}{i+2}(\beta_i(b_{i+1}) + f\alpha_{i+1}(b_{i+1}))$ and extending this action to the whole representation using the commutation relations. Then these constructions are inverse to each other, so any map of $\mathfrak{sl}_2(V_1)$ representations induces a map of quiver representations and vice versa. □

A similar result holds for the Lie superalgebra $\mathfrak{sl}_2^-(V_1)$ [HK01]:

Proposition 2.4.4. *The category of finite-dimensional representations of $\mathfrak{sl}_2^-(V_1)$ is equivalent to the category of finite-dimensional representations of the quiver:*



with relations

$$\alpha_{i+1}\ \alpha_i = \beta_i\ \beta_{i+1} = 0 \textit{ for all } i \in \mathbb{N}$$
$$\beta_{i+1}\ \alpha_{i+1} = \alpha_i\ \beta_i \textit{ for all } i \in \mathbb{N}.$$

The proof of this proposition is entirely similar to that of the previous theorem, though noting that there is a further non-trivial relation in the superalgebra $\mathfrak{sl}_2^-(V_1)$, namely $[v_i, v_i] = 0$ for $i = \pm 1$, which leads to the first set of relations on the quiver.

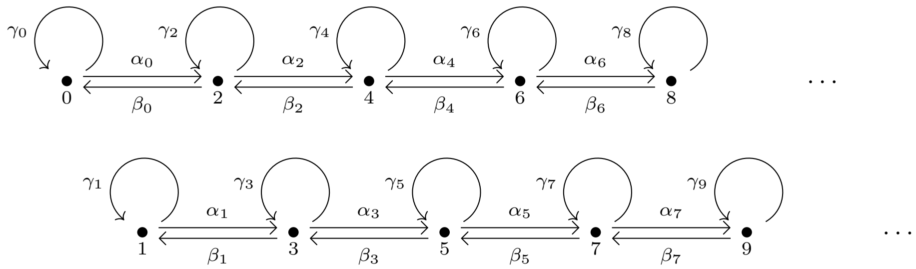
Note that this quiver together with the relations above has path algebra Z which is an example of a zigzag algebra, in particular the zigzag algebra of the chain

Zigzag algebras will play an essential role in the categorification of the adjoint representation seen in section 3.2.

2.4.3 The representation superalgebra $\mathfrak{sl}_2^-(V_2)$

We now give an extension of Loupias' results to describe the finite-dimensional representations of the superalgebra $\mathfrak{sl}_2^-(V_2)$. The quiver corresponding to $\mathfrak{sl}_2^-(V_2)$ is now disconnected, a property that holds for all $\mathfrak{sl}_2^{\pm}(V_n)$ for even n.

Theorem 2.4.5. *The category of finite-dimensional representations of the algebra $\mathfrak{sl}_2^-(V_2)$ is equivalent to the category of finite-dimensional representations of the quiver*



with relations:

$$
\begin{aligned}
&\gamma_{i+2}\ \alpha_i + \alpha_i\ \gamma_i = 0\\
&\beta_i\ \gamma_{i+2} + \gamma_i\ \beta_i = 0\\
&\alpha_{i+2}\ \alpha_i = 0\\
&\beta_i\ \beta_{i+2} = 0\\
&\beta_{i+2}\ \alpha_{i+2} + \alpha_i\ \beta_i = 0\\
&\gamma_i^2 = 4(i+2)\beta_i\ \alpha_i
\end{aligned}
$$

for all $i \in \mathbb{N}$.

Proof. The proof proceeds as for Loupias' theorem. Let the highest weight spaces B_i of the copies of irreducible $\mathfrak{sl}_2$ representations be the vector spaces corresponding to each vertex in the quiver. Define

$$
\begin{aligned}
\alpha_i &: B_i \to B_{i+2} & \beta_i &: B_{i+2} \to B_i & \gamma_i &: B_i \to B_i\\
b_i &\mapsto v_2 \cdot b_i & b_{i+2} &\mapsto y_i \cdot b_{i+2} & b_i &\mapsto x_i \cdot b_i
\end{aligned}
$$

where $x_i = (i+2)v_0 - 2fv_2$ and $y_i = v_{-2} - \frac{1}{(i+2)(i+4)} f x_{i+2} - \frac{2}{(i+2)(i+4)} f^2 v_2$. Thus, the maps β_i and γ_i are projections onto the spaces B_i of v_{-2} and v_0 respectively.

Using the relations on the Lie algebra, we get the relations on the quiver. Note that we are only interested in maps on the B_i, so in all the following computations, we implicitly apply the projection map onto B_i, under which any term of the form fA is sent to zero, for A any composition of elements in $\mathfrak{sl}_2^-(V_2)$.

1. Immediately from $v_2^2 = 0$, we must have $\alpha_{i+2}\alpha_i = 0$ for all i.

2. $v_{-2}^2 = 0$:

$$\begin{aligned}
0 &= v_{-2}^2 \cdot b_{i+4} \\
&= v_{-2}(\phi(i+2)y_{i+2} \cdot b_{i+4} + \psi(i+2)fx_{i+4} \cdot b_{i+4} - \eta(i+2)f^2v_2 \cdot b_{i+4}) \\
&= \phi(i)\phi(i+2)y_iy_{i+2} \cdot b_{i+4} + \psi(i+2)v_{-2}fx_{i+4} \cdot b_{i+4} + \eta(i+2)v_{-2}f^2v_2 \cdot b_{i+4} \\
&= \phi(i)\phi(i+2)i_iy_{i+2}b_{i+4} \text{ since } v_{-2}f = fv_{-2}.
\end{aligned}$$

for all $i \in \mathbb{N}$ and all $b_{i+4} \in B_{i+4}$. Thus, $\beta_i\beta_{i+2} = 0$ for all $i \in \mathbb{N}$.

3. $v_0v_2 + v_2v_0 = 0$:

$$\begin{aligned}
v_2v_0 \cdot b_i &= \frac{1}{i+4}x_{i+2}v_2 \cdot b_i + \frac{2f}{i+4}v_2^2 \cdot b_i \\
&= \frac{1}{i+4}x_{i+2}v_2 \cdot b_i \\
v_0v_2 \cdot b_i &= v_2(\frac{1}{i+2}x_i \cdot b_i + \frac{2}{i+2}fv_2 \cdot b_i) \\
&= \frac{1}{i+2}v_2x_i \cdot b_i + \frac{2}{i+2}v_2fv_2 \cdot b_i \\
&= \frac{1}{i+2}v_2x_i \cdot b_i + \frac{2}{i+2}v_0v_2 \cdot b_i \\
\text{Hence, } 0 &= (v_2v_0 + v_0v_2) \cdot b_i \\
&= \frac{1}{i+2}v_2x_i \cdot b_i + \frac{2+i+2}{(i+2)(i+4)}x_{i+2}v_2 \cdot b_i
\end{aligned}$$

for all $i \in \mathbb{N}$ and all $b_i \in B_i$. This implies the relation $\gamma_{i+2}\alpha_i = \alpha_i\gamma_i$ for all $i \in \mathbb{N}$.

4. $v_{-2}v_0 + v_0v_{-2} = 0$:

$$\begin{aligned}
v_{-2}v_0 \cdot b_{i+2} &= v_{-2}(\frac{1}{i+4}x_{i+2} \cdot b_{i+2} + \frac{2}{i+4}fv_2 \cdot b_{i+2} \\
&= \frac{1}{i+4}y_ix_{i+2} \cdot b_{i+2} + \frac{2}{i+4}v_{-2}fv_2 \cdot b_{i+2} \\
&= \frac{1}{i+4}y_ix_{i+2} \cdot b_{i+2} \\
v_0v_{-2} \cdot b_{i+2} &= v_0(y_ib_{i+2} - \frac{1}{(i+2)(i+4)}fx_{i+2} \cdot b_{i+2} - \frac{2}{(i+2)(i+4)}f^2v_2 \cdot b_{i+2}) \\
&= \frac{1}{i+2}x_iy_i \cdot b_{i+2} - \frac{1}{(i+2)(i+4)}v_0fx_{i+2}b_{i+2} - \frac{2}{(i+2)(i+4)}v_0f^2v_2 \cdot b_{i+2} \\
&= \frac{1}{i+2}x_iy_i \cdot b_{i+2} + \frac{2}{(i+2)(i+4)}v_{-2}x_{i+2} \cdot b_{i+2} \\
&= \frac{1}{i+2}x_iy_i \cdot b_{i+2} + \frac{2}{(i+2)(i+4)}y_ix_{i+2} \cdot b_{i+2} \\
\text{Hence, } 0 &= \frac{1}{i+4}y_ix_{i+2} \cdot b_{i+2} + \frac{1}{i+2}x_iy_i \cdot b_{i+2} + \frac{2}{(i+2)(i+4)}y_ix_{i+2} \cdot b_{i+2} \\
&= \frac{1}{i+2}(y_ix_{i+2} \cdot b_{i+2} + x_iy_i \cdot b_{i+2})
\end{aligned}$$

for all $\imath \in \mathbb{N}$ and $b_{i+2} \in B_{i+2}$. Thus, $\beta_i\gamma_{i+2} + \gamma_i\beta_i = 0$ for all $\imath \in \mathbb{N}$.

5. $v_0^2 = 0$:

$$\begin{aligned}
v_0^2 \cdot b_i &= v_0(\frac{1}{i+2}x_i \cdot b_i + \frac{2}{i+2}fv_2 \cdot b_i \\
&= \frac{1}{(i+2)^2}x_i^2 \cdot b_i + \frac{2}{i+2}v_0 f v_2 \cdot b_i \\
&= \frac{1}{(i+2)^2}x_i^2 \cdot b_i - \frac{4}{i+2}v_{-2}v_2 \cdot b_i \\
\text{Hence, } x_i^2 \cdot b_i &= 4(i+2)v_{-2}v_2 \cdot b_i
\end{aligned}$$

for all $b_i \in B_i$.

6. $v_{-2}v_2 + v_2v_{-2} = 0$:

$$\begin{aligned}
v_{-2}v_2 \cdot b_{i+2} &= y_{i+2}v_2 \cdot b_{i+2} \text{ after projecting onto } B_{i+2} \\
\text{Hence, } x_{i+2}^2 \cdot b_{i+2} &= 4(i+4)y_{i+2}v_2 \cdot b_{i+2}
\end{aligned}$$

for all $b_{i+2} \in B_{i+2}$, so $\beta_i\gamma_{i+2} + \gamma_i\beta_i = 0$ for all $i \in \mathbb{N}$.

$$\begin{aligned}
v_2v_{-2} \cdot b_{i+2} &= v_2(y_i \cdot b_{i+2} - \frac{1}{(i+2)(i+4)}fx_{i+2} \cdot b_{i+2} - \frac{2}{(i+2)(i+4)}f^2v_2 \cdot b_{i+2}) \\
&= v_2y_i \cdot b_{i+2} - \frac{1}{(i+2)(i+4)}v_0x_{i+2} \cdot b_{i+2} + \frac{4}{(i+2)(i+4)}v_{-2}v_2 \cdot b_{i+2} \\
&= v_2y_i \cdot b_{i+2} - \frac{1}{(i+2)(i+4)^2}x_{i+2}^2 \cdot b_{i+2} + \frac{4}{(i+2)(i+4)}y_{i+2}v_2 \cdot b_{i+2} \\
&= v_2y_i \cdot b_{i+2} \\
\text{Hence } 0 &= v_{-2}v_2 \cdot b_{i+2} + v_2v_{-2} \cdot b_{i+2} \\
&= y_{i+2}v_2 \cdot b_{i+2} + v_2y_i \cdot b_{i+2}
\end{aligned}$$

for all $b_{i+2} \in B_{i+2}$. Thus, $\beta_{i+2}\alpha_{i+2} + \alpha_i\beta_i = 0$ for all $i \in \mathbb{N}$.

For the inverse construction, define each vector space associated to the vertex i to be the highest weight space as in the $\mathfrak{sl}_2$ case. Then the action of $\mathfrak{sl}_2$ is determined on each B_i by

$$\begin{aligned}
v_2 \cdot b_i &= \alpha_i b_i \\
v_0 \cdot b_i &= \frac{1}{i+2}(\beta_i + 2f\alpha_i) \cdot b_i \\
v_{-2} \cdot b_i &= \gamma_i + \frac{1}{(i+2)(i+4)}(f\beta_i - 2f^2\alpha_i)
\end{aligned}$$

on the basis vectors of the highest weight spaces, and extend to the total representation by the commutation relations. □

Chapter 3

Categorification

In this chapter we categorify certain irreducible representations of quantum groups. In particular we construct a category $\mathcal{C}$ such that the Grothendieck group $K(\mathcal{C})$ of the category is isomorphic to a chosen representation of the quantum group of a semisimple Lie algebra. More explicitly, given a semisimple Lie algebra $\mathfrak{g}$ with quantum enveloping algebra (quantum group) $U_q(\mathfrak{g})$, and a representation V_λ of $U_q(\mathfrak{g})$ with highest weight λ, we construct categories $\mathcal{C}_\mu$ for each weight μ appearing in the decomposition $V_\lambda = \oplus_\mu V_\lambda(\mu)$ into weight spaces such that the Grothendieck group of $\mathcal{C}_\mu$ is isomorphic to $V_\lambda(\mu)$. Furthermore, we lift the action of the generators of $U_q(\mathfrak{g})$ to functors acting between the categories $\mathcal{C}_\mu$. We also demonstrate how further structure of the quantum group and its representation can be lifted to the categorical level.

In the first example, $\mathfrak{g} = \mathfrak{sl}_n$ and the lifted representation has highest weight $2\omega_k$ where ω_k is a fundamental weight of $\mathfrak{sl}_n$. Since in this case the dimension of the weight spaces $V_\lambda(\mu)$ are Catalan numbers, this is an example of a "bicategorification", that is, replace a number to a vector space and then replace the vector space by a category:

$$\text{number} \underset{\text{dimension}}{\overset{\text{categorification}}{\rightleftarrows}} \text{vector space} \underset{\text{Grothendieck group}}{\overset{\text{categorification}}{\rightleftarrows}} \text{category}$$

More specifically, in this example the number is the mth catalan number $c_m = \frac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix}$, the vector space is a weight space V_λ of a particular representation V of $U_q(\mathfrak{sl}_n)$, and there is a category $\mathcal{C}_\lambda$ corresponding to each weight space V_λ:

$$\frac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix} \underset{\text{dimension}}{\overset{\text{categorification}}{\rightleftarrows}} V_\lambda \underset{\text{Grothendieck group}}{\overset{\text{categorification}}{\rightleftarrows}} \mathcal{C}_\lambda$$

Furthermore, the quantum group $U_q(\mathfrak{sl}_n)$ acts on the representation $V = \bigoplus_\lambda$, and this action lifts to a categorical action on the category $\mathcal{C} := \bigoplus_\lambda V_\lambda \mathcal{C}_\lambda$, with functors lifting the generators of $U_q(\mathfrak{sl}_n)$ mapping between the categories $\mathcal{C}_\lambda$, so that the picture becomes:

$$\left\{ \frac{1}{m(\lambda)+1} \begin{pmatrix} 2m(\lambda) \\ m(\lambda) \end{pmatrix} \right\} \underset{\text{dimension}}{\overset{\text{categorification}}{\rightleftarrows}} \bigoplus_\lambda V_\lambda \circlearrowleft U_q(\mathfrak{sl}_n) \underset{\text{Grothendieck group}}{\overset{\text{categorification}}{\rightleftarrows}} \bigoplus_\lambda \mathcal{C}_\lambda \circlearrowleft \mathcal{U}_Q(\mathfrak{sl}_n)$$

Both examples of categorified representations considered here do not just lift the basic structure of the original representation, but lift further structures on both the representation and the quantum group, satisfying another of the main objectives of categorification.

3.1 Categorifying some level-2 representations of $U_q(\mathfrak{sl}_n)$

Definition A *level two representation* of a quantum group $U_q(\mathfrak{g})$ is an irreducible representation of highest weight $q^{(}\omega_j + \omega_k)$, where ω_j and ω_k are fundamental weights of $\mathfrak{g}$.

We give an exposition of the work of Khovanov and Huerfano in [HK06]. Note that a categorification of the corresponding level-two representation of the Lie algebra $\mathfrak{sl}_n$ is obtained by forgetting the grading in the constructed category.

Let $\omega_1, \dots, \omega_{n-1}$ be the fundamental dominant weights of the Lie algebra $\mathfrak{sl}_n$, and let V be the irreducible representation of the associated quantum group $U_q(\mathfrak{sl}_n)$ with highest weight $q^{2\omega_k}$, for some $0 \leq k \leq n-1$. As seen in chapter 1, any highest weight can be represented by a sequence of n integers, determined by coefficients of fundamental weights. In this case, the highest weight is $2\omega_k = (0, \dots, 0, 2, 0, \dots, 0)$, where the 2 occurs in the kth position in the sequence.

Decompose the representation V into a direct sum of weight spaces V_λ. A weight λ is called *admissible* if $V_\lambda \neq 0$. Then the admissible weights for this representation are represented by sequences of n integers satisfying

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \text{ such that } \sum_{i=1}^{n} \lambda_i = 2k \text{ and } 0 \leq \lambda_i \leq 2$$

Let $m(\lambda)$ be half the number of 1s that appear in the sequence λ. For example, if we are considering $\mathfrak{sl}_5$, and $k = 3$, then an admissible weight is $\lambda = (1, 1, 2, 1, 1)$, corresponding to the weight $(0, -1, 1, 0)$ and $m(\lambda) = 2$.

The requirement that $\sum \lambda_i = 2k$, with $\lambda_i \leq 2$, ensures that $0 \leq m(\lambda) \leq \min\{k, n-k\}$: there can be at most $2k$ 1s in the sequence, so that $\sum \lambda_i \leq 2k$, and the requirement $m(\lambda) \leq n-k$ ensures $\sum_{\lambda_i} \geq 2k$. It is also clear from the requirement that the sum be even that $m(\lambda)$ is an integer.

Lemma 3.1.1. *The dimension of the weight space for admissible λ is dependent only on $m := m(\lambda)$, and is equal to the m-th Catalan number $c_m = \frac{1}{m+1}\begin{pmatrix} 2m \\ m \end{pmatrix}$.*

Proof. By Schur-Weyl duality, the multiplicities of weight spaces V_λ are given by the Kostka numbers, the number of semistandard Young tableaux of a given shape and content. More specifically, given a weight space V_λ, we have $\dim(V_\lambda) = K_{\nu\lambda}$ where $K_{\nu\lambda}$ is the Kostka number associated to the shape ν, and content λ. Here ν is the partition $\nu = (2, 2, \ldots, 2, 0, \ldots, 0)$ consisting of k 2s and $n - k$ 0s. This is the sequence corresponding to the highest weight $2\omega_k$ of the representation V.

It is known that the m-th Catalan number is the number of standard Young tableaux of shape $2 \times m$. The highest weight $2\omega_k$ given by the partition $\nu = (2, \ldots, 2, 0, \ldots, 0)$ as above, corresponds to the Young tableau of shape $2 \times k$. Admissible weights μ consist of ordered n-tuples of integers 0,1, and 2, such that the sum is $2k$. This defines a weight of the tableau by setting the ith coefficient in the sequence to be the number of is in the weight of the tableau. Returning to the previous example for $n = 5$, $k = 3$, let $\lambda = (1, 1, 2, 1, 1)$. This defines the content of the Young tableau to be one 1, one 2, two 3s, one 4 and one 5, so an example of a semistandard Young tableau of shape ν and weight λ is:

1	2
3	3
4	5

To construct a semistandard tableau, any pair of integers corresponding to a 2 in the sequence μ must occur in the same row. Deleting any row consisting of the same number, the Kostka number $K_{\nu\lambda}$ is equal to the simplified Kostka number $K_{\nu',\lambda'}$ where ν' is a $2 \times m$ rectangle and m is half the number of 1s in the sequence λ and λ' is derived from λ by taking λ_i mod 2 for all elements λ_i in the sequence λ. For example, with λ as above, $\lambda' = (1, 1, 0, 1, 1)$. Therefore there are $2m$ non-zero elements in λ', all equal to 1, so $K_{\nu'\lambda'}$ is exactly the m-th Catalan number. □

As in chapter 1, denote the generators of $U_q(\mathfrak{sl}_n)$ by E_i, F_i, K_i and K_i^{-1}.

Let $\varepsilon_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$ be the sequence of n integers with ith entry equal to 1 and $i + 1$th entry equal to -1. Then $E_i \cdot V_\lambda \subset V_{\lambda+\varepsilon_i}$, and $E_i \cdot V_\lambda = 0$ if $\lambda + \varepsilon_i$ is not admissible. Similarly, $F_i \cdot V_\lambda \subset V_{\lambda-\varepsilon_i}$, and F_i acts trivially on V_λ if $\lambda - \varepsilon_i$ is not admissible.

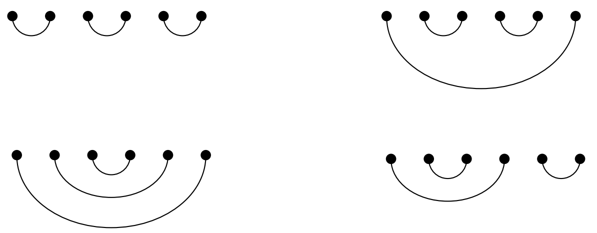
The aim is now to construct a categorification of the level two representation V using a geometric context in which the Catalan numbers arise. As a point of reference, we will construct an abelian category $\mathcal{C} = \bigoplus \mathcal{C}(\lambda)$ where λ ranges over the admissible weights for V and each of the $\mathcal{C}(\lambda)$ are categories of finitely-generated graded H_λ modules, with the action of $U_q(\mathfrak{sl}_n)$ on V lifting to an action on $\mathcal{C}$. Our first step is to construct the H_λ. We will then describe the action of $U_q(\mathfrak{sl}_n)$ on $\mathcal{C}$ and look into some of the structure of this category.

The rings H_λ

Let λ be an admissible weight and let $s = (s_1, s_2, \ldots, s_{2m})$ be the sequence of integers describing the positions of the 1s in λ in order (so that $s_1 < s_2 < \ldots < s_{2m}$): for example, if $\lambda = (1, 2, 1, 1, 2, 2, 1, 1, 1, 0, 2)$ then $s(\lambda) = (1, 3, 4, 7, 8, 9)$. Place marked points on the horizontal axis at each s_i.

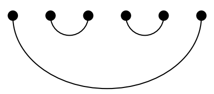
Define B^s to be the set of matchings of the s_i with no quadruple $s_i < s_j < s_k < s_l$ such that i is matched with k and j is matched with l. Then we can visualise B^s as the set of crossingless matchings of the $2m$ points s_i, and the size of B^s is precisely the m-th Catalan number. For simplicity, all figures depict the case where the s_i are the integers $1, 2, \ldots, 2m$, with corresponding set of crossingless matchings denoted B^m.

For example, the set B^3 is:



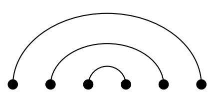
Let a and b be elements in B^m and denote by $R(b)$ the reflection of b about the horizontal axis. Then we can form a closed 1-manifold, denoted $R(b)a$, by gluing together $R(b)$ and a at the endpoints s_i as shown below. Our convention is to read all diagrams from bottom to top.

Let a be

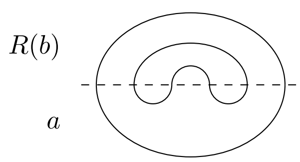


and let b be

So that $R(b)$ is



Resulting in the closed 1-manifold $R(b)a$:



For any pairs $(a, b) \in B^m \times B^m$, $R(b)a$ is a closed one-manifold. To obtain the rings H_λ, we apply a functor $\mathcal{Q}$, more specifically a two-dimensional topological quantum field theory (TQFT) from the category ${}_1\mathbf{Cob}$ of closed 1-manifolds and oriented (1+1)-cobordisms to the category of abelian groups $\mathbf{Ab}$. This method of applying a TQFT will reappear in chapter

5, when discussing the construction of Khovanov homology. The main building block for this functor is the following Frobenius ring $\mathcal{A}$:

Let $\mathcal{A}$ be the cohomology ring $H^*(S^2, \mathbb{Z})$ of the 2-sphere, so that $\mathcal{A} \cong \mathbb{Z}[X]/(X^2)$ (where X is a generator for $H^2(S^2, \mathbb{Z})$). Then the nondegenerate trace form $\mathrm{tr} : \mathcal{A} \to \mathbb{Z}$ on $\mathcal{A}$ is given by:

$$\mathrm{tr}(1) = 0 \qquad\qquad \mathrm{tr}(X) = 1$$

and the unit map $\varepsilon : \mathbb{Z} \to \mathcal{A}$ is defined by $\varepsilon(1) = 0$ and $\varepsilon(X) = 1$. Define a grading on $\mathcal{A}$ by setting $\deg(1) = -1$ and $\deg(X) = +1$. Then the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and comultiplication map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ are degree 1 maps defined as follows:

$$m : \begin{cases} 1 \otimes 1 \mapsto 1 \\ 1 \otimes X \mapsto X \\ X \otimes 1 \mapsto X \\ X \otimes X \mapsto 0 \end{cases} \qquad\qquad \Delta : \begin{cases} 1 \mapsto 1 \otimes X + X \otimes 1 \\ X \mapsto X \otimes X \end{cases}$$

The TQFT $\mathcal{Q}$ is given by the following:

Given a disjoint union $\mathscr{O}_i$ of i circles, $\mathcal{Q}(\mathscr{O}_i) = \mathcal{A}^{\otimes i}$.

Viewing the three-holed sphere $S^2_{1,2}$ as a cobordism from one circle to two circles,

$$\mathcal{Q}(S^2_{2,1}) = \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}.$$

Viewing the three-holed sphere $S^2_{2,1}$ as a cobordism from two circles to one,

$$\mathcal{Q}(S^2_{1,2}) = m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}.$$

Viewing the disc $D^2_{1,0}$ as a cobordism from one circle to the empty manifold,

$$\mathcal{Q}(D^2_{1,0}) = \mathrm{tr} : \mathcal{A} \to \mathbb{Z}.$$

Viewing the disc $D^2_{0,1}$ as a cobordism from the empty manifold to one circle,

$$\mathcal{Q}(D^2_{0,1}) = \varepsilon : \mathbb{Z} \to \mathcal{A},$$

where ε is the unit of $\mathcal{A}$ and tr is the trace of $\mathcal{A}$.

$R(b)a$ is a disjoint union of cycles, hence an object of ${}_1\mathbf{Cob}$, so we apply the functor $\mathcal{Q}$ to it, obtaining $\mathcal{Q}(R(b)a) \cong \mathcal{A}^{\otimes i}$ where i is the number of circles in $R(b)a$.

Define the ring

$$H_\lambda := \bigoplus_{a,b \in B^s} {}_b(H_\lambda)_a \text{ where } \qquad\qquad {}_b(H_\lambda)_a := \mathcal{Q}(R(b)a)\{m(\lambda)\}$$

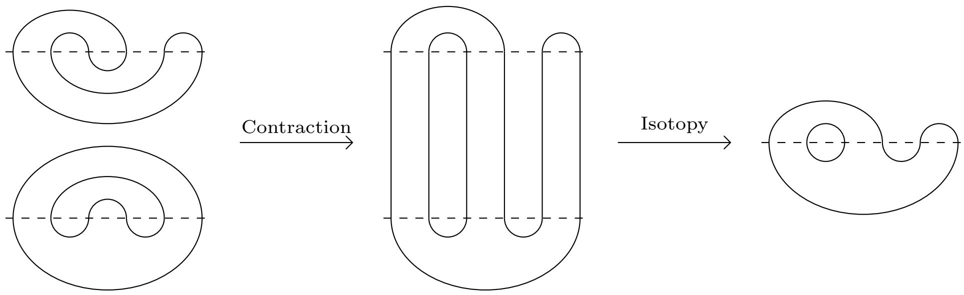


Figure 3.1: Contraction from $R(c)bR(b)a$ to $R(c)a$

and $\{m(\lambda)\}$ denotes an upwards grading shift by $m(\lambda)$. To define multiplication in the ring H_λ, note that for all a, b and c in B^s, there is a canonical cobordism form $R(c)bR(b)a$ to $R(c)a$ given by contracting b with $R(b)$ as shown below:

The contraction cobordism is in fact a surface in $\mathbb{R} \times [0,1] \times [0,1]$, see [Kho02]. This cobordism induces a group homomorphism $\mathcal{Q}(R(c)b) \otimes \mathcal{Q}(R(b)a) \to \mathcal{F}(R(c)a)$. Thus, multiplication in the ring H_λ ${}_d(H_\lambda)_c \otimes {}_b(H_\lambda)_a \to {}_d(H_\lambda)_a$ is defined to be the homomorphism $\mathcal{Q}(R(d)c) \otimes \mathcal{Q}(R(b)a) \to \mathcal{Q}(R(d)a)$ induced by the contraction of c with $R(b)$ if $b = c$ and 0 if $b \neq c$, over all $a, b, c, d \in B^s$. This multiplication is associative and grading-preserving after applying the shift $\{m(\lambda)\}$, where associativity follows from the fact that the one-manifolds corresponding to products $(xy)z$ and $x(yz)$ are isotopic for all $x, y, z \in \mathrm{Ob}_1\mathbf{Cob}$, and the TQFT $\mathcal{Q}$ is a functor from ${}_1\mathbf{Cob}$, hence isotopies in ${}_1\mathbf{Cob}$ induce isomorphisms in $\mathbf{Ab}$.

Furthermore, ${}_a(H_\lambda)_a$ is a subring of H_λ isomorphic to $\mathcal{A}^{\otimes m(\lambda)}$, since each matching in a defines a circle in $R(a)a$ and each of these cycles is disjoint by definition of B^s. Define $1_a := 1^{\otimes m(\lambda)} \in \mathcal{A}^{\otimes m(\lambda)}$. Then 1_a is an idempotent in H_λ, and $\sum_{a \in B^s} 1_a$ is the unit element in H_λ, with ${}_b(H_\lambda)_a = 1_b H_\lambda 1_a$. The reason for studying the rings H_λ is in the categorification of weight spaces V_λ for admissible λ: these will lift to categories of modules over the rings H_λ. Thus, functors between these categories will consist of tensoring with bimodules of the rings, more specifically, (H_λ, H_μ)-bimodules for admissible weights λ and μ.

Let λ and μ be admissible weights, with $m(\lambda) = m$ and $m(\mu) = l$, with associated position vectors given by $s(\lambda) = (s_1, s_2, \ldots, s_{2m})$ and $t(\mu) = (t_1, t_2, \ldots, t_{2l})$. Arrange the points s_i along the horizontal axis and the points t_j along the horizontal line at height 1 in $\mathbb{R} \times [0,1]$. A cobordism between the set of points s_i and the set of points t_j is a disjoint union of arcs and circles (copies of the unit interval and S^1 embedded into $\mathbb{R} \times [0,1]$) such that the endpoints of the arcs have horizontal coordinates s_i and t_i, and all these points are the endpoint of an arc. To ensure that concatenation is smooth, we require the arcs to meet horizontal lines vertically at their endpoints. A cobordism of this form is called a flat tangle T. A flat tangle with $2m$ bottom endpoints given by the position vector s for an admissible weight λ and $2l$ top endpoints given by the position vector t for admissible weight μ is called a flat (l, m)-tangle, or a flat (μ, λ)-tangle if we want to make explicit reference to the $\mathfrak{sl}_n$ weights associated to T.

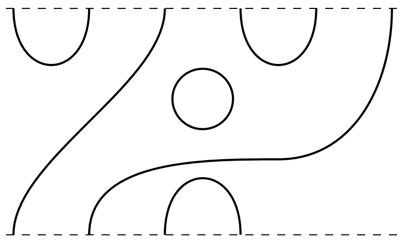


Figure 3.2: A flat (3,2)-tangle

From flat (ν, μ) tangles one defines an (H_ν, H_μ)-bimodule given by

$$\mathcal{Q}(T) = \bigoplus_{\substack{a \in B^s \\ b \in B^t}} \mathcal{Q}(R(b)Ta)\{m(\mu)\}$$

where $R(b)Ta$ is the closed 1-manifold obtained by gluing $R(b)$ and a to the flat tangle T by identifying endpoints.

Note that $\mathcal{Q}(T)$ is well-defined, since each summand consists of a disjoint union of cycles. A (ν, μ) tangle T_1 can be composed with an (η, ν) tangle T_2 by identifying the top endpoints of T_1 with the bottom endpoints of T_2 to form an (η, μ) tangle T_1T_2.

Proposition 3.1.2. *[Kho02] There is a canonical isomorphism of (H_η, H_μ)-bimodules $\mathcal{F}(T_1T_2) \cong \mathcal{F}(T_1) \otimes_{H_\nu} \mathcal{F}(T_2)$.*

The functors between categorified weight spaces will lift the action of the elements E_i and F_i, which take weight spaces V_μ to weight spaces $V_{\mu\pm\varepsilon(i)}$ when these are admissible. The tangles in figures 3.1 and 3.1 demonstrate the action of elementary bimodules on admissible weight spaces that will be used to define the categorical action of E_i and F_i:

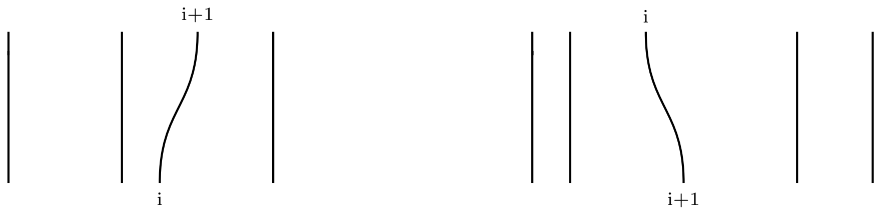


Figure 3.3: The flat tangles Id_i^{i+1} and Id_{i+1}^{i}

To simplify notation, define S to be an integer that is either 0 or 2. The tangles in figures 3.1 and 3.1 above are (ν, μ)-tangles, where μ and ν differ at only two positions, so that $\mu_j = \nu_j$ for all $j \neq i, i+1$. In the case of Id_i^{i+1}, $\mu_i = 1, \mu_{i+1} = S, \nu_i = S$ and $\nu_{i+1} = 1$. Equivalently, i is an element in the sequence $s(\mu)$ and $i+1$ is an element in the sequence $t(\nu)$, so that a (ν, μ) tangle between $s(\mu)$ and $t(\nu)$ must contain an arc from i as a bottom endpoint to $i+1$ as a top endpoint. There is no 1 at the $i+1$-th position of μ or equivalently $s(\mu)$ does not contain

Figure 3.4: The flat tangles $\bigcap_{i,i+1}$ and $\bigcup^{i,i+1}$

$i+1$, hence the gap at $i+1$ on the bottom line. The vertical lines in the tangles above signify a 1 in μ and ν at the same position. The roles of μ_i and ν_i are reversed for Id^i_{i+1}.

For $\bigcap_{i,i+1}$, $\mu_i = \mu_{i+1} = 1$, while $\nu_i = \nu_{i+1} = S$. Thus the points i and $i+1$ are not included in $t(\nu)$ and do not form endpoints for any of the tangles considered here. We therefore require a $(m(\mu), m(\mu)-1)$-tangle, the simplest of which is $\bigcap_{i,i+1}$. Again, reverse the roles of μ_i and ν_i to obtain $\bigcup^{i,i+1}$.

The category $\mathcal{C}$

Having defined the rings H_λ for admissible weights λ, we now construct categories $\mathcal{C}(\lambda)$ and functors between them that categorify the action of U on V. Recall that the decategorification process here consists of taking the Grothendieck group of the constructed category, so we must show that, up to tensoring with the field $\mathbb{Q}(q)$, the Grothendieck group of our category is isomorphic to V, and that the action of functors $\mathcal{E}_i$ and $\mathcal{F}_i$ between the categories $\mathcal{C}(\mu)$ descends to the $U_q(\mathfrak{sl}_n)$ action on the Grothendieck group.

For each admissible weight λ, define $\mathcal{C}(\lambda) := H_\lambda$-**mod**, the category of graded, finitely-generated H_λ modules. Note that if $s(\lambda)$ is empty then H_λ is simply the ground ring $\mathbb{Z}$, so that $\mathcal{C}(\lambda) =_g$ **Ab**, the category of graded finitely-generated abelian groups. Then the categorification of the total vector space V is defined to be

$$\mathcal{C} := \bigoplus_\lambda \mathcal{C}(\lambda)$$

with λ ranging over all admissible weights in V.

To obtain a categorical action of U on $\mathcal{C}$, we must also define functors $\mathcal{E}_i, \mathcal{F}_i : \mathcal{C} \to \mathcal{C}$ that lift the action of E_i and F_i on V. Let $\mathcal{E}_i$ be the sum over all admissible λ in V of the following functors $\mathcal{E}_i^\lambda : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda + \varepsilon_i)$:

- If $\lambda + \varepsilon_i$ is not admissible, $\mathcal{E}_i^\lambda$ is the zero functor.
- If $(\lambda_i, \lambda_{i+1}) = (1,2)$, then $((\lambda+\varepsilon_i)_i, (\lambda+\varepsilon_i)_{i+1}) = (2,1)$ and $\mathcal{E}_i^\lambda = \mathcal{Q}(Id_i^{i+1})$.
- If $(\lambda_i, \lambda_{i+1}) = (0,1)$ then $((\lambda+\varepsilon_i)_i, (\lambda+\varepsilon_i)_{i+1}) = (1,0)$ and $\mathcal{E}_i^\lambda = \mathcal{Q}(Id^i_{i+1})$.

- If $(\lambda_i, \lambda_{i+1}) = (1, 1)$ then $((\lambda + \varepsilon_i)_i, (\lambda + \varepsilon_i)_{i+1}) = (2, 0)$ and $\mathcal{E}_i^\lambda = \mathcal{Q}(\bigcap_{i,i+1})$.
- If $(\lambda_i, \lambda_{i+1}) = (0, 2)$ then $((\lambda + \varepsilon_i)_i, (\lambda + \varepsilon_i)_{i+1}) = (1, 1)$ and $\mathcal{E}_i^\lambda = \mathcal{Q}(\bigcup^{i,i+1})$.

Similarly, let the functor $\mathcal{F}_i$ be the sum over all admissible λ of the functors $\mathcal{F}_i^\lambda : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda - \varepsilon_i)$:

- If $\lambda - \varepsilon_i$ is not admissible, $\mathcal{F}_i^\lambda$ is the zero functor.
- If $(\lambda_i, \lambda_{i+1}) = (1, 0)$, then $((\lambda - \varepsilon_i)_i, (\lambda - \varepsilon_i)_{i+1}) = (0, 1)$ and $\mathcal{F}_i^\lambda = \mathcal{Q}(Id_i^{i+1})$.
- If $(\lambda_i, \lambda_{i+1}) = (2, 1)$ then $((\lambda - \varepsilon_i)_i, (\lambda - \varepsilon_i)_{i+1}) = (1, 2)$ and $\mathcal{F}_i^\lambda = \mathcal{Q}(Id_{i+1}^i)$.
- If $(\lambda_i, \lambda_{i+1}) = (1, 1)$ then $((\lambda - \varepsilon_i)_i, (\lambda - \varepsilon_i)_{i+1}) = (0, 2)$ and $\mathcal{F}_i^\lambda = \mathcal{Q}(\bigcap_{i,i+1})$.
- If $(\lambda_i, \lambda_{i+1}) = (2, 0)$ then $((\lambda - \varepsilon_i)_i, (\lambda - \varepsilon_i)_{i+1}) = (1, 1)$ and $\mathcal{F}_i^\lambda = \mathcal{Q}(\bigcup^{i,i+1})$.

It remains to lift the generators K_i of U to functors from $\mathcal{C}$ to itself. To this end, define the functor $\mathcal{K}_i : \mathcal{C} \to \mathcal{C}$ to be the sum over admissible λ of functors that shift the gradings of objects in $\mathcal{C}(\lambda)$ up by $\lambda_i - \lambda_{i+1}$:

$$\mathcal{K}(M) = M\{\lambda_i - \lambda_{i+1}\} \text{ for } M \in \mathrm{Ob}(\mathcal{C}(\lambda).$$

This functor is clearly invertible, with inverse functor $\mathcal{K}_i^{-1}$ shifting down by $\lambda_i - \lambda_{i+1}$ (or equivalently, shifting up by $\lambda_{i+1} - \lambda_i$.

Note that the functors map between categories in the same way that the generators of $U_q(\mathfrak{sl}_n)$ map between weight spaces. The following propositions show that the functors $\mathcal{E}_i, \mathcal{F}_i$ and $\mathcal{K}_i$ have the same relations up to natural isomorphism as the generators E_i, F_i and K_i (see section **??** for these relations), so that the action of U on V is indeed lifted to a categorical action on $\mathcal{C}$.

Proposition 3.1.3. *There are natural isomorphisms between the following functors:*

(i) $\mathcal{K}_i\mathcal{K}_i^{-1} \cong Id \cong \mathcal{K}_i^{-1}\mathcal{K}_i$

(ii) $\mathcal{K}_i\mathcal{K}_j \cong \mathcal{K}_j\mathcal{K}_i$

(iii) $\mathcal{K}_i\mathcal{E}_j \cong \mathcal{E}_j\mathcal{K}_i\{c_{ij}\}$

(iv) $\mathcal{K}_i\mathcal{F}_j \cong \mathcal{E}_j\mathcal{K}_i\{-c_{ij}\}$

(v) $\mathcal{E}_i\mathcal{F}_j \cong \mathcal{F}_j\mathcal{E}_i$ *if* $i \neq j$

(vi) $\mathcal{E}_i\mathcal{E}_j \cong \mathcal{E}_j\mathcal{E}_i$ *if* $|i - j| > 1$

(vii) $\mathcal{F}_i\mathcal{F}_j \cong \mathcal{F}_j\mathcal{F}_i$ *if* $|i - j| > 1$

(viii) $\mathcal{E}_i^2\mathcal{E}_j \oplus \mathcal{E}_j\mathcal{E}_i^2 \cong \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\{1\} \oplus \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\{-1\}$ *if* $j = i \pm 1$

(ix) $\mathcal{F}_i^2\mathcal{F}_j \oplus \mathcal{F}_j\mathcal{F}_i^2 \cong \mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\{1\} \oplus \mathcal{F}_j\mathcal{F}_i\mathcal{F}_j\{-1\}$ *if* $j = i \pm 1$

where $c_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$

Proof. (i) and (ii) are in fact equalities, since the shift functor $\{1\}$ is an automorphism of the category $\mathcal{C}$, with inverse functor $\{-1\}$, and with composition given by

$$M\{i\}\{j\} = M\{i+j\} \text{ for all } i, j \in \mathbb{Z}$$

and furthermore $M\{0\} = M$.

(iii) In the case $i = j$, let $M \in \mathcal{C}(\lambda)$ for some admissible weight λ. Suppose $\lambda + \varepsilon_i$ is admissible, otherwise the equality is trivial. Then $\mathcal{E}_i(M) \in \mathcal{C}(\lambda)$, say $\mathcal{E}_i(M) = M'$. Then $\mathcal{K}_i(M') = M'\{(\lambda+\varepsilon_i)_i - (\lambda+\varepsilon_i)_{i+1}\} = M'\{\lambda_i + 1 - (\lambda_{i+1} - 1)\} = M'\{\lambda_i - \lambda_{i+1}\}\{2\} = \mathcal{E}_i\mathcal{K}_i(M)\{2\}$. The other cases and (iv) are similar.

When $|i-j| > 1$, (v), (vi) and (vii) follow from the fact that the functors $\mathcal{E}_i$ and $\mathcal{F}_j$ correspond to tangles involving disjoint strands if $i \neq \pm 1$. Thus the tangles associated to $\mathcal{E}_i\mathcal{F}_j$ and $\mathcal{F}_j\mathcal{E}_i$ are isotopic and induce naturally isomorphic functors.

(v) Suppose $j = i+1$. Then consider the three adjacent elements of some admissible λ about λ_{i+1}: $(\lambda_i, \lambda_{i+1}, \lambda_{i+2})$. Under the action of $F_{i+1}E_i$, we are sent to weight spaces with weights that locally are given by

$$(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) \xmapsto{E_i} (\lambda_i + 1, \lambda_{i+1} - 1, \lambda_{i+2}) \xmapsto{F_{i+1}} (\lambda_i + 1, \lambda_{i+1} - 2, \lambda_{i+2} + 1)$$

and similarly,

$$(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) \xmapsto{F_{i+1}} (\lambda_i + 1, \lambda_{i+1} - 1, \lambda_{i+2}) \xmapsto{E_i} (\lambda_i + 1, \lambda_{i+1} - 2, \lambda_{i+2} + 1)$$

so that the only weight λ such that the final weight is admissible has $\lambda_i < 2$, $\lambda_{i+2} < 2$ and $\lambda_{i+1} = 2$. There are four possibilities: $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,2,1), (1,2,0), (0,2,1)$ and $(0,2,0)$. The following figure 3.5 shows the flat tangles associated to $\mathcal{F}_{i+1}\mathcal{E}_i$ and $\mathcal{E}_i\mathcal{F}_{i+1}$ for $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,2,1)$.

Figure 3.5: The flat tangles corresponding to $\mathcal{F}_{i+1}\mathcal{E}_i$ and $\mathcal{E}_i\mathcal{F}_{i+1}$ for $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (1,2,1)$

The flat tangles are isotopic in the plane and thus induce a natural isomorphism of functors.

The other admissible weights and the case $j = i - 1$ are shown in the same way.

(viii) Let $j = i+1$. By a similar argument to part (v), the only admissible weights λ such that at least one of $E_i^2 E_{i+1}$ or $E_{i+1} E_i^2$ sends λ to an admissible weight have $\lambda_i = 0, \lambda_{i+1}, \lambda_{i+2} > 0$. The only possibilities are $(\lambda_i, \lambda_{i+1}, \lambda_{i+2}) = (0,1,2), (0,1,1), (0,2,2)$ and $(0,2,1)$. We consider the case $(0,1,2)$. Under $E_i^2 E_{i+1}$, $(0,1,2) \mapsto (0,2,1) \mapsto (1,1,1) \mapsto (2,0,1)$. Under the action of $E_{i+1}E_i^2$, $(0,1,2)$ gets sent to a weight that is not admissible, so we disregard this action, as the associated functor is zero. Under the action of $E_i E_{i+1} E_i$, $(0,1,2) \mapsto (1,0,2) \mapsto (1,1,1) \mapsto (2,0,1)$. The tangles associated to these maps are:

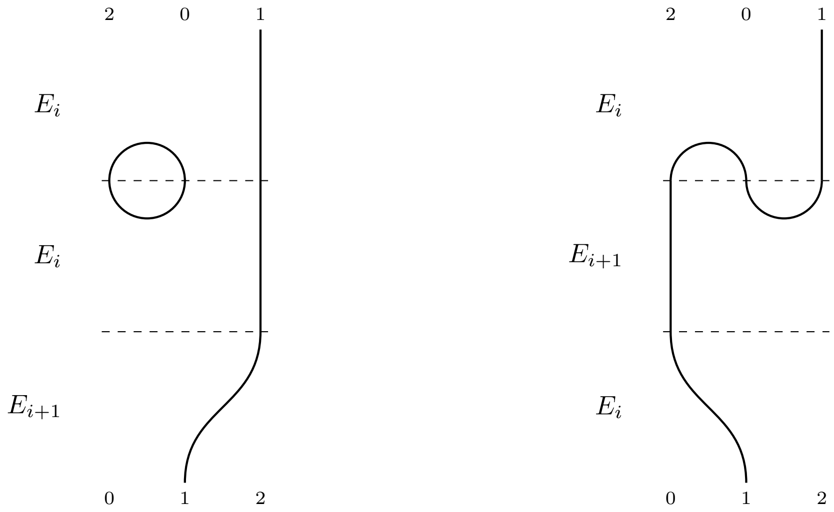


Figure 3.6: The flat tangles associated to $\mathcal{E}_i^2\mathcal{E}_{i+1}$ and $\mathcal{E}_i\mathcal{E}_{i+1}\mathcal{E}_i$

The flat tangle T_1 associated to $\mathcal{E}_i^2\mathcal{E}_{i+1}$ is obtained from the flat tangle T_2 associated to $\mathcal{E}_i\mathcal{E}_{i+1}\mathcal{E}_i$ by planar isotopy and adding in a circle, so $\mathcal{F}(T_1) \cong \mathcal{F}(T_2) \otimes \mathcal{A}$. $\mathcal{A}$ is generated by an element in degree -1 and an element in degree 1. Thus $\mathcal{F}(T_1) \cong \mathcal{F}(T_2)\{1\} \oplus \mathcal{F}(T_2)\{-1\}$, so in terms of the given functors, we have a natural isomorphism

$$\mathcal{E}_i^2\mathcal{E}_{i+1} \oplus \mathcal{E}_{i+1}\mathcal{E}_i^2 \cong \mathcal{E}_i\mathcal{E}_{i+1}\mathcal{E}_i\{1\} \oplus \mathcal{E}_i\mathcal{E}_{i+1}\mathcal{E}_i\{-1\}$$

as required. The other cases and (ix) are similar. □

Proposition 3.1.4. *For any admissible weight λ, there are natural isomorphisms between functors acting on the category $\mathcal{C}(\lambda)$*

(i) $\mathcal{E}_i\mathcal{F}_i \cong \mathcal{F}_i\mathcal{E}_i \oplus Id\{1\} \oplus Id\{-1\}$ *if* $(\lambda_i, \lambda_{i+1}) = (2,0)$

(ii) $\mathcal{E}_i\mathcal{F}_i \cong \mathcal{F}_i\mathcal{E}_i \oplus Id$ *if* $\lambda_i - \lambda_{i+1} = 0$

(iii) $\mathcal{E}_i\mathcal{F}_i \cong \mathcal{F}_i\mathcal{E}_i$ *if* $\lambda_i = \lambda_{i+1}$

(iv) $\mathcal{E}_i\mathcal{F}_i \oplus Id \cong \mathcal{F}_i\mathcal{E}_i$ *if* $\lambda_i - \lambda_{i+1} = -1$

(v) $\mathcal{E}_i\mathcal{F}_i \oplus Id\{1\} \oplus Id\{-1\} \cong \mathcal{F}_i\mathcal{E}_i$ *if* $(\lambda_i, \lambda_{i+1}) = (0,2)$

This proposition in particular shows that the relation $E_iF_i - F_iE_i = \frac{K_i - K_i^{-1}}{q - q^{-1}}$ in U is lifted to the same relation up to natural isomorphism at the categorical level.

Proof. (i) Let $(\lambda_i, \lambda_{i+1}) = (2, 0)$. Then under the action of E_iF_i, $(\lambda_i, \lambda_{i+1}) \mapsto (1, 1) \mapsto (2, 0)$. The tangle associated to $\mathcal{E}_i\mathcal{F}_i$ is a single circle in the plane, while under the action of E_i, $(\lambda_i, \lambda_i) \mapsto (3, -1)$, which is not admissible so $\mathcal{F}_i\mathcal{E}_i$ is associated to the empty tangle and is thus isomorphic to the ground ring $\mathbb{Z}$. Tensoring with the ground ring is equivalent to applying the identity functor. The tangle associated to $\mathcal{E}_i\mathcal{F}_i$ is obtained from the tangle associated to $\mathcal{F}_i\mathcal{E}_i$ by adding in a circle, so as before $\mathcal{E}_i\mathcal{F}_i \cong \mathcal{A} \otimes Id \cong Id\{1\} \oplus Id\{-1\}$. The proof of (v) is identical after swapping $\mathcal{E}_i$ and $\mathcal{F}_i$.

(ii) Let $(\lambda_i, \lambda_{i+1}) = (1, 0)$. $E_iF_i : (1, 0) \mapsto (1, 0)$ and $E_i(1, 0)$ is not admissible, so $\mathcal{F}_i\mathcal{E}_i$ is associated to the empty tangle. The tangle associated to $\mathcal{E}_i\mathcal{F}_i$ is shown below in figure 3.7:

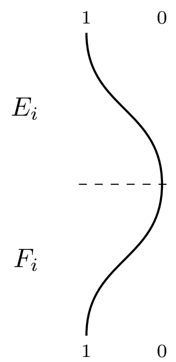


Figure 3.7: The tangle associated to $\mathcal{E}_i\mathcal{F}_i$, $(\lambda_i, \lambda_{i+1}) = (1, 0)$

This tangle is isotopic to the identity (a vertical line), so induces a natural isomorphism from $\mathcal{E}_i\mathcal{F}_i$ to the identity functor. The case $(\lambda_i, \lambda_{i+1}) = (2, 1)$ and the proof of (iv) are shown in the same manner. (iii) Let $(\lambda_i, \lambda_{i+1}) = (1, 1)$. Then $(\lambda_i, \lambda_{i+1}) \xmapsto{E_i} (2, 0) \xmapsto{F_i} (1, 1)$ Similarly,

$$(\lambda_i, \lambda_{i+1}) \xmapsto{F_i} (0, 2) \xmapsto{E_i} (1, 1)$$

so that the tangles associated to $\mathcal{E}_i\mathcal{F}_i$ and $\mathcal{F}_i\mathcal{E}_i$ are identical (figure 3.8:

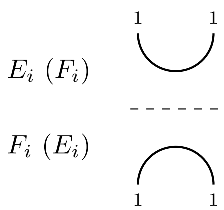


Figure 3.8: The tangle associated to $\mathcal{E}_i\mathcal{F}_i$ and $\mathcal{F}_i\mathcal{E}_i$

The cases $(\lambda_i, \lambda_{i+1}) = (0, 0)$ and $(2, 2)$ are trivial since neither is sent to an admissible weight by either E_i or F_i. □

We may also categorify the quantum divided powers

$$E_i^{(a)} = \frac{E_i^a}{[a]!} \qquad\qquad F_i^{(b)} = \frac{F_i^b}{[b]!}$$

for positive integers a and b where $[a]! = [a][a-1]\dots[1]$ and $[a] = \frac{q^a - q^{-a}}{q-q^{-1}}$. These divided powers are the generators of the integral form of U, the $\mathbb{Z}[q,q^{-1}]$-submodule of U. Note that in the representation V, all weights have coefficients less than or equal to 2, so all the E_i^a and F_i^a are trivial for $a > 2$, and E_i^2 is non-trivial if and only if $(\lambda_i, \lambda_{i+1}) = (0, 2)$ (otherwise $\lambda + 2\varepsilon_i$ is not an admissible weight). If this condition is satisfied, the rings H_λ and $H_{\lambda+2\varepsilon_i}$ are canonically isomorphic, so we define the functors $\mathcal{E}_i^{(2)} : \mathcal{C}(\lambda) \to \mathcal{C}(\lambda + \varepsilon_i)$ and $\mathcal{F}_i^{(2)} : \mathcal{C}(\lambda + 2\varepsilon_i) \to \mathcal{C}(\lambda)$ to be the mutually inverse equivalent functors induced by the isomorphism on rings. The functors $\mathcal{E}_i^{(2)}$ and $\mathcal{F}_i^{(2)}$ are defined to be trivial if the condition on λ is not satisfied. The following proposition shows that the relations between divided powers and regular powers such as the relation $E_i^2 = (q + q^{-1})E_i^{(2)}$ are preserved at the categorified level.

Proposition 3.1.5. *There are natural isomorphisms*

(i) $\mathcal{E}_i^2 \cong \mathcal{E}_i^{(2)}\{1\} \oplus \mathcal{E}_i^{(2)}\{-1\}$

(ii) $\mathcal{F}_i^2 \cong \mathcal{F}_i^{(2)}\{1\} \oplus \mathcal{F}_i^{(2)}\{-1\}$

(iii) $\mathcal{E}_i\mathcal{E}_j\mathcal{E}_i \cong \mathcal{E}_i^{(2)}\mathcal{E}_j \oplus \mathcal{E}_j\mathcal{E}_i^{(2)}$ *if* $|i - j| = 1$

(iv) $\mathcal{F}_i\mathcal{F}_j\mathcal{F}_i \cong \mathcal{F}_i^{(2)}\mathcal{F}_j \oplus \mathcal{F}_j\mathcal{F}_i^{(2)}$ *if* $|i - j| = 1$

Proof. We give a proof of (i). The remaining parts follow since the associated flat tangles are isotopic, up to adding a circle. (i) The flat tangle associated to $\mathcal{E}_i^2$ when $(\lambda_i, \lambda_{i+1}) = (0, 2)$ is:

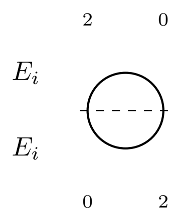


Figure 3.9: The flat tangle associated to $\mathcal{E}_i\mathcal{E}_i$, $(\lambda_i, \lambda_{i+1}) = (0, 2)$

The functor $\mathcal{E}_i^{(2)}$ is an equivalence of categories, and $\mathcal{E}_i^2$ is associated to a flat tangle T that is obtained from the empty tangle by adding a circle, so $\mathcal{E}_i \cong \mathcal{A} \otimes \mathcal{E}_i^{(2)} \cong \mathcal{E}_i^{(2)}\{1\} \oplus \mathcal{E}_i^{(2)}\{-1\}$. Note that since we are not mapping from a category to itself as in previous proofs, the empty tangle is no longer associated to the identity map, rather it is associated to the equivalence of categories induced by the canonical isomorphism of the rings H_λ and $H_{\lambda+2\varepsilon_i}$, when $(\lambda_i, \lambda_{i+1}) = (0, 2)$.

□

Thus, the generators and relations of U are lifted to functors and natural isomorphisms on the category $\mathcal{C}$. However, to show that this is indeed a correct categorification of the representation

V, we must show that the decategorification of $\mathcal{C}$, after tensoring with the ground field $\mathbb{Q}(q)$, is isomorphic to V.

Decategorification

Proposition 3.1.6. *The Grothendieck group K of the category $\mathcal{C}$ is isomorphic to the irreducible representation V of U with highest weight $2\omega_k$:*

$$\bigoplus_{\lambda} K(\mathcal{C}_\lambda) \cong K(\mathcal{C}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong V \cong \bigoplus_{\lambda} V_\lambda$$

where λ ranges over the admissible weights of the representation V.

Proof. Let $\{1\}$ denote the shift functor from $\mathcal{C}$ to itself that shifts the grading of elements in $\mathcal{C}$ up by 1. Then, since the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ do not depend on or modify the grading, they commute with the shift functor. By the equality $M\{i\}\{j\} = M\{i+j\}$, the functor $\mathcal{K}_i$ commutes with the shift functor.

The functors $\mathcal{E}_i$ and $\mathcal{F}_i$ are defined by tensoring with a projective (and hence flat) module, and are therefore exact functors. It is clear that $\mathcal{K}_i$ is also exact. Therefore these functors induce induce well-defined $\mathbb{Z}[q, q^{-1}]$-linear maps $[\mathcal{E}_i], [\mathcal{F}_i]$ and $[\mathcal{K}_i]$ on the Grothendieck group of $\mathcal{C}$. By the functor isomorphisms in propositions 3.1.3 and 3.1.4, these maps satisfy the quantum group relations. $K(\mathcal{C})$ inherits a U-module structure after tensoring with the ground field $\mathbb{Q}(q)$ since the action of $\mathcal{E}_i$, $\mathcal{F}_i$ and $\mathcal{K}_i$ lifted the action of the generators of the quantum group on the representation V.

The weight spaces for $K(\mathcal{C})$ are determined by the elements in Ob$\mathcal{C}$ that are all shifted by the same amount by each of the functors $\mathcal{K}_i$. Recall that these functors shift elements of $\mathcal{C}$ by $\lambda_i - \lambda_{i+1}$, where λ is an admissible weight. Thus the weight spaces of $K(\mathcal{C})$ are exactly the Grothendieck groups of the categories $\mathcal{C}(\lambda)$ for admissible λ, and $K(\mathcal{C}) = \bigoplus_\lambda K(\mathcal{C}(\lambda))$. The dimensions of these weight spaces are precisely the $m(\lambda)$ Catalan numbers, since these enumerate the non-isotopic $(m(\lambda), m(\lambda))$-tangles with no circles, and thus the isomorphism classes of H_λ-modules [Kho02]. Therefore the weight spaces of $K(\mathcal{C})$ have the same dimension as the weight spaces of V, and the representations are isomorphic. □

Structure of the category

Generally, one aims to categorify in a way that lifts as much structure from the original object. This particular categorification lifts several structures from the original representation and quantum group that we will discuss here.

Biadjoint functors

Recall from section **??**, that the algebra U is equipped with an antilinear antiautomorphism

$\tau : U \to U^{op}$ defined by

$$\tau(E_i) = qF_iK_i^{-1},\ \tau(F_i) = qE_iK_i,\ \tau(K_i) = K_i^{-1}$$
$$\tau(f(q)x) = f(q^{-1})\tau(x) \text{ for all } f \in \mathbb{Q}(q) \text{ and } x \in U$$
$$\tau(xy) = \tau(y)\tau(x) \text{ for all } x, y \in U$$

The following proposition shows that the categorification of V lifts the antiautomorphism τ to an operation that sends a functor to its right adjoint functor, if this exists.

Proposition 3.1.7. *The functor $\mathcal{E}_i$ is left adjoint to $\mathcal{F}_i\mathcal{K}_i\{1\}$, $\mathcal{F}_i$ is left adjoint to $\mathcal{E}_i\mathcal{K}_i\{1\}$ and $\mathcal{K}_i$ is left adjoint to $\mathcal{K}_i^{-1}$.*

Proof. The proposition states that, up to grading shifts, the functors $\mathcal{E}_i$ and $\mathcal{F}_i$ are biadjoint. To show this, we find cobordisms between the tangles associated to $\mathcal{E}_i\mathcal{F}_i$ and $\mathcal{F}_i\mathcal{E}_i$ and the identity tangle. These cobordisms involve only two-stranded tangles. First exclude the admissible weights λ such that $\lambda + \varepsilon_i$ is not admissible. For an admissible weight λ, $\lambda + \varepsilon_i$ is admissible only when $(\lambda_i, \lambda_{i+1}) = (1, 1), (0, 1), (0, 2)$ or $(1, 2)$.

1. Let $(\lambda_i, \lambda_{i+1}) = (0, 1)$. Then $((\lambda + \varepsilon_i)_i, (\lambda + \varepsilon_i)_{i+1}) = (1, 0)$ and the tangle associated to $\mathcal{F}_i\mathcal{E}_i$ is the mirror image (about the vertical axis) of the tangle in figure 3.7. There is an obvious planar isotopy $f : \mathbb{R} \times [0, 1] \times [0, 1] \to \mathbb{R} \times [0, 1]$ from this tangle to the identity tangle, which consists here of a single vertical line. This isotopy defines a surface cobordism from the tangle to the identity tangle that induces a natural isomorphism from $\mathcal{F}_i\mathcal{E}_i$ to the identity functor on $\mathcal{C}(\lambda)$. There is a similar cobordism from the identity functor on $\mathcal{C}(\lambda + \varepsilon_i)$ to $\mathcal{E}_i\mathcal{F}_i$. These cobordisms do not introduce any circles into the diagrams so there is no grading shift.

2. Let $(\lambda_i, \lambda_{i+1}) = (1, 1)$. Then the tangle associated to $\mathcal{F}_i\mathcal{E}_i$ is the tangle given in figure 3.8. There is a cobordism from this tangle to the identity tangle, which consists here of two vertical lines. This cobordism induces the multiplication map on H_λ, which increases the degree by one. This shift corresponds to the total shift given by $\mathcal{K}_i\{1\}$.

3. Let $(\lambda, \lambda_{i+1}) = (0, 2)$. Then the tangle associated to $\mathcal{F}_i\mathcal{E}_i$ is a single circle, as in figure 3.9. The empty tangle, associated to the identity on $\mathcal{C}(\lambda + \varepsilon_i)$ is obtained from this circle by taking the trace map, which has degree -1, agreeing with the total degree shift of $\mathcal{K}_i\{1\}$.

The final case is similar. The cobordisms showing the inverse adjunction are the same as in the first case, taken in reverse. ☐

Semilinear form

Given a $\mathbb{Q}(q)$-vector space V, a form $\langle\ ,\ \rangle : V \times V \to \mathbb{Q}(q)$ is called *semilinear*, if it is q-antilinear in the first variable and q-linear in the second: if f is a rational function in q with coefficients in $\mathbb{Q}$, then $\langle f(q)v, w\rangle = f(q^{-1})\langle v, w\rangle = \langle v, f(q^{-1})w\rangle$ for all $v, w \in V$.

Let η be a highest weight vector in the representation V (recall that the weight of η is therefore $2\omega_k$). Then there exists a unique semilinear form on V such that

$$\langle \eta, \eta \rangle = 1 \tag{3.1}$$

$$\langle xv, w \rangle = \langle x, \tau(x)w \rangle \text{ for all } v, w \in V, x \in U \tag{3.2}$$

Now consider the form $(\ , \ ) : K_P(\mathcal{C}) \times K(\mathcal{C}) \to \mathbb{Z}[q, q^{-1}]$, where $K_P(\mathcal{C})$, called the projective Grothendieck group, is the subgroup of $K(\mathcal{C})$ generated by all isomorphism classes $[P]$ of projective objects in $\mathcal{C}$. The form $(\ , \ ,)$ is defined by taking the graded dimension of the Hom space from a projective module P to a general module M:

$$([P], [M]) := q\text{dimHOM}_{\mathcal{C}}(P, M) = \sum_{k \in \mathbb{Z}} q^k \text{dimHom}_{\mathcal{C}}(P\{k\}, M).$$

This is a semilinear form on $\mathbb{Z}[q, q^{-1}]$: applying the shift functor to P a positive number n times induces a multiplication by q^{-n} in $q\text{dimHom}_{\mathcal{C}}(P, M)$, while shifting M up by n induces a multiplication by q^n. Tensoring with $\mathbb{Q}(q)$, $(\ , \ )$ becomes a semilinear form on $\mathbb{Q}(q)$. The isomorphism $K(\mathcal{C}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong V$ can be chosen such that the object $P(2\omega_k)$ in $\mathcal{C}(2\omega_k)$ isomorphic to $\mathbb{Z}$ concentrated in to degree zero is sent to η. Then $P(2\omega_k)$ is projective, and $([P], [P]) = 1$.

Furthermore, lifting the antiautomorphism τ to the operation taking the right adjoint functor, the second relation for the semilinear form in equation 3.1 also holds for $(\ , \ )$. For example:

$$Hom_{\mathcal{C}}(\mathcal{F}_i P, M) = Hom_{\mathcal{C}}(P, \mathcal{E}_i \mathcal{K}_i \{1\} M)$$
$$Hom_{\mathcal{C}}(\mathcal{K}_i P, M) = Hom_{\mathcal{C}}(P, \mathcal{K}_i^{-1} M)$$

Under the isomorphism between $K(\mathcal{C}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ and V, these relations descend to the semilinear form $\langle \ , \ \rangle$ on $V \times V$.

This categorification also lifts other structures on the representation V, such as a unique symmetric bilinear form associated to the same highest weight vector η and the Lusztig canonical basis of V. For further details, see [HK06].

3.2 Categorifying the adjoint representation

Just as we could categorify certain representations of $\mathfrak{sl}_n$, we can also categorify the adjoint representation of any simple, simply-laced Lie algebra $\mathfrak{g}$ of finite type. This again follows the work of Khovanov and Huerfano, as seen in [HK01].

Let $\mathfrak{g}$ be a simple Lie algebra and let $U_q(\mathfrak{g})$ be the quantum group deformation of $\mathfrak{g}$. Let V be the quantum deformation of the adjoint representation of $\mathfrak{g}$. Then V is an irreducible representation that decomposes as a direct sum of the Cartan subalgebra $\mathfrak{h}$ and one-dimensional vector spaces corresponding to the roots of $\mathfrak{g}$. Decategorification here consists of taking the Grothendieck group of a category to return a vector space isomorphic as a $U_q(\mathfrak{g})$ representation to the adjoint representation V. Our aim is therefore to lift the weight spaces of V to abelian categories, with the quantum group generators lifting to functors between the categories in a way that preserves the structure on the quantum group, thus satisfying the decategorification condition. As in the case of level-two representations, one of the categories will be the category of modules over a particular algebra, in this case an algebra constructed from the Dynkin diagram of $\mathfrak{g}$.

Let R be the root system of $\mathfrak{g}$, Π a set of simple roots, $\mathscr{W}$ the Weyl group of $\mathfrak{g}$ and $(\ ,\ )$ the unique $\mathscr{W}$-invariant bilinear form on the real vector space spanned by the roots in R such that $(\alpha, \alpha) = 2)$ for all roots $\alpha \in R$ determined by the Killing form.

3.2.1 The adjoint representation

As noted above, the adjoint representation V is an irreducible representation of $U_q(\mathfrak{g})$ with highest weight the root of $\mathfrak{g}$ lying in the dominant Weyl chamber. It has a canonical basis $\{x_\mu, h_\alpha\}$ with $\mu \in R$ and $\alpha \in \Pi$, with corresponding one-dimensional weight space R_μ for all roots $\mu \in R$ and with weight space R_0 of dimension the rank of $\mathfrak{g}$ (the dimension of the Cartan subalgebra), spanned by the h_α. The action of the quantum group on V is given by

$$\begin{aligned}
K_\alpha x_\mu &= q^{(\alpha,\mu)} x_\mu & K_\alpha h_\beta &= h_\beta & &\text{for all } \alpha, \beta \in \Pi, \mu \in R \\
E_\alpha x_\mu &= 0 & F_\alpha x_\mu &= 0 & &\text{if } (\mu, \alpha) = 0 \\
E_\alpha x_\mu &= 0 & F_\alpha x_\mu &= x_{\mu-\alpha} & &\text{if } (\mu, \alpha) = 1 \\
E_\alpha x_\mu &= x_{\mu+\alpha} & F_\alpha x_\mu &= 0 & &\text{if } (\mu, \alpha) = -1 \\
E_\alpha x_\alpha &= 0 & F_\alpha x_\alpha &= h_\alpha & & \\
E_\alpha x_{-\alpha} &= h_\alpha & F_\alpha x_\alpha &= 0 & &
\end{aligned}$$

for all $\alpha \in \Pi$ and $\mu \in R$.

$$E_\alpha h_\beta = \begin{cases} (q+q^{-1}) x_\alpha & \text{if } \alpha = \beta \\ x_\alpha & \text{if } (\beta, \alpha) = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_\alpha h_\beta = \begin{cases} (q+q^{-1}) x_{-\alpha} & \text{if } \alpha = \beta \\ x_{-\alpha} & \text{if } (\beta, \alpha) = -1 \\ 0 & \text{otherwise} \end{cases}$$

for all $\alpha, \beta \in \Pi$. Note that this action is analogous to the action of the Lie algebra $\mathfrak{g}$ on itself under the adjoint representation: $\mathrm{ad}_X(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$. As in the previous example, the adjoint representation has certain structures that will be lifted by the categorification of V:

Semilinear form

Recall that the quantum group has an antilinear antiautomorphism $\tau : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})^{op}$. There is a q-semilinear form on the adjoint representation: $\langle\ ,\ \rangle : V \times V \to \mathbb{Q}(q)$ that is τ-invariant in the following sense:

$$\langle xv, w\rangle = \langle v, \tau(x)w\rangle \text{ for all } x \in U_q(\mathfrak{g}),\ v, w \in V.$$

In the canonical basis of V, this semilinear form is defined by:

$$\begin{aligned}
&\langle x_\mu, x_\mu\rangle = 1 && \mu \in R\\
&\langle x_\mu, x_\nu\rangle = 0 && \mu \neq \nu \in R\\
&\langle x_\mu, h_\alpha\rangle = 0 && \mu \in R, \alpha \in \Pi\\
&\langle h_\alpha, h_\alpha\rangle = 1 + q^2 && \alpha \in \Pi\\
&\langle h_\alpha, h_\beta\rangle = q && (\alpha, \beta) = \pm 1,\ \alpha, \beta \in \Pi\\
&\langle h_\alpha, h_\beta\rangle = 0 && (\alpha, \beta) = 0,\ \alpha, \beta \in \Pi
\end{aligned}$$

Thus all weight spaces are orthogonal under this semilinear form.

Involutions on V

There are $\mathbb{Q}$-linear involutions ψ_V and ω_V on V:

$$\begin{aligned}
&\psi_V(x_\mu) = x_\mu && \psi_V(h_\alpha) = h_\alpha && \psi_V(f(q)v) = f(q^{-1})\psi_V(v)\\
&\omega_V(x_\mu) = x_{-\mu} && \omega_V(h_\alpha) = h_\alpha && \omega_V(f(q)v) = f(q)\omega_V(v)
\end{aligned}$$

for all $\mu \in R,\ \alpha \in \Pi,\ f \in \mathbb{Q}(q),\ v \in V$.

Claim 3.2.1. *1. The involutions ψ_V and ω_V reverse and preserve the semilinear form on V respectively: $\langle v, w\rangle = \langle \psi_V(w), \psi_V(v)\rangle$ and $\langle v, w\rangle = \langle \omega_V(v), \omega_V(w)\rangle$ for all $v, w \in V$.*

2. $\psi_V(xv) = \psi(x)\psi_V(v)$ and $\omega_V(xv) = \omega(x)\omega_V(v)$ for all $x \in U_q(\mathfrak{g})$ and $v \in V$, where ψ is a $\mathbb{Q}$-linear involution on $U_q(\mathfrak{g})$ and ω is a $\mathbb{Q}(q)$-linear involution on $U_q(\mathfrak{g})$ defined by:

$$\begin{aligned}
&\psi(E_\alpha) = E_\alpha && \psi(F_\alpha) = F_\alpha && \psi(K_\alpha) = K_\alpha^{-1} && \psi(f(q)x) = f(q^{-1})\psi(x)\\
&\omega(E_\alpha) = F_\alpha && \omega(F_\alpha) = E_\alpha && \omega(K_\alpha) = K_\alpha^{-1} && \omega(f(q)x) = f(q)\omega(x)
\end{aligned}$$

for all $x \in U_q(\mathfrak{g}),\ \alpha \in R,\ f \in \mathbb{Q}(q)$.

Proof. 1. The involution ψ_V preserves the canonical basis of V, and since the semilinear form is symmetric in the basis vectors (swapping basis vectors does not affect the semilinear form), the only requirement for such an involution to reverse the semilinear form is that it is q-anitlinear, which is satisfied by ψ_V.

Similarly, ω_V simply swaps the basis vectors x_μ and $x_{-\mu}$, which does not affect the semilinear form since the basis vectors are orthogonal with respect to $\langle\ ,\ \rangle$. Furthermore, ω_V is q-linear, so it preserves the semilinear form.

2. This is clear from the action of the generators of $U_q(\mathfrak{g})$ on the basis vectors of V.

□

Dual canonical basis

From the previously defined semilinear form, one can construct the *dual canonical basis*, which is dual to the canonical basis with respect to the semilinear form $\langle\ ,\ \rangle$: define $\ell_\alpha \in R_0$ by $\langle h_\alpha, \ell_\beta\rangle = \delta_{\alpha,\beta}$. Then the dual canonical basis is $\{x_\mu, \ell_\alpha\}$. Define I to be the $\mathbb{Z}[q,q^{-1}]$-submodule of V generated by the canonical basis vectors and I' the $\mathbb{Z}[q,q^{-1}]$-submodule generated by the dual canonical basis vectors.

Claim 3.2.2. *1. I is a $\mathbb{Z}[q,q^{-1}]$-submodule of I'.*

2. The involutions ψ_V and ω_V preserve the $\mathbb{Z}[q,q^{-1}]$-submodule I'.

Proof. 1. From the semilinear form, and in particular the definition $\langle h_\alpha, \ell_\beta\rangle = \delta_{\alpha,\beta}$, we can determine show that each h_α is a linear combination of the ℓ_β with coefficients in $\mathbb{Z}[q,q^{-1}]$. Explicitly:

$$h_\alpha = (1+q^2)\ell_\alpha + q \sum_{\substack{\beta\in\Pi \\ (\alpha,\beta)=\pm 1}} \ell_\beta.$$

Thus, I is a $\mathbb{Z}[q,q^{-1}]$-submodule of I'.

2. This follows from the previous claims.

□

3.2.2 Building a categorical representation: zigzag algebras

As in the case of level two representations of $\mathfrak{sl}_n$, the categorification of the adjoint representation arises as a direct sum of categories of modules over a particular algebra. Rather than the closed 1-manifolds used to construct the rings H_λ, the object used here to construct the categorification of weight spaces of the adjoint representation is the Dynkin diagram of $\mathfrak{g}$. Specifically, the category associated to the Cartan subalgebra will be the category of modules over the zigzag algebra of the Dynkin diagram. The remaining root spaces will be lifted to copies of the category of graded vector spaces and grading-preserving linear maps. We define the zigzag algebra of a general connected, simply laced graph Γ with no loops and study some of its properties. Note that an example of a zigzag algebra already arose in section **??** as the path algebra of the quiver with relations associated to the Lie algebra $\mathfrak{sl}_2^-(V_1)$.

Definition A graph Γ is *simply-laced* if it has no multiple edges.

Let Γ be a finite simply-laced (undirected) tree.

Definition The *double* of the graph Γ is the directed graph $D\Gamma$ with the same vertex set I as Γ and with each edge e_α of Γ replaced by two directed edges with opposite source and target.

Example. *Examples of a graph* Γ *and its double* $D\Gamma$ *are found in figures 3.10 and 3.11.*

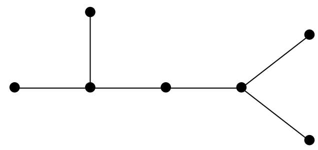


Figure 3.10: Graph Γ

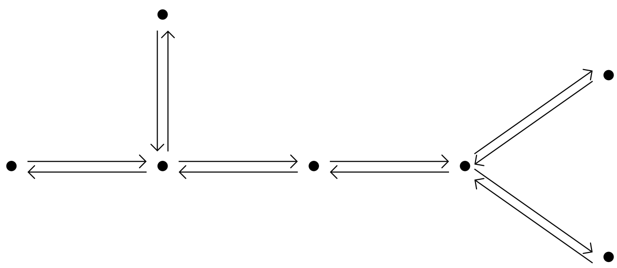


Figure 3.11: Double $D\Gamma$

Let $P(D\Gamma)$ be the path algebra of $D\Gamma$ so that the generators of $P(D\Gamma)$ are paths in $D\Gamma$, with multiplication in $P(D\Gamma)$ given by concatenation of paths, read from left to right. If the target of a path p_1 is not equal to the source of the path p_2, then the product p_1p_2 is zero. For each i in the vertex set I of Γ, let e_i be the length zero path that starts and ends at the vertex v_i. It is clear that in $P(D\Gamma)$ the e_i form a complete set of minimal orthogonal idempotents: $e_ie_j = \delta_{i,j}e_i$ for all $i, j \in I$, and minimality is due to the minimal length of the paths e_i.

Between any two vertices of $D\Gamma$ there are at most two edges, and these edges have opposite orientation. Thus, any edge is completely determined by its source and target, and any path can be uniquely labelled by the vertices through which it passes in order (and for a vertex v_i, we simply write i). For example, the length two loop at v_i passing only through v_j is written $(i|j|i)$.

Definition The *zigzag algebra* $A(\Gamma)$ of a finite, simply-laced tree Γ is a $\mathbb{C}$-algebra defined as follows:

1. If Γ consists of a single vertex, $A(\Gamma) = \mathbb{C}[X]/X^2$.
2. If $\Gamma =$ •——• , then $A(\Gamma) = P(D\Gamma)/J$ where J is the two-sided ideal generated by all paths of length greater than two.
3. If Γ has more than two vertices, then $A(\Gamma) = P(D\Gamma)/J$ where J is the two-sided ideal generated by paths

- $(i|j|k)$ if i and j are connected by an edge, j and k are connected by an edge and $i \neq k$, and
- $(i|j|i) - (i|k|i)$ if i is connected by an edge to both j and k in Γ.

Thus, if Γ has more than two vertices, then paths of the form $\overset{i}{\bullet} \longrightarrow \overset{j}{\bullet} \longrightarrow \overset{k}{\bullet}$ are zero and any vertex has a single equivalence class of length two loops starting and ending at that vertex.

We can define a grading on $A(\Gamma)$ from the length k of paths in $P(D\Gamma)$ if Γ has more than one vertex, and by defining X to be in degree two if Γ has only one vertex. Let I be the vertex set of Γ and J the edge set of Γ.

Claim 3.2.3. *$A(\Gamma) = A(\Gamma)_0 \oplus A(\Gamma)_1 \oplus A(\Gamma)_2$ and*

$$dim(A(\Gamma)_0) = |I|$$
$$dim(A(\Gamma)_1) = 2|J|$$
$$dim(A(\Gamma)_2) = |I|$$

Proof. Any path of length three in $D\Gamma$ is in the form of one of the following:

1. $\overset{i}{\bullet} \longrightarrow \overset{j}{\bullet} \longrightarrow \overset{k}{\bullet} \longrightarrow \overset{l}{\bullet}$
2. $\overset{i}{\bullet} \longrightarrow \overset{j}{\bullet} \rightleftarrows \overset{k}{\bullet}$
3. $\overset{i}{\bullet} \underset{\longrightarrow}{\overset{\longrightarrow}{\longleftarrow}} \overset{j}{\bullet}$

Case 1 is zero in $A(\Gamma)$ since it contains a length two path with distinct start and endpoints. Case 2 is zero for the same reason. If Γ consists of only two vertices, then case 3 is zero by definition of $A(\Gamma)$. If Γ has more than two vertices, then the path in case 3 is equivalent to a path of the same form as case 2 (by equivalence of loops at the same vertex) and is thus also zero. Any longer path must contain a path of length 3 and is hence zero.

The length zero paths in $A(\Gamma)$ are precisely the e_i for $i \in I$ and there are no relations between them, showing a correspondence between I and the generators of $A(\Gamma)_0$. The length one paths correspond to edges in $D\Gamma$, which has two edges for each edge in Γ, and there are no relations on single edges. The length two paths consist exclusively of length two loops at vertices. Since all such loops at a given vertex are equivalent, there is exactly one equivalence class of loops in $A(\Gamma)$ for each vertex in Γ since Γ is connected. □

Definition A *symmetric* $\mathbb{C}$-algebra is an algebra $\mathcal{A}$ that possesses a symmetric non-degenerate trace map, namely a map $\mathrm{tr} : \mathcal{A} \to \mathbb{C}$ satisfying the following:

$$\mathrm{tr}(xy) = \mathrm{tr}(yx) \qquad \text{for all } x, y \in \mathcal{A}$$

and for all $x \in \mathcal{A} \backslash \{0\}$ there exists some $y \in \mathcal{A}$ such that $\mathrm{tr}(xy) \neq 0$

Proposition 3.2.4. *$A(\Gamma)$ is a graded symmetric algebra.*

Proof. Define the map $\mathrm{tr} : A(\Gamma) \to \mathbb{C}$ by sending a path of length two to 1 in $\mathbb{C}$ and all other paths to zero. The only paths not sent to zero under the trace map consist of length 2 loops at some vertex v_i of Γ: $\overset{\mathrm{i}}{\bullet} \underset{y}{\overset{x}{\rightleftarrows}} \overset{\mathrm{j}}{\bullet}$

The path xy is a loop at i and yx is a length two loop at j, so that $\mathrm{tr}(xy) = \mathrm{tr}(yx) = 1$. The only non-trivial product of xy with other paths in $A(\Gamma)$ is with the length zero path at i. Since xy is a loop, it is clear that $ixy = xyi = xy$. All other paths are sent to zero, so the relation $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ holds trivially for these paths and the trace map is symmetric.

Given a length two path $(i|j|i)$, then $\mathrm{tr}((i|j|i)i) = \mathrm{tr}((i|j|i)) = 1$. For a length one path $(i|j)$, $\mathrm{tr}((i|j)(j|i)) = \mathrm{tr}(i|j|i) = 1$. For a length zero path i, if Γ has more than one vertex, then there is a length two loop p at i that is sent to 1 under the trace map. Then $\mathrm{tr}(ip) = \mathrm{tr}(p) = 1$. These are all the non-zero paths in $A(\Gamma)$, so the trace map is non-degenerate. □

Let $A(\Gamma)$-**mod** denote the abelian category of finite-dimensional graded left $A(\Gamma)$-modules and grading-preserving module homomorphisms. Let ${}_g$**Vect** denote the category of graded finite-dimensional $\mathbb{C}$-vector spaces and grading-preserving linear maps. Recall that $\{k\}$ denotes the shift functor on a category of graded algebra modules that shifts the grading of a module up by k: for $M = \bigoplus_n M_n$, $M\{k\}_n = M_{n-k}$.

Let v_i be a vertex in Γ and e_i the minimal idempotent consisting of the length zero path at v_i. Then define the $A(\Gamma)$-modules $P_i := A(\Gamma)i$ and ${}_iP := iA(\Gamma)$.

Lemma 3.2.5. *P_i is a left projective indecomposable $A(\Gamma)$-module spanned by paths ending at v_i. Furthermore, any indecomposable graded projective left $A(\Gamma)$-module is isomorphic, up to a grading shift, to P_i for some $i \in I$.*

An analogous statement holds for right $A(\Gamma)$-modules and ${}_iP$.

Proof. All non-zero paths in P_i must end at v_i, since the product of any other path with i is zero, and for paths p ending at v_i, $pi = p$. Since Γ is finite, the identity element in $A(\Gamma)$ is given by $1_{A(\Gamma)} = \sum_{i\in I} i$ and hence $A(\Gamma) = \bigoplus_{i\in I} P_i$ so the P_i are projective.

By minimality of the idempotents i, the P_i are indecomposable. The uniqueness statement follows from the Krull-Schmidt theorem and the decomposition $A(\Gamma) = \bigoplus_{i\in I} P_i$. □

Lemma 3.2.6.

$${}_iP \otimes_{A(\Gamma)} P_j \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}\{2\} & \textit{if } i = j \\ \mathbb{C}\{1\} & \textit{if } v_i \textit{ and } v_j \textit{ are connected by an edge} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. ${}_iP \otimes_{A(\Gamma)} P_j$ is spanned by paths starting at v_i and ending at v_j. If $i = j$ there are two independent paths starting and ending at v_i: the length zero path e_i and the equivalence class of length two loops at v_i, which accounts for the grading shift in the second summand.

If v_i and v_j are connected by an edge then there is a path starting at v_i and ending at v_j (multiplication by the length zero paths e_i and e_j gives an equivalent path in the path algebra $P(D\Gamma)$), namely $(i|j)$, and the length of the path is one, accounting for the grading shift.

If v_i and v_j are not connected by an edge, there is no path from v_i to v_j in $A(\Gamma)$. □

Let i be an element of the vertex index set I of Γ and define functors

$$\begin{aligned} T_i : A(\Gamma)\text{-}\mathbf{mod} &\to {}_g\mathbf{Vect} \\ M &\mapsto {}_iP \otimes_{A(\Gamma)} M \\ S_i : {}_g\mathbf{Vect} &\to A(\Gamma)\text{-}\mathbf{mod} \\ V &\mapsto P_i \otimes_{\mathbb{C}} V \end{aligned}$$

Lemma 3.2.7. *T_i is right adjoint to S_i and left-adjoint to $S_i\{-2\}$.*

Proof. We define natural transformations $\eta : S_iT_i \Rightarrow \mathrm{Id}_{A(\Gamma)\text{-}\mathbf{Mod}}$ and $\varepsilon : \mathrm{Id}_{{}_g\mathbf{Vect}} \Rightarrow T_iS_i$. The functor $S_iT_i : A(\Gamma)\text{-}\mathbf{Mod} \to A(\Gamma)\text{-}\mathbf{Mod}$ is defined by $M \mapsto P_i \otimes_{\mathbb{C}} {}_iP \otimes_{A(\Gamma)} M$ and we can write the identity map on $A(\Gamma)$-**Mod** as tensoring with $A(\Gamma)$ over itself: $\mathrm{Id}_{A(\Gamma)}(M) = A(\Gamma) \otimes_{A(\Gamma)} M$. Thus the natural transformation η consists of defining a map of $A(\Gamma)$-bimodules, that will abusively also denote $\eta : P_i \otimes_{\mathbb{C}} {}_iP \to A(\Gamma)$. Define $\eta(xe_i \otimes_{A(\Gamma)} e_iy) = xy$, namely η is the multiplication map in $A(\Gamma)$ that concatenates suitable paths in $D\Gamma$ for any paths x ending at v_i (hence xe_i is nonzero) and all paths y starting at v_i.

Similarly, $T_iS_i(W) = {}_iP \otimes_{A(\Gamma)} P_i \otimes_{\mathbb{C}} W$ for all objects W in ${}_g\mathbf{Vect}$ and $\mathrm{Id}_{\mathbb{C}}(W) = \mathbb{C} \otimes_{\mathbb{C}} W$. Thus we define the map $\varepsilon : \mathbb{C} \to {}_iP \otimes_{A(\Gamma)} P_i$ of ${}_g\mathbf{Vect}$-bimodules by $\varepsilon(1) = e_i \otimes_{A(\Gamma)} e_i$. Then

$$\begin{aligned} (\eta \otimes 1_{P_i}) \circ (1_{P_i} \otimes \varepsilon) : P_i &\to P_i \otimes_{A(\Gamma)} {}_iP \otimes_{A(\Gamma)} P_i \to P_i \\ xe_i &\mapsto xe_i \otimes_{A(\Gamma)} e_i \otimes_{A(\Gamma)} e_i \mapsto xe_ie_i \otimes_{A(\Gamma)} e_i \cong xe_i \\ (1_{{}_iP} \otimes \eta) \circ (\varepsilon \otimes 1_{{}_iP}) :{}_i P &\to {}_i P \otimes_{A(\Gamma)} P_i \otimes_{A(\Gamma)} {}_iP \to {}_iP \\ e_ix &\mapsto e_i \otimes_{A(\Gamma)} e_i \otimes_{A(\Gamma)} e_ix \mapsto e_i \otimes_{A(\Gamma)} e_ix \cong e_ix \end{aligned}$$

The bimodule homomorphisms η and ε induce natural transformations of functors and hence T_i is right adjoint to S_i.

To show that T_i is left adjoint to $S_i\{-2\}$, we proceed in a similar manner: define natural transformations $\eta' : T_iS_i\{-2\} \Rightarrow \mathrm{Id}_{{}_g\mathbf{Vect}}$ and $\varepsilon' : \mathrm{Id}_{A(\Gamma)\text{-}\mathbf{Mod}} \Rightarrow S_i\{-2\}T_i$. We have $T_iS_i(W) ={}_i P \otimes_{A(\Gamma)} P_i \otimes_{\mathbb{C}} W$ for all objects W in ${}_g\mathbf{Vect}$ so we define $\eta' :{}_i P \otimes_{A(\Gamma)} P_i \to \mathbb{C}$ to be the trace map of $A(\Gamma)$: $\eta'(e_ix \otimes ye_i) = \mathrm{tr}(yx)$. We also have $S_iT_i(M) = P_i \otimes_{\mathbb{C}} {}_iP \otimes_{A(\Gamma)} M$ for all objects $M \in A(\Gamma)$-**Mod**, so we define $\varepsilon' : A(\Gamma) \to P_i \otimes_{\mathbb{C}} {}_iP$ by setting

$$\varepsilon'(1) = l_i \otimes_{\mathbb{C}} e_i + e_i \otimes_{\mathbb{C}} l_i + \sum_{\substack{j \in I \\ j \leftrightarrow i}} (j|i) \otimes_{\mathbb{C}} (i|j)$$

where l_i denotes the equivalence class of length two loops at the vertex v_i and $j \leftrightarrow i$ signifies "v_j is connected to v_i by an edge". Thus, for right $A(\Gamma)$-modules, $\varepsilon'(e_i) = l_i \otimes_{\mathbb{C}} e_i$, $\varepsilon'(l_i) = e_i \otimes_{\mathbb{C}} (l_i)$ and $\varepsilon'((i|k)) = (k|i) \otimes_{\mathbb{C}} (i|k)$ for any v_k connected to v_i by an edge. Then

$$\begin{aligned}
(\eta' \otimes_{\mathbb{C}} 1_{{}_iP}) \circ (1_{{}_iP \otimes A(\Gamma)}) : {}_iP &\to {}_iP \otimes_{A(\Gamma)} P_i \otimes_{\mathbb{C}} {}_iP \to_i P \\
e_i &\mapsto e_i \otimes_{A(\Gamma)} l_i \otimes_{\mathbb{C}} e_i \mapsto \mathrm{tr}(l_i) \otimes_{\mathbb{C}} e_i = 1 \otimes_{\mathbb{C}} e_i \\
(i|k) &\mapsto (i|k) \otimes_{A(\Gamma)} (k|i) \otimes_{\mathbb{C}} (i|k) \mapsto \mathrm{tr}((k|i)(i|k)) \otimes_{\mathbb{C}} (i|k) = 1 \otimes_{\mathbb{C}} (i|k) \\
l_i &\mapsto l_i \otimes_{A(\Gamma)} e_i \otimes_{\mathbb{C}} l_i \mapsto \mathrm{tr}(l_i) \otimes_{\mathbb{C}} l_i = 1 \otimes_{\mathbb{C}} l_i \\
(1_{P_i} \otimes_{A(\Gamma)} \eta') \circ (\varepsilon' \otimes_{\mathbb{C}} 1_{P_i}) : P_i &\to P_i \otimes_{\mathbb{C}} {}_iP \otimes_{A(\Gamma)} P_i \to P_i \\
e_i &\mapsto e_i \otimes_{\mathbb{C}} l_i \otimes_{A(\Gamma)} e_i \mapsto e_i \otimes_{\mathbb{C}} \mathrm{tr}(l_i) = e_i \otimes_{\mathbb{C}} 1 \\
(k|i) &\mapsto (k|i) \otimes_{\mathbb{C}} (i|k) \otimes_{A(\Gamma)} (k|i) \mapsto (k|i) \otimes_{\mathbb{C}} \mathrm{tr}((k|i)(i|k)) = (k|i) \otimes_{\mathbb{C}} 1 \\
l_i &\mapsto l_i \otimes_{\mathbb{C}} e_i \otimes_{A(\Gamma)} l_i \mapsto l_i \otimes_{\mathbb{C}} 1
\end{aligned}$$

□

3.2.3 Constructing the category $\mathcal{C}$

To categorify the adjoint representation, we proceed in a similar manner to the case of level two representations, namely to each weight μ of the adjoint representation we assign an abelian category $\mathcal{C}_\mu$ and define functors between the categories $\mathcal{C}_\mu$ that lift the action of the quantum group $U_q(\mathfrak{g})$ on the adjoint representation V. Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra, with simply-laced Dynkin diagram as found in figure 1.10.1, R a root system of $\mathfrak{g}$, and Π a set of simple roots of $\mathfrak{g}$. Let Γ be the Dynkin diagram of $\mathfrak{g}$. Then Γ is simply-laced and of finite type and the Killing form on roots takes values in $\{-1, 0, 1\}$ for distinct roots and $(\mu, \alpha) = 2$ for all $\mu \in R$. For every root $\mu \in R$, let $\mathcal{C}_\mu := {}_g\mathbf{Vect}$ and let $\mathbb{C}_\mu \in \mathcal{C}_\mu$ be a one-dimensional vector space concentrated in degree 0. Define $\mathcal{C}_0 := A(\Gamma)$-**mod** and define the category $\mathcal{C}$ to be the direct sum over the weights of the adjoint representation:

$$\mathcal{C} := \bigoplus_{\lambda \in R \bigcup \{0\}} C_\lambda$$

The vertices of the Dynkin diagram Γ of $\mathfrak{g}$ correspond to the simple roots of $\mathfrak{g}$, so from our previous analysis of $A(\Gamma)$, there is a single projective indecomposable left $A(\Gamma)$-module P_α for each simple root α of $\mathfrak{g}$. $A(\Gamma)$ is a finite-dimensional $\mathbb{C}$-vector space and hence an artinian ring, so P_α has a unique simple quotient, denoted L_α.

Define the functors $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ for all $\alpha \in \Pi$ by:

$$\begin{aligned}
\mathcal{E}_\alpha(M) &= ({}_\alpha P \otimes_{A(\Gamma)} M) \otimes_{\mathbb{C}} \mathbb{C}_\alpha \\
\mathcal{F}_\alpha(M) &= ({}_\alpha P \otimes_{A(\Gamma)} M) \otimes_{\mathbb{C}} \mathbb{C}_{-\alpha}
\end{aligned}$$

for all $M \in \mathcal{C}_0$. For what follows, $M \in \mathcal{C}_\mu$ for $\mu \neq 0$ and tensor products are taken over $\mathbb{C}$.

$$
\begin{array}{lll}
\mathcal{E}_\alpha(M) = 0 & \mathcal{F}_\alpha(M) = 0 & \text{if } (\mu,\alpha) = 0 \\
\mathcal{E}_\alpha(M) = 0 & \mathcal{F}_\alpha(M) = M \otimes \mathbb{C}_{\mu-\alpha} & \text{if } (\mu,\alpha) = 1 \\
\mathcal{E}_\alpha(M) = M \otimes \mathbb{C}_{\mu+\alpha} & \mathcal{F}_\alpha(M) = 0 & \text{if } (\mu,\alpha) = -1 \\
\mathcal{E}_\alpha(M) = 0 & \mathcal{F}_\alpha(M) = P_\alpha \otimes M\{-1\} & \text{if } \mu = \alpha \\
\mathcal{E}_\alpha(M) = P_\alpha \otimes M\{-1\} & \mathcal{F}_\alpha(M) = 0 & \text{if } \mu = -\alpha
\end{array}
$$

Define the invertible functor $\mathcal{K}_\alpha : \mathcal{C} \to \mathcal{C}$ to be the shift functor

$$\mathcal{K}_\alpha(M) = M\{(\mu,\alpha)\} \text{ for } M \in \mathcal{C}_\mu$$

with inverse functor $\mathcal{K}_\alpha^{-1}(M) = M\{-(\mu,\alpha)\}$ for $M \in \mathcal{C}_\mu$. These functors lift the relations between the generators of the quantum group $U_q(\mathfrak{g})$, as seen in section **??**, in the sense of the following:

Proposition 3.2.8. *There are natural isomorphisms for $\alpha, \beta \in \Pi$:*

1. $\mathcal{K}_\alpha\mathcal{K}_\alpha^{-1} \cong Id \cong \mathcal{K}_\alpha^{-1}\mathcal{K}_\alpha$
2. $\mathcal{K}_\alpha\mathcal{K}_\beta \cong \mathcal{K}_\beta\mathcal{K}_\alpha$
3. $\mathcal{K}_\alpha\mathcal{E}_\beta \cong \mathcal{E}_\beta\mathcal{K}_\alpha\{(\beta,\alpha)\}$
4. $\mathcal{K}_\alpha\mathcal{F}_\beta \cong \mathcal{F}_\beta\mathcal{K}_\alpha\{-(\beta,\alpha)\}$
5. $\mathcal{E}_\alpha\mathcal{F}_\beta \cong \mathcal{F}_\beta\mathcal{E}_\alpha$ *if* $\alpha \neq \beta$
6. $\mathcal{E}_\alpha\mathcal{E}_\beta \cong \mathcal{E}_\beta\mathcal{E}_\alpha$ *if* $(\alpha,\beta) = 0$
7. $\mathcal{F}_\alpha\mathcal{F}_\beta \cong \mathcal{F}_\beta\mathcal{F}_\alpha$ *if* $(\alpha,\beta) = 0$
8. $\mathcal{E}_\alpha^2\mathcal{E}_\beta \oplus \mathcal{E}_\beta\mathcal{E}_\alpha^2 \cong (Id\{1\} \oplus Id\{-1\})\mathcal{E}_\alpha\mathcal{E}_\beta\mathcal{E}_\alpha$ *if* $(\beta,\alpha) = -1$
9. $\mathcal{F}_\alpha^2\mathcal{F}_\beta \oplus \mathcal{F}_\beta\mathcal{F}_\alpha^2 \cong (Id\{1\} \oplus Id\{-1\})\mathcal{F}_\alpha\mathcal{F}_\beta\mathcal{F}_\alpha$ *if* $(\beta,\alpha) = -1$

Proof. We give only the proof of 5, 8 and 9. The remaining isomorphisms follow from the definition of the functors $\mathcal{E}_\alpha$, $\mathcal{F}_\alpha$ and $\mathcal{K}_\alpha$ and are similar to the previous section on level two representations. Note first that $\mathcal{E}_\alpha(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu+\alpha}$ when $\mu + \alpha \in R \cup \{0\}$ otherwise $\mathcal{E}_\alpha$ is the zero functor. Similarly $\mathcal{F}_\alpha(\mathcal{C}_\mu) \subset \mathcal{C}_{\mu-\alpha}$ if $\mu - \alpha \in R \cup \{0\}$, otherwise $\mathcal{F}_\alpha$ is the zero functor.

In the case where the source and target categories $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$ are associated to non-zero weights and the case where the source category is $\mathcal{C}_0$, 5 follows from the identification of categories $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$ for all roots μ and ν and the canonical isomorphism $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ for any objects in ${}_g\mathbf{Vect}$. When the target category is $\mathcal{C}_0$, the source category is $\mathcal{C}_{-\alpha+\beta}$. Neither $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ can act non-trivially on this category as this would imply $(-\alpha + \beta, \alpha) = -1$, and hence $(\alpha, \beta) = 1$, which contradicts the fact that α and β are simple roots.

The only case in which the functors in 8 can act non-trivially when the source category is $\mathcal{C}_{-\alpha}$. In this case, $\mathcal{E}_\alpha^2\mathcal{E}_\beta$ is the zero functor and for $M \in Ob(\mathcal{C}_{-\alpha})$,

$$\begin{aligned}\mathcal{E}_\beta\mathcal{E}_\alpha^2(M) &= \mathcal{E}_\beta\mathcal{E}_\alpha(P_\alpha \otimes_{\mathbb{C}} M{-}1)\\ &= \mathcal{E}_\beta({}_\alpha P \otimes_{A(\Gamma)} P_\alpha \otimes_{\mathbb{C}} M\{-1\} \otimes_{\mathbb{C}} \mathbb{C}_\alpha)\\ &= {}_\alpha P \otimes_{A(\Gamma)} P_\alpha \otimes_{\mathbb{C}} M\{-1\} \otimes_{\mathbb{C}} \mathbb{C}_\alpha \otimes_{\mathbb{C}} \mathbb{C}_{\alpha+\beta}\\ &\cong (\mathbb{C} \oplus \mathbb{C}\{2\}) \otimes_{\mathbb{C}} M\{-1\}\\ &\cong M\{-1\} \oplus M\{1\}\end{aligned}$$

Furthermore, $(\alpha, \beta) = -1$, hence the vertices corresponding to α and β in the Dynkin diagram of $\mathfrak{g}$ are connected by an edge and by lemma 3.2.6, ${}_\beta P \otimes_{A(\Gamma)} P_\alpha \cong \mathbb{C}\{1\}$. Thus, for $M \in Ob(\mathcal{C}_{-\alpha})$,

$$\begin{aligned}\mathcal{E}_\alpha\mathcal{E}_\beta\mathcal{E}_\alpha(M) &= \mathcal{E}_\alpha\mathcal{E}_\beta(P_\alpha \otimes_{\mathbb{C}} M\{-1\})\\ &= \mathcal{E}_\alpha({}_\beta P \otimes_{A(\Gamma)} P_\alpha \otimes_{\mathbb{C}} M\{-1\} \otimes_{\mathbb{C}} \mathbb{C}_\beta)\\ &= {}_\beta P \otimes_{A(\Gamma)} P_\alpha \otimes_{\mathbb{C}} M\{-1\} \otimes_{\mathbb{C}} \mathbb{C}_\beta \otimes \mathbb{C}_{\alpha+\beta}\\ &\cong \mathbb{C}\{1\} \otimes_{\mathbb{C}} M\{-1\}\\ &\cong M\end{aligned}$$

This shows 8 in the only non-trivial case. The proof for 9 is similar, with the only non-trivial case having source category $\mathcal{C}_\alpha$. □

There is a further relation on the quantum group: for any $\alpha \in \Pi$

$$E_\alpha F_\alpha - F_\alpha E_\alpha = \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}}$$

The only structure that can be lifted to the category $\mathcal{C}$ is positive and integral, so we consider the action of $\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}}$ on individual weight spaces V_μ: K_α acts on V_μ by multiplication by $q^{(\alpha,\mu)}$ and K_α^{-1} is multiplication by $q^{-(\alpha,\mu)}$. Thus, $\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}}$ acts by multiplication by $\frac{q^{(\alpha,\mu)} - q^{-(\alpha,\mu)}}{q-q^{-1}}$. Let $k = (\alpha, \mu) \in \mathbb{Z}$. Then, for $k \geq 0$, we have

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{1}{q}(q^k - q^{-k})(1 + q^{-2} + q^{-4} + \ldots) = q^{k-1} + q^{k-3} + \ldots + q^{1-k}$$

and for $k < 0$, $q^k - q^{-k} = -(q^{k'} - q^{-k'})$, where $k' = -k > 0$, so $\frac{q^k - q^{-k}}{q-q^{-1}} = -[-k]$. Thus, we can rewrite the $\mathfrak{sl}_2$ relation in an integral positive form:

If $(\alpha, \mu) \geq 0$, $E_\alpha F_\alpha = F_\alpha E_\alpha + [(\mu, \alpha)]$ on V_μ.

If $(\alpha, \mu) < 0$, $E_\alpha F_\alpha + [-(\alpha, \mu)] = F_\alpha E_\alpha$ on V_μ.

Define the functor $\mathrm{Id}^{[k]} = \mathrm{Id}\{k-1\} \oplus \mathrm{Id}\{k-3\} \oplus \ldots \oplus \mathrm{Id}\{1-k\}$ for any nonnegative integer k from $\mathcal{C}_\mu$ to itself. Then we can lift the $\mathfrak{sl}_2$ relation as follows:

Proposition 3.2.9. *For $\lambda \in R \cup \{0\}$ there are natural isomorphisms in the category $\mathcal{C}_\mu$*

$$\begin{aligned}&\mathcal{E}_\alpha\mathcal{F}_\alpha \cong \mathcal{F}_\alpha\mathcal{E}_\alpha \oplus Id^{[(\mu,\alpha)]} && \textit{if } (\mu, \alpha) \geq 0\\ &\mathcal{E}_\alpha\mathcal{F}_\alpha \oplus Id^{[-(\mu,\alpha)]} \cong \mathcal{F}_\alpha\mathcal{E}_\alpha && \textit{if } (\mu, \alpha) < 0\end{aligned}$$

Proof. Let M be an object in $\mathcal{C}_0$. Then

$$\begin{aligned}\mathcal{E}_\alpha\mathcal{F}_\alpha(M) &= \mathcal{E}_\alpha({}_\alpha P \otimes_{A(\Gamma)} M) \otimes_{\mathbb{C}} \mathbb{C}_{-\alpha} \\ &= P_\alpha \otimes_{\mathbb{C}} ({}_\alpha P \otimes_{A(\Gamma)} M) \otimes_{\mathbb{C}} \mathbb{C}_{-\alpha}\{-1\} \\ \mathcal{F}_\alpha\mathcal{E}_\alpha(M) &= \mathcal{F}_\alpha({}_\alpha P \otimes_{A(\Gamma)} M) \otimes \mathbb{C}_\alpha \\ &= P_\alpha \otimes_{\mathbb{C}} ({}_\alpha P \otimes_{A(\Gamma)} M) \otimes_{\mathbb{C}} \mathbb{C}_\alpha\{-1\}\end{aligned}$$

These objects are isomorphic as $A(\Gamma)$-modules, identifying the categories $\mathcal{C}_\alpha$ and $\mathcal{C}_{-\alpha}$ with ${}_g\mathbf{Vect}$. We also have $[(0,\alpha)] = [0] = 0$ for all simple roots α.

Let M be an object in $\mathcal{C}_\alpha$. Then

$$\begin{aligned}\mathcal{E}_\alpha\mathcal{F}_\alpha(M) &= \mathcal{E}_\alpha(P_\alpha \otimes_{\mathbb{C}} M\{-1\}) \\ &=_\alpha P \otimes_{A(\Gamma)} P_\alpha \otimes_{\mathbb{C}} M\{-1\} \otimes_{\mathbb{C}} \mathbb{C}_\alpha \\ &\cong (\mathbb{C} \oplus \mathbb{C}\{2\}) \otimes_{\mathbb{C}} M\{-1\} \otimes_{\mathbb{C}} \mathbb{C}_\alpha \\ &\cong M\{-1\} \oplus M\{1\} \\ \mathcal{F}_\alpha\mathcal{E}_\alpha(M) &= 0\end{aligned}$$

where some of the isomorphisms follow from claim 3.2.6. For $\mu = \alpha$, $[(\mu,\alpha)] = [2]$, so $\mathrm{Id}^{[2]} = \mathrm{Id}\{-1\} \oplus \mathrm{Id}\{1\}$.

Let M be an object in $\mathcal{C}_\mu$ with $(\mu,\alpha) = -1$. Then

$$\begin{aligned}\mathcal{E}_\alpha\mathcal{F}_\alpha(M) &= 0 \\ \mathcal{F}_\alpha\mathcal{E}_\alpha(M) &= \mathcal{F}_\alpha(M \otimes_{\mathbb{C}} \mathbb{C}_{\mu+\alpha}) \\ &= M \otimes_{\mathbb{C}} \mathbb{C}_{\mu+\alpha} \otimes_{\mathbb{C}} \mathbb{C}_{\mu-\alpha} \\ &\cong M\end{aligned}$$

and $[(\mu,\alpha)] = [1] = 1$.

The remaining cases are similar. □

3.2.4 Decategorification

Proposition 3.2.10. *The Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$ is isomorphic to the $\mathbb{Z}[q,q^{-1}]$-submodule I' of V generated by the dual canonical basis $\{x_\mu, l_\alpha\}$. This isomorphism restricts to an isomorphism between the projective Grothendieck group $K_P(\mathcal{C})$ of $\mathcal{C}$ and the $\mathbb{Z}[q,q^{-1}]$-submodule I of V generated by the canonical basis $\{x_\mu, h_\alpha\}$.*

Proof. Define $\iota : K(\mathcal{C}) \to I'$ by $\iota([\mathbb{C}_\mu]) = x_\mu$ and $\iota([L_\alpha]) = q\ell_\alpha$. This map identifies isomorphism classes of simple objects in $\mathcal{C}$ with the elements of the dual canonical basis. The subcatetogory $\mathcal{C}_{\neq 0} := \bigoplus_{\mu\in R}\mathcal{C}_\mu$ is a semisimple category (as a direct sum of copies of ${}_g\mathbf{Vect}$), hence any object in $\mathcal{C}_{\neq 0}$ decomposes into a sum of copies of the $\mathbb{C}_\mu$, up to shifts. The following lemma shows that the simple objects L_α are sufficient to generate the Grothendieck group of the category $\mathcal{C}$. Note that the lemma holds more generally for a finite-dimensional k-algebra $\mathscr{A}$ with complete set of orthogonal idempotents $\{e_1, \ldots, e_n\}$ and corresponding set of simple modules $\{L_1, \ldots, L_n\}$ such that $e_i\mathscr{A} \neq e_j\mathscr{A}$ for $i \neq j$.

Lemma 3.2.11. *The Grothendieck group $K(\mathcal{C}_0)$ is a free $\mathbb{Z}[q, q^{-1}]$=module with basis the set of isomorphism classes of simple modules $\{[L_\alpha] | \ \alpha \in \Pi\}$.*

Proof. Let M be an object in $\mathcal{C}_0$. Then M is a finite-dimensional graded $A(\Gamma)$-module, and $A(\Gamma)$ is in particular an Artinian ring, so M has a finite Jordan Hölder series

$$0 = M_0 \subset M_1 \subset \ldots \subset M_n = M.$$

Then we have short exact sequences

$$0 \longrightarrow M_{k-1} \longrightarrow M_k \longrightarrow M_k/M_{k-1} \longrightarrow 0$$

for $1 \leq k \leq n$. In particular, for $k = n$,

$$0 \longrightarrow M_{n-1} \longrightarrow M \longrightarrow M/M_{k-1} \longrightarrow 0$$

Thus, in the Grothendieck group,

$$[M] = [M/M_{n-1}] + [M_{n-1}] = [M/M_{n-1}] + [M_{n-1}/M_{n-2}] + [M_{n-2}] = \sum_{k=0}^{n-1} [M_{n-k}/M_{n-k-1}].$$

Each of the summands are isomorphism classes of simple $A(\Gamma)$-modules up to some shift, hence equal to $q^{m_\alpha}[L_\alpha]$ for some $\alpha \in \Pi$ and some $m_\alpha \in \mathbb{Z}$. Therefore $[M] = \sum_{\alpha \in Pi} n_\alpha f(q)[L_\alpha]$, where the $n_\alpha \in \mathbb{N}$ are the multiplicities of the simple modules $[L_\alpha]$ in the Jordan Hölder series and $f(q) \in \mathbb{Z}[q, q^{-1}]$. □

The simple modules generate the Grothendieck group of $\mathcal{C}$ as a $\mathbb{Z}[q, q^{-1}]$-module, hence under the map ι, $K(\mathcal{C})$ is isomorphic to I'.

The map ι sends isomorphism classes of projectives $[P_\alpha]$ to qh_α, so that restricting the identification between $K(\mathcal{C})$ and I' to the projective Grothendieck group $K_P(\mathcal{C})$ gives an isomorphism between $K_P(\mathcal{C})$ and the submodule I. □

$\mathcal{C}$ is a categorification of the adjoint representation in the sense of the following:

Corollary 3.2.12. *After tensoring with the ground field $\mathbb{Q}(q)$, the Grothendieck group of $\mathcal{C}$ is isomorphic to the adjoint representation V of $U_q(\mathfrak{g})$:*

$$K(\mathcal{C}_0) \oplus (\bigoplus_\mu K(\mathcal{C}_\mu) \cong K(\mathcal{C}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \cong V \cong V_0 \oplus (\bigoplus_\mu V_\mu)$$

Proof. The functors $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ consist of tensoring with projective modules, and are hence exact. The shift functor is also clearly exact, so $\mathcal{K}_\alpha$ is exact. Furthermore the functors $\mathcal{E}_\alpha, \mathcal{F}_\alpha$ and $\mathcal{K}_\alpha$ all commute with the shift functor $\{1\}$, and thus induce well-defined $\mathbb{Z}[q, q^{-1}]$-linear maps on the Grothendieck group of $\mathcal{C}$. The source and target categories of each of the functors $\mathcal{E}_i, \mathcal{F}_i$ and $\mathcal{K}_i$ ensure that the functors descend to maps acting as the generators of the quantum group on the Grothendieck group. The natural isomorphisms in propositions 3.2.8 and 3.2.9 therefore descend to the $U_q(\mathfrak{g})$ relations between the induced maps. The identification of images of simple objects in $\mathcal{C}$ with elements of the dual canonical basis of V shows that the induced maps act as the generators for $U_q(\mathfrak{g})$ on the adjoint representation.

For example: given M an object in $\mathcal{C}_\mu$, $\mathcal{K}_\alpha(M) = M\{(\mu, \alpha)\}$, which descends to $[\mathcal{K}_\alpha]([M]) = [M\{(\mu, \alpha)\}] = q^{(\mu,\alpha)}[M]$ in the Grothendieck group.

If $M = \mathbb{C}_{-\alpha}$, then $\mathcal{E}_\alpha(M) = P_\alpha \otimes_\mathbb{C} \mathbb{C}_{-\alpha}\{-1\}$, which descends to

$$\begin{aligned}[\mathcal{E}_\alpha]([\mathbb{C}_{-\alpha}]) &= [P_\alpha \otimes_\mathbb{C} \mathbb{C}_{-\alpha}\{-1\}] \\ &= q^{-1}[P_\alpha] \\ &= q^{-1} q h_\alpha\end{aligned}$$

after identifying $[P_\alpha]$ with h_α under the isomorphism ι.

Finally, by the previous proposition, the Grothendieck group of $\mathcal{C}$, after tensoring with the ground field, and V are isomorphic as vector spaces, and hence isomorphic as $U_q(\mathfrak{g})$ representations. □

3.2.5 Further lifted structure

The structure on the adjoint representation described in section 3.2.1 is lifted to the category $\mathcal{C}$.

Semilinear form

The semilinear form $\langle\ ,\ \rangle$ on V can be considered as the shadow of higher structure on the category $\mathcal{C}$, namely the graded dimension of homomorphism spaces in $\mathcal{C}$:

$$\langle [P], [M] \rangle = \mathrm{gdimHOM}_\mathcal{C}(P, M) = \sum_{i \in \mathbb{Z}} q^i \mathrm{dimHom}_\mathcal{C}(P\{i\}, M)$$

for any projective object in $\mathcal{C}$ and any module M in $\mathcal{C}$, where $\mathrm{Hom}_\mathcal{C}(-,-)$ is the space of grading-preserving morphisms between pairs of objects, and $\mathrm{HOM}_\mathcal{C}(-,-)$ is the graded vector space of all morphisms between objects in $\mathcal{C}$.

Adjointness

The antiautomorphism τ on $U_q(\mathfrak{g})$ lifts to an operation on functors acting on the categories $\mathcal{C}_\mu$:

Proposition 3.2.13. *The functor $\mathcal{E}_\alpha$ is left adjoint to $\mathcal{F}_\alpha\mathcal{K}_\alpha^{-1}\{1\}$, the functor $\mathcal{F}_\alpha$ is left adjoint to $\mathcal{E}_\alpha\mathcal{K}_\alpha\{1\}$ and $\mathcal{K}_\alpha$ is left adjoint to $\mathcal{K}_\alpha^{-1}$.*

Proof. In the cases where μ and $\mu+\alpha$ are non-zero, the statement is clear, since $\mathcal{E}_\alpha$ and $\mathcal{F}_\alpha$ consist of tensoring with a simple module. In the case where either μ or $\mu+\alpha$ is zero, the functors consist of tensoring with projective $A(\Gamma)$-modules, and this reduces to the statement of lemma 3.2.7. □

Thus, the action of τ at the level of $U_q(\mathfrak{g})$ lifts to an operation at the categorical level of sending a functor to its right adjoint functor. Let T be a product of the functors $\mathcal{E}_\alpha, \mathcal{F}_\alpha, \mathcal{K}_\alpha, \mathcal{K}_\alpha^{-1}$ and the shift functors, and T^{ad} the adjoint functor of T if it exists and let P and M be any objects in $\mathcal{C}$. Then there is an isomorphism of graded vector spaces $\mathrm{HOM}_\mathcal{C}(T(P), M) \cong \mathrm{HOM}_\mathcal{C}(P, T^{ad}(M))$, where $T(P)$ is projective because the functors generating T are exact and P is projective. In particular this holds if $T = \mathcal{E}_\alpha, \mathcal{F}_\alpha$ or $\mathcal{K}_\alpha^{\pm 1}$, so that this isomorphism descends to the τ-invariance of the semilinear form on V.

The antilinear involution ψ_V

Let $\chi : A(\Gamma) \to A(\Gamma)$ be the antiinvolution sending a path $(i_1|i_2|\dots|i_j)$ in $A(\Gamma)$ to the path in the opposite direction $(i_j|\dots|i_2|i_1)$. Then we can define the following functors:

$$\begin{aligned} * :\ {}_g\mathbf{Vect} &\to {}_g\mathbf{Vect} \\ W &\mapsto W^* = \mathrm{Hom}(W, \mathbb{C}) \\ \Psi : \mathcal{C} &\to \mathcal{C} \\ M &\mapsto M^* \text{ if } M \in Ob(\mathcal{C}_\mu) \text{ with } \mu \in R \\ M &\mapsto \chi(M^*) \text{ if } M \in Ob(\mathcal{C}_0) \end{aligned}$$

where by $\chi(M^*)$, we mean consider the right $A(\Gamma)$-module M^* as a left $A(\Gamma)$-module by applying the antiinvolution to $A(\Gamma)$ acting on M^*. Thus, M^* can be considered as an object in $\mathcal{C}_0$. Note that the functors $*$ and Ψ are contravariant.

Proposition 3.2.14. *1. There are natural isomorphisms*

$$\Psi\mathcal{E}_\alpha \cong \mathcal{E}_\alpha \qquad \Psi\mathcal{F}_\alpha \cong \mathcal{F}_\alpha\Psi \qquad \Psi\mathcal{K}_\alpha \cong \mathcal{K}_\alpha^{-1}\Psi \qquad \Psi\{i\} \cong \{-i\}\Psi$$

2. The functor Ψ induces a well-defined involution on the Grothendieck group $K(\mathcal{C})$, acting as the map ψ_V on V.

3. There is a natural isomorphism $\Psi^2 \cong Id$, namely Ψ is an involution.

Proof. 1. This follows from the isomorphism between W and W^* for any finite-dimensional (graded) vector space W. Note that $(\mathbb{C}\{k\})^* \cong \mathbb{C}\{-k\}$ since $\mathrm{Hom}(\mathbb{C}\{k\}, \mathbb{C}$ must consist of homomorphisms that have degree $\{-k\}$. This fact ensures the existence of natural isomorphisms $\Psi\mathcal{K}_\alpha \cong \mathcal{K}_\alpha^{-1}\Psi$ and $\Psi\{k\} \cong \{-1\}\Psi$.

2. Ψ sends objects to isomorphic objects, so it is an exact functor, and induces a well-defined map on the Grothendieck group. From the relation $\Psi\{k\} \cong \{-k\}\Psi$, this map is q-antilinear. For $\mu \neq 0$, $\Psi(\mathbb{C}_\mu) = \mathbb{C}_\mu{}^* \cong \mathbb{C}_\mu$, which descends to $[\Psi][\mathbb{C}_\mu] = [\mathbb{C}_\mu]$, or under the isomorphism ι, $[\Psi](x_\mu) = x_\mu$. Similarly, $\Psi(P_\alpha) = \chi({}_\alpha P) \cong P_\alpha^{op}$, where P_α^{op} denotes P_α with inverse grading. After applying the isomorphism ι, this descends to $[\Psi](qh_\alpha) = q^{-1}h_\alpha$ and simplifies to $[\Psi](h_\alpha) = h_\alpha$. Therefore Ψ induces a map acting on the Grothendieck group of $\mathcal{C}$ as the involution ψ_V, with the isomorphisms in 1 descending to relations at the decategorified level ensuring $[\Psi](ax) = \psi(a)[\Psi](x)$ for all $a \in U_q(\mathfrak{g})$ and all $x \in K(\mathcal{C})$.

3. This is clear from the definition of Ψ.

□

Furthermore, 3 implies that Ψ is an autoequivalence of $\mathcal{C}$, and we have

$$\mathrm{HOM}_{\mathcal{C}}(\Psi(M), \Psi(N)) \cong \mathrm{HOM}_{\mathcal{C}}(\Psi^2(N), M) \cong \mathrm{HOM}_{\mathcal{C}}(N, M)$$

for all objects M and N in $\mathcal{C}$. This descends to the equation $\langle \psi_V[M], \psi_V[N]\rangle = \langle [N], [M]\rangle$ on the Grothendieck group.

The linear involution ω_V

Define the following autoequivalence $\Omega : \mathcal{C} \to \mathcal{C}$ by setting Ω to be the identity functor on $\mathcal{C}_0$ and for any $\mu \in R$, setting Ω to be the equivalence of categories from $\mathcal{C}_\mu$ to $\mathcal{C}_{-\mu}$ obtained by identifying each of the categories with ${}_g\mathbf{Vect}$.

Proposition 3.2.15. *1. There are natural isomorphisms*

$$\Omega\mathcal{E}_\alpha \cong \mathcal{F}_\alpha\Omega \qquad \Omega\mathcal{F}_\alpha \cong \mathcal{E}_\alpha\Omega \qquad \Omega\mathcal{K}_\alpha \cong \mathcal{K}_\alpha^{-1}\Omega$$

2. The functor Ω induces a well-defined involution of the Grothendieck group of $\mathcal{C}$, acting as the involution ω_V does on V.

3. There is a natural isomorphism $\Omega^2 \cong Id$.

Proof. 1. This proof is similar to the previous proposition 3.2.14

2. Ω is clearly an exact functor and since it commutes with the shift functor, it descends to a well-defined q-linear map on the Grothendieck group. Ω preserves the projective objects in $\mathcal{C}_0$ and sends the simple objects $\mathbb{C}_\mu$ to simple objects $\mathbb{C}_{-\mu}$, so under the isomorphism ι, $[\Omega]$ acts as the involution ω_V on the Grothendieck group.

3. This is clear from the definition of Ω.

□

As for the functor Ψ, the functor Ω is an equivalence, so that by 3, there is an isomorphism

$$\mathrm{HOM}_{\mathcal{C}}(\Omega M, \Omega N) \cong \mathrm{HOM}_{\mathcal{C}}(M, \Omega^2 N) \cong \mathrm{HOM}_{\mathcal{C}}(M, N)$$

which descends to the equation $\langle \omega_V[M], \omega_V[N] \rangle = \langle [M], [N] \rangle$ on the Grothendieck group of $\mathcal{C}$ for all objects $M, N \in \mathcal{C}$.

Therefore this is a categorification of the adjoint representation of $U_q(\mathfrak{g})$ that lifts the structure of V to the category $\mathcal{C}$.

The categorical action of quantum groups $U_q(\mathfrak{g})$ suggests that the quantum group itself can be categorified. Further evidence of the existence of categorifications of quantum groups is given by the structures on $U_q(\mathfrak{g})$ itself, such as the semilinear form $\langle\ ,\ \rangle$, or the Beilinson-Lusztig-Macpherson idempotented modifications of quantum groups [BLM90]. Indeed, categorified quantum groups have since been constructed, most notably by Rouquier-Chuang for $\mathfrak{sl}_2$ [CR08], Khovanov-Lauda for $\mathfrak{sl}_n$ [KL09] and Rouquier for symmetrisable Kac-Moody algebras [Rou08].

Chapter 4

Annular Khovanov homology

We construct a modification of Khovanov homology for links contained in the solid torus from the perspective of representation theory. This is a particular example of a homology of links in I-bundles over surfaces defined by Asaeda, PRzytycki and Sikora [APS04]. We then demonstrate some of the rich structure underlying this annular homology. The main result of this chapter is theorem 4.4.1, which states that the current algebra $\mathfrak{sl}_2^-(V_2)$ acts on the annular Khovanov homology of a link.

To discuss link homologies, we use the following definitions and theorem.

Definition A *knot* is a smooth embedding of the circle S^1 into $\mathbb{R}^3$ (or alternatively into S^3).

A *link* is a disjoint union of knots.

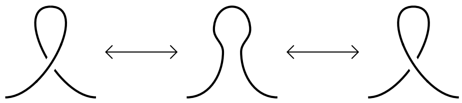
Knots and links are distinguished up to isotopy, where

Definition Let K_1 and K_2 be links given by the respective embeddings f and g from S^1 to $\mathbb{R}^3$. K_1 and K_2 are said to be *isotopic* if there exists a homotopy $H : S^1 \times [0,1] \to \mathbb{R}^3$ from f to g such that $H(x,t)$ is an embedding for all fixed $t \in [0,1]$.

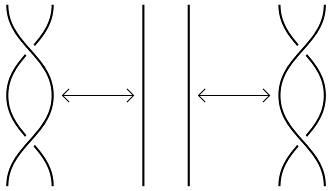
A simpler method of studying links or knots is to consider link diagrams, wherein we project a link onto the plane, taking note of the bottom and top strands at any double points in this projection. This projection is not canonical, and a single link has many different diagrams, so it would seem that the question of whether two link diagrams are obtained from the same link is a very complex one. However, the following theorem of Reidemeister [Rei27] shows that there are in fact only three ways in which two link diagrams of a given link can differ, up to planar isotopies:

Theorem 4.0.16 (Reidemeister)**.** *Two link diagrams D_1 and D_2 correspond to the same link up to isotopy if D_2 can be obtained from D_1 be a sequence of moves of the following types (called Reidemeister moves) and planar isotopies:*

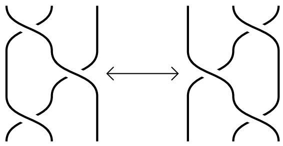
(R1)



(R2)



(R3)



One aim in knot theory is to define properties of links using link diagrams that are intrinsic to the link, namely this property does not depend on the choice of projection. Such a property is called an *invariant* of links, and by the Reidemeister theorem, to check that an object is a link invariant, it is sufficient to prove that it is invariant under Reidemeister moves.

4.1 Khovanov's categorification of the Jones polynomial

First, we provide a construction of Khovanov's link invariant **??**, which will be later modified to suit the annular case. The main objective is to construct a cochain complex $\mathcal{C}(L)$ of graded vector spaces from an oriented link L such that the cohomology groups of $\mathcal{C}(L)$ are link invariants, and the graded Euler characteristic of $\mathcal{C}(L)$ is the unnormalised Jones polynomial. This is another example of categorification, where the decategorified object, a polynomial, is somewhat more complex to the example encountered in chapter 3, where a number was lifted to a vector space. The additional complexity of a polynomial lifts to additional structure at the categorified level, namely a sequence of vector spaces.

Let L be an oriented knot or link in $\mathbb{R}^3$ and choose a projection of L onto the plane, noting the relative heights of strands at double points. We call such a projection a link diagram, denoted D, or $D(L)$ if there is some ambiguity. The choice of projection is restricted so that D has only a finite number n of double points, no triple intersection points, no tangencies and no cusps. We generally follow the notation used by Bar-Natan in his exposition of Khovanov's work [BN02], in particular in our presentation of the Jones polynomial.

4.1.1 The Jones polynomial

The definition of the Jones polynomial here uses the skein relations of the Kauffman brace, a Laurent polynomial with coefficients in $\mathbb{Z}$, permitting a simple construction of the Jones polynomial from a given link diagram D.

The Kauffman bracket is defined axiomatically:

1. The bracket of the constant polynomial 1: $\langle \emptyset \rangle = 1$
2. The bracket of a closed loop with no crossings is the polynomial $q + q^{-1}$, and disjoint diagrams multiply:for any link diagram D, $\langle \bigcirc D \rangle = (q + q^{-1})\langle D \rangle$
3. The bracket for a double point is given by a linear combination of its two resolutions:

where  is called a 0-resolution and  is called a 1-resolution.

The Jones polynomial is a renormalisation of the Kauffman bracket: for a link L with some projection D, the Jones polynomial is obtained from the Kauffman bracket of D by the following relation

$$J(D) = \frac{(-1)^{n_-} q^{n_+ - 2n_-}}{q + q^{-1}} \langle D \rangle$$

Note that while the Jones polynomial is defined for oriented knots and links, the Kauffman bracket does not see this orientation. The relation above reintroduces the orientation to the Jones polynomial. The following theorem shows that the Jones polynomial is an invariant of oriented knots and links [Jon85].

Theorem 4.1.1. *The Jones polynomial is invariant under Reidemeister moves.*

Consequently, for a link or knot L and any choice of projection D of L, we may define the Jones polynomial unambiguously for an isotopy class of links L, denoted $J(L)$, where $J(L) = J(D)$.

4.1.2 Resolution of a knot or link diagram

Given an oriented knot or link L and a choice of link diagram D, assign an ordering to the set X of crossings. The orientation of L is carried over to D and determines the parity of a crossing:

 is a positive crossing

 is a negative crossing

Let n_+ and n_- be the number of positive and negative crossings respectively. As in the categorification of level two representations of $\mathfrak{sl}_n$, the construction of Khovanov homology consists of applying a functor from ${}_1\mathbf{Cob}$ to ${}_g\mathbf{Vect}$. To apply this functor, each double point of D must be resolved as either a 0-resolution or a 1-resolution. A *complete resolution* is a diagram with all double points resolved. There are 2^n different possible complete resolutions of a diagram D with n double points, and these complete resolutions bijectively correspond to the elements of the set of ordered sequences $\{0,1\}^X$. Fix an ordering of X and label each complete resolution by the element $\alpha \in \{0,1\}^X$ corresponding to the choice of resolution at each crossing in this order. Each complete resolution consists of a disjoint union of copies of S^1 embedded in $\mathbb{R}^2$, and we can apply a TQFT as in section 3.1. The height $h(\alpha)$ of a complete resolution is the number of 1-resolutions in α: $h(\alpha) = \sum_{i=1}^{|X|} \alpha_i$ where α_i denotes the ith element of the sequence α, with respect to the choice of ordering on X.

To organise the set of complete resolutions of a diagram D, define the n-dimensional cube of resolutions of D as follows: the 2^n vertices of the cube consist of distinct resolution $\alpha \in \{0,1\}^X$, arranged into columns such that each column consists of resolutions of the same height h, and arrange the columns in ascending order of height, namely, there will be $n+1$ columns, from height 0 to height n. Two vertices are connected by an edge if their resolutions α and β differ at a single position $i \in \{1,\ldots,n\}$:

$$|\alpha_j - \beta_j| = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Edges in the cube of resolution are drawn as arrows from α to β for $h(\alpha) = h(\beta) - 1$. This edge is labelled by d_ξ where $\xi \in \{0,1,\star\}^X$ is a sequence with a single $\star$ such that sending $\star \to 0$ takes ξ to α and sending $\star \to 1$ takes ξ to β. A thorough discussion of edges can be found in section **??**.

4.1.3 Chain groups

The aim is to obtain a homology theory for links from link diagrams, so the main step in the construction of Khovanov homology is to define a chain complex from a given link diagram D. To construct a cochain complex we must now associate a chain group to the resolutions. This is achieved through the use of a topological quantum field theory (TQFT), a functor $\mathcal{Q}$ from the cobordism category ${}_1\mathbf{Cob}$ of closed 1-manifolds to the category ${}_g\mathbf{Vect}$ of finite-dimensional graded vector spaces over $\mathbb{C}$. A more complete description of a similar TQFT is given in section 3.1.

This TQFT sends a disjoint union of cycles (copies of S^1) to a tensor power of a graded vector space W. More concretely, let α be a complete resolution of a diagram D and let k_α be the number of cycles in α. To a single cycle, the functor $\mathcal{Q}$ associates a q-graded vector space W, with basis $\{w_+, w_-\}$, where $\deg(w_+) = q$ and $\deg(w_-) = q^{-1}$, so that $q\dim W = q + q^{-1}$. To a union of k disjoint cycles, $\mathcal{Q}$ associates $W^{\otimes k}$. Hence, to a resolution α, $\mathcal{Q}$ associates the vector space $W^{\otimes k_\alpha}$.

Let $[\![D]\!]$ be a chain complex with chain groups consisting of direct sums over α of equal height of $\mathcal{Q}(\alpha)$, where α is a complete resolution of D shifted by the common height:

$$[\![D]\!]^r = \bigoplus_{\alpha : h(\alpha) = r} W^{\otimes k_\alpha}\{r\},$$

for $r \in \{0, 1, \dots n\}$, otherwise $[\![D]\!]^r = 0$. This is obtained by taking the direct sum over all resolutions in a column of the cube of resolutions after applying the TQFT and shifting up by the height of resolutions in a given column. To ensure $[\![D]\!]$ is an honest to goodness chain complex, it remains to define boundary maps between the chain groups $[\![D]\!]^r$.

4.1.4 Boundary maps

The TQFT $\mathcal{Q}$ is a functor from ${}_1\mathbf{Cob}$ to ${}_g\mathbf{Vect}$, so grading-preserving linear maps between the chain groups arise from cobordisms between one-manifolds.

Recall that the edges of the cube of resolutions are labelled by d_ξ for $\xi \in \{0, 1, \star\}^X$. Each edge of the cube of resolutions corresponds to a change of a single resolution, from a zero-resolution to a one-resolution. This change of a single resolution consists either of two cycles fusing into a single cycle, or a cycle splitting into exactly two distinct cycles, while all other cycles remain the same. In either case, the cobordism consists of a three-holed sphere and a copy of a cylinder for each unchanged cycle. Under the TQFT $\mathcal{Q}$, the three-holed sphere corresponds to one of two types of linear maps at the chain level: a multiplication map $m : W \otimes W \to W$ corresponding to a fusing of two cycles and a comultiplication map $\Delta : W \to W \otimes W$ corresponding to a splitting of a single cycle. There is no canonical ordering of the cycles in each resolution, so m and Δ must be commutative and cocommutative respectively. Furthermore, the maps must be associative and coassociative respectively and satisfy the following identity:

$$\Delta \circ m = (m \otimes id) \circ (id \otimes \Delta).$$

Thus we define $m : W \otimes W \to W$ by

$$m : \begin{cases} w_+ \otimes w_+ \mapsto w_+ \\ w_+ \otimes w_- \mapsto w_- \\ w_- \otimes w_+ \mapsto w_- \\ w_- \otimes w_- \mapsto 0 \end{cases}$$

and $\Delta : W \to W \otimes W$ by

$$\Delta : \begin{cases} w_+ \mapsto w_+ \otimes w_- + w_- \otimes w_+ \\ w_- \mapsto w_- \otimes w_- \end{cases}$$

The maps m and Δ shift the q-grading down by one, so in fact we have $m : W \otimes W \to W\{-1\}$ and $\Delta : W \to W \otimes W\{-1\}$. For example, in the mapping $m(w_+ \otimes w_-) = w_-$, $\deg(w_+ \otimes w_-) = 1 + (-1) = 0$ while $\deg(w_-) = -1$. In the category ${}_g\mathbf{Vect}$, morphisms are grading-preserving, so this shift must be counteracted in the definition of chain groups of $[\![D]\!]$.

The edges d_ξ of the cube are defined in terms of m and Δ. If the edge ξ corresponds to a cobordism from two cycles to one, let $d^\xi := m \otimes \mathrm{Id}^{j_\xi}$, where j_ξ is the number of unchanged cycles. Similarly, if the edge ξ corresponds to a cobordism from one cycle to two, then $d_\xi := \Delta \otimes \mathrm{Id}^{j_\xi}$.

Each boundary map of the chain complex is defined to be the sum over edge maps between two fixed columns of the cube of resolutions. However, the definition of the maps m and Δ ensures that each square of the cube of resolutions commutes, so that the map defined above will not be a boundary map for chain complexes. It is therefore necessary to add in signs to the edge maps so that each square of the cube of resolutions anticommutes. This is achieved by flipping signs of odd numbers of edge maps on each square. A systematic way of doing this is by multiplying each d_ξ by the factor $(-1)^\xi := (-1)^{\sum_{i<j} \xi_i}$ where j is the position of the $\star$ in ξ and ξ_i is the ith element in the sequence ξ. More concretely, we add up the number of 1s in the sequence that occur before the $\star$.

Define the boundary maps on $[\![D]\!]$ by

$$d^r := \sum_{h(\xi)=r} (-1)^\xi d_\xi.$$

Thus, to preserve the q-grading over each boundary map, we must shift up by one degree at each successive chain group $[\![D]\!]^r$ to counteract the shift down by one occurring once in each d_ξ and consequently in each d^r, as seen in the definition of the chain groups.

4.1.5 Final adjustments

The last step is to apply shifts to the homological and q-gradings to obtain a homological link invariant. The resulting chain complex $\mathcal{C}(D)$ is defined by

$$\mathcal{C}(D) = [\![D]\!][-n_-]\{n_+ - 2n_-\}.$$

Note that these shifts are reflected in the normalisation factors in the Jones polynomial.

4.1.6 Khovanov homology

Khovanov homology of a link diagram D is defined by

$$Kh(D) = \bigoplus_{i,j\in\mathbb{Z}} H^{i,j}(\mathcal{C}(D)),$$

where i denotes the homological grading (the grading inherited from the degree of the chain group), and j denotes the degree of the q-grading, and $\oplus_{j\in\mathbb{Z}} H^{i,j}(\mathcal{C}(D)) = \ker d^i / \mathrm{im}\ d^{i-1}$.

Theorem 4.1.2 ([Kho00]). *Khovanov homology $Kh(D)$ is a link invariant. That is, if D and D' are diagrams of inks that are isotopic in S^3, then $Kh(D) \cong Kh(D')$.*

We express this theorem by writing $Kh(L)$ in the place of $Kh(D)$ whenever D is a link diagram for a link L, and call $Kh(L)$ the Khovanov invariant.

The proof that Khovanov cohomology is a link invariant consists of showing that it is invariant under Reidemeister moves. A clear proof of this invariance can be found in [BN02].

4.1.7 Recovering the Jones polynomial

In the case of a chain complex of graded vector spaces, decategorification consists of taking the graded Euler characteristic of that chain complex.

Bar-Natan [BN02] defines the Khovanov polynomial in variables t and q as follows:

$$Kh(L)(t,q) = \sum_{i,j \in \mathbb{Z}} t^i q^j \dim H^{i,j}(\mathcal{C}(L)).$$

The graded Euler characteristic χ_q of the chain complex $\mathcal{C}(L)$ is obtained by setting $t = -1$:

$$\chi_q(\mathcal{C}(L)) = Kh(L)(-1,q)$$

It is clear from the construction of $\mathcal{C}(L)$ that the graded Euler characteristic of the Khovanov invariant of a link L is the unnormalised Jones polynomial of L:

$$Kh(L)(-1,q) = (q + q^{-1})J(L)$$

Hence, Khovanov homology categorifies the Jones polynomial. The construction of annular Khovanov is very similar to the construction of $Kh(L)$, and is illustrated by the example in section 4.3.

4.2 Lee's variant of Khovanov homology

Eun Soo Lee defined a deformation of Khovanov homology for links by introducing a new differential, denoted d_L here. This variant of Khovanov homology plays an important role in defining the current algebra action on annular Khovanov homology in section **??**. This differential induces a degree $(1,4)$ map on Khovanov cohomology, pairing off certain terms in $Kh(L)$. The original purpose of defining d_L was to prove the following theorem:

Theorem 4.2.1 ([Lee05]). *For an alternating knot L, its Khovanov invariants $H^{i,j}(\mathcal{C}(L))$ of degree difference (1,4) are paired except in the 0th cohomology group. More precisely, the equality*

$$Kh(L)(t,q) = q^{-s}(q + q^{-1}) + (q^{-1} + tq^2 \cdot q)P(t,q)$$

holds for some integer s and some polynomial P.

where an alternating link is a link that admits a diagram whose crossings are alternatively positive and negative when traveling along any component of the link.

4.2.1 Defining a new differential

Lee homology retains the chain groups from Khovanov homology, so that boundary maps are again defined in terms of multiplication and comultiplication, corresponding to splits and merges of cycles in resolutions of links. The Lee boundary map has multiplication $m_{Lee} : W \otimes W \to W\{1\}$ and comultiplication $\Delta_L : W \to (W \otimes W)\{1\}$ given by:

$$m_L : \begin{cases} w_+ \otimes w_+ \mapsto 0 \\ w_+ \otimes w_- \mapsto 0 \\ w_- \otimes w_+ \mapsto 0 \\ w_- \otimes w_- \mapsto w_+ \end{cases}$$

$$\Delta_L : \begin{cases} w_+ \mapsto 0 \\ w_- \mapsto w_+ \otimes w_+ \end{cases}$$

From this definition, m_L and Δ_L are (co)associative and (co)commutative, and satisfy the relation

$$\Delta_L \circ m_L = (m_L \otimes id) \circ (id \otimes \Delta_L)$$

These identities follow from the fact the m_L and Δ_L act trivially on most basis elements.

From this new definition of multiplication and comultiplication in Lee cohomology define the boundary maps d_L for the resulting chain complex exactly as for Khovanov cohomology, taking d_L to be the sum down a column of m_L and Δ_{Lee}, up to tensoring with the identity as necessary, for merge and splits respectively when each 0 resolution is changed to a 1-resolution. Signs are added following the same rule as previously to ensure that $d_L^2 = 0$. Lee homology is hence defined to be the cohomology of the chain complex $(C^\bullet), (d_L + d)^\bullet)$, where d is the Khovanov differential and the C^r are the chain groups from Khovanov homology. This is a well-defined chain complex, since d_L and d anti commute, and $d_L^2 = d^2 = 0$, so $(d + d_L)^2 = 0$.

Furthermore, d and d_L are compatible in the sense of the following:

1. $m \circ (m_L \otimes id) + m_L \circ (m \otimes id) = m \circ (id \otimes m_L) + m_L \circ (id \otimes m)$
2. $(\Delta \otimes id) \circ \Delta_L + (\Delta_L \otimes id)\Delta = (id \otimes \Delta) \circ \Delta_L + (id \otimes \Delta_L) \otimes \Delta$
3. $\Delta \circ m_L + \Delta_L \circ m = (m \otimes id) \circ (id \otimes \Delta_L) + (m_L \otimes id) \circ (id \otimes \Delta)$

4.2.2 Properties of Lee's modified cohomology

Lee's cohomology is particularly simple to compute: its total dimension is dependent only on the number of components of the link in question and the homological degree of the non-zero cohomology groups are given by linking numbers of components of the link, where the linking number of two components of a link.

We consider the chain complex $(C^\bullet, (d+d_L)^\bullet)$, where the C^r are again unchanged. To facilitate the computation of homology, Lee forms a new basis (that does not preserve the q-grading), consisting of

$$a = x + 1 \qquad\qquad b = x - 1$$

The multiplication and comultiplication maps for $d + d_L$ become:

$$m_{d+d_L} : \begin{cases} a \otimes a \mapsto 2a \\ a \otimes b \mapsto 0 \\ b \otimes a \mapsto 0 \\ b \otimes b \mapsto -2b \end{cases} \qquad \Delta_{d+d_L} : \begin{cases} a \mapsto a \otimes a \\ b \mapsto b \otimes b \end{cases}$$

This chain complex also gives a new cohomology theory that is a link invariant, called the Lee invariant.

The following theorem of Lee follows from a useful result from Hodge theory. We can define an inner product on the chain complex produced when constructing the Lee invariant such that monomials a and b form an orthonormal basis, so that there exists a well-defined adjoint $(d + d_L)^*$ of the sum of the Khovanov and Lee differentials $d + d_L$. The adjoint is defined by:

$$m_{(d+d_L)^*} : \begin{cases} a \otimes a \mapsto a \\ a \otimes b \mapsto 0 \\ b \otimes a \mapsto 0 \\ b \otimes b \mapsto b \end{cases} \qquad \Delta_{(d+d_L)^*} : \begin{cases} a \mapsto 2a \otimes a \\ b \mapsto -2b \otimes b \end{cases}$$

This adjoint differential facilitates the description of Lee homology:

Theorem 4.2.2. *Let $(d + d_L)^*$ be the adjoint of $d + d_L$. Then*

$$H_i(D) \cong \ker(d + d_L : C^i(D) \to C_{i+1}(D)) \cap \ker(d + d_L{}^* : C^i(D) \to C_{i-1}(D)).$$

Proof. We have by definition

$$H^i(D) = \frac{\ker((d + d_L) : C^i(D) \to C^{i+1}(D))}{\operatorname{im}((d + d_L) : C^{i-1} \to C^i(D))}$$

We use the inner product, denoted $(\ , \ )$, on the chain complex $C^\bullet(D)$ to decompose C^i into $C^i(D) = \operatorname{im}(d + d_L) \oplus (\operatorname{im}(d + d_L))^\perp$.

$(\operatorname{im}(d+d_L))^\perp = \ker(d+d_L)^*$ since $(d+d_L)^*(\alpha) = 0 \iff 0 = ((d+d_L)^*(\alpha), \beta) = (\alpha, (d+d_L)(\beta))$ for all $\beta \in C^i$, so $\alpha \perp \operatorname{im}(d + d_L)$.

Let $K = \ker((d+d_L) : C^i(D) \to C_{i+1}(D)) \cap \ker((d+d_L)^* : C^i(D) \to C_{i-1}(D))$ and $K' = H^i(D)$. Since $K \subset \ker(d + d_L)$ we can define the map $\Phi : K \to K'$ by $\Phi(\alpha) = \alpha + \operatorname{im}(d + d_L)$.

Then Φ is injective: let $c \in K \cap \operatorname{im}(d + d_L) \subset \ker(d + d_L)^* \cap \operatorname{im}(d + d_L)$. Then we have seen that $c = 0$ in K.

Φ is surjective: let $\beta = \alpha + \operatorname{im}(d + d_L) \in K'$. If $\alpha \in \ker(d + d_L)^*$ then $\beta = \Phi(\alpha)$ and we are done. If not, then $\alpha \in (\ker(d + d_L)^*)^\perp = \operatorname{im}(d + d_L)$ (since C^i is finite-dimensional), so $\alpha = 0$, and $\beta = \Phi(0)$. ☐

The linking number ℓ_{12} of two knots S_1 and S_2 is an invariant of links given by the following formula:

$$\ell_{12} = \frac{n_+ - n_-}{2}$$

where n_+ and n_- are the total numbers of positive and negative crossings between S_1 and S_2 respectively, defined in section 4.1.2

Theorem 4.2.3 (Lee)**.** *The Lee homology ring $H(L) = \oplus_{i\in\mathbb{Z}} H^i(L)$ for an oriented link L with n components $S_1, \ldots, S_n$ has dimension 2^n. If the linking number of S_j and S_k is ℓ_{jk}, then*

$$dim H^i(L) = 2 \cdot |\{E \subset \{2, \ldots, n\} : (\sum_{j\in E, k\notin E} 2\ell_{jk}) = i\}|$$

Lee's proof [Lee05] consists of distinguishing the 2^{n-1} orientation-preserving resolutions of all choices of relative orientation on the diagram D, showing that these contribute exactly two basis vectors to Lee homology and showing that these are the only contributiona using the long exact sequence on homology:

$$\ldots \to H^{i-1}(D(\star 0)) \to H^{i-1}(D(\star 1)) \to H^i(D) \to H^i(D \star 0)) \to H^i(D(\star 1)) \to \ldots$$

where $D(\star 0)$ is a link diagram D of L with the last crossing resolved to a 0-resolution and $D(\star 1)$ is D with the last crossing resolved to a 1-resolution.

4.3 The annular case

Asaeda, Przytycki and Sikora defined a variant of Khovanov homology, wherein knots and links are contained within I-bundles M over surfaces $F \neq \mathbb{RP}^2$ [APS04]. A specific case of this variant considers *annular knots and links*: knots and links restricted to the thickened annulus $A \times I$, where A is a closed, oriented annulus and I is the closed interval $[0, 1]$. In particular, we consider annular braid closures. Annular links are isotopic if and only if their link diagrams differ by a sequence of annular Reidemeister moves and isotopies, namely isotopies that do not pass through the boundaries of the thickened annulus. The thickened annulus is parametrised by the following:

$$A \times I = \{(r, \theta, z) : r \in [1, 2], \theta \in [0, 2\pi], z \in [0, 1]\} \subset S^3 = \mathbb{R}^3 \cup \infty$$

and annular links admit link diagrams $D(L)$ by projecting any representative of the isotopy class of L onto $A \times \{\frac{1}{2}\}$. From such a projection one constructs a triply-graded chain complex CKh in a similar way to regular Khovanov homology.

Regard the link diagrams $D(L)$ as being contained within $S^2 \backslash \{x_0, x_\infty\}$ where x_0 and x_∞ are considered to be basepoints corresponding to the inner and outer bounding circles of the annulus A respectively. In general the second basepoint x_∞ will be excluded or implicit in what follows. An example of an annular link diagram with a choice of ordering on crossings is given below in figure 4.1. Crossings are labeled by (n, e_n), where $n \in \mathbb{N}$ is the number of the crossing in the order chosen, and $e_n \in \{+, -\}$ is the parity of that crossing. The basepoint x_0 is denoted by a $*$ in each link diagram. To return to regular Khovanov homology, we simply forget the basepoint.

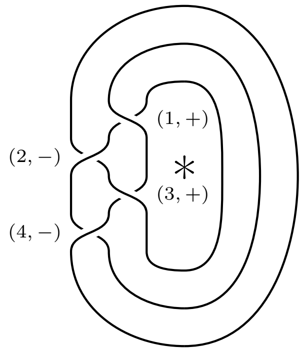


Figure 4.1: The annular link diagram of the figure-eight knot

4.3.1 Chain groups

Let D be a link diagram for an annular link L. The cube of resolutions of D as in regular Khovanov homology, noting the location of the basepoint in each of the resolutions. Thus there are three types of cycles in the resolutions of D: those that do not enclose the basepoint, called trivial cycles, and positive and negative cycles that do enclose the basepoint, called non-trivial cycles. The sign of non-trivial cycles is determined by the following process:

1. Construct a line ℓ between the basepoints x_0 and x_∞.
2. Starting from x_0 enumerate the crossings between ℓ and the cycles of the resolution.
3. A non-trivial cycle is positive if its crossing number is even and negative if its crossing number is odd. Note that a non-trivial cycle can only have an odd number of crossings with ℓ. If the number of crossings between a non-trivial cycle and ℓ is greater than one, the parity of the crossing number is the parity of the first crossing number.

This sign allocation clearly involves making several choices (the basepoints x_0 and x_∞, the direction of ℓ, and associating positive cycles to even crossing numbers), and is thus non-canonical - isotopic diagrams can have different choices of positive and negative non-trivial cycles. However this choice does not carry down to the level of homology.

As in regular Khovanov homology, chain groups are obtained from resolutions by applying a TQFT from $\mathbf{Cob}_1$ to $\mathbf{gVect}_{\mathbb{C}}$. Here the vector spaces are two-dimensional and bigraded, where the first grading is the usual q-grading that returns the Jones polynomial, and the second s-grading relates to a Lie algebra-module structure defined shortly.

- To trivial cycles associate the vector space W with basis $\{w_+, w_-\}$, such that $\deg(w_+) = (1, 0)$ and $\deg(w_-) = (-1, 0)$. This vector space is precisely the one defined in regular annular Khovanov homology.
- To positive non-trivial cycles associate the vector space V with basis $\{v_1, v_{-1}\}$ such that $\deg(v_1) = (1, 1)$ and $\deg(v_{-1}) = (-1, -1)$.

- To negative non-trivial cycles associate the dual V^* of V, with basis $\{\overline{v_1}, \overline{v_{-1}}\}$ such that $\deg(\overline{v_1}) = (1, 1)$ and $\deg(\overline{v_{-1}}) = (-1, -1)$.

Note that for this grading to be consistent on V^*, the natural pairing $\langle\, ,\, \rangle$ on $V \times V^*$ determines the basis of V^* by

$$\langle v_1, \overline{v_1}\rangle = 0 \qquad \langle v_1, \overline{v_{-1}}\rangle = 1$$
$$\langle v_{-1}, \overline{v_1}\rangle = 1 \qquad \langle v_{-1}, \overline{v_{-1}}\rangle = 0$$

Let α be a complete resolution of the link diagram D. Define k_α to be the number of trivial cycles in α, l_α to be the number of positive non-trivial cycles and m_α to be the number of negative non-trivial cycles. Then we define the chain groups in the chain complex $(\mathrm{CKh}^\bullet, d^\bullet)$ to be

$$\mathrm{CKh}^i(D) = \bigoplus_{\alpha : h(\alpha)=i} W^{\otimes k_\alpha} \otimes V^{\otimes l_\alpha} \otimes V^{*m_\alpha}\{(i, 0)\}$$

and it remains to define the boundary map d.

For example, consider the annular knot in figure 4.2.

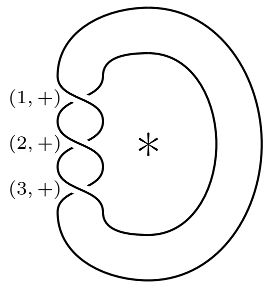


Figure 4.2: The annular link diagram for the positive trefoil

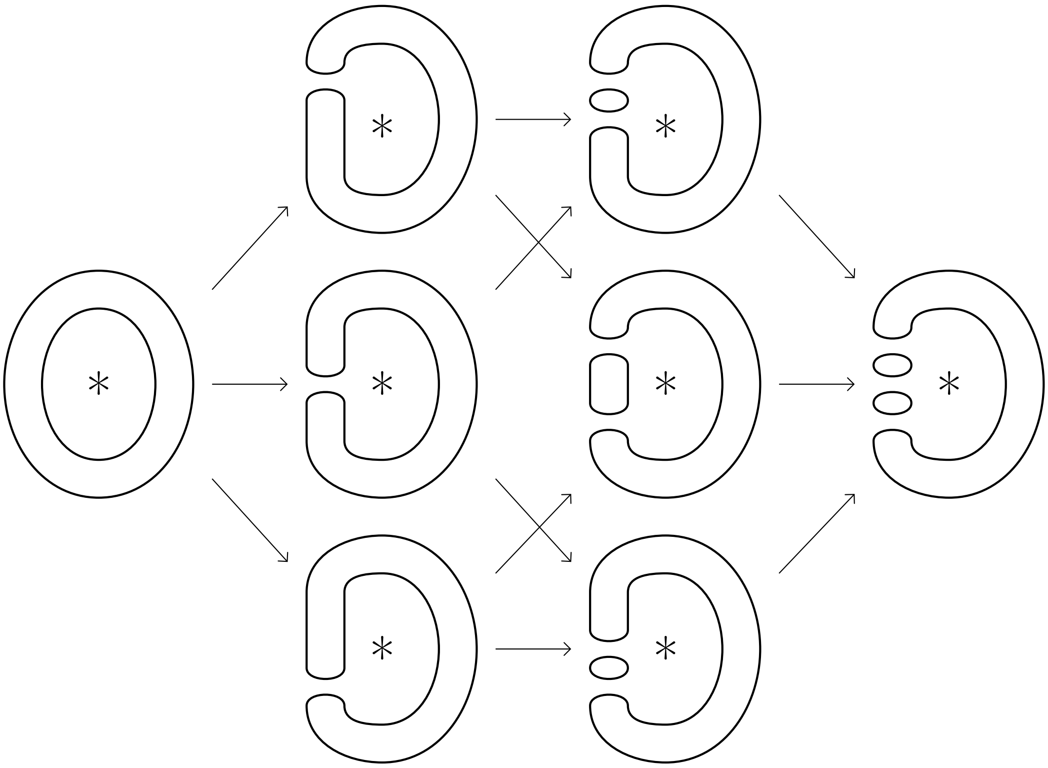
Then the cube of resolutions for the trefoil is shown in figure **??**, with the corresponding chain groups underneath.

The only modification here from the regular case is in homological degree zero. To pass from the annular case to the regular case, we replace each of the representations V and V^* by the trivial representation W.

4.3.2 Boundary maps

To define the boundary maps on CKh, one first distinguishes the multiplication and comultiplication maps found on individual edges of the cube of resolutions. We first note that the only types of splits and merges that are possible between two resolutions are:

1. Two trivial cycles merging into a single trivial cycle and a trivial cycle splitting into two trivial cycles: $W \otimes W \leftrightarrow W$,



$$V \otimes V^* \longrightarrow \bigoplus_{i=1}^{3} W \longrightarrow \bigoplus_{i=1}^{3} W \otimes W \longrightarrow W \otimes W \otimes W$$

Figure 4.3: Cube of resolutions for the trefoil

2. A trivial and a non-trivial cycle merging to form a non-trivial cycle of the same sign and vice versa: $V \otimes W \leftrightarrow V$ or $V^* \otimes W \leftrightarrow V^*$,

3. Two non-trivial cycles of opposite sign merging to form a trivial cycle and vice versa: $V \otimes V^* \leftrightarrow W$.

The first case corresponds to regular Khovanov homology: when there is no interaction with the basepoint, annular Khovanov homology reduces to the usual form. The second case reflects the fact that (up to isotopy, so that no cycle lies tangent to the line ℓ) a trivial cycle will always have an even number of crossings with ℓ or no crossings with ℓ, so the non-trivial cycles formed must always be of the same type as the original cycles. Note that there cannot be a merging of two non-trivial cycles of the same type: by the definition of sign on cycles two adjacent non-trivial cycles will always have opposing signs.

The relevant multiplication and comultiplication maps are defined from regular Khovanov edge maps, with terms deleted if they do not preserve the second s-grading. We will take note of these deleted terms for later use. Case 1 is trivially the same as the regular case, since the s-grading is always zero.

d	$V \otimes W \leftrightarrow V$	$V^* \otimes W \leftrightarrow V^*$	$V \otimes V^* \leftrightarrow W$
m	$v_1 \otimes w_+ \mapsto v_1$	$\overline{v_1} \otimes w_+ \mapsto \overline{v_1}$	$v_1 \otimes \overline{v_1} \mapsto 0$
	$v_1 \otimes w_- \mapsto 0$	$\overline{v_1} \otimes w_- \mapsto 0$	$v_1 \otimes \overline{v_{-1}} \mapsto w_-$
	$v_{-1} \otimes w_+ \mapsto v_{-1}$	$\overline{v_{-1}} \otimes w_+ \mapsto \overline{v_{-1}}$	$v_{-1} \otimes \overline{v_1} \mapsto w_-$
	$v_{-1} \otimes w_- \mapsto 0$	$\overline{v}_{-1} \otimes w_- \mapsto 0$	$v_{-1} \otimes \overline{v_{-1}} \mapsto 0$
Δ	$v_1 \mapsto v_1 \otimes w_-$	$\overline{v_1} \mapsto \overline{v_1} \otimes w_-$	$w_+ \mapsto v_1 \otimes \overline{v_{-1}} + v_{-1} \otimes \overline{v_1}$
	$v_{-1} \mapsto v_{-1} \otimes w_-$	$\overline{v_{-1}} \mapsto \overline{v_{-1}} \otimes w_-$	$w_- \mapsto 0$

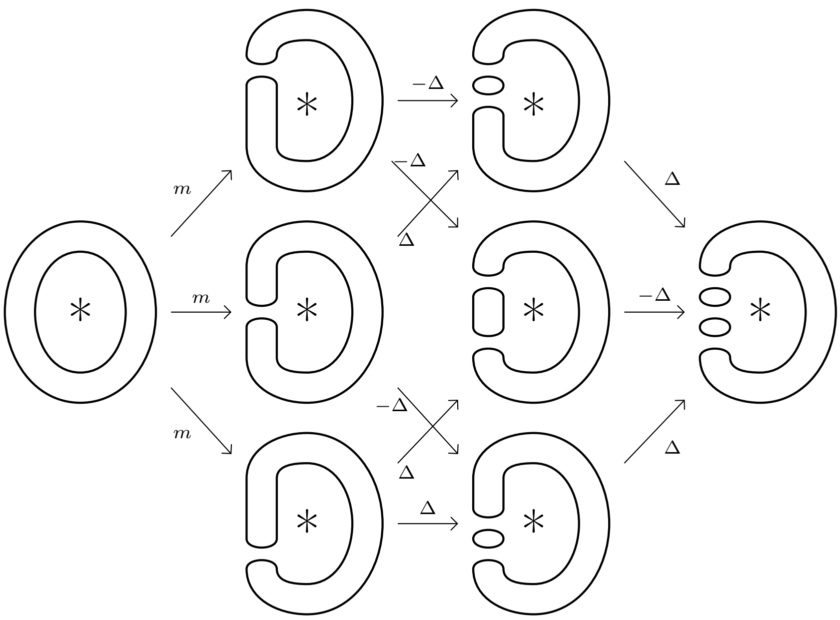
The deleted terms are

$$m_{del}(v_1 \otimes w_-) = v_{-1} \qquad m_{del}(\overline{v_1} \otimes w_-) = \overline{v_{-1}} \qquad m_{del}(v_1 \otimes \overline{v_1}) = w_+$$

$$\Delta_{del}(v_1) = v_{-1} \otimes w_+ \qquad \Delta_{del}(\overline{v_1}) = \overline{v_{-1}} \otimes w_+ \qquad \Delta_{del}(w_-) = v_{-1} \otimes \overline{v_{-1}}$$

Each edge of the cube of resolutions will be assigned one of the multiplication or comultiplication maps tensored with copies of the identity map for each unchanged cycle and with signs flipped following the same process as in the regular case. Then the differential for $(\mathrm{CKh}^\bullet, d^\bullet)$ is the sum of each edge map down columns between vertices α with the same height $h(\alpha)$.

Returning to the example of the trefoil, the cube of resolutions with relevant edge maps is given in figure 4.4.



$$V \otimes V^* \xrightarrow{d^0} \bigoplus_{i=1}^3 W \xrightarrow{d^1} \bigoplus_{i=1}^3 W \otimes W \xrightarrow{d^2} W \otimes W \otimes W$$

Figure 4.4: Boundary maps for the trefoil

The annular Khovanov homology of the trefoil can be found in section 4.5.

We finally add in the grading shifts, following the convention of Bar-Natan, so that the final chain complex is given by:

$$\mathcal{CKh}(D) = \mathrm{CKh}(D)[-n_-]\{(n_+ - 2n_-, 0)\}$$

with differential d defined above. Let $AKh^i(D) = \ker(d^i)/\mathrm{im}(d^{i-1})$ be the ith cohomology group of $\mathcal{CKh}$. Then Akh^i is in fact independent of the link diagram D chosen:

Theorem 4.3.1 ([Rob13])**.** *The tri-graded annular Khovanov cohomology $AKh(L) = \oplus_{i\in\mathbb{Z}} AKh^i(L)$ of an oriented knot or link L with the differential defined above is an invariant of annular knots and links.*

4.3.3 An $\mathfrak{sl}_2$ action on $\mathcal{CKh}$

One aspect of interest in studying annular Khovanov homology comes from a representation theoretic perspective. There is a rich and beautiful structure on this homology that is partially described by the following preliminary result:

Theorem 4.3.2. *There is an $\mathfrak{sl}_2\mathbb{C}$ action on $\mathcal{CKh}$ and thus on annular Khovanov homology.*

This means that there is a natural way of viewing the chain groups $\mathcal{CKh}^i$ as representations of $\mathfrak{sl}_2\mathbb{C}$ and the differentials d^i as $\mathfrak{sl}_2\mathbb{C}$-module homomorphisms, so that by definition of cohomology groups (using the fact that the kernel and image of a module homomorphism are sub-modules), these can also be naturally viewed as $\mathfrak{sl}_2\mathbb{C}$ representations.

Proof. We first define each of the two-dimensional bi-graded vector spaces W, V and V^* as $\mathfrak{sl}_2$ representations. Define W to be the direct sum of two copies of the trivial representation $V_0 \cong \mathbb{C}$ so that $W = V_0\{(1,0)\} \oplus V_0\{(-1,0)\} \cong \mathbb{C}\{(1,0)\} \oplus \mathbb{C}\{(-1,0)\}$. Let V be the standard representation, with v_1 a highest weight vector, and v_{-1} a weight vector associated to the weight -1. Finally, let V^* be the dual representation of V. While V and V^* are isomorphic as representations of $\mathfrak{sl}_2$, the isomorphism is non-trivial, and introduces signs that will be used in further results. The action of $\mathfrak{sl}_2$ on V^* is therefore:

$$\begin{aligned} &e \cdot \overline{v_1} = 0 && f \cdot \overline{v_1} = -\overline{v_{-1}} && h \cdot \overline{v_1} = \overline{v_1} \\ &e \cdot \overline{v_{-1}} = -\overline{v_1} && f \cdot \overline{v_{-1}} = 0 && h \cdot \overline{v_{-1}} = -\overline{v_{-1}} \end{aligned}$$

Using this definition, it is possible to interpret the s-grading as the weight-space grading of representations of $\mathfrak{sl}_2$: all elements in W have s-grading 0, reflecting the fact that W is the trivial representation, so $x \cdot W = 0$ for all $x \in \mathfrak{sl}_2$. Furthermore, the s-grading of 1 for both v_1 and $\overline{v_1}$ is consistent with the fact that these are both highest weight vectors for the two-dimensional representation. Finally, the s-grading of -1 for v_{-1} and $\overline{v_{-1}}$ is also consistent with their associated weights. Note that as a particular case of annular Khovanov homology, regular Khovanov homology also has a natural $\mathfrak{sl}_2$ action, but since all cycles are trivial in this case, it consists exclusively of the trivial representation. Chain groups consist of direct sums of tensor products of the $\mathfrak{sl}_2$ representations V, V^* and W, and since $U(\mathfrak{sl}_2)$ is a Hopf algebra, the chain groups are themselves $\mathfrak{sl}_2$ representations.

The boundary maps d^i consist of sums of tensor products of m and Δ with the identity map. Hence to show that the d^i are $\mathfrak{sl}_2$-intertwining maps, it suffices to show that m and Δ commute with the $\mathfrak{sl}_2$ action on chain groups.

We consider the multiplication and comultiplication maps between W and $V \otimes V^*$. For $\Delta : W \to V \otimes V^*$, it is clear that the $\Delta \circ x = 0$ for all $x \in \mathfrak{sl}_2$, since W is the trivial representation. We must therefore check that $x \circ \Delta = 0$ for all x. This verification is taken on the basis vectors of W and generators of $\mathfrak{sl}_2$. Note that Δ is trivial on w_-, so commutativity automatically holds on Span$\{w_-\}$.

$$\begin{aligned}
e \circ \Delta(w_+) &= e(v_1 \otimes \overline{v_{-1}} + v_{-1} \otimes \overline{v_1}) \\
&= e \cdot v_1 \otimes \overline{v_{-1}} + v_1 \otimes e \cdot \overline{v_{-1}} + e \cdot v_{-1} \otimes \overline{v_1} + v_{-1} \otimes e \cdot \overline{v_1} \\
&= 0 - v_1 \otimes \overline{v_1} + v_1 \otimes \overline{v_1} + 0 \\
&= 0 \\
f \circ \Delta(w_+) &= f(v_1 \otimes \overline{v_{-1}} + v_{-1} \otimes \overline{v_1}) \\
&= v_{-1} \otimes \overline{v_{-1}} + 0 + 0 - v_{-1} \otimes \overline{v_{-1}} \\
&= 0 \\
h \circ \Delta(w_+) &= h(v_1 \otimes \overline{v_{-1}} + v_{-1} \otimes \overline{v_1}) \\
&= v_1 \otimes \overline{v_{-1}} - v_1 \otimes \overline{v_{-1}} - v_{-1} \otimes \overline{v_1} + v_{-1} \otimes \overline{v_1} \\
&= 0
\end{aligned}$$

Thus Δ commutes with the $\mathfrak{sl}_2$ action. One similarly checks that the multiplication commutes with the $\mathfrak{sl}_2$ action. Since $m(v_1 \otimes \overline{v_1}) = m(v_{-1} \otimes \overline{v_{-1}}) = 0$, the composition $x \circ m$ is zero for any $x \in \mathfrak{sl}_2$. Furthermore, by definition of the weights associated to each of the v_i and $\overline{v_i}$, $e(v_1 \otimes \overline{v_1}) = f(v_{-1} \otimes \overline{v_{-1}}) = 0$.

$$\begin{aligned}
m \circ f(v_1 \otimes \overline{v_1}) &= m(v_{-1} \otimes \overline{v_1}) - m(v_1 \otimes \overline{v_{-1}}) & m \circ e(v_{-1} \otimes \overline{v_{-1}}) &= m(v_1 \otimes \overline{v_{-1}}) - m(v_{-1} \otimes \overline{v_1}) \\
&= w_- - w_- = 0 & &= w_- - w_- = 0 \\
m \circ h(v_1 \otimes \overline{v_1}) &= 2m(v_1 \otimes \overline{v_1}) = 0 & m \circ h(v_{-1} \otimes \overline{v_{-1}}) &= -2m(v_{-1} \otimes \overline{v_{-1}}) = 0
\end{aligned}$$

Hence $m \circ x(v_1 \otimes \overline{v_1}) = x \circ m(v_1 \otimes \overline{v_1}) = 0$ and $m \circ x(v_{-1} \otimes \overline{v_{-1}}) = x \circ m(v_{-1} \otimes \overline{v_{-1}}) = 0$ and m commutes with the $\mathfrak{sl}_2$ action on Span$\{v_1 \otimes \overline{v_1}, v_{-1} \otimes \overline{v_{-1}}\}$. For the remaining basis vectors, it is also clear that $x \circ m$ will always be the zero map, since $m : V \otimes V^* \to W$ and W is the trivial representation.

$$\begin{aligned}
m \circ e(v_1 \otimes \overline{v_{-1}}) &= -m(v_1 \otimes \overline{v_1}) = 0 & m \circ e(v_{-1} \otimes \overline{v_1}) &= m(v_1 \otimes \overline{v_1}) = 0 \\
m \circ f(v_1 \otimes \overline{v_{-1}}) &= m(v_{-1} \otimes \overline{v_{-1}}) = 0 & m \circ f(v_{-1} \otimes \overline{v_1}) &= -m(v_{-1} \otimes \overline{v_{-1}}) = 0
\end{aligned}$$

Furthermore, $h(v_1 \otimes \overline{v_{-1}}) = h(v_{-1} \otimes \overline{v_1}) = 0$, so $m \circ x$ is the zero map on $V \otimes V^*$ and hence m commutes with the $\mathfrak{sl}_2$ action on $V \otimes V^*$.

For the edge maps between $V \otimes W$ and V, we note that since the action of $\mathfrak{sl}_2$ is trivial on W, and that $m|_{V \otimes \mathrm{Span}(w_+)}$ is an isomorphism of $\mathfrak{sl}_2$ representations, commutativity of the multiplication map and the $\mathfrak{sl}_2$ action holds on this subspace. On the subspace $V \otimes \mathrm{Span}(w_-)$, $m \circ x$ is also trivial for all $x \in \mathfrak{sl}_2$, since $x \cdot (V \otimes \mathrm{Span}(w_-) \subseteq V \otimes \mathrm{Span}(w_-)$, which is in the kernel of m. Therefore m commutes with the $\mathfrak{sl}_2$ action on $V \otimes W$. Similarly, Δ is an isomorphism between V and $V \otimes \mathrm{Span}(w_-)$, and by the trivial action of $\mathfrak{sl}_2$ on W, Δ is also a map of representations.

The proof that the $\mathfrak{sl}_2$ action commutes with the edge maps m and Δ on $V^* \otimes W \leftrightarrow V^*$ follows by the same reasoning.

Therefore there is a natural way of viewing the chain groups as representations of the Lie algebra $\mathfrak{sl}_2$ and the boundary maps as intertwining maps. This carries naturally down to an action on cohomology AKh, since both $\ker(d^i)$ and $\mathrm{im}(d^{i-1})$ are subrepresentations of $\mathcal{CK}h^i$ for all $i \in \mathbb{Z}$, so their quotient inherits an action of $\mathfrak{sl}_2$ as well. □

Remark. *The $\mathfrak{sl}_2$-action on homology is an annular link invariant: any annular isotopy or Reidemeister move will induce an isomorphism of $\mathfrak{sl}_2$-modules on homology.*

The $\mathfrak{sl}_2$ action on annular Khovanov homology is not however the only underlying structure. In the following section, we demonstrate a much richer structure, given by the current algebra $\mathfrak{sl}_2^-(V_2)$.

4.4 A current algebra action on annular Khovanov homology

The aim of this section is to give a proof of the following theorem:

Theorem 4.4.1 (Grigsby-Licata-Wehrli). *There is an $\mathfrak{sl}_2^-(V_2)$ action on annular Khovanov homology.*

We begin by defining a current algebra action on individual cohomology groups then demonstrate that this definition is compatible with the previous structure.

We have seen that the chain complex $\mathcal{CK}h$ is a triply-graded vector space, consisting of the homological grading, the usual Khovanov q-grading and the $\mathfrak{sl}_2$ weight space s-grading. We fix the following notation: an element v in $\mathcal{CK}h$ has degree (a, b, c), where a is the homological degree, b is the q-grading and c is the s-grading.

4.4.1 Decomposing the annular Khovanov and Lee differentials

Just as the Khovanov differential was modified in the annular case to preserve the s-grading at the chain level, one can also define an annular variant of Lee homology that preserves the s-grading using individual edge maps. We denote the s-grading preserving annular Lee multiplication and comultiplication edge maps by m_{Lee} and Δ_{Lee} respectively. The s-grading preserving annular Lee differential is denoted d_{Lee}. As before, the case where trivial cycles split or merge to form other trivial cycles corresponds exactly to the regular Lee homology case. Note that in the non-trivial case, none of the multiplication and comultiplication maps for Lee homology preserve the $\mathfrak{sl}_2$-weight grading, so all terms are deleted. These deleted terms are:

$$m_{Lee,del}(v_{-1} \otimes w_-) = v_1 \qquad \Delta_{Lee,del}(v_1) = v_1 \otimes w_+$$
$$m_{Lee,del}(v_{-1} \otimes \overline{v_{-1}}) = w_+ \qquad \Delta_{Lee,del}(v_1) = v_1 \otimes w_+$$
$$\Delta_{Lee,del}(\overline{v_{-1}}) = \overline{v_1} \otimes w_+ \qquad \Delta_{Lee,del}(w_-) = v_1 \otimes \overline{v_1}$$

The s-grading is shifted up by 2 in all the non-trivial cases. We can therefore decompose the annular Lee differential into two parts:

$$\partial_{Lee} = d_{Lee} + d^+_{Lee}$$

where ∂_{Lee} denotes the total annular differential, and d^+_{Lee} is the s-grading-lowering component consisting of those terms that were initially deleted from d_{Lee}.

Similarly, we recall the deleted terms from the annular Khovanov differential:

$$m_{del}(v_+ \otimes w_-) = v_- \qquad m_{del}(\overline{v_+} \otimes w_-) = \overline{v_-} \qquad m_{del}(v_+ \otimes \overline{v_+}) = w_+$$
$$\Delta_{del}(v_+) = v_- \otimes w_+ \qquad \Delta_{del}(\overline{v_+}) = \overline{v_-} \otimes w_+ \qquad \Delta_{del}(w_-) = v_- \otimes \overline{v_-}$$

All of the deleted terms have s-grading shifts that lower the degree by 2, so we may also decompose the annular Khovanov differential as:

$$\partial = d + d^-$$

where ∂ is the total annular Khovanov differential, d is the grading-preserving component and d^- is the s-grading-lowering component.

As seen previously, the Lee differential ∂_{Lee} anticommutes with the Khovanov differential, though each pair of the components may not strictly commute at the chain level.

The aim is to determine the current algebra action on homology using the components d^- and d^+_{Lee}. To this end we define the action at the chain level, and show that up to homotopy, this definition is compatible with the s-grading-preserving Khovanov differential, which is simply the annular differential. We have seen that d^- is a degree (1,0,-2) map, and that d^+_{Lee} is a degree (1,4,2) map. Thus, these maps behave on the s-grading as one would expect the e and f elements of $\mathfrak{sl}_2$ would on the weight spaces of a representation. Similarly, from our analysis of the current algebra $\mathfrak{sl}_2^-(V_2)$, the vectors v_2 and v_{-2} have a similar property of shifting between weight spaces. For this reason we make the following definition of the action of the vectors v_i from $\mathfrak{sl}_2^-(v_2)$, noting also that they shift the homological degree by 1. The vectors v_i act by zero on all basis vectors not shown in the following table.

$\mathfrak{sl}_2^-(V_2)$	$V \otimes W \leftrightarrow V$	$V^* \otimes W \leftrightarrow V^*$	$V \otimes V^* \leftrightarrow W$
v_2	$v_{-1} \otimes w_- \mapsto v_1$	$\overline{v_{-1}} \otimes w_- \mapsto -\overline{v_1}$	$v_{-1} \otimes \overline{v_{-1}} \mapsto w_+$
v_{-2}	$v_1 \otimes w_- \mapsto v_{-1}$	$\overline{v_1} \otimes w_- \mapsto -\overline{v_{-1}}$	$v_1 \otimes \overline{v_1} \mapsto w_+$
v_0	$v_1 \otimes w_- \mapsto v_1$	$\overline{v_{-1}} \otimes w_- \mapsto -\overline{v_{-1}}$	$v_1 \otimes \overline{v_{-1}} \mapsto w_+$
v_0	$v_{-1} \otimes w_- \mapsto -v_{-1}$	$\overline{v_1} \otimes w_- \mapsto \overline{v_1}$	$v_{-1} \otimes \overline{v_1} \mapsto -w_+$
v_2	$v_{-1} \mapsto v_1 \otimes w_+$	$\overline{v_{-1}} \mapsto -\overline{v_1} \otimes w_+$	$w_- \mapsto v_1 \otimes \overline{v_1}$
v_{-2}	$v_1 \mapsto v_{-1} \otimes w_+$	$\overline{v_1} \mapsto -\overline{v_{-1}} \otimes w_+$	$w_- \mapsto v_{-1} \otimes \overline{v_{-1}}$
v_0	$v_1 \mapsto v_1 \otimes w_+$	$\overline{v_{-1}} \mapsto -\overline{v_{-1}} \otimes w_+$	$w_- \mapsto v_1 \otimes \overline{v_{-1}}$
v_0	$v_{-1} \mapsto -v_{-1} \otimes w_+$	$\overline{v_1} \mapsto \overline{v_1} \otimes w_+$	$-v_{-1} \otimes \overline{v_1}$

4.4.2 Proof of theorem 4.4.1

Proof. To show that there is a current algebra action, we must show that the maps we have just defined satisfy the current algebra relations and square to zero, at least up to some chain homotopy. Furthermore, we must show that the differential map d anticommutes with the current algebra action on individual chain groups so that this action carries down to the cohomology level. This proof also shows that the s-grading preserving components of the Khovanov and Lee differentials are themselves differentials on $\mathcal{CK}h$, namely they square to zero.

We can decompose the chain groups $\mathcal{CK}h^i$ into a direct sum of $\mathfrak{sl}_2$ weight spaces:

$$\mathcal{CK}h^i = \bigoplus_{s \in \mathbb{Z}} \mathcal{CK}^i(s)$$

where s is the $\mathfrak{sl}_2$ weight.

Denoting the s-degree, the shift in s-grading, of a map by $s$deg, we have seen that both ∂ and ∂_{Lee} split into two homogeneous components, where $s\deg(d) = s\deg(d_{Lee}) = 0$, $s\deg(d^-) = -2$, and $s\deg(d^+_{Lee}) = 2$.

We now use the fact that the total Lee and Khovanov maps are differentials, to obtain:

$$0 = \partial^2 = (d + d^-)^2 = d^2 + dd^- + d^-d + (d^-)^2$$

and

$$0 = \partial_{Lee}^2 = d_{Lee}^2 + d_{Lee}d^+_{Lee} + d^+_{Lee}d_{Lee} + (d^+_{Lee})^2$$

Each of these maps can be decomposed into homogeneous s-degree components:

$$s\deg(d^2) = 0 \qquad s\deg(d_{Lee}^2) = 0$$
$$s\deg(dd^- + d^-d) = -2 \qquad s\deg(d_{Lee}d^+_{Lee} + d^+_{Lee}d_{Lee}) = 2$$
$$s\deg((d^-)^2) = -4 \qquad s\deg((d^+_{Lee})^2) = 4$$

From the weight space decomposition of the chain groups, the maps ∂^2 and ∂Lee^2 are both zero if and only if their homogeneous s-degree components are all zero:

$$\partial^2 = 0 \iff \begin{cases} d^2 &= 0 & (4.1a) \\ dd^- + d^- d &= 0 & (4.1b) \\ (d^-)^2 &= 0 & (4.1c) \end{cases}$$

Similarly,

$$\partial_{Lee}^2 = 0 \iff \begin{cases} d_{Lee}^2 &= 0 & (4.2a) \\ d_{Lee}d_{Lee}^+ + d_{Lee}^+ d_{Lee} &= 0 & (4.2b) \\ (d_{Lee}^+)^2 &= 0 & (4.2c) \end{cases}$$

We have also seen that the Lee and Khovanov differentials anticommute, so that

$$\begin{aligned} 0 &= \partial\partial_{Lee} + \partial_{Lee}\partial \\ &= (dd_{Lee}^+ + d_{Lee}^+ d) + (d^- d_{Lee} + d_{Lee}d^-) + (dd_{Lee} + d_{Lee}d + d^- d_{Lee}^+ d_{Lee}^+ d^-) \end{aligned}$$

where each component in brackets is homogeneous in s-degree, of s-degrees 2, -2 and 0 respectively. As before, we have:

$$\partial\partial_{Lee} + \partial_{Lee}\partial = 0 \iff \begin{cases} dd_{Lee}^+ + d_{Lee}^+ d &= 0 & (4.3a) \\ d^- d_{Lee} + d_{Lee}d^- &= 0 & (4.3b) \\ dd_{Lee} + d_{Lee}d + d^- d_{Lee}^+ d_{Lee}^+ d^- &= 0 & (4.3c) \end{cases}$$

From (1a) and (2a), we conclude that the s-grading preserving components of the Khovanov and Lee differentials are themselves differentials. From (1b) and (3a), the annular differential d anticommutes with the current algebra maps d^- and d_{Lee}^+, so the current algebra has a well-defined action on cohomology. (1c) and (2c) consist of current algebra relations: $v_2^2 = v_{-2}^2 = 0$. Equation (3c) shows that $v_2 v_{-2} + v_{-2}v_2 \simeq 0$, where d_{Lee} is considered as a chain homotopy.

It therefore remains to show the final current algebra relation

$$[e, v_{-2}] = -[f, v_2]$$

and that this matches our definition for v_0.

For the $V \otimes W \leftrightarrow V$ case:

$$\begin{aligned} [e, v_{-2}](v_1 \otimes w_-) &= e(v_{-1}) = v_1 & -[f, v_2](v_1 \otimes w_-) &= v_2(v_{-1} \otimes w_-) = v_1 \\ [e, v_{-2}](v_{-1} \otimes w_-) &= -v_{-2}(v_1 \otimes w_-) = -v_{-1} & -[f, v_2](v_{-1} \otimes w_-) &= -f(v_1) = -v_{-1} \\ [e, v_{-2}](v_{-1}) &= -v_{-2}(v_1) = -v_{-1} \otimes w_+ & -[f, v_2](v_{-1}) &= -f(v_1 \otimes w_+) = -v_{-1} \otimes w_+ \\ [e, v_{-2}](v_1) &= e(v_{-1} \otimes w_+ = v_1 \otimes w_+ & -[f, v_2](v_1) &= v_2(v_{-1}) = v_1 \otimes w_+ \end{aligned}$$

For the $V \otimes V^* \leftrightarrow W$ case:

$$\begin{aligned}
[e, v_{-2}](v_{-1} \otimes \overline{v_{-1}}) &= -v_{-2}(v_1 \otimes \overline{v_{-1}} - v_{-1} \otimes \overline{v_1}) = 0 \\
-[f, v_2](v_{-1} \otimes \overline{v_{-1}}) &= f(w_+) = 0 \\
[e, v_{-2}](v_{-1} \otimes \overline{v_1}) &= -v_{-2}(v_1 \otimes \overline{v_1}) = -w_+ \\
-[f, v_2](v_{-1} \otimes \overline{v_1}) &= v_2(v_{-1} \otimes (-\overline{v_{-1}})) = -w_+ \\
[e, v_{-2}](v_1 \otimes \overline{v_{-1}}) &= -v_{-2}(v_1 \otimes (-\overline{v_1})) = w_+ \\
-[f, v_2](v_1 \otimes \overline{v_{-1}}) &= v_2(v_{-1} \otimes \overline{v_{-1}}) = w_+ \\
[e, v_{-2}](v_1 \otimes \overline{v_1}) &= e(w_+) = 0 \\
-[f, v_2](v_1 \otimes \overline{v_1}) &= v_2(v_{-1} \otimes \overline{v_1} - v_1 \otimes \overline{v_{-1}}) = 0 \\
[e, v_{-2}](w_-) &= e(v_{-1} \otimes \overline{v_{-1}}) = v_1 \otimes \overline{v_{-1}} - v_{-1} \otimes \overline{v_1} \\
-[f, v_2](w_-) &= -f(v_1 \otimes \overline{v_1}) = -v_{-1} \otimes \overline{v_1} + v_1 \otimes \overline{v_{-1}}
\end{aligned}$$

□

4.5 Computing annular Khovanov homology

4.5.1 Stabilised unknots

We compute the annular homology of positively stabilised unknots: we perform a Reidemeister 1 move that interacts with the basepoint x_0, such that the resulting crossing is positive. In the regular Khovanov homology case, the resulting link would be isotopic to the unknot, and hence would have isomorphic homology. In the annular case, a positive stabilisation introduces a non-trivial modification of the homology groups. Knowledge of the resulting representations could lead to determining whether a link diagram is minimally presented, since representations of positive stabilisations may be characterisable. This question is still open.

We compute annular Khovanov homology for the twice-stabilised unknot:

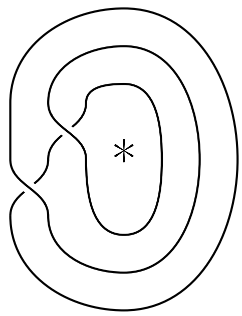
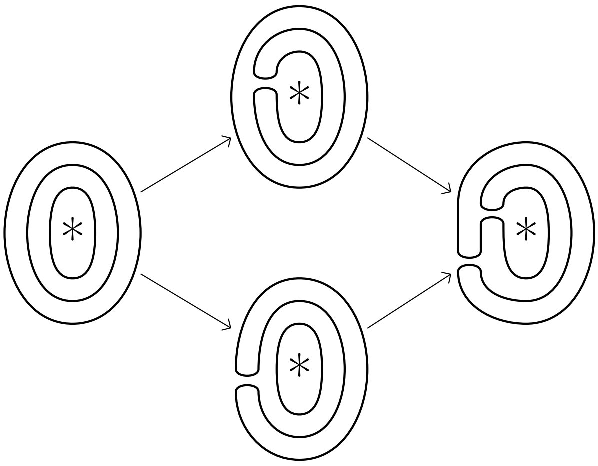


Figure 4.5: Twice-stabilised unknot

The corresponding cube of resolutions is:

We determine the kernel and image of each boundary map d^i.



$$V \otimes V^* \otimes V \to (W \otimes V) \oplus (V \otimes W) \longrightarrow V$$

For $d^0 : V \otimes V^* \otimes V \to (V \otimes W) \oplus (V \otimes W)$, we choose a basis for each vector space: for $V \otimes V^* \otimes V$, choose the basis $\{v_1 \otimes \overline{v_1} \otimes v_1,\ v_1 \otimes \overline{v_1} \otimes v_{-1},\ v_1 \otimes \overline{v_{-1}} \otimes v_1,\ v_1 \otimes \overline{v_{-1}} \otimes v_{-1},\ v_{-1} \otimes \overline{v_1} \otimes v_1,\ v_{-1} \otimes \overline{v_1} \otimes v_{-1},\ v_1 \otimes \overline{v_{-1}} \otimes v_1,\ v_{-1} \otimes \overline{v_{-1}} \otimes v_{-1}\}$. For $(V \otimes W) \oplus (V \otimes W)$, choose the basis $\{(v_1 \otimes w_+, 0), (v_1 \otimes w_-, 0), (v_{-1} \otimes w_+, 0), (v_{-1} \otimes w_-, 0), (0, v_1 \otimes w_+), (0, v_1 \otimes w_-), (0, v_{-1} \otimes w_+), (0, v_{-1} \otimes w_-)\}$. Then the first component of d^0 corresponds to the multiplication map on the inner two circles tensored with the identity on the outer circle and the second component is the identity on the inner circle and the multiplication map on the outer two circles: $d^0 = (m \otimes \mathrm{Id}, \mathrm{Id} \otimes m)$. In our chosen basis, d^0 is described by the following matrix:

$$d^0 = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&0 \end{bmatrix} \sim \begin{bmatrix} 0&1&0&0&-1&0&0&0 \\ 0&0&1&0&1&0&0&0 \\ 0&0&0&1&0&0&-1&0 \\ 0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

Hence,

$$\begin{aligned} \ker d^0 = \mathrm{Span}\{ & v_1 \otimes \overline{v_1} \otimes v_1, \\ & v_1 \otimes \overline{v_{-1}} \otimes v_1 - v_1 \otimes \overline{v_1} \otimes v_{-1} - v_{-1} \otimes \overline{v_1} \otimes v_1, \\ & v_{-1} \otimes \overline{v_1} \otimes v_{-1} - v_{-1} \otimes \overline{v_{-1}} \otimes v_1 - v_1 \otimes \overline{v_{-1}} \otimes v_{-1}, \\ & v_{-1} \otimes \overline{v_{-1}} \otimes v_{-1}\} \end{aligned}$$

and

$$\mathrm{im}\ d^0 = \mathrm{Span}\{(v_1 \otimes w_-, 0), (v_{-1} \otimes w_-, 0), (0, v_1 \otimes w_-), (0, v_{-1} \otimes w_-)\}$$

For $d^1 : (V \otimes W) \oplus (V \otimes W) \to V$, we have

$$
\begin{aligned}
(v_1 \otimes w_+, 0) &\mapsto -v_1 \\
(v_1 \otimes w_-, 0) &\mapsto 0 \\
(v_{-1} \otimes w_+, 0) &\mapsto -v_{-1} \\
(v_{-1} \otimes w_-, 0) &\mapsto 0 \\
(0, v_1 \otimes w_+) &\mapsto v_1 \\
(0, v_1 \otimes w_-) &\mapsto 0 \\
(0, v_{-1} \otimes w_+) &\mapsto v_{-1} \\
(0, v_{-1} \otimes w_-) &\mapsto 0
\end{aligned}
$$

So that

$$
\begin{aligned}
\ker d^1 = \mathrm{Span}\{&(v_1 \otimes w_+, v_1 \otimes w_+), (v_{-1} \otimes w_+, v_{-1} \otimes w_+), \\
&(v_1 \otimes w_-, 0), (v_{-1} \otimes w_-, 0), (0, v_1 \otimes w_-), (0, v_{-1} \otimes w_-)\}
\end{aligned}
$$

and im $d^1 = V$.

Therefore,

$$
\begin{aligned}
\mathcal{AK}h^0 &= \ker d^0 \cong V_3 \\
\mathcal{AK}h^1 &= \ker d^1/\mathrm{im}\ d^0 \cong \mathrm{Span}\{(v_1 \otimes w_+, v_1 \otimes w_+), (v_{-1} \otimes w_+, v_{-1} \otimes w_+)\} \cong V_1\{(1,0)\} \\
\mathcal{AK}h^2 &= \ker d^2/\mathrm{im}\ d^1 = V/V = 0
\end{aligned}
$$

where the isomorphisms $\mathcal{AK}h^0 \cong V_3$ and $\mathcal{AK}h^1 \cong V_1\{(1,0)\}$ are determined by the $\mathfrak{sl}_2$-weight gradings on basis elements. Since each weight space is one-dimensional, each of the homology groups is an irreducible representation of $\mathfrak{sl}_2$, given by the highest $\mathfrak{sl}_2$-weight that appears in the grading.

The full current algebra action can be described, up to multiplication by scalars, by the following diagram 4.6:

The nodes of the diagram denote the $\mathfrak{sl}_2$-weight spaces of the homology groups, and the arrows demonstrate how the elements v_i of $\mathfrak{sl}_2^-(V_2)$ act on these weight spaces: all the v_i raise the homological degree by one. v_2 raises the $\mathfrak{sl}_2$-grading by two, v_0 maintains the $\mathfrak{sl}_2$-grading and v_{-2} lowers the grading by two.

This result for annular homology can be simply contrasted with Khovanov homology: forgetting the basepoint, the chain complex becomes $W^{\otimes 3} \to (W \otimes W) \oplus (W \otimes W) \to W$, and since the stabilised unknot is isotopic to the unknot, its Khovanov homology is given by a copy of the trivial representation W concentrated in homological degree zero, and this holds for any number of stabilisations, since they consist of Reidemeister 1 moves.

Note that up to homological and q-gradings, the annular homology of the twice stabilised unknot is isomorphic as a representation of $\mathfrak{sl}_2$ to $V_1 \otimes V_1$. In fact, a similar statement holds for an

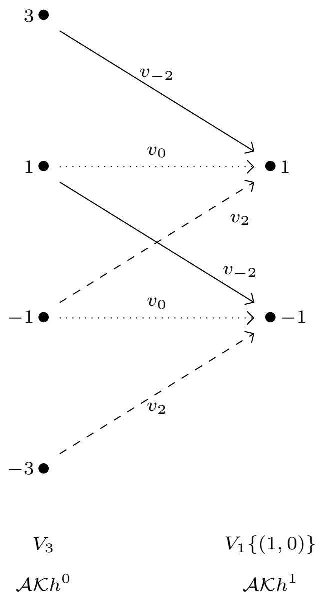


Figure 4.6: The current algebra action on $\mathcal{AK}h$ of the twice-stabilised unknot

n-times stabilised unknot: given an unknot positively stabilised around the basepoint n times, the resulting annular Khovanov homology is isomorphic, up to shifts in homological degree and q-grading to $V_n \otimes V_1 \cong V_{n-1} \oplus V_{n+1}$.

4.5.2 The positive trefoil

To compute the annular homology of the trefoil, we determine the homology groups as in the previous examples by choosing a basis for each of the vector spaces and computing the boundary maps in each of these bases, to find:

$$\mathcal{AK}h^0 \cong V_2\{3\}$$
$$\mathcal{AK}h^1 \cong V_0\{5\}$$
$$\mathcal{AK}h^2 \cong V_0\{5\}$$
$$\mathcal{AK}h^3 \cong V_0\{9\}$$

From the cube of resolutions of the trefoil, it is clear that annular homology reduces to regular Khovanov homology for homological degrees 2 and 3, since the cycles in each resolution are all trivial, except in the height zero resolution. Indeed, the Khovanov homology polynomial for the positive trefoil knot is $Kh(\text{trefoil}) = q + q^3 + q^5t^2 + q^9t^3$ [BN02], which agrees with annular Khovanov homology in homological degrees 2 and 3.

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