Fenchel-Nielsen Coordinates
Trouser Decomposition of Riemann surfaces

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What is a surface?
Examples of surfaces

- **sphere**
- **real projective plane**
- **torus**
- **Klein bottle**
How many different surfaces are there?
How do we classify them?
Fact: Any compact connected 2-manifold can be obtained from a polygon in the plane by gluing corresponding sides of the boundary together.

For example, the $g$-holed torus $T^2 \# T^2 \# \ldots T^2$ can be represented by $4g$-gon by identifying edges using symbol $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$. 
Classification of compact surfaces

**Figure 1. Handle**

**Figure 2. Crosshandle**
Classification of compact surfaces

**Figure 3. Cap**

**Figure 4. Crosscap**
They are all we need
Actually, more than enough

Lemma

A crosshandle is homeomorphic to two crosscaps.
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Classification of surfaces

Dyck’s Theorem

Handles and crosshandles are equivalent in the presence of a crosscap.
Classification of surfaces

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Let $a$ be the connected sum with a torus (or putting a handle) and let $b$ be the connected sum with $\mathbb{RP}^2$ (or putting a cross-cap). Then this is a sort of semigroup generated by $a$ and $b$ with a relation $ba = b^3$. 
Riemann surface

Definition #1 Riemann surface

A Riemann surface is a one-complex-dimensional connected complex manifold; that is, a two-real-dimensional connected manifold $M$ with a maximal set of charts $\{U_\alpha, \phi_\alpha\}_{\alpha \in A}$ on $M$ such that $\phi_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism onto an open subset of the complex plane $\mathbb{C}$ and the transition functions are holomorphic maps.
Theorem #1 Uniformization theorem for Riemann surfaces

A simply connected Riemann surface is isomorphic to either the Riemann sphere $\mathbb{P}^1$, then complex plane $\mathbb{C}$, or the open unit disc $\mathbb{D} \subset \mathbb{C}$.
**Definition #2 Möbius transformation**

A Möbius transformation is a mapping $\mathbb{P}^1 \to \mathbb{P}^1$ of the form

$$z \mapsto \frac{ax+b}{cz+d} \text{ with } ad - bc \neq 0.$$ 

**Theorem #2 Automorphisms of simply connected Riemann surfaces**

- $\text{Aut}(\mathbb{P}^1)$ is the group of all Möbius transformations.
- $\text{Aut}(\mathbb{C})$ is the group of Möbius transformations of the form $z \mapsto az + b$ with $a \neq 0$.
- $\text{Aut}(\mathbb{D})$ is the group of Möbius transformations of the form $z \mapsto \frac{az+b}{bz+a}$ with $|a|^2 > |b|^2$. 
Möbius transformation

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Universal covering space of Riemann surfaces

Theorem #3 Universal covering space of Riemann surfaces

- The Riemann sphere \( \mathbb{P}^1 \) is the universal cover only of itself.
- The plane \( \mathbb{C} \) is the universal cover of itself, of the punctured plane \( \mathbb{C} - \{a\} \), and of all compact Riemann surfaces homeomorphic to a torus.
- All other Riemann surfaces have universal covering space analytically isomorphic to \( \mathbb{D} \).

Definition #3 Hyperbolic Riemann surface

A Riemann surface is hyperbolic if its universal covering space is isomorphic to \( \mathbb{D} \).
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Definition #3 Hyperbolic Riemann surface

A Riemann surface is hyperbolic if its universal covering space is isomorphic to $\mathbb{D}$. 
What do we know about hyperbolic Riemann surfaces?

- A Fuchsian group is a discrete subgroup of $\text{Aut}(\mathbb{H}^2)$, which is isomorphic $\text{PSL}_2(\mathbb{R})$.
- A hyperbolic Riemann surface $X$ could be identified with $\mathbb{H}^2/\Gamma$ for some Fuchsian group $\Gamma$ which is isomorphic to $\pi_1(X)$.
- A Fuchsian group acts on $\mathbb{H}^2$ properly discontinuously. (Conversely, any subgroup of $\text{Isom}^+(\mathbb{H}^2)$ which acts on $\mathbb{H}^2$ properly discontinuously is discrete.)
What do we know about hyperbolic Riemann surfaces?

Theorem #4 Universal covering space of Riemann surfaces

If a Fuchsian group $\Gamma$ is torsion-free, then $X := \mathbb{H}^2/\Gamma$ has a unique structure of a Riemann surface for which the projection is a local isomorphism.

So, the torsion-free Fuchsian groups and hyperbolic Riemann surfaces are essentially the same subject.
Let $X$ be a surface and $\gamma : [a, b] \rightarrow X$ be a curve on $X$.

- $\gamma$ is *simple* if it is injective.
- $\gamma$ is *closed* if $\gamma(a) = \gamma(b)$.
- $\gamma$ is *essential* if $\gamma$ is not homotopic to a point or a boundary component, if there is any.
Proposition Geodesic representative

For an essential simple closed curve $c$, there exists a unique geodesic $\gamma$ in the free homotopy class of $c$.

Why? Let’s lift $c$ to the universal cover and look for the deck transformation which maps one end point to another. Is it elliptic? Parabolic? Hyperbolic? Wait, what are they?
Let $f$ be an isometry of hyperbolic plane.

- $f$ is *elliptic* if it fixes one point in the hyperbolic plane.
- $f$ is *parabolic* if it fixes one point on the boundary of hyperbolic plane.
- $f$ is *hyperbolic* if it fixes two points on the boundary of hyperbolic plane.
Cut your Riemann surface along a family of disjoint closed geodesics. Until when?

**Definition #5 filling**

A family $\mathcal{F}$ of essential simple closed curves on the surface $S$ is said to fill $S$ if any other essential simple closed curve should intersect an element of $\mathcal{F}$.

When this family fills the surface, call it a loop system. And by the previous result, we may assume that each element in the loop systems is a closed geodesic.
What the complement of a loop system looks like?
Decomposition
Theorem #5 Pants-decomposition

A hyperbolic Riemann surface $R$ of genus $g$ without boundary always contains a loop system of $3g - 3$ disjoint simple closed geodesics. Regardless of which loop system we choose, cutting $R$ along the geodesics in the system always decomposes $R$ into $2g - 2$ pairs of pants.

Proof? First, let’s assume that each component of $R$ after cutting is a pair of pants, and count the number of pairs of pants in the maximal decomposition.
Maximal Essential curves

- $N =$ the number of disjoint simple closed geodesics in any such system on $R$
- $M =$ the number of pairs of pants in a decomposition of $R$
- $L =$ an element of our loop system
- $n_1 =$ the number of connected components of $R - L$
- $g_1 =$ the sum of the genera of the connected components of $R - L$

Then $g_1 - n_1 = (g - 1) - 1$ and $R - L$ has two boundary components.
We proceed inductively, cutting $R$ successively along loops in our set of disjoint simple closed geodesics. Whenever we cut along a new loop,

- the resulting set has two more boundary components,
- the sum of the genera less the number of connected components decreases by one.
Thus after we cut along $N$ closed geodesics, we get

$$3M = 2N \text{ and } g_N - n_N = (g - 1) - N.$$ \[
\]
But $g_N = 0$ and $n_N = M$ so that $M = N - g + 1$. Then $2N = 3M = 3N - 3g + 3$ so that

$$N = 3g - 3 \text{ and } M = 2g - 2.$$ \[
\]
Definition #6 A pair of pants (or a trouser)

A pair of pants is a complete hyperbolic surface with geodesic boundary, whose interior is homeomorphic to the complement of three points in the 2-sphere.
A pair of pants!
Why do we care about trousers?

**Proposition**

Let $X$ be a compact connected hyperbolic surface with geodesic boundary. If all simple closed geodesics of $X$ are boundary components, then $X$ is homeomorphic to a pair of pants.

This tells us why pairs of pants are natural building blocks for Riemann surfaces: they are the only compact hyperbolic surfaces with geodesic boundary that cannot be further simplified by cutting along simple geodesics.
Consider various number of boundary components of $X$. 0, 1, or 2..
In case, where $X$ has at least two boundary components $A, B$, let $C$ be a simple arc joining $A$ and $B$.
Intersection number? why geodesics are in minimal position?
Proposition

Let $X$ be a noncompact connected hyperbolic surface with compact geodesic boundary, perhaps empty; assume that every simple closed curve in $X$ is either homotopic to a point, or bounds a punctured disc, or is homotopic to a boundary component. Then one of the following six possibilities holds:

- $X$ is a trouser with one, two, or three cusps
- $X$ is a half-annulus $\{z \in \mathbb{C} | 1 \geq |z| < R\}$ for some $1 < R < \infty$
- $X$ is isomorphic to the punctured disc $D^*$
- $X$ is isomorphic to $D$. 

Proof is similar with the previous one.
First note that there is a unique common perpendicular to two geodesics which do not intersect (even on the boundary) in hyperbolic plane. Let’s lift two boundary components of a trouser. What can we know from this?
A pair of pants with three seams
Proposition

For any given positive numbers \( l_1, l_2, l_3 \), there exists a unique hyperbolic right-angled hexagon with alternating edge lengths \((l_1, l_2, l_3)\).
A pair of pants is decomposed into two right-angled hexagons.
A pair of pants is decomposed into two right-angled hexagons
What’s left?

What did we get? Let $X$ be a hyperbolic Riemann surface with genus $g$ having no boundary.

- We have a loop system with $3g - 3$ simple closed geodesics.
- Each connected component of the complement of the loop system is a trouser.
- The hyperbolic structure of a trouser is completely determined by the lengths of boundary components (or cuffs!).

How about the hyperbolic structure of the entire Riemann surface?
Dehn Twist

\[ f : S \times I \rightarrow S \times I, \ (s, t) \mapsto (se^{2\pi it}, t). \]
Dehn Twist
Let $S$ be a closed, oriented surface of genus $g \geq 2$. A *marked hyperbolic surface* is a pair $(\phi, X)$ consisting of an oriented compact hyperbolic surface $X \cong H/\Gamma$ and an orientation-preserving homeomorphism $\phi : S \to X$.

Two marked surfaces $(\phi_i, X_i), \ i = 1, 2$ are *equivalent* if there exists an isometry $\alpha : X_1 \to X_2$ such that $\phi_2^{-1} \circ \alpha \circ \phi_1 = \psi$ is isotopic to the identity.

**Definition #6 Teichmüller Space**

The space of such equivalence classes is the *Teichmüller space* $T_g = Teich(S)$. 
Intuitively, what we have shown is that any point Teichmüller space of a hyperbolic Riemann surface of genus $g$ may be specified by $3g - 3$ nonnegative real numbers (lengths of simple closed geodesics) and $3g - 3$ real numbers (degree of twisting between glued pairs of pants). Thus $T_g$ is equivalent (as sets) to

$$\mathbb{R}^{3g-3} \times \mathbb{R}^{3g-3}.$$
But the pants-decomposition of a surface is not unique!
Coordinate Change

S-move

A-move
Coordinate Change

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We let $Mod(S)$ denote the group of orientation-preserving homeomorphisms $\psi : S \to S$, modulo those isotopic (equivalently, homotopic) to the identity. It acts on $Teich(S)$ by $\psi \cdot (\phi, X) = (\phi \circ \psi^{-1}, X)$. The quotient space is the moduli space

$$\mathcal{M}_g = \mathcal{M}(S) = Teich(S)/Mod(S).$$

Let $S$ be the space of complex structures on $S$.

- $\mathcal{M}(S) = S/\text{Homeo}^+(S)$
- $\mathcal{T}(S) = S/\text{Homeo}_0(S)$
(If time permits) Collar Theorem
Let $l_i$ and $t_i$ be denote the length coordinates and twisting coordinates, respectively. They are not well-defined on the Moduli space, but their derivatives are:

Define the 2-form on Teichmüller space $\omega = \sum_i dl_i \wedge dt_i$.

Wolpert showed that this 2-form does not depend on the choice of coordinates, so it descends to a 2-form on Moduli space.
Classification of Surfaces
Riemann Surfaces
Simple Closed Curves on Surfaces
Fenchel-Nielsen coordinates

Few words on Complex analytic theory

- Conformal vs. quasi-conformal maps
- Beltrami differential $\mu(z) \frac{dz}{dz}$
- $M(R)$ - space of Beltrami differentials with $||\mu||_\infty < 1$.
- $\{\text{conformal classes of quasi-conformal deformations of a given Riemann surface}\} \leftrightarrow \{\text{elements in } M(R)\}$.
- Two quasiconformal maps $f_0 : R \rightarrow R_0, f_1 : R \rightarrow R_1$ are Teichmüller equivalent if there exists a conformal map $c : R_0 \rightarrow R_1$ and $f_1^{-1} \circ c \circ f_0$ is homotopic to the identity on $R$ such that boundary of $R$ is fixed point-wise under homotopy.
- The space of equivalence classes in $M(R)$ with above relation is $\mathcal{T}(R)$.
References

- Teichmüller theory and application to Geometry, Topology, and Dynamics, John. H. Hubbard
- Three dimensional Geometry and Topology, William P. Thurston
- On groups generated by two positive multi-twists: Teichüller curves and Lehmer’s number, Christopher J Leininger
- The Fenchel-Nielsen Coordinates of Teichmüller Space, Kathy Paur