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ON THE CONVEX HULL GENUS OF SPACE CURVES†

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LET $K \subset \mathbf{R}^3$ be a simple closed curve, and $\tilde{K}$ be its convex hull. In [1], Almgren and Thurston define the (oriented) convex hull genus of K to be the minimal genus of an (oriented) surface contained in $\tilde{K}$ and bounded by K. They give examples showing that even if K is unknotted both the orientable and non-orientable convex hull genus of K may be arbitrarily large.

In §3 of this paper we show that their argument can be modified to apply to the class of almost-convex curves; these are roughly curves which are on the boundary of their convex hull except for small dips into the interior to avoid intersections. The formula we obtain gives a lower bound for the (oriented) convex hull genus of almost convex curves which is essentially independent of the topology of $\mathbf{R}^3 - K$; it is expressed in terms of the projection of K onto the boundary of K, and does not take into account which strand passes over the other.

Moreover, we show in §2 that Seifert's construction, appropriately modified, can be carried out within $\tilde{K}$, giving an upper bound for the convex hull genus of K in terms of a spherical projection. For almost-convex curves, the upper and lower bounds coincide, giving an exact formula for both their orientable and non-orientable convex hull genus.

The examples we obtain are much the same as those in [1] in the orientable case, but substantially simpler in the non-orientable case.

§1. PRELIMINARIES

(i) The genus of surfaces

Every compact surface without boundary is either:

a sphere	genus 0
a connected sum of n tori	genus n
a connected sum of n projective planes	genus $u/2$.

For a surface X with boundary, the genus is by definition the smallest genus of a surface without boundary in which X can be embedded.

LEMMA 1. *The Euler characteristic of a compact surface X of genus g with n boundary components is*

$$\chi(X) = 2 - 2g - n.$$

†We work throughout in the polyhedral category because it is easier to carry out Seifert's construction polyhederally than smoothly, and also because the transversality theorem we need is obvious. However, all the statements are valid without modification for smooth curves and surfaces.

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Proof. Obvious.

Remark. The more usual convention for the genus of a non-orientable surface is twice ours. But ours will avoid separate consideration of the orientable and non-orientable cases several times. With our convention, the "number of handles" of a non-orientable surface of genus g is $g-1$ or $g-(1/2)$ depending on whether g is an integer or a half-integer.

(ii) Tailoring a curve to a pattern

Let C be a polyhedral closed curve immersed in a polyhedral surface S. We will call C *generic* if (a) it has at worst double points, and (b) these occur only in the interior of edges of C and in the interior of faces of S.

Let $p \in C$ be such a double point. A *pattern at* p is a choice of one of the pairs of opposite quadrants at p defined by C; a *pattern for* C is a pattern at each double point of C.

To *tailor* C *to a pattern* σ we perform the following operation at each double point p:

Pick points a, b, c and d near p, one on each of the four rays of C emanating from p, and circularly ordered;

Erase the segments connecting a, b, c and d to p, and add that pair of segments (ab), (cd) or (bc), (ad) which lies in the quadrants specified by σ at p. See Fig. 1.

The tailored curve will be called C_σ, it is a union of a certain number $c(\sigma)$ of simple closed curves.

Let C have n double points. Of the 2^n patterns for C, one is of particular interest: the *canonical pattern*. It is the unique pattern σ such that both C and C_σ can be oriented, and the orientations coincide on $C \cap C_\sigma$. The canonical pattern can be found by orienting C, and picking at each double point those quadrants bounded by one inward pointing and one outward pointing ray.

§2. SEIFERT'S CONSTRUCTION

Let $K \subset \mathbf{R}^3$ be a polyhedral simple closed curve, and Δ be a 3-simplex in the interior of $\tilde{K}$ such that $\Delta \cap K = \emptyset$. (The interior of $\tilde{K}$ is empty only if K is contained in a plane; we will assume from now on that this does not occur.) Pick p in Δ such that the projection from p of K onto the boundary S of Δ is a generic curve C.

Tailor C to some pattern σ and for each quadruple a, b, c, d chosen on C near a double point let a', b', c', d' be the corresponding points on K. Moreover suppose the new segments of C_σ are (ab) and (cd).

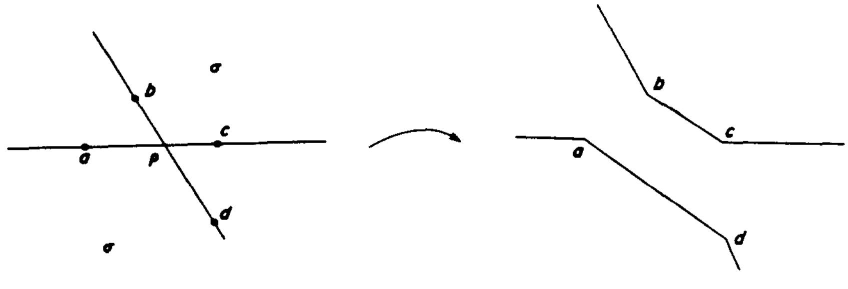

Fig. 1.

Define the Seifert surface X_σ to be $X_1 \cup X_2 \cup X_3$, where X_1 is a union of disjoint discs in Δ with boundary C_σ; X_2 is that part of the cone over C_σ centered at p lying between C_σ and either K or a segment $(a'b')$ or $(c'd')$, whichever is appropriate; X_3 is the union, for each double point, of the triangles $(a'b'c')$ and $(b'c'd')$ (the triangles $(a'b'd')$ and $(a'c'd')$ would also do).

Figure 2 should convince the sceptics that the space X_σ constructed is in fact a polyhedral surface with boundary K, and is contained in $\tilde{K}$.

For the sake of completeness we sketch a proof that X_1 exists.

Lemma 2. *Let $C_1, \ldots, C_n$ be disjoint simple closed curves on the boundary S of a 3-simplex $\Delta \subset \mathbf{R}^3$. Then there exist disjoint polyhedral discs $D_1, \ldots, D_n$ embedded in Δ such that the boundary of D_i is C_i.*

Proof. Order the C_i and choose one component S_i of $S \backslash C_i$ so that if $j > i$, then $S_j \subset S_i$. Choose concentric simplices $\Delta_1 \subset \Delta_2 \subset \cdots \subset \Delta_n \subset \Delta$. Now let D_i be the union of that part of the cone over C_i between Δ and Δ_i and the projection of S_i on $\partial\Delta_i$. The space D_i is a polyhedral disc by the Shoenfliess theorem, and the ordering makes the D_i disjoint. Q.E.D.

Lemma 3. *The genus of X_σ is $(I - c(\sigma) + 1)/2$, where I is the number of double points of C.*

Proof. Up to homotopy, $X_1 \cup X_2$ is just $c(\sigma)$ points, and X_3 is I edges connecting them. Thus $\chi(X_\sigma) = c(\sigma) - I$, and since $\partial K_\sigma = K$, the formula follows from Lemma 1. Q.E.D.

Lemma 4. *The surface X_σ is orientable if and only if σ is the canonical pattern.*

Proof. If σ is the canonical pattern, orient C_σ in a way compatible with an

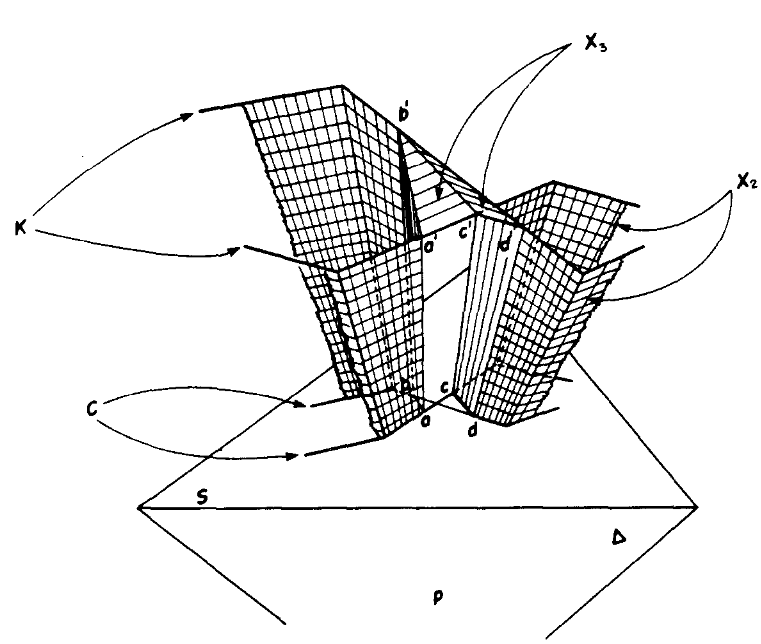

Fig. 2.

orientation of C; orient each component of X_1 so that C_σ has the boundary orientation, and extend this orientation to $X_1 \cup X_2$. It is straightforward to show that each component of X_3 respects the orientations of the two components of X_2 it connects.

Conversely, if X_σ is orientable, orient $X_2 \cup X_3$ and give the boundary $K \cup C_\sigma$ the boundary orientation. It is clear from Fig. 2 that the given orientation of C_σ and the orientation of C opposite to that induced by K are compatible. Q.E.D.

§3. ALMOST-CONVEX CURVES

A curve $K \subset \mathbf{R}^3$ is called *almost-convex* if every point of K is on the boundary of $\tilde{K}$ except for I disjoint closed edges C_i, called inner segments, through each of which passes a plane P_i with $C_i \subset P_i$ such that in one of the half-spaces H_i determined by P_i, K consists of two edges as in Fig. 3. Call H'_i the other half space.

Remark. It is easy to generate many almost-convex curves; on the surface of a sphere or an ellipsoid, draw any curve, dipping occasionally into the interior to avoid intersections. Such curves can easily be approximated by almost-convex polyhedral curves.

Notation. For each inner segment C_i, let C'_i, C''_i be the adjacent edges along K, and let A'_i, A''_i be the edges of K which intersect H_i.

Pick $\Delta \subset (\bigcap_i H'_i) \cap \tilde{K}$ a small 3-simplex centered at p, with boundary S. For a generic such p, the projection of K onto S from p is a generic curve C with exactly one double point p_i in the image of each inner segment C_i, and no others.

Let $X \subset \tilde{K}$ be a polyhedral surface bounded by K. For each inner segment C_i pick a plane P'_i parallel and close to P_i, in H'_i. Then $P'_i \cap K$ consists of four points a_i, b_i, c_i, d_i on C'_i, A'_i, C''_i, A''_i respectively forming a convex quadrilateral. If X is in general position with respect to all the P'_i, then each $P'_i \cap X$ will consist of two polyhedral intervals B'_i, B''_i ending at the points a_i, b_i, c_i, d_i and a certain number of simple closed curves, all disjoint and contained in the convex hull of a_i, b_i, c_i, d_i. In particular, *the intervals B'_i and B''_i joint either a_i to b_i and c_i to d_i or a_i to d_i and b_i to c_i but not a_i to c_i and b_i to d_i*; they therefore define a pattern σ for C.

THEOREM 5. *The genus of X is at least as great as the genus of X_σ.*

Proof. Pick a small closed tubular neighborhood Y in X of $K \bigcup_i (B'_i \cup B''_i)$. Then Y is seen to have $I + c(\sigma) + 1$ boundary components, as follows:

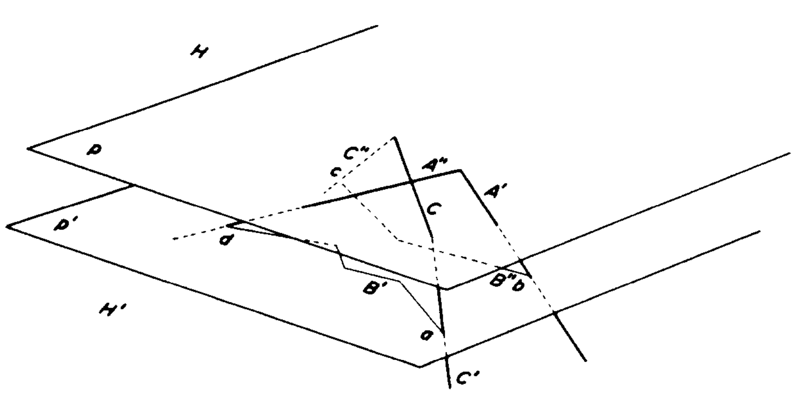

Fig. 3.

One for each inner sugment C_i, going from a_i to b_i along B'_i (say), then from b_i to d_i along A'_i and A''_i, then from d_i to c_i along B''_i and back to a_i along C''_i, C_i and C'_i.

One for each component of C_σ, running along K until it nearly meets one of the planes P'_i, then following the appropriate B_i curve until it meets K again, etc. And finally K.

But $\chi(Y) = -2I$, since up to homotopy Y is obtained from K by adding $2I$ 1-simplices. So Lemma 1 gives that the genus of Y is $1/2(I - c(\sigma) + 1)$, which is the genus of X_σ by Lemma 3. Since Y is embedded in X, the genus of X is at least as large as the genus of Y. Q.E.D.

PROPOSITION 6. *If X is orientable, the pattern σ is the canonical one.*

Proof. Orient K as the boundary of X, and let X_i be the part of X which lies outside of P'_i. Then X_i is bounded by two segments of K, the curves B'_i and B''_i and perhaps some other simple closed curves. If B'_i and B''_i are oriented as parts of the boundary of X_i, this must be compatible with the orientation of K. This is precisely what is needed to show that σ is the canonical pattern.

Conclusion. If K is an almost convex curve, project it generically onto a small sphere well in the interior of its convex hull (i.e. in $\tilde{K} \cap \bigcap_i H'_i$). This gives a curve C with a certain number I of double points. Compute c_1 the largest $c(\sigma)$ for all patterns σ, and $c_2 = c(\sigma)$ for σ the canonical pattern. Then the not necessarily orientable convex hull genus of K is $1/2(I - c_1 + 1)$, and the orientable convex hull genus of K is $1/2(I - c_2 + 1)$.

I know of no algorithm for finding c_1 other than examining the 2^I cases, but c_2 is always easily computable.

§4. EXAMPLES

Example 1. For the unknotted curve drawn in Fig. 4, we find $I = 4m$ and $c_1 = c_2 = 2m + 1$, so that both the orientable and the non-orientable genus is m.

Remarks. (i) There are in fact two patterns σ besides the canonical one with $c(\sigma) = 2m + 1$, so that there is a non-orientable surface realizing the minimal genus as well as an orientable one. I know of no case with positive orientable genus for which the non-orientable genus is larger; i.e. I know of no curve C for which the *unique* minimum of $c(\sigma)$ is the canonical pattern, except for some cases where $I + 1 =$

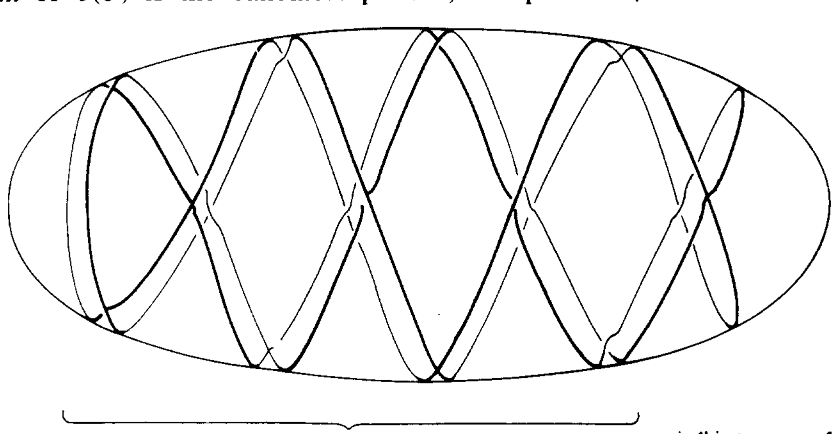

Fig. 4.

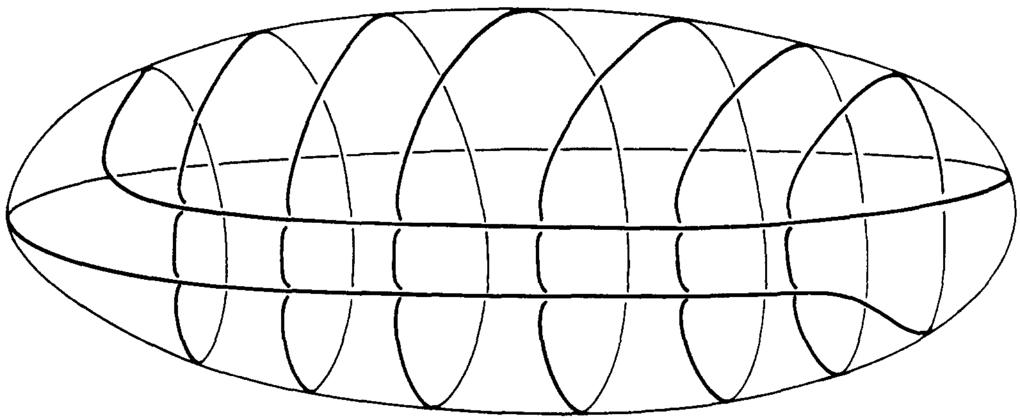

Fig. 5.

sup $c(\sigma)$. (ii) It is remarkable that for the curve drawn, and more generally for all curves drawn on the boundary of a convex body except for occasional dips into the interior to avoid intersections, the genus does not depend on which strand passes under the other. Thus the convex hull genus of an almost convex curve K is not sensitive to the topology of the knot K. (iii) If the ellipsoid of Fig. 4 is very elongated, K is nearly straight except near the ends of the ellipsoid, where each of the four strands does an about-turn. Thus for any ϵ, the total curvature of K can be made less than $4\pi + \epsilon$.

Example 2. The curve in Fig. 5 is a slightly modified version of the one appearing in ([1], p. 529). Applying our formulas, we find it has orientable convex hull genus 6 and non-orientable convex hull genus 9/2. Using appropriate generalizations of the techniques in this paper, one can show that the orientable convex hull genus of the curve actually given in [1] is 4, and its non-orientable genus is (1/2), as seen in [1]. Notice the jump that the non-orientable convex hull genus underwent as a result of the slight modification making the curve almost convex.

REFERENCES

1. F. Almgren and W. Thurston: Examples of unknotted curves which bound only surfaces of high genus within their convex hull. *Ann. Math.* **105** (1977) 527–538.
2. Fox: A quick trip through knot theory. In *Topology of 3-manifolds and related topics*, Proc. of the University of Georgia Inst. (1961).

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