Superattractive Fixed Points in $\mathbf{C}^n$

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1. Introduction. Many of the most important algorithms of mathematics are iterative, and in the good cases quadratically convergent. The most typical of these is Newton's method. If x_0 is a root of the equation $f(x) = 0$ and $f'(x_0) \neq 0$, then if you start iterating Newton's algorithm at some y_0 sufficiently close to x_0, generating the sequence $y_0, y_1, y_2, \ldots$, you can expect the errors $E_n = |y_n - x_0|$ to satisfy approximately $E_{n+1} \approx (E_n)^2$, so that $E_n \approx (E_0)^{2^n}$.

Note that this is very much more rapid than geometric convergence, where $E_n \approx \alpha^n E_0$ for some α satisfying $0 < \alpha < 1$: in one case you double the number of correct digits, in the other you add approximately $-\log_m \alpha$ correct digits in base m at each iteration.

A fixed point x_0 of a mapping f will be called superattractive if there is a neighborhood U of x_0 such that

$$|f^{\circ n}(y) - x_0| \leq C|f^{\circ(n-1)}(y) - x_0|^2$$

for some constant C, and all $n > 0$, $y \in U$.

In this paper we will try to analyze the behavior of an analytic mapping f near a superattractive fixed point x_0; the condition above shows that the linear terms of f vanish at x_0.

In dimension one, there is a very clean way of saying essentially all there is to say about the local behavior of f near x_0. If $f(x_0 + u) = x_0 + au^k + \cdots$ with $a \neq 0$, then there is an analytic local coordinate φ at x_0 such that $\varphi(f(x)) = \varphi(x)^k$. The local coordinate is called the *Böttcher coordinate*; it is unique up to multiplication by a $(k-1)^{\text{th}}$ root of 1 [M and DH].

The analogous statement is false in higher dimensions. Let $f : U \to \mathbf{C}^n$ be an analytic map defined on an open subset $U \subset \mathbf{C}^n$, and x_0 a superattractive fixed point of f. When $n \geq 2$, the map f is not in general locally conjugate, even topologically, to its terms of lowest degree; the local geometry near such a point is much too rich for anything like that to be true.

To see this, consider the case $n = 2$, and suppose $x_0 = 0$. Generically such a mapping f has near 0 a critical locus which consists of two transversal non-singular curves. Since the critical locus is exactly the set of points at which the

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mapping is not a local homeomorphism, it must be preserved by any conjugacy, even topological. Moreover, all of its forward and inverse images must be preserved. These together have an intricate geometric structure. Specifically, the problem we will focus on is that for homogeneous mappings the forward images of the critical curves are smooth curves, but for non-homogeneous mappings they tend to have critical points at the origin. By a theorem of Mumford [Mu], no local homeomorphism of $\mathbf{C}^2$ near the origin can map a smooth curve to a singular curve, hence this prevents even topological conjugacy.

Example. Consider the mapping

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 + y^3 \\ y^2 \end{bmatrix}.$$

The critical locus is easily seen to be the union of the two axes, but the y-axis is mapped to the parametrized curve

$$t \mapsto \begin{bmatrix} t^3 \\ t^2 \end{bmatrix},$$

i.e., the curve of equation $x^2 = y^3$, which has a singularity at the origin, and intersects a 3-sphere centered at the origin in a trefoil knot.

Still, the quadratic terms influence the dynamics in an essential way; the object of this paper is to describe this phenomenon.

Outline of the paper. Suppose that $F : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{n+1}, 0)$ is an analytic mapping defined near 0, with power series $F = F_k + F_{k+1} + \cdots$ with lowest degree terms of degree $k \geq 2$. We will require throughout that F_k be non-degenerate in the sense that $F_k^{-1}(0) = 0$.

In Section 2 we prove that the "potential function"

$$h_F(\mathbf{x}) = \lim_{m \to \infty} \frac{1}{k^m} \log \|F^{\circ m}(\mathbf{x})\|$$

exists on the basin of 0, and we explore some of its properties. To go further, we need some properties of currents, sketched in Section 3. Sections 4–8 are devoted to the homogeneous case. In Section 4, we reprove the existence of the Brolin measure for arbitrary rational functions, with a potential theoretic interpretation. This proof is considerably simpler than the present ones [L, FLM] , and easily generalizes (Section 5) to provide an analog of the Brolin measure to arbitrary endomorphisms of $\mathbb{P}^n$. It turns out the right generalization is an invariant (1,1)-form, and although there is also an invariant measure, its properties are not quite as clear.

Section 6 exhibits this $(1,1)$-form in a special case.

Sections 7 and 8 examine the boundary of the basin of attraction in the homogeneous case. The "Levi current" of this boundary is closely related to the $(1,1)$-form above, and leads to remarkable foliations of the 3-sphere, and the proof eventually exhibits the invariant $(1,1)$-form as the curvature of a connection on a line bundle over $\mathbb{P}^n$.

Section 9 generalizes these results to the non-homogeneous case. Essentially, they say that the support of $dd^c h_F$ closely resembles the support of $dd^c h_{F_k}$, providing what is presumably the best generalization of Böttcher's theorem that one can hope for in higher dimensions.

A future paper [HP] will apply these results to Newton's method in two dimensions, exhibiting how the geometry of $dd^c h_F$ inside the basins of attraction connects the structure of the boundary to the structure of certain Julia sets of rational functions.

The results in this paper, more particularly Sections 2–6, have analogs in the theory of polynomial diffeomorphisms of $\mathbf{C}^2$ [Hu, FM, BS1, BS2, BS3, FS, HO], and we have clearly benefitted from the techniques that have been developed in that field, more particularly the use of plurisubharmonic functions, currents and the Monge-Ampère operator.

Many people have helped during the writing of this paper. Curt McMullen prompted the whole investigation by asking about the situation of Section 8 John Milnor suggested the results in Section 9, and Bill Thurston suggested the interpretation of the Levi form in Proposition 7.2. Adrien Douady suggested the example in Section 6, and was helpful on many other points. Nessim Sibony prompted us to find a geometric interpretation of the Levi-current. Eric Bedford suggested the relevance of the Monge-Ampère operator, and helped with pull-backs of currents. JianGuo Cao helped with Section 5. Conversations with Misha Lyubich, Monique Hakim, Dennis Sullivan, Jiaqi Luo and John Smillie all contributed.

Ramin Farzaneh made the computer pictures in Figures 5 and 6. The reader who tries to rewrite the program will discover that there are major computational difficulties, which he overcame.

We must particularly thank the referee. He (or she) read the paper with great care, finding many minor mistakes, and one important one, involving the meaning of f-invariant; the discussion in Remark 5.2 was inspired by a comment by the referee.

Since the first version of this paper was distributed in 1991, there has been much further work on this topic. There has been an explosion of papers on holomorphic dynamics in several variables, by Bedford, Fornaess, He, Klimek, Lyubich, Sibony, Smillie, Ueda and others. Fornaess and Sibony [FS3], Ueda [U] and Klimek [K] have studied endomorphisms of $\mathbb{P}^n$. Ueda and independently Fornaess and Sibony have proved that the converse of Proposition 5.4 is true, which we asked as a problem.

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2. The potential function. Provide $\mathbf{C}^{n+1}$ with some norm, let U be a neighborhood of $\mathbf{0}$ in $\mathbf{C}^{n+1}$ and $F : U \to \mathbf{C}^{n+1}$ a holomorphic mapping with power series

$$F(\mathbf{x}) = F_k(\mathbf{x}) + F_{k+1}(\mathbf{x}) + \cdots$$

where each F_m is homogeneous of degree m and the leading term F_k has degree $k \geq 2$.

We will say that F_k is non-degenerate if $F_k(\mathbf{x}) = \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$.

Theorem 2.1. *If F_k is non-degenerate, then*

(a) *The limit*

$$h_F(\mathbf{x}) = \lim_{m\to\infty} \frac{1}{k^m} \log \|F^{\circ m}(\mathbf{x})\|$$

exists in $[-\infty, \infty)$ for $\mathbf{x}$ in some neighborhood V of $\mathbf{0}$. The function h_F is pluri-subharmonic in V, continuous on $V - \{\mathbf{0}\}$, and the convergence is uniform on compact subsets of $V - \{\mathbf{0}\}$.

(b) *The function h_F satisfies $h_F\big(F(\mathbf{x})\big) = kh_F(\mathbf{x})$.*

(c) *There exists a constant C such that*

$$\log \|\mathbf{x}\| - C \leq h_F(\mathbf{x}) \leq \log \|\mathbf{x}\| + C.$$

Proof. Clearly the origin is an attracting fixed point: let V be its basin. Since F_k is non-degenerate, there exists a constant $K > 1$ such that

$$\frac{1}{K} \leq \|F_k(\mathbf{x})\| \leq K$$

whenever $\|\mathbf{x}\| = 1$. It follows that $\|F_k(\mathbf{x})\| \geq K^{-1}\|\mathbf{x}\|^k$ for any $\mathbf{x}$, and in particular, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|(F - F_k)(\mathbf{x})\| \leq \varepsilon \|F_k(\mathbf{x})\|$$

when $\|\mathbf{x}\| \leq \delta$, and hence

(1)

$$K^{-1}(1-\varepsilon)\|\mathbf{x}\|^k \leq (1-\varepsilon)\|F_k(\mathbf{x})\| \leq \|F(\mathbf{x})\| \leq (1+\varepsilon)\|F_k(\mathbf{x})\| \leq K(1+\varepsilon)\|\mathbf{x}\|^k.$$

To prove the existence of the limit in part (a), and part (c), it is enough to show that the series with general term

$$\frac{1}{k^{n+1}} \log \|F^{\circ(n+1)}(\mathbf{x})\| - \frac{1}{k^n} \log \|F^{\circ n}(\mathbf{x})\|$$
$$= \frac{1}{k^n} \left(\frac{1}{k} \log \|F\big(F^{\circ n}(\mathbf{x})\big)\| - \log \|F^{\circ n}(\mathbf{x})\| \right)$$

is uniformly convergent in V. We may assume that $\|F^{\circ n}(\mathbf{x})\| < \delta$, and taking a logarithm of (1), we find

$$\left| \frac{1}{k} \log \|F\big(F^{\circ n}(\mathbf{x})\big)\| - \log \|F^{\circ n}(\mathbf{x})\| \right| \leq |\log(1-\varepsilon)| + \log K$$

which is bounded. Now take $\varepsilon = \frac{1}{2}$ for instance; the coefficient $1/k^n$ makes the series converge. This proves the existence of the limit. Since a limit of pluri-subharmonic functions is pluri-subharmonic, h_F is plurisubharmonic.

Part(b) is obvious from the definition of h_F. □

Corollary 2.2. *If F is as above, and u is a bounded function on a neighborhood of* 0 *in* $\mathbf{C}^{n+1}$, *then the limit*

$$\lim_{m \to \infty} \frac{1}{k^m} \left(\log \|F^{\circ m}(\mathbf{x})\| + u\big(F^{\circ m}(\mathbf{x})\big) \right)$$

exists and is equal to $h(\mathbf{x})$.

Remark. There is a useful geometric way of interpreting the non-degeneracy condition. If the map F_k is non-degenerate, it maps lines in $\mathbf{C}^{n+1}$ to lines, and hence naturally induces a mapping on the associated projective space

$$f_k : \mathbb{P}^n \to \mathbb{P}^n,$$

which is simply F_k in homogeneous coordinates. In particular, when $n = 1$, this is a rational function of degree k, and we will try to use the extensive work on iteration of rational functions in dimension 1.

Let $\tilde{\mathbf{C}}^{n+1}$ be the blow-up of $\mathbf{C}^{n+1}$ at the origin, and $p : \tilde{\mathbf{C}}^{n+1} \to \mathbf{C}^{n+1}$ the canonical projection. (The blow-up is a standard construction in algebraic geometry, see for instance [GH, p. 182].) Of course, $p^{-1}(\mathbf{0}) = \mathbb{P}^n$. Let $\tilde{U} = p^{-1}(U)$ (recall that $U \subset \mathbf{C}^{n+1}$ is the domain of F).

Proposition 2.3. *If the mapping F_k is non-degenerate, then $F : U \to \mathbf{C}^{n+1}$ induces an analytic mapping $\tilde{F} : \tilde{U} \to \tilde{\mathbf{C}}^{n+1}$, and the restriction f_k of $\tilde{F}$ to $\mathbb{P}^n = p^{-1}(\mathbf{0})$ has positive degree.*

Proof. It is easy to see that F on $\tilde{U} - \mathbb{P}^n$ and f_k on $\mathbb{P}^n$ together form a continuous mapping $\tilde{U} \to \tilde{\mathbf{C}}^{n+1}$. But a mapping which is continuous and analytic except on an analytic subset of codimension 1 is analytic. $\square$

3. The formalism of currents. In this section we will choose some notation and summarize a few well-known facts about currents; we will follow [F, DR and BT].

Generalities on currents. On any oriented differentiable manifold X of dimension n let $\mathscr{A}^p(X)$ denote the space of p-forms of class C^∞, and $\mathscr{D}^p(X) \subset \mathscr{A}^p(X)$ the subspace of forms with compact support. Further denote $\mathscr{D}_p(X)$ the dual of $\mathscr{D}^{n-p}(X)$, called the space of p-currents. Clearly the pairing

$$\mathscr{A}^p(X) \times \mathscr{D}^{n-p}(X) \to \mathbf{C}$$

given by

$$(\varphi, \omega) \mapsto \int_X \varphi \wedge \omega$$

induces an inclusion $\mathscr{A}^p(X) \subset \mathscr{D}_p(X)$.

Integration over the fiber. Currents have two functorial properties:

- they can be pushed forward by smooth proper mappings;
- they can be pulled back by submersions.

The first operation is fairly transparent, but the second is a bit delicate. If $f : X \to Y$ is a submersion from an oriented manifold of dimension n to an oriented manifold of dimension $m \leq n$. Then the fibers are manifolds of dimension $r = n - m$ and integration over the fibers gives a map

$$f_* : \mathscr{D}^p(X) \to \mathscr{D}^{p-r}(Y)$$

defined as follows.

Any p-form φ on X with compact support can be written $\varphi = \psi \wedge f^*\omega$, where ψ is an r-form with compact support on X and ω is a $(p-r)$-form on Y. To see this, use a partition of unity to write φ as a sum of forms with support in a coordinate neighborhood, and in local coordinates the decomposition becomes obvious.

We can then consider the function $f_*\psi$ on Y with compact support defined by

$$f_*\psi(y) = \int_{f^{-1}(y)} \psi$$

and define $f_*\varphi = f_*\psi \wedge \omega$. That all of this is well defined is proved in [BoT, pp. 61–62].

The pullback $f^* : \mathscr{D}_{n-p}(Y) \to \mathscr{D}_{n-p}(X)$ on currents is by definition the transpose of f_*.

This leaves a problem of ambiguity: a smooth current has apparently two different pull-backs, one as a form and one as a current. Fortunately, there is no conflict, because of the following result:

Lemma 3.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathscr{A}^p(Y) & \xrightarrow{f^*} & \mathscr{A}^p(X) \\ \downarrow & & \downarrow \\ \mathscr{D}_p(Y) & \xrightarrow{f^*} & \mathscr{D}_p(X) \end{array}$$

Proof. This is the transpose of Prop. 6.15 in [BoT]. □

We will require the following result.

Proposition 3.2. *Let $f : X \to Y$ be a submersion as above. Then*

$$f_* : \mathscr{D}^p(X) \to \mathscr{D}^{p-r}(Y)$$

is a split surjection, and

$$f^* : \mathscr{D}_p(Y) \to \mathscr{D}_p(X).$$

is a split injection; in particular it is an isomorphism onto its image.

Proof. The second part is an immediate consequence of the first. For that part, choose a partition of unity (U_i, u_i) on Y, such that for each open subset $U_i \subset Y$ there exists a subset $V_i \subset f^{-1}(U_i) \subset X$, an open subset W_i of $\mathbf{R}^r$ and a diffeomorphism $V_i \to U_i \times W_i$ commuting with the projections. Choose an r-form ψ_i with compact support on each W_i such that

$$\int_{W_i} \psi = 1.$$

Now the formula

$$\varphi \mapsto \sum_i f^*(u_i\varphi) \wedge \psi_i$$

defines a mapping $f^{\#} : \mathscr{D}^{p-r}(Y) \to \mathscr{D}^p(X)$, and it is immediate that $f_* \circ f^{\#} =$ id. □

The precise statement we will need is the following.

Corollary 3.3. *Let $f : X \to Y$ be a submersion as above. If a sequence $\{T_i\} \subset \mathscr{D}_p(Y)$ has the property that the sequence $\{f^*(T_i)\}$ converges in $\mathscr{D}_p(X)$, then the sequence $\{T_i\}$ converges.*

Pullbacks by ramified mappings. We will also need sometimes to pull back currents by holomorphic mappings which are not submersions; we will of course want this construction to extend the pull-back defined above at the regular values. Such a construction is not possible in general, as the following examples show.

Example. Consider the mapping $f : \mathbf{C} \to \mathbf{C}$ given by $w = f(z) = z^2$. Then the push-forward of the function $|z|^2$, which is C^∞, is the function $2|w|$, which is continuous, but not differentiable. Thus there is no mapping

$$f_* : \mathscr{D}^0(\mathbf{C}) \to \mathscr{D}^0(\mathbf{C}),$$

and hence no mapping

$$f^* : \mathscr{D}_2(\mathbf{C}) \to \mathscr{D}_2(\mathbf{C}).$$

With the same mapping, the push-forward of $dz \wedge d\bar{z}$ is $(dw \wedge d\bar{w})/2|w|$, which is not even continuous. Thus there is no push-forward

$$f_* : \mathscr{D}^2(\mathbf{C}) \to \mathscr{D}^2(\mathbf{C}),$$

and by duality no pull-back

$$f^* : \mathscr{D}_0(\mathbf{C}) \to \mathscr{D}_0(\mathbf{C}).$$

Thus we are forced into ad-hoc constructions, which are sufficient for our case. There are two cases where we can extend the definition of push-forward in the differentiable context, which we state as (A) and (B) below, and a further case which applies only for analytic manifolds, which we will describe at the end of the next section.

(A) If $f : X \to Y$ is any smooth mapping of differentiable manifolds, and φ is a closed 1-current on Y which is the differential du of a continuous function, we can define $f^*\varphi = df^*u$. This is well defined since f^*u is continuous and unique up to an additive constant. With this definition, $f^*\varphi$ clearly coincides with the previous definition above the regular values.

(B) If X and Y have the same dimension n and $f : X \to Y$ is proper, and such that for any $y \in Y$, the set $f^{-1}(y)$ is finite, the mapping

$$f_* : C(X) \to C(Y) \quad \text{given by} \quad (f_*\varphi)(y) = \sum_{x \in f^{-1}(y)} \deg_x f \varphi(x)$$

is well defined (and more specifically, $f_*\varphi$ is continuous). Moreover, it extends the push-forward given by integration over the fiber, as above. Let $\mathscr{M}_n(X) \subset \mathscr{D}_n(X)$ be the n-currents on X of order 0, in other words, the dual of the continuous function with compact support, or in still other words, the regular measures on X. By transposition, we get a pullback

$$f^* : \mathscr{M}_n(Y) \to \mathscr{M}_n(X).$$

The only case where both of these constructions are defined is when both X and Y are 1-dimensional, $f : X \to Y$ is differentiable, with the derivative vanishing at a discrete set, and we wish to pull-back a 1-current of the form du, where $u : Y \to \mathbf{R}$ is continuous and locally of bounded variation (so that du is a measure). We leave it to the reader to show that in this case the constructions agree.

The case of complex manifolds. In the case of complex manifolds, all the constructions above work for forms of type (p,q). More specifically, if X and Y are complex manifolds of complex dimension n and $m < n$ respectively, $r = n - m$, and $f : X \to Y$ is an analytic submersion, then integration over the fiber gives a mapping

$$f_* : \mathscr{D}^{p,q}(X) \to \mathscr{D}^{p-r,q-r}(Y),$$

and the transpose defines a pullback on currents

$$f^* : \mathscr{D}_{n-p,n-q}(Y) \to \mathscr{D}_{n-p,n-q}(X)$$

which again extends the pull-back of forms.

For complex manifolds, the differential operators

$$\partial : \mathscr{D}^{p,q} \to \mathscr{D}^{p+1,q}$$

and

$$\bar\partial : \mathscr{D}^{p,q} \to \mathscr{D}^{p,q+1}$$

can be extended to currents in the standard way, inspired by integration by parts:

$$\langle \partial T, \varphi \rangle = -\langle T, \partial\varphi \rangle$$

and

$$\langle \bar\partial T, \varphi \rangle = -\langle T, \bar\partial\varphi \rangle.$$

We can also define the operators $d = \bar\partial + \partial$ and $d^c = i(\bar\partial - \partial)$. These are real operators (i.e., they take real forms to real forms), and the operator dd^c is a variant of the Laplacian well adapted to complex analysis. In one dimension, we have

$$dd^c f = \Delta f \, dx \wedge dy.$$

The Monge-Ampère operator. When u is a plurisubharmonic function (psh) on $\mathbf{C}^n$, we will need to define the currents $(dd^c u)^j$ for $j = 1, \ldots, n$. The case $j = n$ is called the *Complex Monge-Ampère Operator*. There is an extensive literature [BT, L2] about this, but we require only the simplest case, where u is a continuous psh. The following paragraph sums up what we need to know. The material is lifted from [BT].

If u is a continuous psh function on $U \subset \mathbf{C}^n$, then $dd^c u$ is a positive $(1,1)$-current (in fact, for a continuous function, this is the definition of a psh function). In particular the current $dd^c u$ is of order 0 (this means it can be evaluated on continuous test forms). We can define $(dd^c u)^2$, and more generally $dd^c u_1 \wedge dd^c u_2$, by

$$\langle (dd^c u_1 \wedge dd^c u_2, \varphi\rangle = \langle dd^c u_1, u_2 dd^c \varphi\rangle$$

since $u_2 dd^c \varphi$ is a continuous test form. With this definition, $dd^c u_1 \wedge dd^c u_2$ is a positive $(2,2)$-form, hence again of order 0 and we can define recursively

$$\langle (dd^c u_1 \wedge \cdots \wedge dd^c u_m, \varphi\rangle = \langle (dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1}, u_m dd^c \varphi\rangle.$$

Furthermore, if v_m is a sequence of continuous subharmonic functions converging uniformly to v, then the sequence $(dd^c v_m)^j$ converges to $(dd^c v)^j$ in the topology of distributions [BT, Prop. 2.3].

Pullbacks by analytic ramified mappings. In general, pull-backs of currents by analytic mappings which are not submersions are no better behaved than pull-backs by smooth mappings, and we are forced into the same sorts of ad-hoc constructions. Still, the resulting constructions are more interesting.

Just as for differentiable mappings, we can pull back closed 1-currents of the form du when u is a continuous function; by the same construction, we can also pull back 1-currents of the form $d^c u$ when u is a continuous function. Having both the operators d and d^c gives us the following generalization of the construction in (A).

(A′) If $f : X \to Y$ is any analytic mapping of analytic manifolds, and φ is a closed $(1,1)$-current on Y which can be written $dd^c u$ for some continuous function u, we can define $f^*\varphi = dd^c f^* u$. This is well defined since $f^* u$ is continuous and unique up to a pluri-harmonic function by Weyl's lemma. Again, with this definition, $f^*\varphi$ clearly coincides with the previous definition above the regular values.

This construction applies in particular to *positive* closed (1,1)-currents which can be written $dd^c u$ for a pluri-subharmonic function u. This is the case in which we will be most interested.

For differentiable mappings, this construction does not seem to extend to currents of higher degree. In the complex case, we saw above that wedge products of positive (1,1)-currents can be defined, and using this we can further set

$$f^*(dd^c u_1 \wedge \cdots \wedge dd^c u_m) = dd^c f^* u_1 \wedge \cdots \wedge dd^c f^* u_m$$

when $u_1, \ldots, u_m$ are continuous pluri-subharmonic functions.

(B′) Just as in the differentiable case, if X and Y are of dimension n and $f : X \to Y$ is proper with finite fibers, we can define the pull-back of (n,n)-currents of order 0, i.e., of measures.

Both constructions are now defined when we pull-back an (n,n)-current which can be written

$$\omega = dd^c u_1 \wedge \cdots \wedge dd^c u_n$$

where $u_1, \ldots, u_n$ are continuous pluri-subharmonic function on Y, and more particularly when X and Y are Riemann surfaces and $f : X \to Y$ is a ramified covering map. Again we leave the verification that the two constructions coincide to the reader.

An example. We will need the following computation of a Monge-Ampère operator in Section 6. It is given as a remark in [BT, Cor. 2.5]; the authors think it may be useful to include some details.

Let U be open in $\mathbf{C}^2$, and f and g be two analytic functions on U, with linearly independent derivatives. Set $u = \sup\{\operatorname{Re} f, \operatorname{Re} g, 0\}$.

Lemma 3.5.

(a) *The current* $dd^c u \in \mathscr{D}_{1,1}$, *evaluated on the test-form* $\varphi \in \mathscr{D}^{1,1}$, *gives*

$$\langle dd^c u, \varphi \rangle = \int_{-\infty}^{\infty} \left(\int_{f = g + it, \operatorname{Re} f \geq 0} \varphi \right) dt + \int_{-\infty}^{\infty} \left(\int_{f = it, \operatorname{Re} g \leq 0} \varphi \right) dt$$
$$+ \int_{-\infty}^{\infty} \left(\int_{g = it, \operatorname{Re} f \leq 0} \varphi \right) dt$$

(b) *The current* $(dd^c u)^2 \in \mathscr{D}_{2,2}$, *evaluates on the test function* $\varphi \in \mathscr{D}^{0,0}$, *gives*

$$\langle (dd^c u)^2, \varphi \rangle = \int_{\operatorname{Re} f = \operatorname{Re} g = 0} \varphi \, d(\operatorname{Im} f) \wedge d(\operatorname{Im} g),$$

where the surface $\operatorname{Re} f = \operatorname{Re} g = 0$ *is oriented so that* $d(\operatorname{Im} f) \wedge d(\operatorname{Im} g)$ *is a positive 2-form.*

Proof. Without loss of generality, we may use f and g as coordinates, i.e., set $f = x = x_1 + ix_2$ and $g = y = y_1 + iy_2$. Divide the plane $\operatorname{Re} x, \operatorname{Re} y$ into three regions V_1, V_2 and V_3 as shown, and let U_1, U_2 and U_3 be the corresponding regions of C^2.

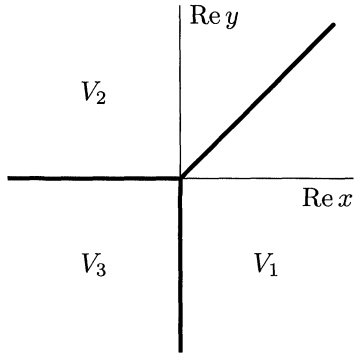

FIGURE 1

We can write

$$\langle dd^c u, \varphi\rangle = \int_{U_1} x_1\, dd^c\varphi + \int_{U_2} y_1\, dd^c\varphi + \int_{U_3} (0)\, dd^c\varphi.$$

We will transform these integrals by integration by parts in the standard way, to put the dd^c on the function u; this term will vanish since u is pluriharmonic on each of the U_i, leaving just the boundary terms.

Since

$$d(x_1 d^c\varphi) = dx_1 \wedge d^c\varphi + x_1 dd^c\varphi,$$

we find by Stokes' theorem

$$\int_{U_1} x_1\, dd^c\varphi = -\int_{U_1} dx_1 \wedge d^c\varphi + \int_{\partial U_1} x_1\, d^c\varphi.$$

After writing $dx_1 \wedge d^c\varphi$ in terms of ∂ and $\bar\partial$, noting that there are no forms of type $(3,1)$ or $(1,3)$ on $\mathbf{C}^2$, and using $d^c x_1 = dx_2$, we get

$$-\int_{U_1} dx_1 \wedge d^c\varphi = \int_{U_1} d^c x_1 \wedge d\varphi = \int_{U_1} dx_2 \wedge d\varphi = -\int_{U_1} d(dx_2 \wedge \varphi);$$

integrating by parts again gives

$$-\int_{U_1} dx_1 \wedge d^c\varphi = -\int_{\partial U_1} dx_2 \wedge \varphi,$$

and there is a similar terms for U_2.

Now let $X_{i,j} = \partial U_i \cap \partial U_j$, oriented as the boundary of U_i.

The formula above gives

$$\langle dd^c u, \varphi\rangle = \int_{X_{1,2}} \big((x_1 - y_1) d^c\varphi - dx_2 \wedge \varphi\big) \\ + \int_{X_{1,3}} (x_1 d^c\varphi - dx_2 \wedge \varphi) + \int_{X_{2,3}} (y_1 d^c\varphi - dy_2 \wedge \varphi).$$

In each of the integrals above, the first term vanishes, so the sum can be rewritten

$$\langle dd^c u, \varphi\rangle = \int_{-\infty}^{\infty} \left(\int_{y=x+it, \mathrm{Re} x \geq 0} \varphi\right) dt + \int_{-\infty}^{\infty} \left(\int_{y=it, \mathrm{Re} x \leq 0} \varphi\right) dt \\ + \int_{-\infty}^{\infty} \left(\int_{x=it, \mathrm{Re} y \leq 0} \varphi\right) dt.$$

This proves (a).

To prove (b), we need to evaluate

$$\langle dd^c u, u dd^c\varphi\rangle = \int_{-\infty}^{\infty} \left(\int_{y=x+it, \mathrm{Re} x \geq 0} x_1\, dd^c\varphi\right) dt \\ + \int_{-\infty}^{\infty} \left(\int_{x=it, \mathrm{Re} y \leq 0} x_1\, dd^c\varphi\right) dt + \int_{-\infty}^{\infty} \left(\int_{y=it, \mathrm{Re} x \leq 0} y_1\, dd^c\varphi\right) dt.$$

The last two terms vanish, since u is zero there. To understand the first term, it is easier to switch to the coordinates $v = y + x$, $w = y - x$.

To integrate the first term by parts, we find it easier to pass entirely to real notation. Parametrizing our domain of integration by v_2, w_1 and w_2, we can write our integral

$$\int_{-\infty}^{\infty} \left(\int_{w=it, v_1 \geq 0} \frac{v_1}{2}\, dd^c\varphi\right) dw_2.$$

Since dw_1 vanishes on the domain of integration, and there already is a term in dw_2, only the partials of φ with respect to v_1 and v_2, contribute to the integral. Remembering that in one dimension we have

$$dd^c f = \Delta f\, dx \wedge dy$$

this gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_0^{\infty} \frac{v_1}{2} \left(\frac{\partial^2\varphi}{\partial v_1^2} + \frac{\partial^2\varphi}{\partial v_2^2}\right) dv_1\right) dv_2\, dw_2.$$

If we integrate with respect to v_2 first, we see that the term $\partial^2\varphi/\partial v_2^2$ integrates to 0, and the term $x_1(\partial^2\varphi/\partial v_1^2)$, integrated by parts once and then simply integrated, gives

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(iv_2, iw_2)\, dv_2\, dw_2.$$

Finally, note that $dv_2\, dw_2 = 2\, dx_2\, dy_2$. □

4. The homogeneous case in dimension 2. Let $F : \mathbf{C}^2 \to \mathbf{C}^2$ be a non-degenerate homogeneous polynomial map of degree $k > 1$, and let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the associated rational function.

We will say that a subset $X \subset \mathbb{P}^1$ is f-exceptional if $\operatorname{card}\big(f^{-n}(X)\big)$ is bounded. Such a subset exists only in the two following cases:

(a) the mapping f is conjugate to a polynomial and X corresponds to the point at ∞ under the conjugacy;
(b) the mapping f is conjugate to $z \mapsto z^m$ for some m with $m \neq -1$, 0, 1 and X corresponds to a subset of $\{0, \infty\}$ under the conjugacy.

A measure on $\mathbb{P}^1$ will be called f-exceptional if it assigns positive measure to an f-exceptional set.

Further let $\pi : \mathbf{C}^2 - \{\mathbf{0}\} \to \mathbb{P}^1$ be the canonical projection.

Theorem 4.1.

(a) *For any non-exceptional probability measure μ, the sequence of measures*

$$\mu_m = \frac{1}{k^m}(f^m)^*\mu$$

converges to a measure μ_f independent of μ

(b) *The function h_F satisfies the identity*

$$\frac{1}{2\pi} dd^c h_F = \pi^* \mu_f,$$

where π^* denotes the transpose of integration over the fiber (cf. Section 3). In particular, h_F is pluriharmonic except on $\pi^{-1}J_f$.

Proof. This result is easy to prove for measures μ which are Laplacians of bounded functions; the general case is a bit fussier.

First set

$$\mu_0 = \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

to be the ordinary element of area on the sphere, normalized to area 1. Observe that

$$dd^c \frac{1}{2\pi}\log(\sqrt{|x|^2+|y|^2}) = \pi^*\mu_0.$$

If we set

$$h_m = \frac{1}{k^m}\log\|F^{\circ m}\| = \frac{1}{k^m}(F^{\circ m})^*(\log\|\ \|),$$

we see that

$$\begin{aligned}\frac{1}{2\pi} dd^c h_m &= \frac{1}{2\pi}\frac{1}{k^m}(F^{\circ m})^*(dd^c \log\| \ \|)\\ &= \frac{1}{k^m}(F^{\circ m})^*(\pi^*\mu_0)\\ &= \frac{1}{k^m}\pi^*(f^{\circ m})^*\mu_0 = \pi^*\mu_n.\end{aligned}$$

Since the sequence $dd^c h^m$ converges in $\mathscr{D}_{1,1}(\mathbf{C}^2 - \{0\})$, the result follows from Corollary 3.3.

Now suppose that ν is a probability measure on $\mathbb{P}^1$, with the property that there exists a bounded measurable function u on $\mathbb{P}^1$ with

$$dd^c u = \nu - \mu_0.$$

It isn't very difficult to see that such measures are exactly those which assign measure 0 to all sets of capacity 0, but we will not need to know this; all we will need is that smooth probability measures belong to this class.

The function

$$h_{\nu,0}(\mathbf{x}) = \pi^* u(\mathbf{x}) + \frac{1}{2\pi}\log\|\mathbf{x}\|$$

satisfies $dd^c h_{\nu,0} = \pi^*\nu$, and so the same argument as above can be used to show that

$$h_{\nu,m}(\mathbf{x}) = \frac{1}{k^m} h_{\nu,0}\big(F^{\circ m}(\mathbf{x})\big)$$

and

$$\nu_m = \frac{1}{k^m}(f^{\circ m})^*\nu$$

are related by $dd^c h_{\nu,m} = \pi^*(\nu_m)$. By Corollary 2.2, we see that the result follows as above.

Extending this result to arbitrary non-exceptional measures is a bit more elaborate. An argument analogous to the one below occurs in [L] and [FLM].

We need to show that a non-exceptional measure μ is a limit of smooth measures ν_n such that

$$\lim_{n\to\infty}\lim_{m\to\infty}(f^{\circ m})^*(\nu_n) = \lim_{m\to\infty}\lim_{n\to\infty}(f^{\circ m})^*(\nu_n).$$

Indeed, the left-hand side is μ_f, whereas the right-hand side the limit of the pull-backs of μ.

From the standard formula to construct an inverse of the Laplacian by convolving with $\log|z|$, we see that it is enough to consider the Dirac measure at a single non-exceptional point.

We will derive the result from the following lemma.

Lemma 4.2. *For any non-exceptional point $z \in \mathbb{P}^1$ and any $\varepsilon > 0$, there is a neighborhood U of z such that for all $n > 0$, all but εk^n of the k^n inverse images of z, counted with multiplicity, are contained in components of $f^{-n}(U)$ of diameter less than ε.*

Proof of Theorem 4.1 from Lemma 4.2. Choose a sequence $\varepsilon_n \to 0$, a sequence of neighborhoods U_n of z with diameters tending to 0 and satisfying the condition of Lemma 4.2. Then choose smooth probability measures ν_n with support in u_n, which converge to the Dirac measure δ_z at z.

Moreover, consider a continuous "test" function φ on $\mathbb{P}^1$; this function is uniformly continuous, so there exists a sequence $\eta_n \to 0$ such that $|\varphi(\zeta_1) - \varphi(\zeta_2)| < \eta_n$ when $d(\zeta_1, \zeta_2) < \varepsilon_n$.

Then from Lemma 4.2, we have

$$\left|\left\langle \left((f^{\circ m})^* \nu_n - (f^{\circ m})^* \delta_z\right), \varphi \right\rangle\right| \leq \varepsilon_n \sup \varphi + \delta_n.$$

The first term on the right-hand side is contributed by the components of $(f^{\circ m})^{-1}(U_n)$ with diameter greater than ε_n, which have total measure at most ε_n. The second term comes from the other components, in which φ is almost constant since their diameter is small.

Passing to the limit as $m \to \infty$, we get

$$\lim_{m \to \infty} \left|\left\langle \left((f^{\circ m})^* \nu_n - (f^{\circ m})^* \delta_z\right), \varphi \right\rangle\right| \leq \varepsilon_n \sup \varphi + \delta_n,$$

and now taking a limit as $n \to \infty$,

$$\lim_{n \to \infty} \lim_{m \to \infty} (f^{\circ m})^* \nu_n = \lim_{m \to \infty} (f^{\circ m})^* \delta_z = \lim_{m \to \infty} \lim_{n \to \infty} (f^{\circ m})^* \nu_n.$$

Proof of Lemma 4.2. Let μ be the Dirac mass at z, $\mu_n = (f^n)^* \mu$, which is a measure supported on $X_n = f^{-n}\{z\}$, and for any neighborhood W of z and $x \in X_n$ let $W_n(x)$ be the component of $f^{-n}(W)$ containing x.

No point of $f^{-3}(z)$ has weight greater than $(k-1)/k$, and it follows that for any $\delta > 0$, there exists m such that no point of X_m has weight greater than δ. Choose a neighborhood V of z such that for all $x \in X_m$, the set $V_m(x)$ is simply connected and contains no other element of X_m.

Further choose for each $x \in X_m$ two concentric round discs $\Delta'(x) \subset \Delta''(x) \subset V_m(x)$ centered at x with the radius of $\Delta'(x)$ half the radius of $\Delta''(x)$, and let

$$U = \bigcap_{x \in X_m} f^m(\Delta'(x)).$$

We need to show that for δ sufficiently small, the set U satisfies the conditions of the lemma, i.e., that for all $n \geq m$, the subset of X_n contained in components of $f^{-n}(U)$ of diameter at most ε has weight at least $1-\varepsilon$ under μ_n.

Define recursively $Z_0 = X_m$, $Z_i = Z_i' \cup Z_i''$ by

$$Z_i' = \{x \in Z_i \mid V_{m+i}(x) \text{ contains a critical value}\},$$

$Z_i'' = Z_i - Z_i'$ and $Z_{i+1} = f^{-1}Z_i''$.

For any $x \in Z_i$, we have

- $\mu_{m+i}(\{x\}) \leq \delta/k^i$;
- the set $U_{m+i}(x)$ is homeomorphic to a disc, and there is a univalent branch

$$g_x : U_m\big(f^i(x)\big) \to U_{m+i}(x)$$

of f^{-i}.

Since there are at most $2k-2$ critical values, we have $\operatorname{card}(Z_i') \leq 2k-2$ for all i, so an argument using a geometric series shows that

$$\mu_{m+i}(Z_i) \geq 1 - 2k\delta.$$

At most $1/\varepsilon^2$ points of Z_i have neighborhoods of area greater than ε^2, and these together have mass at most $\delta/(\varepsilon^2 k^i)$.

Finally, by the Koebe distortion theorem, the sets $g_x\big(\Delta'(f^i(x))\big)$ are approximately round discs, in the sense that their diameters are bounded by a fixed constant times the square root of the area. □

5. Invariant forms in several dimensions. One nice feature of the proof above is that it generalizes almost word for word to arbitrary dimensions, giving an analog of the Brolin measure for endomorphisms f of $\mathbb{P}^n$. The theorem is somewhat weaker than when $n = 1$, in the sense that it only applies to reasonably nice forms. In fact, we think that this is the right attitude: there is an "easy" theorem which says that given a smooth $(1,1)$-form ω , the sequence of $(1,1)$-forms

$$\omega_m = \frac{1}{k^m}(f^m)^*\omega$$

converges to an f-invariant $(1,1)$-current ω_f independent of ω. The issue of what $(1,1)$-currents ω are exceptional, in the sense that they dont satisfy this is a hard problem, best considered later. The determination of which measures are exceptional in dimension 1 required the Koebe distortion lemma, of which no analog exists in higher dimensions, so it seems likely that the determination of the exceptional currents may prove difficult even if $n = 2$.

Let $F : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ be homogeneous of degree $k \geq 2$ and non-degenerate.

Theorem 5.1.

(a) *For any smooth closed* $(1,1)$*-form* ω *on* $\mathbb{P}^n$*, representing the positive generator of* $H^1(\mathbb{P}^n,\mathbf{Z})$*, the sequence of* $(1,1)$*-forms,*

$$\omega_m = \frac{1}{k^m}(f^m)^*\omega,$$

converges to a $(1,1)$*-current* ω_f *independent of* ω*, and satisfying* $f^*\omega_f = k\omega_f$*.*

(b) *The function* h_F *satisfies the identity*

$$\frac{1}{2\pi}dd^c h_F = \pi^*\omega_f.$$

Remark 5.2. The pullback $f^*\omega_f$ was defined in Section 3, in the subsection on pullbacks by analytic mappings. This is what we mean when we say that ω_f is f-invariant. It follows that $f_*\omega_f = k^{n-1}\omega_f$, but this is a weaker statement. The following computation explains why the coefficients are what they are.

A closed (p,q)-current on an n-dimensional compact complex manifold X determines both an element of $H^{p+q}(X,\mathbf{C})$ and an element of $H_{2n-(p+q)}(X,\mathbf{C})$, which is reasonable since these spaces are isomorphic by Poincaré duality. The more natural identification is with the homology group, since the very natural push-forward of currents induces the natural homomorphism in homology.

A mapping $f : \mathbb{P}^n \to \mathbb{P}^n$ given by homogeneous polynomials of degree k is of topological degree k^n. The cohomology of $\mathbb{P}^n$ is

$$H^*(\mathbb{P}^n,\mathbf{Z}) = \mathbf{Z}[x]/(x^{n+1}),$$

where x is the element of $H^2(\mathbb{P}^n,\mathbf{Z}) \equiv H_{2n-2}(\mathbb{P}^n,\mathbf{Z})$ corresponding to the homology class of a hyperplane. The statement about the degree says that $f^*x^n = k^n x^n$, so that $f^*x = kx$. Hence the mapping

$$f^* : H^{2p}(\mathbb{P}^n,\mathbf{Z}) \to H^{2p}(\mathbb{P}^n,\mathbf{Z})$$

is multiplication by k^p, and by duality, so is

$$f_* : H_{2p}(\mathbb{P}^n,\mathbf{Z}) \to H_{2p}(\mathbb{P}^n,\mathbf{Z}).$$

Our $(1,1)$-current ω_f represents either the positive generator of $H_{2n-2}(\mathbb{P}^n,\mathbf{Z})$ or the positive generator of $H^2(\mathbb{P}^n,\mathbf{Z})$. Thinking of it as a homology class, the only way it can be "invariant" is to require $f_*\omega_f = k^{n-1}\omega_f$, whereas as a cohomology class the appropriate invariance is $f^*\omega_f = k\omega_f$.

Proof. First consider a hermitian inner product on $\mathbf{C}^{n+1}$, and let ω_0 be the associated Kähler 2-form on $\mathbb{P}^n$, satisfying $\pi^*\omega_0 = (1/2\pi)dd^c \log\| \ \|$ (see [C]). Then we can write

$$\pi^* \frac{1}{k^m}\big((f^{\circ m})^*\omega_0\big) = (F^{\circ m})^* \frac{1}{k^m}\pi^*\omega_0 = \frac{1}{2\pi}(F^{\circ m})^* \frac{1}{k^m}\, dd^c \log\| \ \|$$
$$= \frac{1}{2\pi}\, dd^c \frac{1}{k^m} \log\|F^m\|.$$

Since $1/k^m \log\|F^m\|$ converges to h_F uniformly on compact subsets, the last term converges as a sequence of currents to $(1/2\pi)dd^c h_F$ as $m \to \infty$. Using 3.3, this shows that Theorem 5.1 is true if ω comes from a hermitian metric.

The case of a general ω requires the following lemma.

Lemma 5.3. *If ω is an arbitrary closed $(1,1)$-form on $\mathbb{P}^n$ representing c_1, there exists a smooth function u on $\mathbb{P}^n$ such that*

$$\omega - \omega_0 = dd^c u.$$

Proof. This is standard Hodge theory; a proof using elliptic operators appears as Proposition 7.1 of [KM]. It can also be proved as follows. First observe that by the Poincaré and Dolbeault lemmas any closed $(1,1)$-form ω can be written locally $\omega = dd^c u$ [C, Chap. 7, (C)]. Take a covering $\mathscr{U}$ of $\mathbb{P}^n$ such that on each $U_i \in \mathscr{U}$, we can write $\omega - \omega_0 = dd^c u_i$; the functions $u_{i,j} = u_i - u_j$ now form a 1-cocycle with values in the sheaf $\mathscr{H}$ of germs of harmonic functions.

Consider the exact sequence of sheaves

$$0 \to \mathbf{R} \to \mathscr{O} \to \mathscr{H} \to 0,$$

given by the inclusion of the locally constant purely imaginary functions, and by taking the real part. The corresponding exact sequence gives in part

$$\cdots \to H^1(\mathbb{P}^n, \mathbf{R}) \to H^1(\mathbb{P}^n, \mathscr{O}) \to H^1(\mathbb{P}^n, \mathscr{H}) \to H^2(\mathbb{P}^n, \mathbf{R}) \to \cdots.$$

Both of the first two terms vanish in our case (for a general Kähler manifold, the map between them is surjective). The map $H^1(\mathbb{P}^n, \mathscr{H}) \to H^2(\mathbb{P}^n, \mathbf{R})$ gives the first Chern class.

This shows that the 1-cocycle $u_{i,j}$ is a coboundary, so that refining the cover if necessary, there exist harmonic functions v_i on U_i with $v_i - v_j = u_{i,j}$. Now the function u given on each U_i by $u = u_i - v_i$ is a global function with $dd^c u = \omega - \omega_0$. □

We can now finish the proof of Theorem 5.1 as follows. Let $v = \pi^* u = u \circ \pi$, so that v is a bounded function, and set

$$\pi^* \frac{1}{k^m}(f^{\circ m}\omega) = (F^{\circ m})^* \frac{1}{k^m}\pi^*\omega = (F^{\circ m})^* \frac{1}{k^m}\left(dd^c(\log \| \ \| + v)\right)$$

$$= dd^c\left(\frac{1}{k^m}\log\|F^m\| + \frac{1}{k^m} v \circ F^m\right).$$

The term $dd^c(1/k^m \log\|F^m\|)$ converges to $dd^c h_F$, and $dd^c(1/k^m v \circ F^m$ converges to 0 since v is bounded. The result follows from Lemma 3.4. □

This result allows us to define a sequence of invariant forms. According to [CLN] (there is a detailed discussion in [BT]), each of the sequences

$$\lim_{m\to\infty} \frac{1}{k^{pm}} (dd^c \log\|F^{\circ m}\|)^p$$

converges to closed (p,p)-current. As above, we have

$$\pi^* \frac{1}{k^{pm}} (f^{\circ m})^* \omega_0^p = \frac{1}{k^{pm}} (dd^c \log\|F^{\circ m}\|)^p,$$

and by Proposition 3.3, we see that the sequence of forms

$$\frac{1}{k^{pm}} (f^{\circ m})^* \omega_0^p$$

converges to a closed positive (p,p)-current, which we will call ω_f^p. This current is $f-invariant$, in the sense that

$$f^*\omega_f^p = k^p \omega_f^p.$$

It also satisfies $f_*\omega_f^p = k^{(n-p)}\omega_f^p$, which again is a strictly weaker statement. For the definition of the pull-back, see Section 3, subsection on pull-backs by analytic maps; for the coefficients, see above the discussion in Remark 5.2. Note that these forms do not vanish for $p \le n$, since ω_f^p represents the positive generator of $H^{2p}(\mathbb{P}^n, \mathbf{Z})$. This is of particular interest when $p = n$, and ω_f^n is a measure, f-invariant in the sense that $f_*\omega_f^n = \omega_f^n$.

Define $J_f^p = \operatorname{supp}(\omega_f^p)$. Clearly

$$J_f^1 \supset J_f^2 \supset \cdots \supset J_f^n.$$

Further we define the Julia set J_f to be the set of points such that f is not normal on any neighborhood.

Proposition 5.4. *We have* $J_f \supset J_f^1$.

Proof. Choose $\mathbf{x} \in \mathbb{P}^n - J_f$, and a neighborhood V of $\mathbf{x}$ such that the subsequence $f^{\circ m_i}$ is uniformly convergent on V, converging to g. By shrinking V if necessarily, we may choose a norm $\| \ \|$ such that $\log \| \ \|$ is pluriharmonic on $\pi^{-1} g(V)$. Thus on $\pi^{-1}(V)$, the sequence

$$\frac{1}{k^{m_i}} \log \| F^{\circ m_i} \|$$

is a sequence of functions, uniformly convergent on compact subsets, and eventually pluriharmonic on every compact set. Thus h_F is pluriharmonic on $\pi^{-1}(V)$, and hence ω_f vanishes on V by Theorem 5.1 (b). □

Problems. Is the inclusion $J_f^1 \subset J_f$ an equality?

Is ω_f^n the unique measure of maximal entropy invariant under f?

Another natural problem is the following: if η is an arbitrary smooth (p,p)-form on $\mathbb{P}^n$ representing c_1^p, then does the sequence $(f^{\circ m})^* \eta$ converge to ω_f^p? We only know something about this problem when $p = n$, i.e., when studying measures. Reasoning exactly as above, we consider a smooth probability measure μ, and write $\mu - \omega_0^n = (dd^c u)^n$. The fact that this is possible is guaranteed by the Calabi conjecture proved by Yau [Y]. To complete the proof, we need to show the convergence of the sequence

$$\left(dd^c \frac{1}{k^m} \log \| F^{\circ m} \| + dd^c \frac{1}{k^m} u \circ \pi \circ F^{\circ m} \right)^n .$$

As far as we know, this sequence may fail to converge if u is not plurisubharmonic: the inequality of [CLN] only holds for plurisubharmonic functions.

6. Two examples. There is a drastic shortage of examples of endomorphisms of $\mathbb{P}^n$ for $n \geq 2$. We present in this section two examples; the second really just shows that the constructions of [BS], [FS] and [HO] about Hénon mappings can also be understood in the present context. Fornaess and Sibony have recently found examples where the critical locus is strictly preperiodic, and have shown that in that case the Julia set is all of $\mathbb{P}^n$.

Example 1. Let p_1, p_2 be 2 monic polynomials of degree k, with Julia sets J_i, filled in Julia sets K_i, potentials G_i. Further let $\tilde{p}_1(x,z)$ and $\tilde{p}_2(y,z)$ be the homogenizations of p_1 and p_2.

Consider the mapping

$$F : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \tilde{p}_1(x,z) \\ \tilde{p}_2(y,z) \\ z^k \end{bmatrix}$$

and the induced map $f : \mathbb{P}^2 \to \mathbb{P}^2$. Of course, the mapping f is just

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p_1(x) \\ p_2(y) \end{bmatrix}$$

in the finite plane.

Proposition 6.1.

(a) *The function h_F is given by*

$$h_F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \sup\{G_1(x/z) + \log|z|, G_2(y/z) + \log|z|\}.$$

(b) *The restriction of ω_f to the finite plane is $1/2\pi\, dd^c \sup\{G_1(x), G_2(y)\}$.*

Proof. The formula for F gives immediately

$$F^{\circ m}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z^{k^m} p_1^{\circ m}(x/z) \\ z^{k^m} p_2^{\circ m}(y/z) \\ z^{k^m} \end{bmatrix}.$$

Using the sup norm, this gives

$$h_F = \sup\left\{\log|z| + \frac{1}{k^m}\log\left|P_1^{\circ m}\left(\frac{x}{z}\right)\right|, \log|z| + \frac{1}{k^m}\log\left|P_2^{\circ m}\left(\frac{y}{z}\right)\right|\right\},$$

which clearly converges to the formula (a). For part (b), it is enough to set $z = 1$. □

Suppose that K_1 and K_2 are connected, and let $\varphi_i : \mathbf{C} - K_i \to \mathbf{C} - \bar{D}$ be the corresponding Böttcher coordinates. Let X_ϑ be the Riemann surface of equation $\varphi_1(x) = e^{i\vartheta}\varphi_2(y)$.

Proposition 6.2.

(a) *The restriction of ω_f to the finite plane is given more explicitly by*

$$\langle \omega_f, \alpha\rangle = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{X_\vartheta}\alpha\right)d\vartheta + \int_{J_1}\left(\int_{\{x\}\times K_2}\alpha\right)\mu_1(dx)$$
$$+ \int_{J_2}\left(\int_{K_1\times\{y\}}\alpha\right)\mu_2(dy).$$

(b) *The current ω_f^2 is $\mu_1 \times \mu_2$.*

Proof. Set

$$G_\varepsilon = \sup\{G_1(x), G_2(y), \varepsilon\}.$$

Clearly

$$\sup\{G_1(x), G_2(y)\} = \lim_{\varepsilon\to 0} G_\varepsilon = \lim_{\varepsilon\to 0} \sup\{\operatorname{Re}\log\varphi_1(x), \operatorname{Re}\log\varphi_2(y), \varepsilon\},$$

where any branch of the logarithm can be used, since the result is local. Using Lemma 3.5 (a), we see that

$$\begin{aligned}\langle dd^c G_\varepsilon, \eta\rangle = &\int_0^{2\pi} \left(\int_{\varphi_1(x)=e^{it}\varphi_2(y), \log|\varphi_1(x)|\geq\varepsilon} \eta \right) dt \\ &+ \int_0^{2\pi} \left(\int_{\varphi_1(x)=e^{\varepsilon+it}, \log|\varphi_2(y)|\leq\varepsilon} \eta \right) dt \\ &+ \int_0^{2\pi} \left(\int_{\varphi_2(y)=e^{\varepsilon+it}, \log|\varphi_1(x)|\leq\varepsilon} \eta \right) dt.\end{aligned}$$

Since the Brolin measure on J_i is also the harmonic measure, clearly the three terms above tend to the terms given in the statement 6.2 (a) (except for the factor of 2π).

For part (b), note first that $(dd^c G_\varepsilon)^2$ tends to $(dd^c G)^2$, since the convergence is uniform. We have as above, applying Lemma 3.5 (b), that

$$\langle (dd^c G_\varepsilon)^2, \eta\rangle = \int_0^{2\pi}\int_0^{2\pi} \eta(\varphi_1^{-1}(e^{it_1}), (\varphi_2^{-1}(e^{it_2}))\, dt_1\, dt_2.$$

Again, this tends (except for a factor of $(2\pi)^2$, to the product of the Brolin measures. □

Remark. The form $1/(2\pi)\, dd^c G$ represents the fundamental class c_1 of of $\mathbb{P}^2$, and the product of the Brolin measures represents c_1^2.

Example 2. In this example we will freely use the notation of [HO]. Let $p(x) = x^k + q(x)$ be a monic polynomial of degree k. The Hénon mapping $f : \mathbf{C}^2 \to \mathbf{C}^2$ given by

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) - ay \\ x \end{bmatrix}$$

does not extend to an endomorphism of $\mathbb{P}^2$, but if we modify it slightly to

$$f_\varepsilon : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) - ay \\ x + \varepsilon y^k \end{bmatrix},$$

then it does, and in fact if we use the slope $s = y/x$ as a coordinate on the line at ∞, then the induced mapping is just $s \mapsto \varepsilon s^k$.

From [HO], define

$$V_+ = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; |y| \le |x| \text{ and } |x| \ge \alpha \right\}, \quad \text{and}$$

$$W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; |x| \le \alpha \text{ and } |y| \le \alpha \right\},$$

and set $W_+ = W \cup V_+$. The following figure describres these subsets.

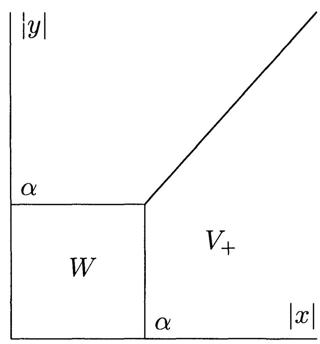

FIGURE 2

Then $f_\varepsilon(W_+) \subset W_+$ when α is sufficiently large and ε is sufficiently small. Since the leading term of the first coordinate of f is x^k and the leading term of the second coordinate is εy^k, the first term is bigger than the second in V_+ if α is sufficiently large and ε sufficiently small. On the other hand, in [HO] we show that if we choose α at least as large as the largest root of $|x|^k - |q(x)| - (|a| + 2) = 0$, then

$$f_{\mathbf{0}}(\bar{W}) \subset W_+,$$

and since $\bar{W}$ is compact, f_ε is uniformly close to f_0 on W when ε is small, so we will still have

$$f_\varepsilon(\bar{W}) \subset W_+.$$

So we can choose α large enough and ε small enough that both inclusions are true.

The reader should check that when α is chosen as above the function

$$G_+ \begin{bmatrix} x \\ y \end{bmatrix} = \lim_{n \to \infty} \frac{1}{k^n} \log_+ \left\| f_\varepsilon^{\circ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\|$$

exists and is pluri-subharmonic in W_+.

Proposition 6.4.

(a) *The functions* G_+ *and* h_{F_ε} *are related by the formula*

$$h_{F_\varepsilon} = G_+ \circ \pi + \log|z|$$

on $\pi^{-1}(W_+)$.

(b) *On* W_+ *we have* $\omega_{f_\varepsilon} = \dfrac{1}{2\pi} dd^c G_+$.

Proof. The mapping F_ε is given by the formula

$$F_\varepsilon\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \tilde{P}(x,z) - ayz^{k-1} \\ xz^{k-1} + \varepsilon y^k \\ z^k \end{bmatrix} = z^k \begin{bmatrix} P(x/z) - ay/z \\ x/z + \varepsilon(y/z)^k \\ 1 \end{bmatrix},$$

so that

$$F_\varepsilon^{\circ n}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = z^{k^n} \begin{bmatrix} f_\varepsilon^{\circ n}(x/z, y/z) \\ 1 \end{bmatrix}.$$

Taking logarithms, we find

$$\frac{1}{k^n} \log \left| F_\varepsilon^{\circ n}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right| = \log|z| + \frac{1}{k^n} \log \left\| \begin{bmatrix} f_\varepsilon^{\circ n}(x/z, y/z) \\ 1 \end{bmatrix} \right\|.$$

The second term is clearly $\log_+ \|f_\varepsilon^{\circ n}(\pi(x,y,z))\|$. This proves (a).

On $\pi^{-1}(W_+)$, we have $z \neq 0$, so that

$$dd^c h_\varepsilon = \pi^* dd^c G_+$$

and the result follows. □

This shows that on W_+, the current ω_{f_ε} corresponds to the current μ_+ [BS1, BS2, BS3, FS]. In the paper [BS3], the authors prove that $\mu_+^2 = 0$, so that $\omega_{f_\varepsilon}^2$ vanishes on W_+. It is quite possible for a Hénon mapping, hence f_ε, to have a repelling periodic point in W_+, and this show that the repelling periodic points are not necessarily in the support of ω_f^n, which the first example might have suggested.

7. The Basin of attraction and its boundary. Let $F : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ be a non-degenerate homogeneous mapping of degree k as above. Then the origin is a super-attracting fixed point; let Ω_F be its basin. We will frequently need to consider sets of the form $\pi^{-1}X$ where $X \subset \mathbb{P}^n$; we will denote $CX = \pi^{-1}X$, and speak of the cone over X.

Everything in the statement below is obvious; however, the Levi form is usually used only for domains with C^2 boundaries, and our use of the notion is unconventional. Of course, it corresponds formally to the standard definition of the Levi form [H], [G], and we give below in Proposition 7.2 a geometric justification for this usage.

Proposition 7.1.

(a) *The domain Ω_F is circled: if $\mathbf{x} \in \Omega_F$ and $\alpha \in \mathbf{C}$ satisfies $|\alpha| \leq 1$, then $\alpha\mathbf{x} \in \Omega$. In particular, Ω_F is contractible.*

(b) *We have*

$$\Omega_F = \{\mathbf{x} \in \mathbf{C}^{n+1} \mid h_F(\mathbf{x}) < 0\},$$

in particular, Ω_F is pseudo-convex.

(c) *The "Levi current" of $\partial\Omega_F$ is $-(i\pi)\pi^*\omega_f$, and since h_F is pluriharmonic on $\mathbf{C}^{n+1} - CJ_f^1$, the part of the boundary of Ω_F which does not lie above the Julia set must be foliated by Riemann surfaces.*

To justify our use of the term "Levi current" in this setting, where the boundary is not of class C^2, we will show that it has an analogous geometric interpretation to the Levi-form. If $X \subset \mathbf{C}^{n+1}$ is a C^1 real hypersurface, then it naturally carries an n-dimensional complex differential system. This means that every real vector space T_xX (of real dimension $2n+1$) contains a complex n-dimensional subspace, namely $T_xX \cap iT_xX$. The Levi form is the *curvature* of this system; it measures the non-integrability of the system. In particular, X is foliated by n-dimensional complex leaves if and only if the Levi form vanishes identically.

In our setting, Ω_F doesn't have a smooth boundary $X = \partial\Omega_F$, so that most of the above does not make sense. But $Y = X \cap C(\mathbb{P}^n - J_f^1)$ is smooth, and hence does carry a differential system, in fact integrable. Moreover,

$$\pi : X \to \mathbb{P}^1$$

is a principal circle bundle, and this differential system defines an integrable principal connection on the restriction $\pi|_Y : Y \to \mathbb{P}^1 - J_f^1$. In order for our terminology to be reasonable, the Levi-current should correspond to the curvature of this connection; and in particular, integrals of the curvature should measure the holonomy of the connection, as in the Gauss-Bonnet theorem.

Proposition 7.2. *If $\eta : [0,1] \to \mathbb{P}^n - J_f^1$ is a simple closed curve, and $\tilde{\eta} : [0,1] \to Y$ is a horizontal lift of η, then*

$$\tilde{\eta}(1) = \alpha\tilde{\eta}(0),$$

where

$$\alpha = e^{2\pi i \int_{D_\eta} \omega_f},$$

and $D_\eta \subset \mathbb{P}^1$ is a disc having η as its oriented boundary.

Remark. Since $\mathbb{P}^n$ is simply connected, there exist smooth singular discs bounded by η. For any particular choice of D_η, the integral is well defined, for instance as the limit of the integrals of the forms

$$\frac{1}{k^m}(f^{\circ m})^* \omega_0.$$

Of course, $H_2(\mathbb{P}^n)$ is isomorphic to $\mathbf{Z}$, and the integral does depend on the relative homology class of the disc. But since ω_f represents c_1, two different discs will lead to integrals which differ by an integer, and the exponential will remain unchanged.

Proof. This is actually a familiar statement in an unfamiliar setting. The blow-up $\tilde{\mathbf{C}}^{n+1}$ becomes with the obvious projection a line-bundle $E \to \mathbb{P}^n$, in fact the tautological line-bundle, dual to the hyperplane bundle. Suppose that Ω is a circled domain in $\mathbf{C}^{n+1}$, such that the function

$$N(\mathbf{x}) = \inf\left\{ r > 0 \mid \frac{1}{r}\mathbf{x} \in \Omega \right\}$$

is of class C^2. Then N is the norm for a hermitian metric on E. A standard result [C, 6.2] says that there is a unique connection on E compatible with this metric, and of type (1,0). In any holomorphic trivialization defined by a non-vanishing section $\sigma \in \Gamma(E)$, the connection is given by the 1-form

$$2\frac{\partial(N \circ \sigma)}{N \circ \sigma} = 2\partial h_\sigma,$$

where we have set $h_\sigma = \log(N \circ \sigma)$.

The requirement that the connection be of type (1,0) means precisely that the horizontal subspace at a point $\mathbf{x}_0 \in E$ is the complex hyperplane contained in the tangent space to the $2n+1$ real dimensional manifold of equation $N(\mathbf{x}) =$

$N(\mathbf{x}_0)$. Now again from [C, 6.5], we find that the curvature of this connexion is given by

$$\frac{1}{i\pi}\bar{\partial}\partial h_\sigma,$$

which is clearly a (1,1)-form on $\mathbb{P}^n$, independent of the chosen trivialization. Another way of saying this is to say that there is a (1,1)-form ω on $\mathbb{P}^n$ such that

$$\frac{1}{i\pi}\bar{\partial}\partial h = \pi^*\omega.$$

The holonomy of the connexion around a curve η in $\mathbb{P}^n$ is given by $e^{\int_\eta \partial h_\sigma}$, and the fact that this is independent of the choice of σ is part of the theory of connexions.

The Gauss-Bonnet formula, which in this setting is just Stokes' theorem, says that

$$2\int_\eta \partial h_\sigma = 2\int_{D_\eta} d\partial h_\sigma = 2\int_{D_\eta} \bar{\partial}\partial h_\sigma = 2i\pi\int_{D_\eta}\omega.$$

The argument above does not quite apply on our case, because h_F is not of class C^2, but it can be uniformly approximated by functions of class C^2, which are moreover pluri-harmonic on η. The result follows by passing to the limit. □

Remark. It seems likely that even if a curve in $\mathbb{P}^n$ intersects the Julia set in a Cantor set, there should be a unique continuous lift which is horizontal off this Cantor set. One might hope that Proposition 7.2 will still hold in that case.

8. The Basin of attraction in dimension 2. We will now describe the foliation from part (c) when $n = 1$ and f is a polynomial; in that case there is an entertaining relationship between the Böttcher coordinate at infinity and the leaves of the foliation.

Suppose that $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a polynomial; which we will assume without loss of generality to be monic. In this context, Brolin proved the existence of the Brolin measure using potential theory.

Define the Brolin potential

$$G_f(z) = \lim_{n\to\infty}\frac{1}{k^n}\log_+|f^{\circ n}(z)|,$$

and recall that G_f vanishes identically on

$$K_f = \{z \mid f^{\circ n}(z) \text{ is bounded}\},$$

and is harmonic on $\mathbf{C} - K_f$; it is therefore the Green's function of K_f. The essence of Brolin's proof is that $\mu_f = (1/2\pi)dd^cG_f$, so that μ_f has a potential-theoretic interpretation on $\mathbb{P}^1$, not just on $\mathbf{C}^2$.

Since this result and Theorem 4.1 both use potential theory, we can expect that there is a close relationship, which is spelled out in the next proposition.

Proposition 8.1. *We have*

$$h_F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = G_f\left(\frac{x}{y}\right) + \log|y|.$$

Proof. By the homogeneity of F, we have

$$F^{\circ n}\begin{bmatrix} x \\ y \end{bmatrix} = y^{k^n}\begin{bmatrix} f^{\circ n}(x/y) \\ 1 \end{bmatrix}.$$

This leads to

$$\frac{1}{k^n}\log\left\|F^{\circ n}\begin{bmatrix} x \\ y \end{bmatrix}\right\| = \log|y| + \frac{1}{k^n}\log\left\|\begin{bmatrix} f^{\circ n}(x/y) \\ 1 \end{bmatrix}\right\|.$$

If $x/y \in K_f$, clearly the second summand of the last term tends to 0, so the formula is correct since $G_f(x/y) = 0$. If $x/y \notin K_f$, then the second summand of the last terms tends to $G_f(x/y)$, so again the formula is correct. □

With this formula, we can get back to studying the foliation of $\partial\Omega_F$. We will require the Böttcher coordinate [M, DH]

$$\varphi_f : (\mathbf{C}, \infty) \to (\mathbf{C}, \infty),$$

chosen tangent to the identity at ∞, where the notation $(\mathbf{C}, \infty)$ stands for some neighborhood of ∞ in $\mathbf{C}$. The function φ_f satisfies $\varphi_f\big(f(z)\big) = \varphi_f(z)^k$, and $\log|\varphi_f(z)| = G_f(z)$ wherever φ_f is defined. Let $\psi_f(z) = \varphi_f^{-1}(z)$; the mapping ψ_f is defined on the $\bar{\mathbf{C}} - \bar{D}_R$, where

$$R = \sup_{f'(z)=0} e^{G_f(z)}.$$

Proposition 8.2. *For any complex number a, the parametrized curve $\gamma_a : \bar{\mathbf{C}} - \bar{D}_R \to \mathbf{C}^2$ given by*

$$\gamma_a(\zeta) = \begin{bmatrix} a\psi_f(\zeta)/\zeta \\ a/\zeta \end{bmatrix}$$

when $\zeta \neq \infty$ and

$$\gamma_a(\infty) = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

satisfies $h_F\big(\gamma_a(\zeta)\big) = \log|a|$. In particular, when $|a| = 1$, these parametrized curves are parts of the foliation of $C(\mathbb{P}^1 - K_f) \cap \partial\Omega_F$.

Proof. Proposition 8.1 gives

$$h_F(\gamma_a(\zeta)) = G_f\big(\psi_f(\zeta)\big) + \log\left|\frac{a}{\zeta}\right| = \log|a|. \qquad \square$$

It is clear from Proposition 8.2 that understanding the analytic continuations of ψ_f and the leaves of the foliation are the same problem. This allows a complete description of the foliation of $\partial\Omega_F \cap C(\mathbb{P}^1 - K_f)$ when K_f is connected, so that ψ_f is defined on D.

Proposition 8.3. *If K_f is connected, then the leaves of the foliation of*

$$\partial\Omega_F \cap C(\mathbb{P}^1 - K_f)$$

are the images of γ_a when $|a| = 1$; in particular they are closed discs in $\mathbf{C}^2 - C(K_f)$.

Proof. This is all obvious from the previous proposition. $\square$

The description of the foliation when K_f is not connected is more complicated, and we will attempt it only when f is a quadratic polynomial.

Proposition 8.4. *If f is the quadratic polynomial $f(z) = z^2 + c$ and K_f is not connected, then $C(\mathbb{P}^1 - K_f) \cap \partial\Omega_F$ is foliated by Riemann surfaces of infinite genus, each of which is dense.*

Proof. Choose $R_0 > R$ such that $R_0^{1/2} < R$, and set $R_i = R_0^{1/2^i}$. Set

$$U_i = \{z \in \mathbb{P}^1 \mid G_f(z) > \log R_i\} = f^{-i}(U_0).$$

Then for each i, we have that $V_i = C(U_i) \cap \partial\Omega_F$ is the complement of 2^i solid tori in $\partial\Omega_F$, embedded so as to be unknotted, and each one linked with linking number 1 with each of the others.

Clearly the leaves of the foliation of V_0 are simply the discs $X_a = \gamma_a(\bar{\mathbf{C}} - \bar{D}_0)$, for $|a| = 1$. It is also easy to see that the $X_a^i = F^{-i}X_a$ are the leaves of the foliation of V_i. We will be done when we can show the following facts.

Lemma 8.5.

(a) *Each X_a^i is connected, and contains*

$$\bigcup_{a'^{2^i} = a^{2^i}} X_{a'}.$$

(b) *Each X_a^i contains 2^{i+1} critical values of F, above each of which are two ordinary double points.*

(c) *Each X_a^i has 2^i boundary components, one on each boundary torus of V_i.*

Proof of Proposition 8.4 using Lemma 8.5. The fact that the leaves are dense follows from (a); the angles a' such that $a'^{2^i} = a^{2^i}$ for some i are dense in the unit circle.

To see that they have infinite genus, we will use the Riemann- Hurwitz formula, which gives

$$\chi(X_a^i) = 4\chi(X_a^{i-1}) - 2 \cdot 2^i$$

using part (b). Using $\chi(X_a^0) = 1$, it is easy to show by induction that

$$\chi(X_a^i) = -2^{i+1}(2^{i-1} - 1).$$

Now the genus $g_i =$ genus X_a^i satisfies

$$g_i = 1 - \frac{1}{2}(\chi(X_a^i) + \text{ number of boundary components of } X_a^i),$$

which gives in this case $g_i = 1 + 2^{i-1}(2^i - 3)$, and clearly tends to infinity. $\square$

Proof of Lemma 8.5. To see the connectedness of X_a^i, it is enough to show that $X_{a'}^1$ is connected for any a'. Indeed, suppose X_a^i is not connected; let Y be a component; then Y maps to its image with degree less than 4. Consider a component of $Y \cap V_1$, which will be $X_{a'}^1$ for some a'. Since F will be of degree 4 on. $X_{a'}^1$, we will be done.

The proof that $X_{a'}^1$ is connected is a scissors and glue construction: consider two copies of $\bar{\mathbf{C}} - D_{R_1}$, both cut along the radial lines from 0 to the square roots of $\varphi_f(c)$ (recall that c is the critical value of f). We invite the reader to check that if each copy is glued to the other along the cut, then γ_a on one and γ_{-a} on the other agree on the identification lines.

To see the second part of (a), it is enough to consider the following formula:

$$F\big(\gamma_a(\zeta)\big) = \gamma_{a^2}(\zeta^2).$$

Part (b) is straightforward: all the critical values are in V_0, and each $\gamma_a(D_{R_0})$ contains exactly two.

Part (c) The set U_i has complement made up of discs $X_{i,1}, \ldots, X_{i,2^i}$, and f is injective on each of these discs. Since $\pi^{-1}(X_{i,j})$ is a bundle of circles over a disc, it is trivial, and hence a solid torus. The homology of the boundary then has a canonical basis, represented by a curve which bounds a disc in the solid torus, and a fiber $\pi^{-1}(z) \cap \partial\Omega_F$ for some $z \in \partial X_{i,j}$.

Clearly, in this basis the action of F on the homology is given by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

In this basis, let us see that a leaf intersect the boundary of V_0 in a $(1,1)$-curve. Since the leaves are given explicitly by γ_a, this is just an easy computation: the first coordinate is just

$$\frac{x}{y} = \psi_f(\zeta)$$

for $|\zeta| = R_0$ which goes once around $X_{0,1}$. For the second coordinate, we may project on the y-axis, and also find a circle which turns once around a fiber.

It follows from this that a component of ∂V_i intersects a leaf in a $(2^i, 1)$-curve, which is of course a connected curve. $\square$

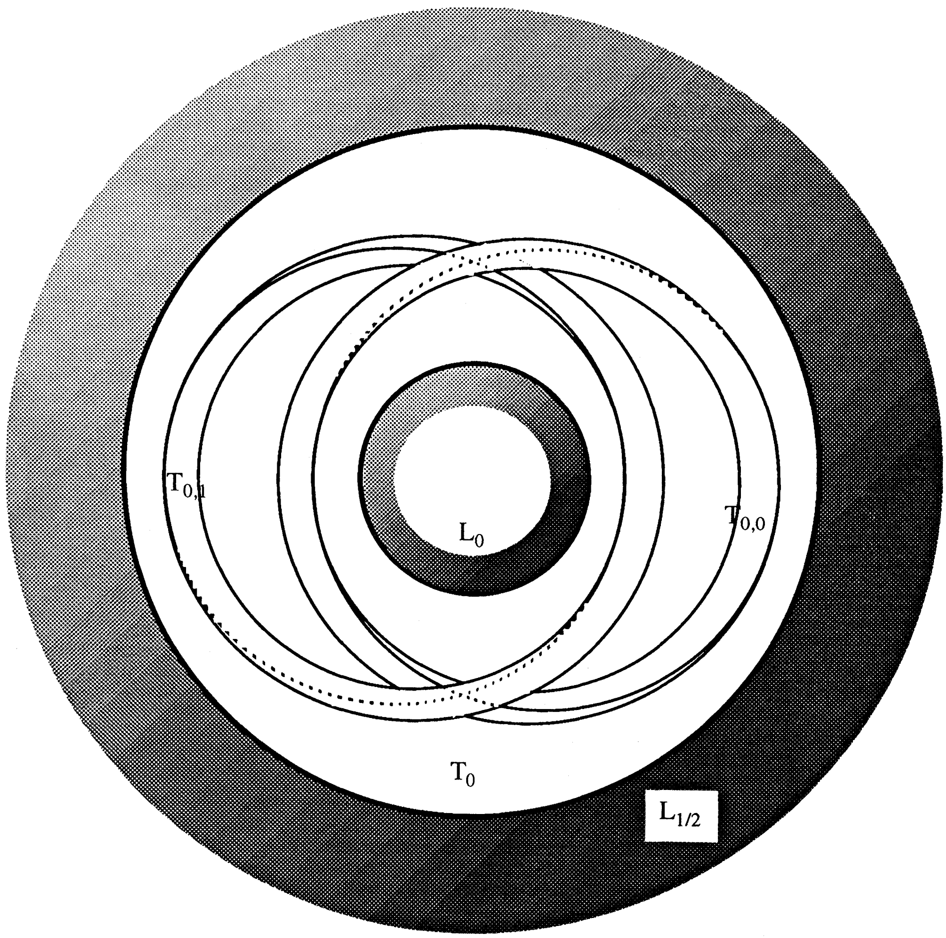

FIGURE 3

Remark. Part (c) gives a nice example of Proposition 7.2. Indeed, the integral of the Brolin measure over $X_{i,j}$ is 2^{-i}. But that is just the monodromy of a $(2^i, 1)$-curve.

The foliation of $\partial\Omega_F - \pi^{-1}J_f$ is a bit difficult to visualize, and Figures 3 and 4 are intended to suggest how it appears. Figure 3 represents a solid torus T_0, with two linked solid tori $T_{0,0}$ and $T_{0,1}$ inside it. Clearly one can put two linked solid tori inside each of these, and so forth. The decreasing intersection K of this collection of tori is homeomorphic to a circle cross a Cantor set, embedded into S^3 so that each path component is unknotted, and each pair of path components links with linking number 1.

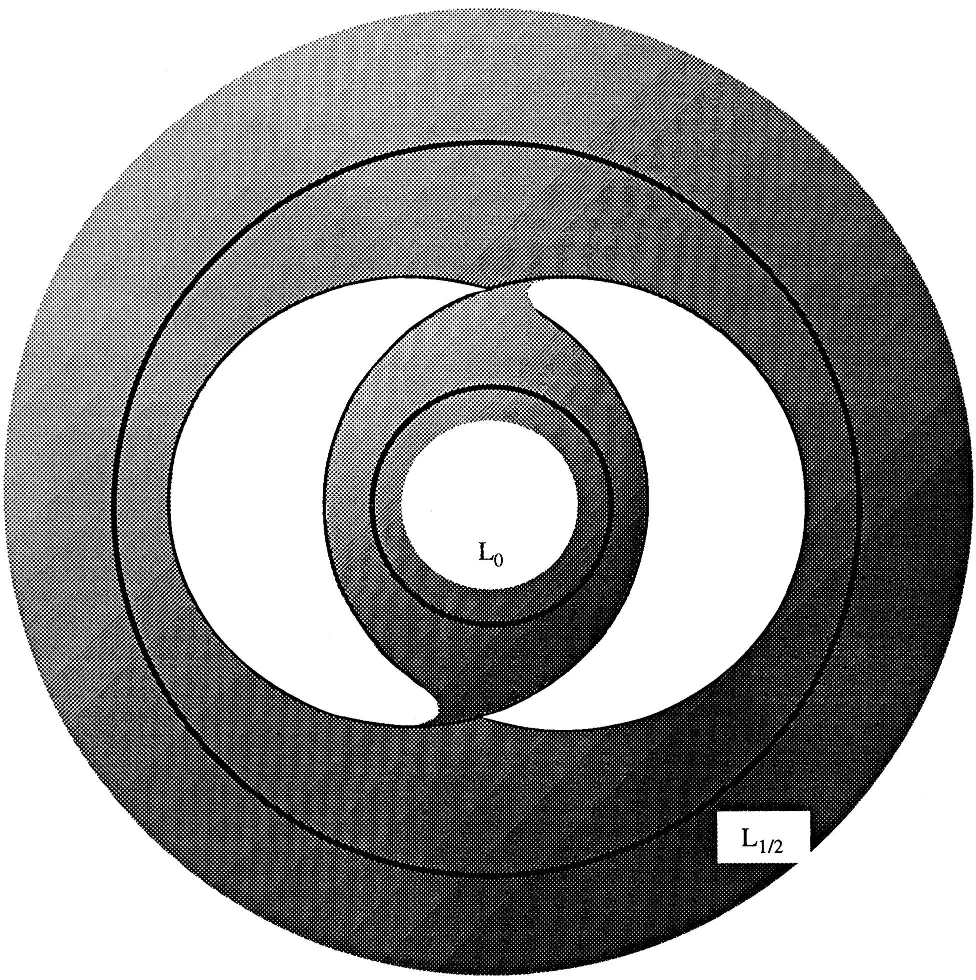

FIGURE 4

The set K will correspond to $\pi^{-1}J_f$, and its complement is foliated by dense surfaces of infinite genus. We indicate stages 0 and 1 of their construction. At stage 0, the outside of T_0 is foliated by discs L_t parametrized by $\mathbf{R}/\mathbf{Z}$ in the standard way, so that the inner disc L_0 and the outer disc $L_{1/2}$ are leaves.

At stage 1, L_0 and $L_{1/2}$ become part of a single leaf $L_{0,1/2}$, whose boundary consists of one simple closed curve on $\partial T_{0,0}$ and one on $\partial T_{0,1}$. The part of this leaf inside T_0 is drawn in figure 4; it is a sphere with four holes, so that the leaf at that stage is an annulus.

At the next stage, $L_{0,1/2}$ and $L_{1/4,3/4}$ join up to form a single leaf, etc.

9. The non-homogeneous hyperbolic case. For a general map $F = F_k + \cdots : U \to \mathbf{C}^{n+1}$, starting with terms of degree $k \geq 2$, we do not understand the structure of $dd^c h_F$. Still there is one fact which is easy to prove and provides a lot of insight. It is best stated in terms of $\tilde{F} : \tilde{U} \to \tilde{\mathbf{C}}^{n+1}$, the lift to the blowup (see Proposition 2.3). If $\mathbf{z} \in \mathbb{P}^n$ is a fixed point with eigenvalues $\lambda_1, \ldots, \lambda_n$ for f, then $\mathbf{z}$ is also a fixed point of $\tilde{F}$, with eigenvalues $0, \lambda_1, \ldots, \lambda_n$. In particular, if $\mathbf{z}$ is attractive in $\mathbb{P}^n$, then it is still attractive in $\tilde{U}$.

Proposition 9.1. *Let $\mathbf{z} \in \mathbb{P}^n$ be an attractive fixed point of $\tilde{F}$, and let $V \subset \tilde{U}$ be its basin. Then h_F is pluriharmonic on $V - \mathbb{P}^n$.*

Proof. The proof of Proposition 5.4 can easily be adapted to the present case. □

Let $F : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ be a non-degenerate homogeneous mapping of degree k, inducing $f : \mathbb{P}^n \to \mathbb{P}^n$. A closed set $Y \subset \mathbb{P}^n$ satisfying $f(Y) = Y$ will be called *f-expanding* if Y has a neighborhood W and there exists a metric ρ inducing the ordinary topology on W which is strongly expanding on W in the sense that there exists $\varepsilon > 0$ and $C > 1$ such that we have

$$\rho\big(f(x), f(y)\big) \geq C\rho(x,y),$$

for all x, $y \in W$ with $\rho(x,y) \leq \varepsilon$.

Examples.

(a) A repelling periodic cycle clearly is an expanding subset, for the ordinary metric.

(b) If $n = 1$ and f is a hyperbolic rational function, then J_f is expanding, for the Poincaré metric on an appropriate neighborhood W of J_f [DH].

(c) If $n = 2$ and we take the first example of Section 6, then if p_1 and p_2 are hyperbolic, the set $J_1 \times J_2$ is expanding. It may be that one reasonable definition of "hyperbolic" in higher dimensions is that f is expanding on the support of ω_f^n.

Let $H : U \to \mathbf{C}^{n+1}$ be an analytic mapping satisfying $H(\mathbf{x}) \in o(\|\mathbf{x}\|^k)$, and let $G = F + H$. Further define the sets

$$X = \{\mathbf{x} \mid G^m(\mathbf{x}) \in CW \text{ for all } m\}$$

and

$$X_\delta = \{\mathbf{x} \in X \mid h_G(\mathbf{x}) \leq \log \delta\}.$$

Proposition 9.2. *For all sufficiently small δ, there exists a unique mapping $\pi_G : X_\delta \to Y$ such that the diagram*

$$\begin{array}{ccc} X_\delta & \xrightarrow{G} & X_\delta \\ {\scriptstyle \pi_G}\Big\downarrow & & {\scriptstyle \pi_G}\Big\downarrow \\ Y & \xrightarrow{f} & Y \end{array}$$

commutes.

Proof. Sullivan [Su] gives the following construction. Define an ε-telescope on Y to be a sequence of closed subsets $B_i \subset W$, $i = 0, 1, \ldots$, of diameter $\leq \varepsilon$ such that $f(B_i) \supset B_{i+1}$. Then for ε sufficiently small, there exists a unique $z \in B_0$ such that $f^m(z) \in B_m$ for all m.

Of course,

$$z \in B_0 \cap f^{-1} B_1 \cap f^{-2} B_2 \cap \ldots,$$

and the intersection is a single point since f is uniformly expanding on W.

Now extend the metric ρ on W to a metric on $\tilde{U}$. Choose δ small enough so that for any $\mathbf{x} \in X_\delta$, the intersections $W \cap B_\varepsilon\big(G^m(\mathbf{x})\big)$ form an ε-telescope. Define $\pi_G : X_\delta \to Y$ by setting $\pi_G(\mathbf{x})$ to be the point specified by the telescope.

This mapping clearly makes the diagram commute. □

We understand X quite well if the following extra assumption is made: There exist neighborhoods $W_1 \subset W$ of Y such that $f(W_1) = W$, W_1 is relatively compact in W and $f : W_1 \to W$ is a covering map. Note that in dimension 1, this is already implied by the expanding condition, but not in higher dimensions.

Let H_0, $H_1 : U \to \mathbf{C}^{n+1}$ be two mappings satisfying $H_i(\mathbf{x}) \in o(\|\mathbf{x}\|^k)$, and let us define $G_i = F + H_i$. Further define the sets

$$X_i = \{\mathbf{x} \mid G_i^m(\mathbf{x}) \in CW \text{ for all } m\}$$

and

$$X_{i,\delta} = \{\mathbf{x} \in X_i \mid h_{G_i}(\mathbf{x}) \leq \log \delta\}.$$

Theorem 9.3.

(a) *There exists* $\delta > 0$ *and a unique homeomorphism*

$$\varphi : X_{0,\delta} \to X_{1,\delta}$$

conjugating $G_0|_{X_{0,\delta}}$ *to* $G_1|_{X_{1,\delta}}$.

(b) *The fibers of* π_{G_0} *and* π_{G_1} *are Riemann surfaces.*

(c) *The mapping* φ *satisfies* $\pi_{G_0} = \pi_{G_1} \circ \varphi$, *and is analytic on the fibers.*

Before giving the proof, we will illustrate the meaning of this theorem. Suppose we are in case (b) of the examples above: $n = 1$, the mapping f is a hyperbolic rational function and $Y = J_f$.

Corollary 9.4. *In a sufficiently small neighborhood of* $\mathbf{0}$, *the support of* $dd^c h_F$ *is contained in* X.

Proof. This is clear from Proposition 9.1: in a small neighborhood of $\mathbb{P}^1 \subset \tilde{U}$ all points are attracted to $\mathbb{P}^1$, hence are either in X or attracted to an attractive cycle of f. □

Example. Consider the mapping

$$F : (x,y) \mapsto (x^2 - y^2, y^2 + x^3).$$

The associated homogeneous map is $(x,y) \mapsto (x^2 - y^2, y^2)$, and so if we use $z = x/y$ as a local coordinate on $\mathbb{P}^1$, we find

$$f(z) = z^2 - 1.$$

The Julia set of this polynomial is well-known, and represented in Figure 5. According to Theorem 9.3 and Corollary 9.4, we expect the support of $dd^c h_F$ to be a set which near the origin looks like the cone over J_f, but further away is deformed by the terms x^3 in the definition of F. Represented in Figures 6, (a, b, c, d) are the lines $y = .2$, $.5$, 1 and 2. In each one of these sections we have represented the domain of attraction of the origin, and within it the support of $dd^c h_F$. You see indeed a set just like J_f in the first section and near the origin, but it gets badly deformed as you get further away from $\mathbf{0}$.

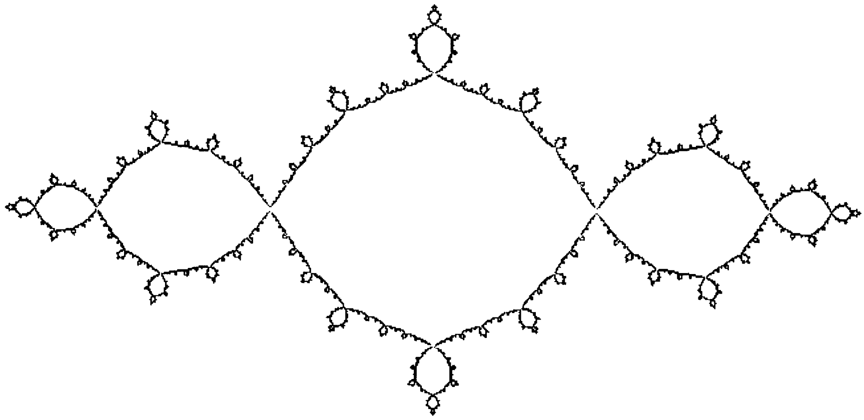

FIGURE 5

Proof of 9.3. We will attempt, as in the one-dimensional case, to use scattering theory, i.e., to give a meaning to

$$\lim_{m\to\infty} G_1^{-m} G_0^m(\mathbf{x}).$$

Before examining the existence of the limit, we must make sense of the inverse images, since every point has k^2 inverse images counted with multiplicity. In the one-dimensional case, this is proved by rewriting the limit above as an infinite product, and using the principal branch of the k^{th} root as the main tool to lift the ambiguity. Here these analytic tricks do not work, and we need to use a more topological approach, embodied in Lemma 9.6.

Let $G_\tau = F + (1-\tau)H_0 + \tau H_1$, and make the obvious definitions of X_τ and $X_{\tau,\delta}$. Set $H = H_1 - H_0$.

The next lemma collects various facts which will be useful later.

Lemma 9.5. *There exist $R > 0$ and $C > 1$ such that on the ball B_R we have*

(a)

$$\left(\frac{\|\mathbf{x}\|}{C}\right)^{k^m} \leq \|G_\tau^m(\mathbf{x})\| \leq (C\|\mathbf{x}\|)^{k^m};$$

(b) *The set $G_\tau^{-1}(C(W)\cap B_R)\cap C(W)$ is non-empty and relatively compact in $C(W)$;*

(c) $\|H(\mathbf{x})\| \leq C\|\mathbf{x}\|^{k+1}$;

(d) *On $C(W)\cap B_R$ we have $\|(d_{\mathbf{x}}G_\tau)^{-1}\| \leq C\|\mathbf{x}\|^{1-k}$.*

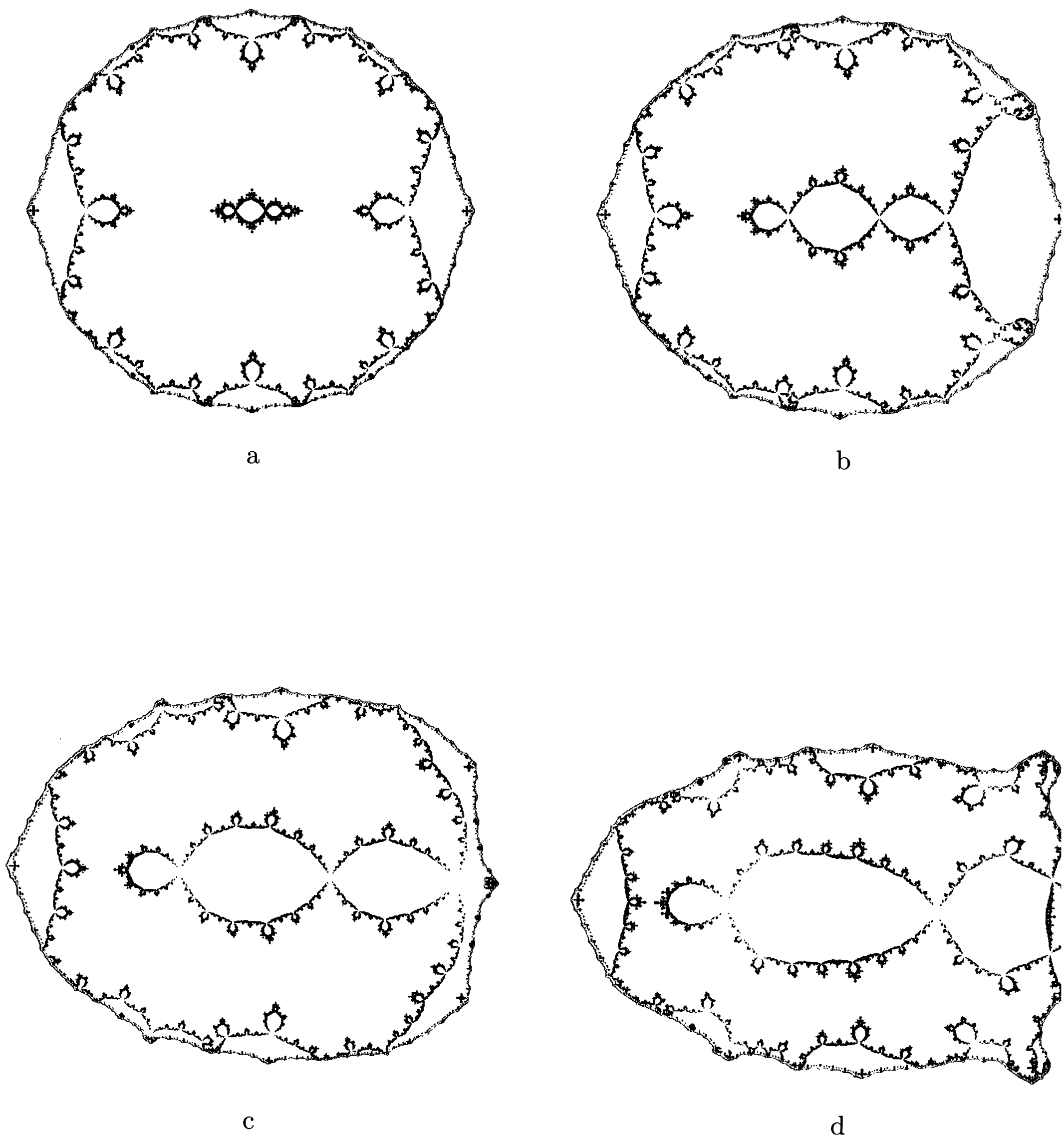

FIGURE 6

Remark. If the expanding set Y is totally invariant, i.e., satisfies $f^{-1}(Y) = Y$, for instance when Y is the Julia set of a rational function, then part (b) can be changed to read

$$G_\tau^{-1}\big(C(W) \cap B_R\big) \subset C(W)$$

and is relatively compact in $C(W)$. In other cases, for instance when Y is a repelling cycle, there may be other components, which will be irrelevant to our discussion.

Proof of 9.5. (a) By Thm. 2.1 (b), we have $h_\tau\big(G_\tau^m(\mathbf{x})\big) = k^m h_\tau(\mathbf{x})$, and using Theorem 2.1 (c) four times, we get

$$k^m(\log\|\mathbf{x}\| - \log C_\tau) \leq h_\tau\big(G_\tau^m(\mathbf{x})\big) \leq \log\|G_\tau^m(\mathbf{x})\| + \log C_\tau$$

and

$$\log\|G_\tau^m(\mathbf{x})\| - \log C_\tau \leq h_\tau\big(G_\tau^m(\mathbf{x})\big) \leq k^m(\log\|\mathbf{x}\| + \log C_\tau).$$

If you set $C = \sup_\tau C_\tau^2$ and exponentiate, this gives (a).

Part (b) is clearly true for the homogeneous map F, since $F^{-1}(W)$ is relatively compact in W. So it is true for all G_τ when R is sufficiently small.

Part (c) is clear: $\|H\| \in O(\|\mathbf{x}\|^{k+1})$.

Part (d) can be seen as follows: use the formula for the inverse as the matrix of cofactors divided by the determinant. The determinant Δ is a convergent power series which starts with terms of degree $(n+1)(k-1)$. The restriction of Δ to a line through 0 in $\mathbf{C}^{n+1}$ is a power series in one variable, which starts with a non-vanishing term of degree $2k-2$ unless the line is tangent to the curve $\Delta = 0$. The lines with this property are precisely those which correspond to the critical points of f, in particular the lines in $\bar{C}(W)$ are not among them. So there is a constant $C_1 > 0$ such that $|\Delta(\mathbf{x})| \geq C_1\|\mathbf{x}\|^{2k-2}$ on $C(W)$. Now the cofactors are power series starting with terms of degree at least $n(k-1)$, and the result follows. □

Set $\delta = R/C^3$.

Lemma 9.6. *If* $\|\mathbf{x}\| < \delta$, *there exist unique curves* $\{\gamma_{m,\mathbf{x}}(\tau)\}_{m=0,1,\dots}$ *satisfying*

(a) $\gamma_{m,\mathbf{x}}(\tau)$ *is a branch of* $G_1^{-m}G_\tau^{-1}G_0^{m+1}(\mathbf{x})$;
(b) $\gamma_{m,\mathbf{x}}(0) = \gamma_{m-1,\mathbf{x}}(1)$;
(c) $\gamma_{0,\mathbf{x}}(0) = \mathbf{x}$.

The following diagram may help the reader understand what is happening in this proof.

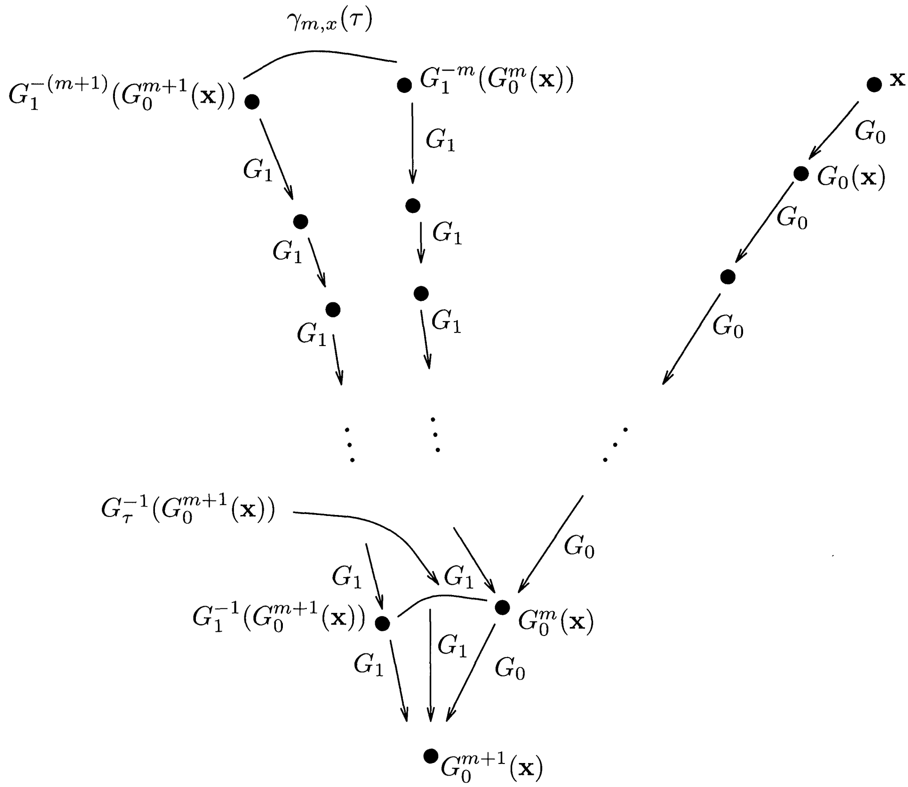

FIGURE 7

Proof of Lemma 9.6. As the reader will easily imagine, this is essentially an application of the path-lifting property of covering spaces. However, defining exactly what is a covering space of what is a bit messy.

Sublemma 9.7.

(a) *The mapping*

$$G_1^m : B_R \cap G_1^{-m}\big(B_{C(C\delta)^{k^m}} \cap C(W)\big) \cap C(W) \to \big(B_{C(C\delta)^{k^m}} \cap C(W)\big)$$

is a covering map.

(b) *The mapping*

$$G_\tau \times id : B_{C(C\delta)^{k^{m+1}}} \cap \Big((G_\tau \times id)^{-1} \big((B_{(C\delta)^{k^{m+1}}} \cap C(W)) \times I \big) \Big) \cap \big(C(W) \times I \big)$$
$$\to \big((B_{(C\delta)^{k^{m+1}}} \cap C(W)) \times I \big)$$

is a covering map.

Proof. A mapping which is proper and a local homeomorphism is a finite covering map. Both maps above are local homeomorphisms by the implicit function theorem.

The proof that they are proper is an application of Lemma 9.5 (a). Indeed,

$$B_R \cap G_1^{-m} \big(B_{C(C\delta)^{k^m}} \cap C(W) \big)$$

will have compact closure in B_R unless there exists $y \in \partial B_R$ with $G_1(y) \in B_{C(C\delta)^k} \cap C(W)$. But if $\|y\| = R$, we have

$$\|G_1^m(y)\| \geq \left| \frac{R}{C} \right|^{k^m} \geq |C^2 \delta|^{k^m} \geq C(C\delta)^{k^m}.$$

The proof of part (b) is similar. □

Proof of 9.6. The curve $\tau \to \big(G_0^{m+1}(\mathbf{x}), \tau \big)$ is contained in $(B_{(C\delta)^{k^{m+1}}} \cap CW) \times I$ by Lemma 9.5 (a). So by Sublemma 9.7 (b), there exists a unique curve $\tau \mapsto \eta_{m,\mathbf{x}}(\tau)$ such that $G_\tau\big(\eta_{m,\mathbf{x}}(\tau)\big) = G_0^{m+1}(\mathbf{x})$ and $\eta_{m,\mathbf{x}}(0) = G_0^m(\mathbf{x})$. Again by Lemma 9.5 (a) and (b), we have

$$\eta_{m,\mathbf{x}}(\tau) \in B_{C(C\delta)^{k^m}} \cap C(W).$$

Now applying Sublemma 9.7 (a), and induction on m, we see that there exists unique curve $\gamma_{m,\mathbf{x}}(\tau)$ in B_R with $G_1^m\big(\gamma_{m,\mathbf{x}}(\tau)\big) = \eta_{m,\mathbf{x}}(\tau)$ and $\gamma_{m,\mathbf{x}}(0) = \gamma_{m-1,\mathbf{x}}(1)$. □

In order to prove the theorem, we need to show that

$$\sum_{m=0}^{\infty} \mathrm{length}(\gamma_{m,\mathbf{x}}) < \infty.$$

This is an exercise in the use of the mean value theorem.

Lemma 9.8. *We have*

$$\operatorname{length}(\eta_{m,\mathbf{x}}) \leq C^{2k(k^m+1)} \|\mathbf{x}\|^{2k^m}.$$

Proof. The equation $G_\tau\big(\eta_{m,\mathbf{x}}(\tau)\big) = G_0^{m+1}(\mathbf{x})$ gives

$$(C\|\eta_{m,\mathbf{x}}(\tau)\|)^k \geq \|G_\tau \eta_{m,\mathbf{x}}(\tau)\| = \|G_0^{m+1}(\mathbf{x})\| \geq \left(\frac{\|\mathbf{x}\|}{C}\right)^{k^{m+1}},$$

which gives

$$(*) \qquad \|\eta_{m,\mathbf{x}}(\tau)\| \geq \frac{1}{C}\left(\frac{\|\mathbf{x}\|}{C}\right)^{k^m}.$$

Using the opposite inequalities in the same way gives

$$\|\eta_{m,\mathbf{x}}(\tau)\| \leq C(C\|\mathbf{x}\|)^{k^m}.$$

Differentiating $G_\tau\big(\eta_{m,\mathbf{x}}(\tau)\big) = G_0^{m+1}(\mathbf{x})$ with respect to τ gives

$$\eta'_{m,\mathbf{x}}(\tau) = -(d_{\eta_{m,\mathbf{x}}(\tau)} G_\tau)^{-1} H\big(\eta_{m,\mathbf{x}}(\tau)\big).$$

Using Lemma 9.5 (c) and (d), together with the bounds above gives

$$\|\eta'_{m,\mathbf{x}}(\tau)\| \leq \left(\frac{1}{C}\left(\frac{\|\mathbf{x}\|}{C}\right)^{k^m}\right)^{1-k} \left(C(C\|\mathbf{x}\|)^{k^m}\right)^{k+1}.$$

Collecting terms and integrating with respect to τ gives the result. □

Lemma 9.9. *We have the inequality*

$$\|(d_{\gamma_{m,\mathbf{x}}(\tau)} G_1^m)^{-1}\| \leq C^{k^m-1+2m(k-1)} \|\mathbf{x}\|^{1-k^m}.$$

Proof. Let $\mathbf{y} = \gamma_{m,\mathbf{x}}(\tau)$, and $\mathbf{y}_j = G_1^j(\mathbf{y})$ for $j = 0, \dots, m-1$. In order to use Lemma 9.5 (d), we need lower bounds on the $\|\mathbf{y}_j\|$, which we get as follows.

From the equation $G_1^{m-j}(\mathbf{y}_j) = \eta_{m,\mathbf{x}}(\tau)$, Lemma 9.5 (a) and the inequality $(*)$ above, we get

$$(C\|\mathbf{y}_j\|)^{k^{m-j}} \geq \|G_1^{m-j}(\mathbf{y}_j)\| = \|\eta_{m,\mathbf{x}}(\tau)\| \geq \frac{1}{C}\left(\frac{\|\mathbf{x}\|}{C}\right)^{k^m},$$

which gives

$$\|\mathbf{y}_j\| \geq C^{-(2+k^j)} \|\mathbf{x}\|^{k^j}.$$

Now by Lemma 9.5 (d) and the chain rule, we have

$$\|(d_{\gamma_{m,\mathbf{x}}(\tau)} G_1^m)^{-1}\| \leq \left(\prod_{j=0}^{m-1} \|\mathbf{y}_j\|\right)^{1-k} \leq \left(\prod_{j=0}^{m-1} C^{-(2+k^j)} \|\mathbf{x}\|^{k^j}\right)^{1-k}$$
$$= C^{k^m-1+2m(k-1)} \|\mathbf{x}\|^{1-k^m}. \qquad \square$$

We can now prove the main step in the theorem.

Lemma 9.10. *We have the inequality*

$$\operatorname{length}(\gamma_{m,\mathbf{x}}) \leq C^{k^m(2k+1)+2k+2m(k-1)-1}\|\mathbf{x}\|^{k^m+1}.$$

Proof. This is just a matter of putting together Lemmas 9.8 and 9.9, and collecting terms. □

Proof of Theorem 9.3. If we choose $\|\mathbf{x}\| \leq \delta/(2k+1)$, we see that the series

$$\sum_{m=0}^{\infty} \operatorname{length}(\gamma_{m,\mathbf{x}})$$

is convergent. Therefore the limit

$$\varphi(\mathbf{x}) = \lim_{m\to\infty} \gamma_{m,\mathbf{x}}(1)$$

exists, and defines a continuous function of $\mathbf{x} \in X_0$. By definition we have that $\gamma_{m,\mathbf{x}}(1)$ is a branch of $G_1^{-(m+1)}G_0^{m+1}(\mathbf{x})$, so we certainly have the conjugacy relation $\varphi \circ G_0 = G_1 \circ \varphi$. Further, reversing the roles of G_0 and G_1 shows that φ is a homeomorphism $X_0 \to X_1$.

This proves part (a) of Theorem 9.3. To see part (b), take $H_0 = 0$. In that case, the fibers of π_{G_0} are of course straight lines, and since φ is a uniform limit of analytic mappings on each fiber, the limit is also analytic, and an isomorphism of a fiber of π_{G_0} onto an analytic curve.

To prove (c), let φ_0 conjugate the mapping G_0 to F on X_0, and let φ_1 conjugate F to G_1 on $X = \pi^{-1}(Y)$. Then $\varphi = \varphi_1 \circ \varphi_0$, and the result follows. □

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