HAIRS FOR THE COMPLEX EXPONENTIAL FAMILY

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In this paper we consider both the dynamical and parameter planes for the complex exponential family $E_\lambda(z) = \lambda e^z$ where the parameter λ is complex. We show that there are infinitely many curves or "hairs" in the dynamical plane that contain points whose orbits under E_λ tend to infinity and hence are in the Julia set. We also show that there are similar hairs in the λ-plane. In this case, the hairs contain λ-values for which the orbit of 0 tends to infinity under the corresponding exponential. In this case it is known that the Julia set of E_λ is the entire complex plane.

1. Introduction

Our goal in this paper is to describe the dynamics and the parameter plane for the family of complex exponential functions

$$E_\lambda(z) = \lambda e^z$$

where $\lambda \in \mathbb{C}$ is a parameter. This family has been discussed extensively when $\lambda \in \mathbb{R}$ (see [Devaney & Krych, 1984]) Our main goal is to prove the existence of "hairs" in both the dynamical and parameter planes for this family.

1.1. The quadratic family

The hairs in the exponential family are the analogues of the *external rays* for the family of quadratic polynomials

$$Q_c(z) = z^2 + c$$

as introduced by Douady and Hubbard [1982]. We first recall the definition of these rays, and then describe their analogues in the exponential family.

For Q_c, the filled Julia set is defined as

$$K_c = \left\{ z \in \mathbb{C} | Q_c^n(z) \text{ is bounded}, \quad n = 1, 2, \ldots \right\}.$$

The boundary of the filled Julia set, J_c, is called the Julia set of Q_c. Equivalently,

$$J_c = \{ z \in \mathbb{C} | \{Q_c^n\}$$
$$\text{fails to be a normal family at } z \}.$$

It is known that all the interesting dynamics of Q_c occurs on J_c. In particular, J_c is the closure of the set of repelling periodic points for Q_c. See [Blanchard, 1984; Carleson & Gamelin, 1993; Douady & Hubbard, 1982] for general background on the dynamics of the quadratic family.

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Near ∞, the dynamics of Q_c are equivalent to those of $z \to z^2$. More precisely, we may find a neighborhood U_c of ∞ and an analytic conjugacy

$$\phi_c : U_c \to \mathbb{C}$$

such that

$$(\phi_c(z))^2 = \phi_c(Q_c(z)).$$

The preimage under ϕ_c of a straight ray of the form $t \to te^{2\pi i\theta}$ for $t > 1$ is called an external ray for Q_c and denoted as $\gamma_\theta(t)$. By the conjugacy, $Q_c(\gamma_\theta) = \gamma_{2\theta}$.

The filled Julia sets for the quadratic family exhibit a fundamental dichotomy as determined by the fate of the orbit of the critical point, 0. Either K_c is connected, in which case the orbit of 0 under Q_c is bounded, or K_c is a Cantor set, in which case $Q_c^n(0) \to \infty$. When K_c is connected, the conjugacy ϕ_c may be extended to $\mathbb{C} - K_c$. In the case that $\lim_{t\to 1} \gamma_\theta(t)$ exists, we call this point the *landing point* of the ray γ_θ.

1.2.   *The Mandelbrot set*

The fundamental dichotomy above is also reflected in the parameter plane for Q_c. The picture in the parameter plane, the well-known Mandelbrot set M, consists of all c-values for which K_c is connected or, equivalently, for which $Q_c^n(0)$ is bounded. See Fig. 1.

According to Douady and Hubbard [1982], there is a uniformization of the exterior of M

$$\Phi : \mathbb{C} - M \to \mathbb{C} - \{z||z| > 1\}.$$

As in the case of the dynamical plane, we define the external ray

$$R_\theta(t) = \Phi^{-1}(te^{2\pi i\theta}),$$

for $t > 1$. Again as in the case of K_c, some of these rays are known to land at points of M. There is a beautiful relationship between the dynamics of the landing points for certain of the external rays in M and the doubling map $\theta \to 2\theta \mod 1$. For example, suppose the binary expansion of θ is either periodic or preperiodic under angle doubling, i.e. doubling modulo one. Then it is known that the ray R_θ lands at a point in M. Moreover, if θ is periodic, R_θ lands at a root point of a hyperbolic component of M (a bifurcation point). If θ is
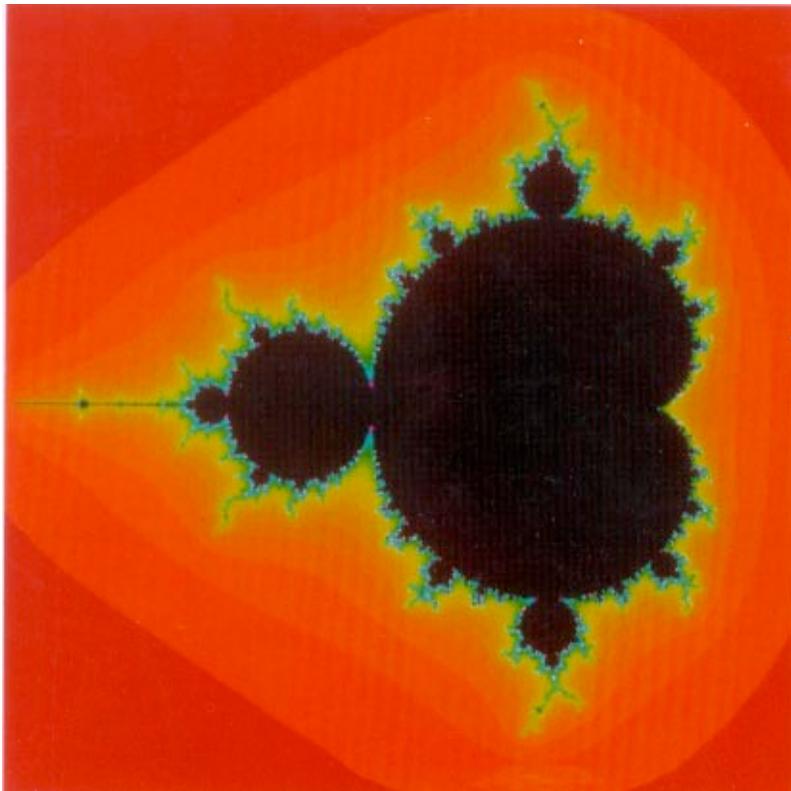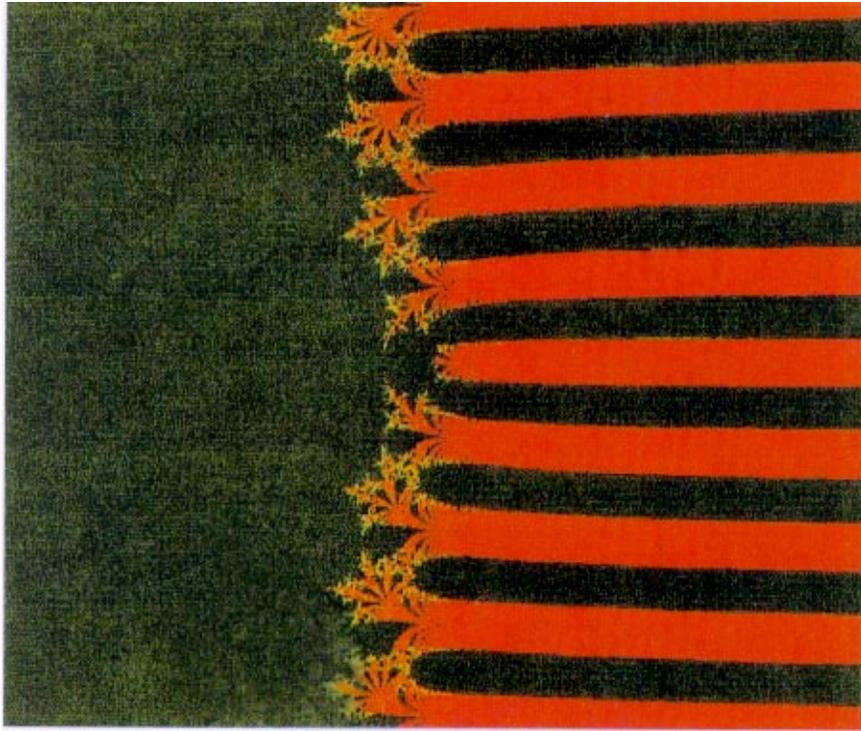

Fig. 1.   The Mandelbrot set for $Q_c(z)$.

Fig. 2.   The parameter plane for E_λ.

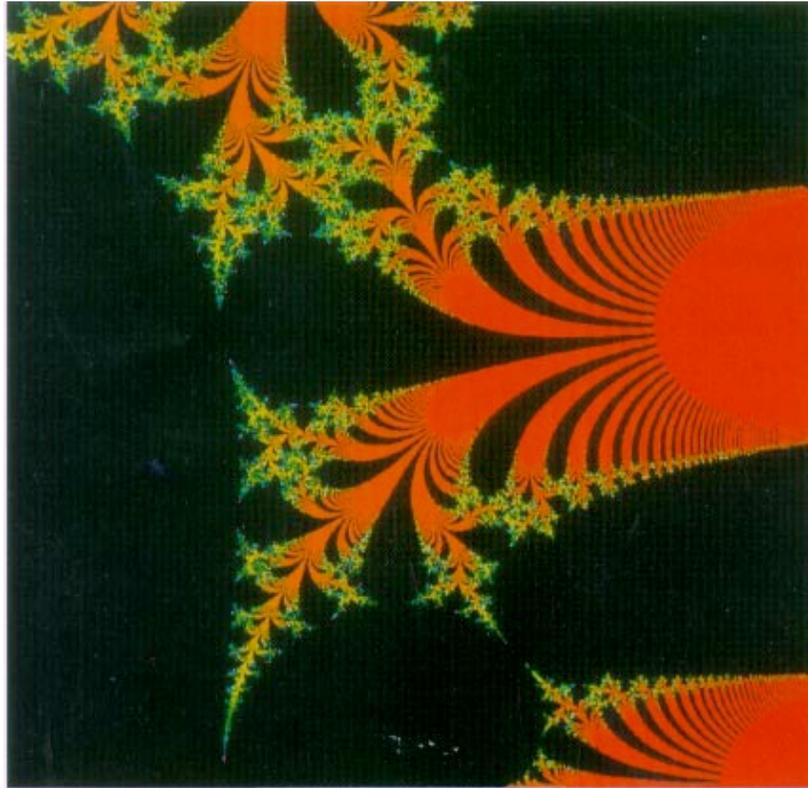Fig. 3.   Detail of the parameter plane near $\lambda = 0$.

Fig. 4.   More detail of the hairs of the parameter plane.

preperiodic, then R_θ lands at a Misiurewicz point in M, i.e. a c-value for which the orbit of 0 eventually cycles and $K_c = J_c$ is a dendrite. See [Douady & Hubbard, 1982] for details.

1.3.   *The exponential family*

We now turn to the exponential family $E_\lambda(z) = \lambda e^z$ where $\lambda \in \mathbb{C}$ is a parameter. The Julia set of E_λ is defined as before as

$$J(E_\lambda) = \{z \in \mathbb{C} | \{E_\lambda^n\}$$

$$\text{fails to be a normal family at } z\} \,.$$

It is known that $J(E_\lambda)$ is also the closure of the set of repelling periodic points of E_λ [Baker, 1968]. It is also known that if $E_\lambda^n(z) \to \infty$, then $z \in J(E_\lambda)$. This follows immediately from the fact that complex exponentials cannot have wandering domains or domains at ∞. See [Baker, 1984; Eremenko & Lyubich, 1985; Goldberg & Keen, 1986]. Recall that, for the quadratic family, points whose orbits escape do not belong to the Julia set.

This is one difference between the exponential and quadratic families, but there are many others. Chief among them is the fact that ∞ is an attracting

fixed point for Q_c, but ∞ is an essential singularity for E_λ. As we show below, $J(E_\lambda)$ contains ∞ in its closure. As a consequence, there is no analogue of the conjugacies ϕ_c or Φ in either the dynamical or parameter planes for E_λ.

However, we may still identify certain curves in the dynamical plane that play the role of the external rays γ_θ. We will show that for each $\lambda \in \mathbb{C}' = \mathbb{C} - \{\text{nonpositive reals}\}$ and $K \in \mathbb{Z}^+$ there is an invariant Cantor set contained in the Julia set of E_λ on which the dynamics of E_λ are conjugate to the one-sided shift map on $2K$ symbols. Attached to each point in this Cantor set is a curve or "hair" that extends to ∞. The point in the Cantor set is called the endpoint of the hair. The orbit of any point on a hair (excluding the endpoint) tends to ∞ under iteration of E_λ and, moreover, E_λ permutes these hairs as dictated by the shift map.

Due to this, the hairs are the analogues of the external rays γ_θ. However, there are several differences. First, the hairs for E_λ lie in the Julia set of E_λ whereas the γ_θ lie in the complement of J_c for Q_c. Secondly, the hairs do not foliate an open neighborhood of ∞ for E_λ as they do for Q_c.

For the quadratic family, the orbit of the critical point plays a crucial role in determining the

dynamics. For E_λ, there is no such critical point. However, E_λ has a unique asymptotic value (or omitted value) 0. This orbit plays a somewhat similar role for E_λ. For example, it is known [Baker & Rippon, 1984; Eremenko & Lyubich, 1992] that the Julia set of E_λ is the entire plane if $E_\lambda^n(0) \to \infty$. Also, if E_λ admits an attracting cycle, then the orbit of 0 must tend to this cycle, showing that E_λ may have at most one attracting cycle. In this case it is known that $J(E_\lambda)$ is a nowhere dense subset of the plane.

1.4. *The parameter plane for the exponential family*

We do not have a fundamental dichotomy for E_λ based on the fate of the orbit of 0 as we do for Q_c, for there are many values of λ for which the orbit of 0 is bounded yet the Julia set of E_λ is the entire plane. Nevertheless, we may begin to paint the picture of the parameter plane for E_λ based on the fate of the orbit of 0. In Fig. 2, we show a picture of this plane. The visible black regions are "hyperbolic components", C_k, i.e. λ-values for which E_λ has an attracting cycle of period k. The cardioid-shaped region in the center of the picture is the attracting fixed point region. The large black region to the left is C_2. The large horizontal strips extending to the right comprise C_3. It is known that for each $k > 2$, C_k consists of infinitely many distinct components and each component extends to ∞ in the direction of the right half plane [Baker & Rippon, 1984].

Between these components sit the analogues of the external rays in the parameter plane. See Figs. 2–4. In these images, these rays appear to occupy open sets. However, we will show that there are actually infinitely many distinct curves (or hairs) in the complement of the hyperbolic components. These curves consist of λ values for which $E_\lambda^n(0) \to \infty$ and so the Julia set of E_λ is $\mathbb{C}$. As in the case of the dynamical plane, we will study the arrangement of these hairs using symbolic dynamics.

2. The Dynamical Plane for $E_\lambda(z)$

In this section we discuss the dynamics of $E_\lambda(z) = \lambda e^z$. We assume throughout that $\lambda \in \mathbb{C}'$. The dynamics of E_λ when $\lambda \in \mathbb{R}^-$ are well understood; we exclude this case merely to gain continuity of the Markov partitions discussed below.

We will prove the existence of an invariant Cantor set in $J(E_\lambda)$ for all $\lambda \in \mathbb{C}'$. A similar construction was obtained for real λ in [Aarts & Oversteegen, 1993; Devaney & Krych, 1984; Devaney & Tangerman, 1986].

2.1. *A Markov partition for E_λ*

For each integer k, we define horizontal strips $R(k) = R_\lambda(k)$ by

$$R(k) = \{z \in \mathbb{C} | (2k-1)\pi$$
$$- \arg \lambda < \text{Im } z < (2k+1)\pi - \arg \lambda\}.$$

Here $\arg \lambda$ is the principal branch of the argument, taking values between $\pm \pi$. Note that E_λ maps the boundary of $R(k)$ onto the negative real axis. Consequently, E_λ maps $R(k)$ onto $\mathbb{C}'$ for each k, and $\mathbb{R}^- \cup \{0\} \subset R(0)$ for each λ. In particular, $E_\lambda(R(k)) \supset R(j)$ provided $j \neq 0$. Hence the $R(k)$ provide a type of Markov partition for E_λ, at least when $k \neq 0$.

Definition 2.1. For $z \in \mathbb{C}$, the *itinerary* of z under E_λ is the sequence of integers $s(z) = s_0 s_1 s_2 \ldots$ where $s_n = k$ iff $E_\lambda^n(z) \in R(k)$. We do not define the itinerary of z if $E_\lambda^n(z) \in \mathbb{R}^-$ for some n.

Definition 2.2. An itinerary $s_0 s_1 s_2 \ldots$ is called *regular* if $s_j \neq 0$ for all j.

Let Σ_K consist of all one-sided sequences $s_0 s_1 s_2 \ldots$ where each $s_j \in \mathbb{Z}$, $s_j \neq 0$, and $|s_j| \leq K$, i.e. all regular sequences. The one-sided shift σ on Σ_K (also called the shift on $2K$ symbols) is defined by

$$\sigma(s_0 s_1 s_2 \ldots) = (s_1 s_2 s_3 \ldots).$$

It is well known that σ has dense periodic points in Σ_K, has dense orbits, and exhibits sensitive dependence on initial conditions.

2.2. *Invariant Cantor sets for E_λ*

In this section we will construct invariant Cantor sets for E_λ on which E_λ is conjugate to the one-sided shift on Σ_K for each integer $K > 0$.

For each $\lambda \in \mathbb{C}'$, we will construct an invariant Cantor set homeomorphic to Σ_K. Toward that end, we define L_{λ, s_j} to be the inverse of E_λ defined on $\mathbb{C}'$ and taking values in the strip $R_\lambda(s_j)$. Note that

$$L_{\lambda, s_j}(z) = -\log \lambda + \log z + 2\pi i s_j$$

where log is the principal branch of the logarithm.

Given λ, define

$$M_K = \{\sup |\mathrm{Im}\, z||z \in R_\lambda(\pm K)\}$$

$$m_K = \{\inf |\mathrm{Im}\, z||z \in R_\lambda(\pm 1)\}$$

M_K and m_k give the maximal and minimal vertical distances to the real line in the union of the strips $R_\lambda(s_i)$ when $0 < |s_i| \le K$. Note that $M_K > 2\pi$.

Define

$$V_\alpha^\beta(s_i) = \left\{ z \in \overline{R(s_i)} | \alpha \le \mathrm{Re}\, z \le \beta \right\}.$$

Each $V_\alpha^\beta(s_i)$ is a closed rectangle whose interior lies in $R_\lambda(s_i)$. Throughout this section we fix α and β so that

$$0 < |\lambda|e^\alpha < m_K$$

$$|\lambda|e^\beta > M_K + |\beta| + |\alpha|\,.$$

With this choice of α and β, the image of each $V_\alpha^\beta(s_i)$ under E_λ is an annular region given by

$$|\lambda|e^\alpha \le |z| \le |\lambda|e^\beta\,.$$

Our choice of α and β guarantees that $E_\lambda(V_\alpha^\beta(s_i))$ covers $V_\alpha^\beta(s_j)$ for each i and j (assuming $0 < |s_i|, |s_j| \le K$). See Fig. 5.

Let

$$V_\alpha^\beta = \bigcup_{0 < |j| \le K} V_\alpha^\beta(j)\,.$$

V_α^β depends on K and λ. Let $\Lambda_{K,\lambda}$ be the set of points whose orbits remain for all time in V_α^β.

Theorem 2.3. *Suppose $K > 0$. For each $\lambda \in \mathbb{C}'$, $\Lambda_{K,\lambda}$ is homeomorphic to Σ_K and $E_\lambda|\Lambda_{K,\lambda}$ is conjugate to the shift map on Σ_K.*

Proof. Let $s = (s_0 s_1 s_2 \ldots) \in \Sigma_K$. Let $z \in V_\alpha^\beta$. Define

$$L_{\lambda,s}^n(z) = L_{\lambda,s_0} \circ \cdots \circ L_{\lambda,s_{n-1}}(z)$$

We claim that

$$\lim_{n \to \infty} L_{\lambda,s}^n(z)$$

exists and is independent of z.

To see this, note that $E_\lambda(\mathrm{Int}\, V_\alpha^\beta(s_j))$ is a simply connected region in $\mathbb{C}$ for each j. Also, L_{λ,s_j} maps this region strictly inside itself and so each L_{λ,s_j} is a strict contraction in the Poincaré metric on $E_\lambda(\mathrm{Int}\, V_\alpha^\beta(s_j))$. In particular, each L_{λ,s_j} is
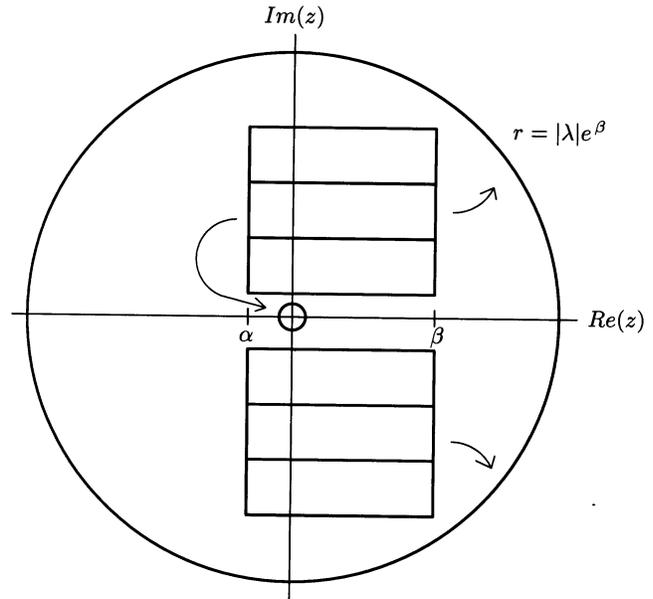

Fig. 5. $E_\lambda(V_\alpha^\beta(s_j)) \supset V_\alpha^\beta(s_k)$ for all k.

a strict contraction in the Poincaré metric on the subset V_α^β. As a consequence, the sets $L_{\lambda,s}^n(V_\alpha^\beta)$ are nested and decrease in diameter to 0 as $n \to \infty$. Hence $\lim_{n\to\infty} L_{\lambda,s}^n(z)$ exists and is independent of z.

We therefore define $\Phi(s) = \lim_{n\to\infty} L_{\lambda,s}^n(z)$ for any $z \in V_\alpha^\beta$. Standard arguments (see [Blanchard, 1984, Theorem 9.9]) then show that Φ is a homeomorphism which gives the conjugacy between E_λ and the shift map.   ∎

Given λ, K, we define $z_\lambda(s)$ to be the unique point in $\Lambda_{K,\lambda}$ whose itinerary under E_λ is s.

Corollary 2.4. *Suppose $s = \overline{s_0 \cdots s_{n-1}}$ is a repeating sequence in Σ_K. Then $z_\lambda(s)$ is a repelling periodic point of period n for E_λ.*

Proof. Since $L_{\lambda,s}^n$ is a composition of analytic maps, it follows that $L_{\lambda,s}^n$ is analytic. Also, $L_{\lambda,s}^n(V_\alpha^\beta(s_0))$ is contained in the interior of $V_\alpha^\beta(s_0)$. Since $L_{\lambda,s}^n$ is a strict contraction in the Poincaré metric on $V_\alpha^\beta(s_0)$, it follows that $L_{\lambda,s}^n$ has a unique fixed point in this rectangle and that this fixed point is attracting for $L_{\lambda,s}^n$, hence repelling for E_λ. Since this point has itinerary s for E_λ, it must be $z_\lambda(s)$ and the result follows.   ∎

Corollary 2.5. *Let $s \in \Sigma_K$. Then $z_\lambda(s)$ lies in the Julia set of E_λ.*

Proof. By the previous corollary, $z_\lambda(s)$ is a limit of repelling periodic points given by the conjugacy with the shift map. By a result of Baker [1968], $J(E_\lambda)$ is the closure of the set of repelling periodic points. Hence $z_\lambda(s) \in J(E_\lambda)$. ∎

Remarks

1. The hypothesis that the $s_j \neq 0$ cannot be dropped in the theorem or its corollaries. For example, if $\lambda \in \mathbb{R}$ and $\lambda > 1/e$, then E_λ has two repelling fixed points in the strip $R_\lambda(0)$. See [Devaney & Krych, 1984]. However, when $0 < \lambda < 1/e$, we may allow sequences with 0's, as we show in the next section.
2. It is important to note that $z_\lambda(s)$ is not the only point in the strip $R_\lambda(s_0)$ that has itinerary s. In fact, there are infinitely many points in this strip that share the itinerary s. These are the hairs that we deal with in the next chapter.
3. It is possible to give a more complete symbolic description of the dynamics of E_λ using both unbounded and nonregular itineraries. See [Devaney & Krych, 1984], where this is shown for e^z. However, we will not make use of this fact in this paper.

2.3. The case $0 < \lambda < 1/e$

The case where $0 < \lambda < 1/e$ will play a special role in the remainder of this paper. In this case we do not need to restrict to regular sequences. All of the results below appear elsewhere [Aarts & Oversteegen, 1993; Devaney & Tangerman, 1986], but we include them here for completeness.

Let Σ'_K denote the set of all one-sided sequences $(s_0 s_1 s_2 \ldots)$ where each $s_j \in \mathbb{Z}$ and $|s_j| \leq K$. We specifically allow 0 as an entry for sequences in Σ'_K.

When $\lambda = 1/e$, E_λ has a fixed point at 1 and $E'_\lambda(1) = 1$. If $\lambda \in (0, 1/e)$ then E_λ has two fixed points in $\mathbb{R}^+$. We denote them by a_λ and r_λ where $0 < a_\lambda < 1 < r_\lambda$. One checks easily that a_λ is attracting while r_λ is repelling. Also, $E_\lambda(1) < 1$. See Fig. 6.

Since λ is real, the horizontal strips $R(k)$ are bounded above and below by Im $z = (2k \pm 1)\pi$ for each k. The quantity M_K is therefore given by $M_K = (2K + 1)\pi$.

For any $\tau > 1$, we define the collection of rectangles $\tilde{V}_1^\tau$ by

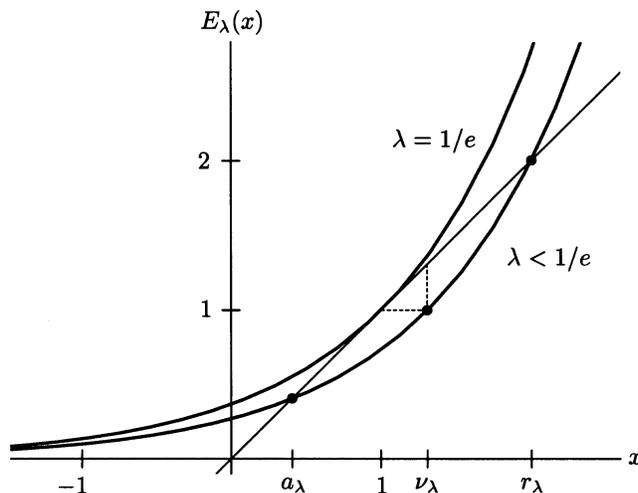$$\tilde{V}_1^\tau = \{z \mid 1 \leq \text{Re } z \leq \tau, \, |\text{Im } z| \leq M_K\}.$$

Fig. 6. The graphs of E_λ for $\lambda = 1/e$ and $\lambda < 1/e$.

The only difference between $\tilde{V}_1^\tau$ and V_1^τ is that we now include $V_1^\tau(0)$, the rectangle in the 0-strip $R_\lambda(0)$, in $\tilde{V}_1^\tau$.

Note also that if $\tau = \tau(\lambda)$ is chosen so that $\lambda e^\tau > \tau + M_K$, then E_λ maps the vertical line Re $z = \tau$ to a circle of radius λe^τ centered at 0. This circle contains all of $\tilde{V}_1^\tau$ in its interior. Also, E_λ maps the vertical line Re $z = 1$ to a circle of radius smaller than 1. It follows that $L_{\lambda, s_i}(\tilde{V}_1^\tau)$ is contained in the interior of $\tilde{V}_1^\tau$ for each i. See Fig. 7.

Let $\nu_\lambda = -\log \lambda$ so that $E_\lambda(\nu_\lambda) = 1$. Note that $1 < \nu_\lambda < r_\lambda$. Hence E_λ maps the strip $1 \leq \text{Re } z < \nu_\lambda$ to the annular region $\lambda e \leq |z| < 1$ which lies in the exterior of $\tilde{V}_1^\tau$. Therefore, if $z \in \tilde{V}_1^\tau$, then Re $L_{\lambda, s_i}(z) \geq \nu_\lambda$ for each i. Also, if Re $z \geq \nu_\lambda$, then $0 < |L'_{\lambda, s_i}(z)| < \omega_\lambda$ for some constant $\omega_\lambda < 1$. In fact, $\omega_\lambda = 1/\nu_\lambda$. It follows that for any sequence $s_0 s_1 s_2 \ldots$ with $|s_j| \leq K$ and any $z \in \tilde{V}_1^\tau$, we have

$$\left| \frac{d}{dz}(L_{\lambda, s}^n)(z) \right| < \omega_\lambda^{n-1}$$

since it takes at most one iteration of L_{λ, s_i} to guarantee that Re $L_{\lambda, s_i}(z) \geq \nu_\lambda$.

Let $\Lambda'_{K, \lambda}$ denote the set of points whose orbit remains for all time in $\tilde{V}_1^\tau$. Using the above facts, we may mimic the proof of Theorem 2.3 to prove:

Theorem 2.6. *Suppose τ is chosen so that $\lambda e^\tau > \tau + M_K$. Then $\Lambda'_{K, \lambda}$ is homeomorphic to Σ'_K and $E_\lambda | \Lambda'_{K, \lambda}$ is conjugate to the shift map on Σ'_K.*
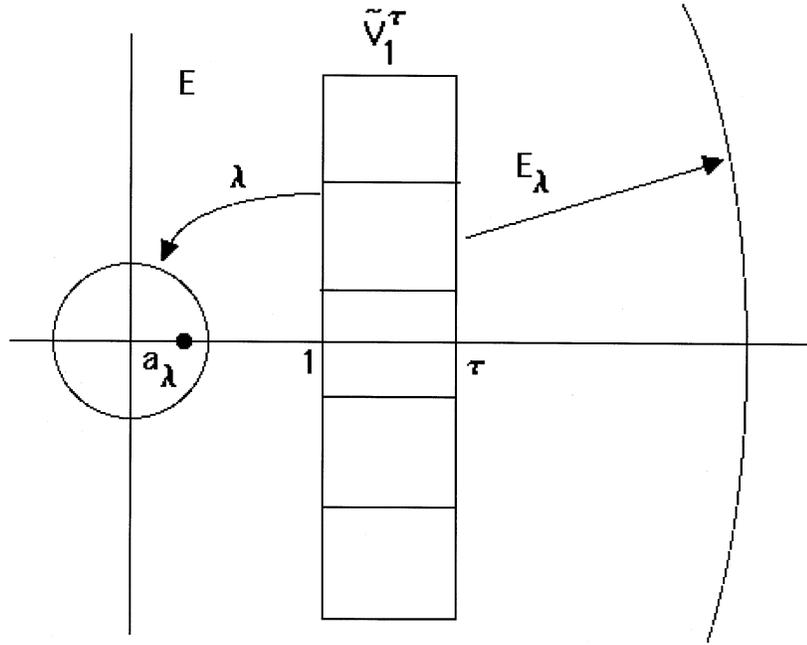
Fig. 7.   Image of $\tilde{V}_1^\tau$ under E_λ.

3. Hairs in the Dynamical Plane

Our goal for this section is to show that each point in $\Lambda_{K,\lambda}$ (or $\Lambda'_{K,\lambda}$, if $0 < \lambda < 1/e$) actually lies at the endpoint of a continuous curve, all points of which share the same itinerary.

Definition 3.1.   A continuous curve $h_{\lambda,s} : [1, \infty) \to R_\lambda(s_0)$ is called a *hair* attached to $z_\lambda(s)$ if

1. $h_{\lambda,s}(1) = z_\lambda(s)$.
2. For each $t \geq 1$, the itinerary of $h_{\lambda,s}(t)$ under E_λ is s.
3. If $t > 1$, then $\lim_{n \to \infty} \operatorname{Re} E_\lambda^n(h_{\lambda,s}(t)) = \infty$.
4. $\lim_{t \to \infty} \operatorname{Re} h_{\lambda,s}(t) = \infty$.

Roughly speaking, a hair attached to $z_\lambda(s)$ is a continuous curve stretching from $z_\lambda(s)$ to infinity in the right half plane. Any point z on this hair with $z \neq z_\lambda(s)$ shares the same itinerary as $z_\lambda(s)$ and has an orbit which tends to infinity in the right half plane. Thus each hair lies in the Julia set of E_λ. On the other hand, the orbit of the endpoint $z_\lambda(s)$ remains bounded.

We will show that each point $z_\lambda(s) \in \Lambda_{K,\lambda}$ has a hair attached to it, and that, up to parametrization, this hair is unique.

We will often encounter the case where $\lambda = 1/e$ in the sequel, so we choose a simplified notation in this case. Define $E(z) = (1/e)e^z$. Note that E has

a fixed point at 1 and $E'(1) = 1$. Also, $E^n(t) \to \infty$ as $n \to \infty$ if $t > 1$. See Fig. 8.

For any sequence $s = s_0 s_1 s_2 \ldots$ we define the functions $G_s^n : \mathbb{C}' \times [1, \infty) \to \mathbb{C}$ by

$$G_s^n(\lambda, t) = L_{\lambda,s}^n \circ E^n(t).$$

Note that G_s^n is well defined provided that $L_{\lambda,s}^n \circ E^n(t)$ does not lie in $\mathbb{R}^- \cup \{0\}$ for each j. This is one of the reasons for our assumption in Sec. 3.2 that each $s_j \neq 0$, as this implies that $L_{\lambda,s_j}(z)$ never
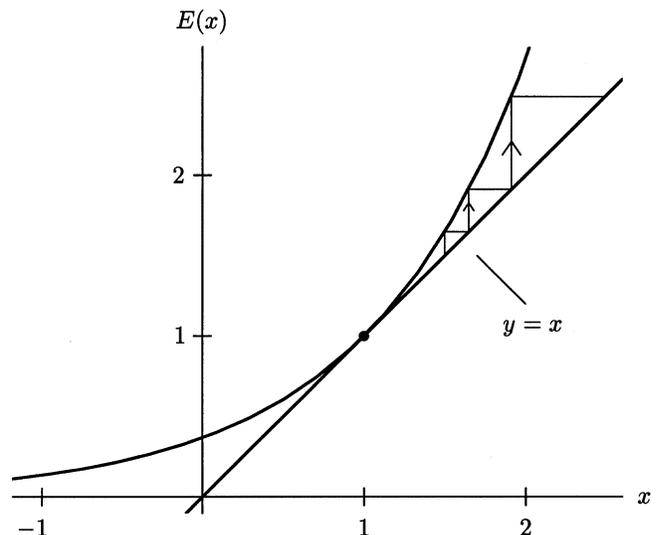

Fig. 8.   The graph of $E(x) = (1/e)e^x$.

meets $\mathbb{R}^- \cup \{0\}$. Our goal is to show that, for each $t \geq 1$, the G_s^n converge uniformly as a function of λ to an analytic function. For each fixed λ, this limit function will give a parametrization of the hair $h_{\lambda,s}$ as a function of t.

3.1. *The case* $0 < \lambda < 1/e$

In this section, we will consider hairs for the special case where $\lambda \in [\alpha, 1/e)$ for some $0 < \alpha < 1/e$. In the next section we will consider the general case. Fix a sequence $s = (s_0 s_1 s_2 \ldots) \in \Sigma'_K$.

Proposition 3.2. Suppose $\lambda \in [\alpha, 1/e)$ with $\alpha > 0$. Let $s = s_0 s_1 s_2 \cdots \in \Sigma'_K$. There is a constant $M = M(\lambda)$ such that, for all $n \geq 0$ and $t \geq 1$,

$$t \leq \operatorname{Re} G_s^n(\lambda, t) \leq t + M.$$

Proof. Recall that the constant $M_K = (2K + 1)\pi$ satisfies

$$|\operatorname{Im} G_{\sigma^i(s)}^n(\lambda, t)| \leq M_K$$

for all i and n since the itinerary s is bounded. Here, $\sigma^i(s) = (s_i s_{i+1} s_{i+2} \ldots)$. Choose $M = M(\lambda) \geq M_K$ so that, in addition,

$$M > \log(2M) - \log \lambda + \log 2.$$

Now we use induction. We have

$$L_{\lambda, s_0}(z) = -\log \lambda + \log |z| + 2\pi i s_0 + i \arg z.$$

Therefore

$$\begin{aligned}
\operatorname{Re} L_{\lambda, s_0}((1/e)e^t) &= -\log \lambda + \log((1/e)e^t) \\
&= -\log \lambda - 1 + t \\
&\geq t
\end{aligned}$$

since $\lambda \leq 1/e$. Also,

$$\begin{aligned}
\operatorname{Re} L_{\lambda, s_0}((1/e)e^t) &= -\log \lambda - 1 + t \\
&< -\log \lambda + \log 2 + t \\
&< t + M - \log(2M) \\
&\leq t + M
\end{aligned}$$

by our assumption on M. This gives the result when $n = 1$. Now let us assume that

$$t \leq \operatorname{Re} G_u^{n-1}(\lambda, t) \leq t + M$$

for any sequence u with $|u_j| \leq K$ for all j. We write $G_{\sigma(s)}^{n-1}(\lambda, E(t)) = \beta + i\nu$.

By induction we have

$$E(t) \leq \beta \leq E(t) + M.$$

Hence $1 + \log \beta \geq t$. Also $|\nu| \leq M_K < M$ by hypothesis.

Thus

$$\begin{aligned}
\operatorname{Re} G_s^n(\lambda, t) &= \operatorname{Re} L_{\lambda, s_0} G_{\sigma(s)}^{n-1}(\lambda, E(t)) \\
&= -\log \lambda + \log |\beta + i\nu| \\
&\geq -\log \lambda + \log \beta \\
&\geq 1 + \log \beta \\
&\geq t.
\end{aligned}$$

For the other inequality, we have

$$\operatorname{Re} G_{\sigma(s)}^{n-1}(\lambda, E(t)) = \beta \leq E(t) + M.$$

Then

$$\begin{aligned}
\operatorname{Re} G_s^n(\lambda, t) &= \operatorname{Re} L_{\lambda, s_0} G_{\sigma(s)}^{n-1}(\lambda, E(t)) \\
&= -\log \lambda + \log |\beta + i\nu| \\
&\leq -\log \lambda + \log(E(t) + M + M).
\end{aligned}$$

This follows since $\beta \leq E(t) + M$ and $|\nu| < M_K < M$. Recall that $\log(a + b) < \log a + \log b$ if both $a, b > 2$. Using this we have, provided $E(t) > 2$,

$$\begin{aligned}
\operatorname{Re} G_s^n(\lambda, t) &\leq -\log \lambda - 1 + t + \log(2M) \\
&\leq -\log \lambda + \log 2 + \log(2M) + t \\
&\leq t + M.
\end{aligned}$$

If $E(t) \leq 2$, then we have

$$\begin{aligned}
\operatorname{Re} G_s^n(\lambda, t) &\leq -\log \lambda + \log(E(t) + M + M) \\
&\leq -\log \lambda + \log(2(M + 1)) \\
&\leq -\log \lambda + \log 2 + \log(M + 1) \\
&\leq -\log \lambda + \log 2 + \log(2M) \\
&\leq t + M. \quad \blacksquare
\end{aligned}$$

This proposition shows that $G_s^n(\lambda, t)$ lies in a bounded region in the plane, since $|\operatorname{Im} G_s^n(\lambda, t)| \leq M_K$ by our assumption that $|s_j| \leq K$ for each j.

Proposition 3.3. Let $\lambda \in [\alpha, 1/e)$ with $\alpha > 0$. For any sequence $s \in \Sigma'_K$ and any $t \geq 1$, the family of functions

$$\lambda \to G_s^n(\lambda, t)$$

converges uniformly in λ on compact subintervals of $[\alpha,\, 1/e)$ for fixed $t \in [\alpha,\, 1/e)$ as $n \to \infty$.

Proof. Choose β such that $\alpha < \beta < 1/e$ and let $\varepsilon > 0$. We will demonstrate the uniform convergence of the G_s^n on the interval $[\alpha,\, \beta]$ for each $t \geq 1$. Let $M^* = \max M(\lambda)$ for $\lambda \in [\alpha,\, \beta]$, where $M(\lambda)$ is as in Proposition 3.2.

Since we have $|s_j| \leq K$, it follows that $|\operatorname{Im} z| \leq M_K$ for any z lying in one of the strips $R_\lambda(s_j)$. By Proposition 3.2, we have, for each $\lambda \in [\alpha,\, \beta]$,

$$t \leq \operatorname{Re} G_s^n(\lambda,\, t) \leq t + M^*$$

for any s and all $t \geq 1$. Thus we have, for any $n,\, m \geq 0$

$$|G_u^n(\lambda,\, t) - G_u^m(\lambda,\, t)| \leq M^* + 2M_K$$

for any sequence $u = u_0 u_1 u_2 \ldots$ with $|u_j| \leq K$.

Now recall that we may choose τ' large enough so that for any $\tau \geq \tau'$

(1) $L_{\lambda, s_i}(\tilde{V}_1^\tau) \subset$ interior $\tilde{V}_1^\tau$ for each $\lambda \in [\alpha,\, \beta]$, and
(2) There exists $\omega = \max_{\lambda \in [\alpha,\, \beta]} \omega(\lambda) < 1$ so that

$$|(L_{\lambda, s}^N)'(z)| < \omega^{N-1}$$

for all $z \in \tilde{V}_1^\tau$.

Suppose $t > 1$. We first choose N so that $(M^* + 2M_K)\omega^{N-1} < \varepsilon$. We then choose $\tau > \tau'$ so that $E^N(t) + M^* < \tau$. Then for any $n,\, m \geq 0$, we have

$$
\begin{aligned}
&|G_s^{N+n}(\lambda,\, t) - G_s^{N+m}(\lambda,\, t)| \\
&= |L_\lambda^N G_{\sigma^N(s)}^n(\lambda,\, E^N(t)) - L_\lambda^N G_{\sigma^N(s)}^m(\lambda,\, E^N(t))| \\
&\leq \omega^{N-1} |G_{\sigma^N(s)}^n(\lambda,\, E^N(t)) - G_{\sigma^N(s)}^m(\lambda,\, E^N(t))| \\
&\leq \omega^{N-1}(M^* + 2M_K) < \varepsilon.
\end{aligned}
$$

For the case $t = 1$, we set

$$\tau^* = \max_{\lambda \in [\alpha,\, \beta]} \tau(\lambda)$$

(in fact, choosing τ^* such that $\log(\tau^*/\alpha) < \tau^*$ will suffice).

Since $L_{\lambda, s_i}(V_1^\tau) \subset V_1^\tau$, by induction we have $L_{\lambda, s}^n(V_1^\tau) \subset V_1^\tau$. This allows us to infer that

$$|L_{\lambda, s}^n(1) - L_{\lambda, s}^m(1)| \leq 2(\tau^* + M_K)$$

for each $n,\, m \geq 0$. Now choose $N > 0$ to satisfy

$$2\omega^{N-1}(\tau^* + M_K) < \varepsilon.$$

Then the proof follows as above. ∎

Since the functions $G_s^n(\lambda,\, t)$ converge uniformly on compact subintervals of $[\alpha,\, 1/e)$ for each fixed t, we define $h_{\lambda, s}(t) = \lim_{n \to \infty} G_s^n(\lambda,\, t)$. In the next section we will prove that $h_{\lambda, s}$ has the properties of a hair as listed in Definition 3.1. In the next section we will also use this result to prove uniform convergence for all $\lambda \in \mathbb{C}'$.

3.2. Hairs in the dynamical plane: the general case

Our goal in this section is to prove that if $s = s_0 s_1 s_2 \ldots$ is a regular, bounded sequence, then there is a unique hair attached to $z_\lambda(s)$ for each $\lambda \in \mathbb{C}'$. We define $G_s^n(\lambda,\, t)$ exactly as in the previous section. Since we can no longer guarantee that $L_{\lambda, 0}(z)$ does not meet 0 or the negative real axis, we suppose that $s_j \neq 0$ for each j, i.e. our sequence is regular.

We first prove

Proposition 3.4. *Suppose $s = s_0 s_1 s_2 \cdots \in \Sigma_K$. For each fixed $t \geq 1$ the family of functions $\{\lambda \to G_s^n(\lambda,\, t)\}$ is a normal family of functions on $\mathbb{C}'$. Moreover, this family converges uniformly as $n \to \infty$ to an analytic function $\lambda \to h_{\lambda, s}(t)$ for each t.*

Proof. Since each $s_j \neq 0$, we have that $G_s^n(\lambda,\, t)$ is well defined. For each n, $G_s^n(\lambda,\, t)$ takes values in the s_0-strip for E_λ. The union of these strips as λ varies over $\mathbb{C}'$ forms an open horizontal strip of width 4π. Hence $\lambda \to G_s^n(\lambda,\, t)$ omits many more than three values in $\mathbb{C}$. By Montel's theorem, it follows that $\{G_s^n(\lambda,\, t)\}$ is a normal family of functions.

Thus, any sequence of the G_s^n contains a subsequence that converges uniformly to an analytic function or to ∞. However, by the results given in the previous section, the sequence $G_s^n(\lambda,\, t)$ converges uniformly for λ in compact subintervals of the form $[\alpha,\, 1/e)$ with $0 < \alpha < 1/e$. Since an analytic function is completely determined by its values along any arc in an open region, it follows that the $G_s^n(\lambda,\, t)$ for general λ converge uniformly to an analytic function on all of $\mathbb{C}'$. ∎

For each $s \in \Sigma_K$ and $t \geq 1$, we now have defined a function

$$h_{\lambda, s}(t) = \lim_{n \to \infty} G_s^n(\lambda,\, t)$$

which is analytic as a function of λ for $\lambda \in \mathbb{C}'$. Note that, when $t = 1$, we have $G_s^n(\lambda, 1) = L_{\lambda,s}^n(1)$. It follows from the results in Sec. 3.1 that $h_{\lambda,s}(1) = z_\lambda(s)$.

We claim now that $t \to h_{\lambda,s}(t)$ is continuous on $[1, \infty)$ and parametrizes the hair attached to $z_\lambda(s)$. To prove this we need a preliminary result, which generalizes Proposition 3.2 in the last section.

Proposition 3.5. *Suppose $s = s_0 s_1 s_2 \cdots \in \Sigma_K$. Given $\lambda \in \mathbb{C}'$, there exists τ, $M > 0$ such that, for all $t \geq \tau$ and $n \geq 1$, we have*

$$t - M \leq \operatorname{Re} G_s^n(\lambda, t) \leq t + M.$$

Proof. Since $|s_i| \leq K$, there exists $M_K > 2\pi$ such that $|\operatorname{Im} L_{\lambda,s_i}(z)| < M_K$ for each s_i and all $z \in \mathbb{C}'$.

Given λ, let us choose $M > 2$ so that the following inequalities hold:

$$M \geq \operatorname{Re}(\log \lambda) + 1 + \log M \qquad (1)$$

$$M \geq \operatorname{Re}(-\log \lambda) - 1 + \log M + \log M_K. \qquad (2)$$

Suppose τ satisfies

$$E(\tau) = \frac{1}{e} e^\tau > \frac{M^2}{M-1} + 1 > M + 2.$$

If $t \geq \tau$, we have $E(t) \geq E(\tau)$, so

$$E(t) > \frac{M^2}{M-1} + 1 > M + 2.$$

Now let us use induction to prove that, for any sequence s with $|s_i| \leq K$,

$$t - M \leq \operatorname{Re} G_s^n(\lambda, t)$$

for all n and all $t \geq \tau$.

We have

$$
\begin{aligned}
\operatorname{Re} G_s^1(\lambda, t) &= \operatorname{Re} L_{\lambda,s_0}(E(t)) \\
&= \operatorname{Re}(-\log \lambda) + \operatorname{Re}(\log (E(t)) \\
&= \operatorname{Re}(-\log \lambda) - 1 + t \\
&\geq t - M + \log M
\end{aligned}
$$

by Eq. (1). Hence $\operatorname{Re} G_s^1(\lambda, t) \geq t - M$, which gives the result when $n = 1$.

Now let us assume that

$$\operatorname{Re} G_u^n(\lambda, t) \geq t - M$$

for any bounded sequence $u = u_0 u_1 u_2 \ldots$ with $|u_j| \leq K$ and all $t \geq \tau$. We thus have

$$\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \geq E(t) - M$$

so

$$
\begin{aligned}
\operatorname{Re} G_s^{n+1}(\lambda, t) &= \operatorname{Re} L_{\lambda,s_0}(G_{\sigma(s)}^n(\lambda, E(t))) \\
&= \operatorname{Re}(-\log \lambda) + \operatorname{Re}(\log G_{\sigma(s)}^n(\lambda, E(t))) \\
&= \operatorname{Re}(-\log \lambda) + \log|G_{\sigma(s)}^n(\lambda, E(t))| \\
&\geq \operatorname{Re}(-\log \lambda) + \log|\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))| \\
&= \operatorname{Re}(-\log \lambda) + \log(\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))).
\end{aligned}
$$

This last equality follows since $\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \geq E(t) - M > 2$.

Therefore

$$
\begin{aligned}
\operatorname{Re} G_s^{n+1}(\lambda, t) &\geq \operatorname{Re}(-\log \lambda) + \log(E(t) - M) \\
&\geq \operatorname{Re}(-\log \lambda) + \log E(t) - \log M.
\end{aligned}
$$

This inequality follows from the fact that

$$\log(a - b) \geq \log a - \log b$$

provided $a > b > 1$ and $a > b^2/(b-1)$. In our case, we have $E(t) > E(\tau) > M > 1$ and $E(t) > M^2/(M-1)$ by assumption.

Thus

$$\operatorname{Re} G_s^{n+1}(\lambda, t) \geq \operatorname{Re}(-\log \lambda) - 1 + t - \log M \geq t - M$$

because of assumption (1).

Now we turn to the proof that

$$\operatorname{Re} G_s^n(\lambda, t) \leq t + M$$

for all $n \geq 1$ and all $t \geq \tau$.

We have

$$
\begin{aligned}
\operatorname{Re} G_s^1(\lambda, t) &= \operatorname{Re} L_{\lambda,s_0}(E(t)) \\
&= \operatorname{Re}(-\log \lambda) + \log(E(t)) \\
&= \operatorname{Re}(-\log \lambda) - 1 + t \\
&\leq t + M - \log M - \log M_K \\
&\leq t + M
\end{aligned}
$$

by (2). (Recall both M, $M_K > 1$.) This proves the case $n = 1$.

Now suppose that

$$\operatorname{Re} G_u^n(\lambda, t) \leq t + M$$

for any bounded sequence $u = u_0 u_1 u_2 \dots$ with $|u_j| \leq K$.

Then we have

$$\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \leq E(t) + M.$$

Now

$$
\begin{aligned}
\operatorname{Re} G_s^{n+1}(\lambda, t) &= \operatorname{Re}(L_{\lambda, s_0}(G_{\sigma(s)}^n(\lambda, E(t)))) \\
&= \operatorname{Re}(-\log \lambda) + \operatorname{Re}(\log G_{\sigma(s)}^n(\lambda, E(t))) \\
&= \operatorname{Re}(-\log \lambda) + \log|G_{\sigma(s)}^n(\lambda, E(t))| \\
&\leq \operatorname{Re}(-\log \lambda) + \log(|\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))| \\
&\quad + |\operatorname{Im} G_{\sigma(s)}^n(\lambda, E(t))|) \\
&\leq \operatorname{Re}(-\log \lambda) + \log(\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t))) \\
&\quad + \log M_K.
\end{aligned}
$$

This last inequality follows from the fact that $\log(a+b) \leq \log a + \log b$ provided both $a, b > 2$ and the fact that

$$\operatorname{Re} G_{\sigma(s)}^n(\lambda, E(t)) \geq E(t) - M \geq E(\tau) - M \geq 2.$$

This uses the previous induction.

We thus have

$$
\begin{aligned}
\operatorname{Re} G_s^{n+1}(\lambda, t) &\leq \operatorname{Re}(-\log \lambda) + \log(E(t) + M) \\
&\quad + \log M_K.
\end{aligned}
$$

Now, as above, $E(t) > M > 2$, so

$$\log(E(t) + M) \leq \log E(t) + \log M.$$

Thus

$$
\begin{aligned}
\operatorname{Re} G_s^{n+1}(\lambda, t) &\leq \operatorname{Re}(-\log \lambda) - 1 + t + \log M \\
&\quad + \log M_K \leq t + M
\end{aligned}
$$

by (2).   ∎

Remark. For later use, note that the previous proposition holds for all sequences in Σ_K', not just regular sequences. Indeed, for t large enough, the L_{λ, s_j} in the proof are bounded away from the half plane $\operatorname{Re} z \leq 0$, and hence these inverse maps are well defined. With this result in hand, we can now prove continuity with respect to t.

Proposition 3.6.   *Let $s \in \Sigma_K$. Then for each $\lambda \in \mathbb{C}'$, $h_{\lambda, s}(t)$ is continuous as a function of $t \in (1, \infty)$.*

Proof.   Let $\varepsilon > 0$ and choose ω with $0 < \omega < 1$. Let τ and M be as given in Proposition 3.5. Choose

$T > \tau + 2M$ so that, if $\operatorname{Re} z > T$ and $|\operatorname{Im} z| < M_K$, then

$$|L_{\lambda, s_i}'(z)| < \omega.$$

By the previous proposition, if $t > T$, then

$$E^k(t) - M \leq \operatorname{Re} G_s^n(\lambda, E^k(t)) \leq E^k(t) + M$$

for all $n, k \geq 0$. Now choose k so that $\omega^k(3M + 2\pi) < \varepsilon$.

Case 1.   Given $t_0 > T$, choose δ such that, if $|t - t_0| < \delta$, then $|E^k(t) - E^k(t_0)| < M$. We claim that, if $|t - t_0| < \delta$, then $|h_{\lambda, s}(t) - h_{\lambda, s}(t_0)| < \varepsilon$. To see this, we note that for each $n \geq 0$ we have

$$|G_{\sigma^k(s)}^n(\lambda, E^k(t)) - G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| < 3M + 2\pi.$$

This follows since

$$
\begin{aligned}
&|\operatorname{Re} G_{\sigma^k(s)}^n(\lambda, E^k(t)) - \operatorname{Re} G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| \\
&\quad < |E^k(t) - E^k(t_0)| + 2M < 3M
\end{aligned}
$$

and

$$|\operatorname{Im} G_{\sigma^k(s)}^n(\lambda, E^k(t)) - \operatorname{Im} G_{\sigma^k(s)}^n(\lambda, E^k(t))| < 2\pi.$$

Therefore

$$
\begin{aligned}
&|G_s^{n+k}(\lambda, t) - G_s^{n+k}(\lambda, t_0)| \\
&\quad = |L_{\lambda, s}^k \circ G_{\sigma^k(s)}^n(\lambda, E^k(t)) - L_{\lambda, s}^k \circ G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| \\
&\quad \leq \omega^k |G_{\sigma^k(s)}^n(\lambda, E^k(t)) - G_{\sigma^k(s)}^n(\lambda, E^k(t_0))| \\
&\quad \leq \omega^k(3M + 2\pi) < \varepsilon.
\end{aligned}
$$

It follows that the function $t \to h_{\lambda, s}(t)$ is continuous for any $s \in \Sigma_K$ and $t > T$.

Note that we did not use the assumption that $s_i \neq 0$ in this part of the proof.

Case 2.   To prove continuity for $1 < t \leq T$, we will assume from now on that $|s_i| > 0$ for all i. Observe that, if $1 < t < T$, then there exists k such that $E^k(t) > T$. Then we have

$$t \to L_{\lambda, s}^k \circ h_{\lambda, \sigma^k(s)}(E^k(t))$$

is continuous provided that each inverse is well defined and continuous. This is true by the assumption that $s_i \neq 0$ so that

$$L_{\lambda, s}^k \circ h_{\lambda, \sigma^k(s)} \circ E^k(t)$$

never meets 0 or $\mathbb{R}^-$ for each i. But this map is given by

$$t \to \lim_{n\to\infty} L^k_{\lambda,s} \circ G^n_{\sigma^k(s)} \circ E^k(t) = h_{\lambda,s}(t).$$

Hence we have continuity for all $t > 1$. ∎

Corollary 3.7. *Let s be any sequence in Σ'_K. For each $\lambda \in \mathbb{C}'$, $h_{\lambda,s}(t)$ exists and is continuous as a function of $t \in (T, \infty)$ for T sufficiently large.*

Proof. Choose τ as in Proposition 3.5 so that in addition, $\tau > M$. This assures that the family of functions $G^n_s(\lambda, t)$ is well defined for any $t > \tau$. Then Proposition 3.4 goes through exactly as before and the value

$$h_{\lambda,s}(t) = \lim_{n\to\infty} G^n_s(\lambda, t)$$

exists for each $t > \tau$. For continuity, we proceed exactly as in Proposition 3.6, choosing T as indicated and then following the proof in Case 1 to obtain continuity when $t > T$. ∎

The case of continuity at $t = 1$ is more delicate.

Proposition 3.8. *Let $s = (s_0 s_1 s_2 \ldots) \in \Sigma_K$. Then for each $\lambda \in \mathbb{C}'$, $h_{\lambda,s}$ is continuous at $t = 1$.*

Proof. Recall from Sec. 2.2 that for any β sufficiently large, $L_{\lambda,s_i}(z)$ maps the set V^β_α strictly inside itself. As in the proof of Theorem 2.3, each L_{λ,s_i} is a strict contraction on V^β_α. For a given t value sufficiently close to one, there exists a first integer N such that $E^N(t)$ is larger than the τ given to us in Proposition 3.5. For any $n > N$, we can invoke the inequality of Proposition 3.5 so that $G^n_{\sigma^N(s)}(\lambda, E^N(t)) \in V^\beta_\alpha$ (provided β is sufficiently large). Then we can use the Poincaré metric to show that the distance between $h_{\lambda,s}(1) = z_s(\lambda)$ and

$$h_{\lambda,s}(t) = \lim_{N\to\infty} L^N_{\lambda,s} \circ G^n_{\sigma^N(s)}(\lambda, E^N(t))$$

can be made arbitrarily small. ∎

Propositions 3.6 and 3.8 combine to show that the function $h_{\lambda,s}(t)$ is continuous for $t \in [1, \infty)$. We now prove that this function actually parametrizes a hair.

Theorem 3.9. *Suppose $s \in \Sigma_K$ is a regular sequence. Then there is a unique hair attached to* $z_\lambda(s)$ *and, moreover, $t \to h_{\lambda,s}(t)$ is a parametrization of this hair.*

Proof. We first verify that $h_{\lambda,s}$ is in fact a hair. We claim that $h_{\lambda,s}(t)$ has itinerary s. Note that

$$E_\lambda \circ h_{\lambda,s}(t) = \lim_{n\to\infty} E_\lambda \circ G^n_s(\lambda, t)$$
$$= \lim_{n\to\infty} G^{n-1}_{\sigma(s)}(\lambda, E(t))$$
$$= h_{\lambda,\sigma(s)}(E(t)).$$

It follows that

$$E^n_\lambda \circ h_{\lambda,s}(t) = h_{\lambda,\sigma^n(s)} \circ E^n(t).$$

Hence $E^n_\lambda(h_{\lambda,s}(t)) \in R_\lambda(s_n)$ as required. Also,

$$E^n(t) - M \le \operatorname{Re} E^n_\lambda \circ h_{\lambda,s}(t) \le E^n(t) + M$$

for n sufficiently large, where M is as given in Proposition 3.5. Therefore, $\operatorname{Re} E^n_\lambda h_{\lambda,s}(t) \to \infty$ as $n \to \infty$ as long as $t > 1$. Finally, since $t - M \le \operatorname{Re} h_{\lambda,s}(t) \le t + M$ for $t > \tau$, it follows that $\operatorname{Re} h_{\lambda,s}(t) \to \infty$ as $t \to \infty$. This proves that $h_{\lambda,s}$ parametrizes a hair.

We finally show that this hair is unique. Suppose this is not true. Then there are two hairs attached to $z_\lambda(s)$. Let U be the open set contained in $R_\lambda(s_0)$ bounded by these two hairs. The images of U under E^n_λ are contained within the images of the hairs attached to $E^n_\lambda(z_\lambda(s))$, which themselves are contained strictly between the lines $\operatorname{Im} z = (2s_i \pm 1)\pi - \arg \lambda$ for each s_i. Moreover, if $z \in U \cap \{\operatorname{Re} z > \nu\}$ for large ν, then $\operatorname{Re} E_\lambda(z) >> \operatorname{Re} z$, and so the orbit of z tends to ∞. This yields a wandering domain or domain at ∞ for E_λ, neither of which can occur (see [Baker, 1984; Eremenko & Lyubich, 1984; Goldberg & Keen, 1986]). ∎

Remarks

1. The hypothesis that s is regular cannot be removed, as it is known that there may be more than one hair attached to a point with bounded orbit if not all of the s_i are nonzero. A classification of which hairs attach to which points in the case where λ is a Misiurewicz point may be found in [Devaney & Jarque, 1997]. A similar program has been carried out for λ values for which E_λ has an attracting cycle in [Bhattarcharjee & Devaney, preprint].

2. In certain cases (e.g. $0 < \lambda \leq 1/e$, $\lambda = 1$) the existence of hairs with regular, unbounded itineraries has been demonstrated using other methods (see [Aarts & Oversteegen, 1993; Devaney & Krych, 1984]). The proof above does not extend to itineraries that grow too quickly. However, the parametrizations given above are necessary for our proof of existence of hairs in the parameter plane, to which we turn in the next section.

4. Hairs in the Parameter Plane

Our goal in this section is to show that there also exist hairs in the parameter plane for the exponential family. These hairs consist of λ-values for which the orbit of 0 under E_λ has a specified itinerary in Σ_K. The hairs have an endpoint which determines a λ-value for which the orbit of 0 is bounded, since $E_\lambda(0) = \lambda$. All other λ-values on the hair have the property that the orbit of 0 tends to ∞ with the specified itinerary. As a consequence, $J(E_\lambda) = \mathbb{C}$ for these λ-values.

Definition 4.1. Let $s = s_0 s_1 s_2 \ldots$. A continuous curve $H_s : [1, \infty) \to \mathbb{C}$ is called a *hair* with itinerary s if H_s satisfies:

1. If $\lambda = H_s(t)$ and $t > 1$, then $\operatorname{Re} E_\lambda^n(0) \to \infty$ and the itinerary of λ under E_λ is s.
2. If $\lambda = H_s(1)$, then $E_\lambda(0) = \lambda = z_\lambda(s)$, and so the orbit of λ under E_λ is bounded and has itinerary s.
3. $\lim_{t \to \infty} \operatorname{Re} H_s(t) = \infty$.

Remark. We use the term "hair" for curves in both the dynamical plane and parameter plane. When necessary, we use the terms dynamical hair and parameter hair to distinguish between them.

Our goal in this section is to prove the following result.

Theorem 4.2. *Suppose s is a bounded, regular sequence. Then there exists a hair in parameter space with itinerary s. Moreover, if s is periodic or preperiodic, then 0 is preperiodic under E_λ for $\lambda = H_s(1)$.*

The proof of the theorem depends upon several lemmas in which we will jump back and forth from dynamical to parameter plane, so we first sketch the main idea.

Given the regular itinerary s, we will construct a simply connected region Q_s in parameter space. Consider the map $F_t(\lambda) = h_{\lambda,s}(t)$. Note that F_t is a function of λ, and assigns to λ the point on the dynamical hair with itinerary s and time parameter t. According to Proposition 3.4, F_t is an analytic function of λ. We will show that F_t maps the closure of Q_s strictly into its interior so therefore F_t has a unique fixed point in Q_s. This fixed point is a λ-value that satisfies $\lambda = h_{\lambda,s}(t)$, so $\lambda = E_\lambda(0)$ lies on the hair in the dynamical plane that is attached to $z_\lambda(s)$. We therefore define the point $H_s(t)$ on the hair in the parameter plane as the unique fixed point of F_t for each $t \geq 1$. If $t > 1$ it follows that $E_\lambda^n(0) \to \infty$, whereas, if $t = 1$, 0 maps after one iteration of E_λ onto $z_\lambda(s)$ and so this orbit is bounded. As we vary t, the fixed point of F_t varies, and this curve of fixed points produces the hair in the parameter plane.

We define Q_s to be the union of the horizontal strips $R_\lambda(s_0)$ for $\lambda \in \mathbb{C}'$. Q_s is an open horizontal strip in $\mathbb{C}$ with height 4π bounded by horizontal lines $\operatorname{Im} z = (2s_0 - 2)\pi$ and $\operatorname{Im} z = (2s_0 + 2)\pi$. For each t, the dynamical hair $h_{\lambda,s}(t)$ lies in $R_\lambda(s_0) \subset Q_s$.

We now think of Q_s as a subset of the parameter plane. Given $\lambda \in Q_s$, $F_t(\lambda) = h_{\lambda,s}(t)$ is a point in Q_s, and so F_t maps Q_s into itself. Now this map either has a unique fixed point in the interior of Q_s or else the F_t-orbits of all points tend to a fixed point on the boundary or to ∞. We will show that the latter cases are impossible.

Lemma 4.3. *The map F_t has no fixed points on the horizontal boundaries of Q_s.*

Proof. Suppose λ_* is a fixed point for F_t on a horizontal boundary of Q_s. Then $\operatorname{Im} \lambda_* = (2s_0 \pm 2)\pi$. However, $R_{\lambda_*}(s_0)$ is given by $\{z | (2s_0 - 1)\pi - \arg \lambda_* < \operatorname{Im} z < (2s_0 + 1)\pi - \arg \lambda_*\}$. If $s_0 \neq \pm 1$, we have $|\arg \lambda_*| < \pi$. It follows that λ_* lies outside $R_{\lambda_*}(s_0)$, and so λ_* cannot lie on a hair lying in this strip. If $s_0 = \pm 1$, we have $\arg \lambda_* = \pm \pi$ along one of the horizontal boundaries of Q_s. But this boundary is the negative real axis, and it is easy to check that all such points lie in the basin of attraction of a real attracting fixed point or two-cycle. Consequently, none of these points lie on a hair. ∎

For each $\nu < 0$, $\eta > 0$, define

$$Z_{\nu,\eta} = \{\lambda \in \mathbb{C} | \operatorname{Re} \lambda \leq \nu, \ |\operatorname{Im} \lambda| \leq \eta\}.$$

$Z_{\nu,\eta}$ is a closed half strip in the parameter plane.

$E_\lambda^2(W_\lambda)$
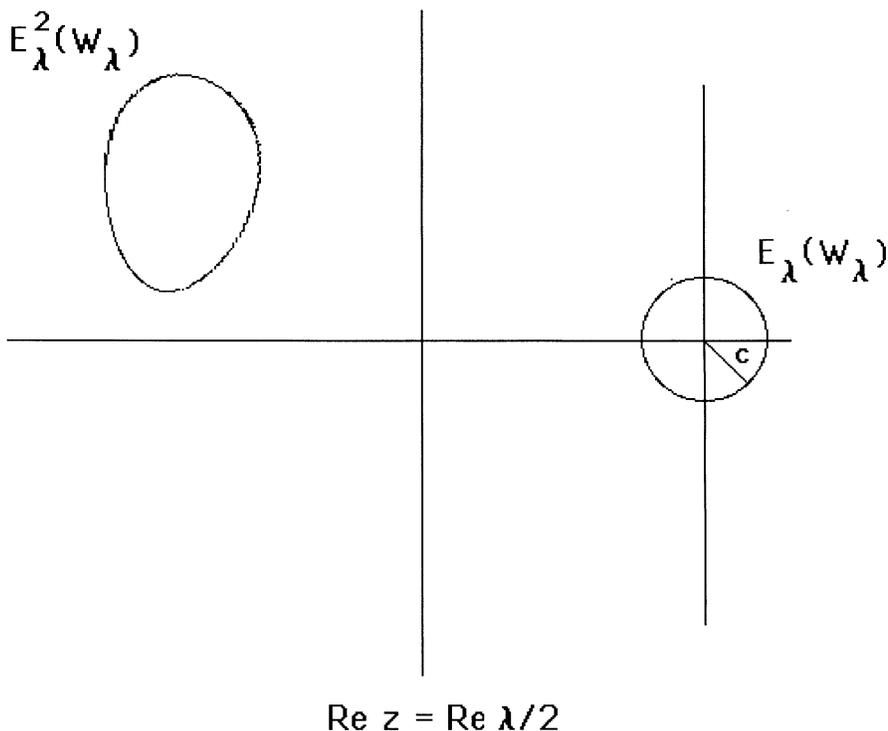
$E_\lambda(W_\lambda)$

c

Re z = Re λ/2

Fig. 9. $E_\lambda^2(W_\lambda) \subset W_\lambda$.

Lemma 4.4.   *For each $\eta \in \mathbb{R}^+$, there is $\nu = \nu(\eta) < 0$ such that if $\lambda \in Z_{\nu,\eta}$, then*

1. *E_λ has an attracting two-cycle, and*
2. *$W_\lambda = \{z | \mathrm{Re}\, z \leq \mathrm{Re}(\lambda/2)\}$ is contained in the basin of attraction of the two-cycle.*

Proof.   Fix $\eta > 0$ and suppose that $|\mathrm{Im}\, \lambda| \leq \eta$. Define

$$c = c(\lambda) = |\lambda| \exp\left(\mathrm{Re}\left(\frac{\lambda}{2} \right) \right).$$

Note that $|\lambda| c(\lambda) \to 0$ and $|\mathrm{Re}\, \lambda| c(\lambda) \to 0$ as $\mathrm{Re}\, \lambda \to -\infty$ since $|\mathrm{Im}\, \lambda|$ is bounded.

We may choose $\nu = \nu(\eta)$ such that, if $\lambda \in Z_{\nu,\eta}$, then

$$|\lambda| c e^c \leq (|\mathrm{Re}\, \lambda| + \eta) c e^c \leq \frac{|\mathrm{Re}\, \lambda|}{4}$$

since $(|\mathrm{Re}\, \lambda| + \eta) c e^c$ tends to 0 as $\mathrm{Re}\, \lambda \to -\infty$.

We claim that E_λ^2 maps W_λ inside itself. To see this, first note that $E_\lambda(W_\lambda)$ is a punctured disk of radius c centered at 0. On this disk

$$|E_\lambda'(z)| = |E_\lambda(z)| = |\lambda| \exp(\mathrm{Re}\, z)$$

which is at most $|\lambda| e^c$. Clearly, $E_\lambda(0) = \lambda$, so points

$z \in E_\lambda(W_\lambda)$ are mapped a maximum distance of

$$|E_\lambda(z) - \lambda| \leq \max_{z \in E_\lambda(W_\lambda)} |E_\lambda'(z)| \cdot |z - 0|$$

$$\leq |\lambda| e^c \cdot c$$

$$\leq \frac{|\mathrm{Re}\, \lambda|}{4}$$

away from λ under E_λ, and thus well to the left of the line $\mathrm{Re}\, z = \mathrm{Re}\, \lambda/2$ (see Fig. 9).

As a consequence, $E_\lambda^2(W_\lambda)$ is contained in the interior of W_λ provided $\lambda \in Z_{\nu,\lambda}$. By the Schwarz lemma, E_λ^2 has an attracting fixed point in W_λ and, moreover, each point in W_λ tends to this point under iteration of E_λ^2. This fixed point gives an attracting two-cycle for E_λ.   ∎

Corollary 4.5.   *Let $s \in \Sigma_K$.   Let $\eta = \sup \{|\mathrm{Im}\, z| | z \in Q_s\}$. Then, if $\lambda \in Z_{\nu(\eta),\eta}$, we have*

$$\mathrm{Re}\, F_t(\lambda) = \mathrm{Re}\, h_{\lambda,s}(t) > \mathrm{Re}\, \lambda$$

for all t.

Proof.   By the previous lemma, any point in the Julia set lies in the half plane $\mathrm{Re}\, z > \mathrm{Re}\, \lambda/2$. Since $\lambda \in Z_{\nu(\eta),\eta}$, we have $\mathrm{Re}\, \lambda < \mathrm{Re}\, \lambda/2 < 0$. It follows that $\mathrm{Re}\, z > \mathrm{Re}\, \lambda$ for all points in the Julia set of E_λ. Since $h_{\lambda,s}(t)$ lies on a hair in the Julia set, it

follows that this point lies to the right of the line $\operatorname{Re} z = \operatorname{Re} \lambda$ as well. ∎

Recall from Sec. 2.2 that

$$V_\alpha^x(s_i) = \{z \in \overline{R(s_i)} | \alpha \le \operatorname{Re} z \le x\}$$

and that $V_\alpha^x = \cup_{0<|j|\le K} V_\alpha^x(j)$. Recall also from Sec. 2.2 that m_K is the smaller of the vertical distances from 0 to R($\pm$1) and that M_K is the largest. Note that $(2K+2)\pi > M_K$. Let $\alpha = \alpha(\lambda) = \log(m_K/(2|\lambda|))$. Finally, we write $E(z) = (1/e)e^z$ as before.

For a given $t \ge 1$, we may choose $\lambda_1 = \lambda_1(t)$ so that the following three properties hold:

1. $\lambda_1(t) > t \ge 1$,
2. for each $x > t$, we have $\lambda_1 e^x > E(x)+(2K+2)\pi$, and
3. if $\operatorname{Re} \lambda > \lambda_1$, then $|\lambda| > \log 2 + \log|\lambda|+(2K+2)\pi$.

Then, if $\lambda \in Q_s$, $\operatorname{Re} \lambda > \lambda_1$, and $\operatorname{Re} z \ge t$, it follows that

$$|E_\lambda(z)| = |\lambda| \exp(\operatorname{Re} z) \ge (\operatorname{Re} \lambda) \exp(\operatorname{Re} z)$$
$$> \lambda_1 \exp(\operatorname{Re} z)$$
$$> E(\operatorname{Re} z) + (2K+2)\pi > E(\operatorname{Re} z) + M_K.$$

Lemma 4.6. *Given $t \ge 1$, let $\lambda \in Q_s$ satisfy $\operatorname{Re} \lambda > \lambda_1(t)$. Then for any n we have*

$$\alpha \le \operatorname{Re} G_s^n(\lambda, t) < t.$$

Proof. Consider V_α^x for any $x \ge t$. We claim that

$$E_\lambda(V_\alpha^x) \supset V_\alpha^{E(x)}.$$

To see this, we first note that the horizontal boundaries of V_α^x are mapped to $\mathbb{R}^-$, hence into the exterior of $V_\alpha^{E(x)}$. The left hand boundary of V_α^x is mapped to the circle of radius $m_K/2$ centered at 0, which also lies in the exterior of $V_\alpha^{E(x)}$. The right hand boundary of V_α^x is mapped to the circle of radius $|\lambda|e^x$ centered at 0. Since

$$|\lambda|e^x > (\operatorname{Re} \lambda)e^x \ge \lambda_1 e^x \ge E(x) + M_K,$$

this circle is outside the furthermost right corner of $V_\alpha^{E(x)}$. We need to check that the furthermost left corners of $V_\alpha^{E(x)}$ are also within the circle of radius $|\lambda|e^x$. If $\alpha > 0$, then clearly $|\lambda|e^x > E(x) + M_K >$

$\alpha + M_K$ since $x \ge 1 > \alpha$. If $\alpha < 0$, then since $\operatorname{Re} \lambda > \lambda_1 > 1$, we have $|\arg \lambda| < \pi/2$, so $m_K > \pi/2$ and $\log m_K > 0$. Hence by condition 3 above,

$$|\lambda|e^x > |\lambda| > \log 2 + \log|\lambda| + M_K$$
$$> \log 2 + \log|\lambda| - \log(m_K) + M_K$$
$$= -\alpha + M_K.$$

This implies that the circle of radius $|\lambda|e^x$ contains the furthermost left corner of $V_\alpha^{E(x)}$. Combining these results, we see that V_α^x is mapped to an annular region that contains $V_\alpha^{E(x)}$ in its interior.

Recall that L_{λ,s_i} is the inverse of E_λ that takes values in $R(s_i)$. Using the above fact, we have $L_{\lambda,s_i} V_\alpha^{E(x)} \subset V_\alpha^x$ for any $x \ge t$ and any i, provided $\operatorname{Re} \lambda > \lambda_1$. Also

$$L_{\lambda,s}^n(V_\alpha^{E^n(x)}) \subset V_\alpha^x.$$

Now recall that

$$G_s^n(\lambda, t) = L_{\lambda,s}^n(E^n(t)).$$

Therefore we have

$$\alpha \le \operatorname{Re} G_s^n(\lambda, t) < t$$

for each n. ∎

Now we complete the proof of Theorem 4.2. It follows from Lemma 4.6 that $\operatorname{Re} F_t(\lambda) \le t$ for all $\lambda \in Q_s$ with $\operatorname{Re} \lambda > \lambda_1 > t$. Thus we see that F_t moves λ-values in the far right portion of Q_s to the left (see Fig. 10). From Corollary 4.5, F_t maps λ-values in the far left region of Q_s to the right. And from Lemma 4.3, F_t has no fixed points on the horizontal boundaries of Q_s. Therefore it follows that F_t has a unique attracting fixed point in Q_s. This fixed point is the λ-value for which $\lambda = E_\lambda(0) = h_{\lambda,s}(t)$, and so λ lies on the appropriate hair. We therefore define $H_s(t)$ to be this λ-value. Clearly, the itinerary of this λ-value is s. Now F_t depends continuously on t. Since the fixed point for F_t is attracting, it follows that these fixed points move continuously with t. Hence H_s is continuous in t.

When $t > 1$, $\operatorname{Re} E_\lambda^n(h_{\lambda,s}(t)) \to \infty$ as $n \to \infty$. When $t = 1$, $h_{\lambda,s}(1) = z_\lambda(s)$ lies at the endpoint of a hair in the dynamical plane. This verifies conditions 1 and 2 in Definition 4.1.

Thus all that remains to prove is condition 3: $H_s(t)$ extends to ∞ in the right half plane as $t \to \infty$.
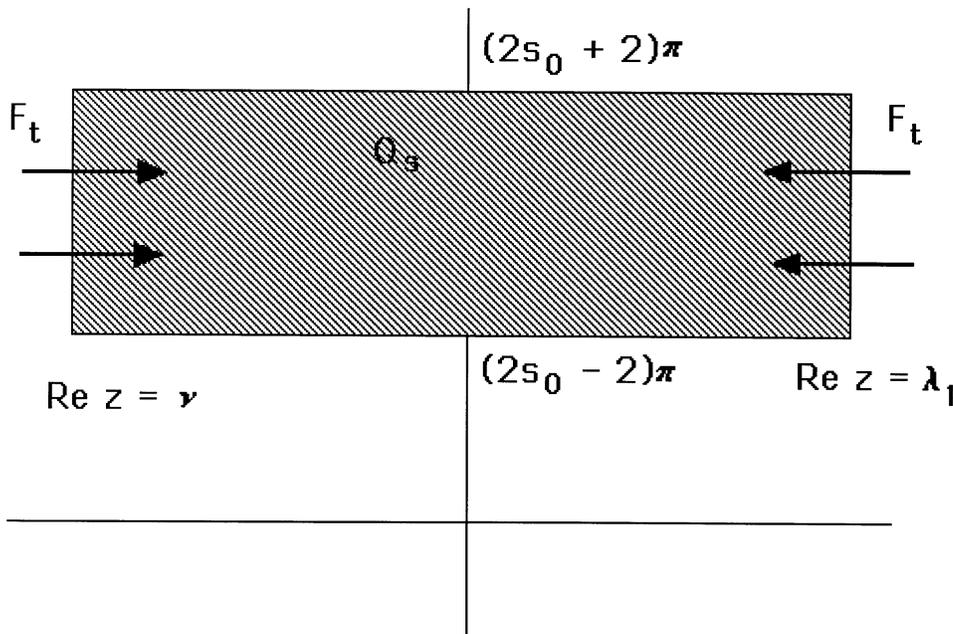
Fig. 10.   The rectangle Q_s.

To show this, suppose $\lim_{t \to \infty} \operatorname{Re} H_s(t) \neq \infty$. This means that there exist sequences $t_i \to \infty$ and $\lambda_i \to \lambda^*$ in Q_s with $|\operatorname{Re} \lambda^*| < \infty$, where $\lambda_i = H_s(t_i)$ for each i. From Proposition 3.5, given λ, there exists τ, M such that, for $t \geq \tau$, we have

$$t - M \leq \operatorname{Re} G_s^n(\lambda, t) \leq t + M. \qquad (3)$$

where $M = M(\lambda)$ was chosen to satisfy (1) and (2) as given in the proof of the Proposition and $\tau(\lambda)$ was chosen so that $E(\tau) > M + 2$. In our case, since λ_i are bounded, so are $\operatorname{Re}(\log \lambda_i)$ and $\operatorname{Re}(-\log \lambda_i)$, which implies that $M(\lambda_i)$ and $\tau(\lambda_i)$ are bounded too. Letting $t_i \to \infty$, we see from (3) that $\operatorname{Re} G_s^n(\lambda_i, t_i) \to \infty$ for any n. This implies that $h_{\lambda_i, s}(t_i) = \lambda_i \to \infty$, contradicting our assumption that λ_i are bounded. This completes the proof of Theorem 4.2.

Remark. As before, it is possible to relax the assumption that all s_i be nonzero. In this case we get the existence of hairs in the parameter plane, but they are only defined in the far right half plane, that is, for t sufficiently large. In particular, these hairs do not terminate at λ-values for which 0 is preperiodic.

5. Conclusion

In this paper we have shown the existence of hairs in both the dynamical and parameter planes for the exponential family $E_\lambda(z) = \lambda e^z$. We emphasize that we have not obtained the complete picture for these hairs. First of all, we have proved the existence of hairs only for bounded, regular itineraries. For these itineraries we showed that each hair was attached to a unique point in either the dynamical or parameter plane.

When the itinerary is not regular, we have only shown that there is a tail for each hair. The existence of the full hairs for these types of itineraries remains a problem. In the dynamical plane, this question has been partially addressed in the case where λ is a Misiurewicz point [Devaney & Jarque, 1997] or E_λ admits an attracting cycle [Bhattarcharjee & Devaney, preprint]. However, the situation in the parameter plane is completely open.

It is also known that there are hairs in the dynamical plane corresponding to unbounded itineraries for certain λ-values [Aarts & Oversteegen, 1993; Devaney & Krych, 1984]. We do not have a general proof of the existence of hairs in either plane in this case.

As is well known, the family of polynomials

$$P_{d,\lambda}(z) = \lambda \left(1 + \frac{z}{d}\right)^d$$

converges uniformly on compact subsets to E_λ. Moreover, this family has unique critical point at $-d$ and critical value at 0. Using techniques

developed by Douady and Hubbard, one can uniformize the exterior of (connected) filled Julia sets for these polynomials, just as in the case of the quadratic maps discussed in the Introduction. Hence one gets a theory of external rays for these maps. In a subsequent paper [Bodelon *et al.*, to appear], we plan to use the results in this paper to discuss the convergence of these external rays as $d \to \infty$ to the hairs for the exponential family. These results depend heavily on the parametrizations of the hairs discussed herein.

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