The Convergence of an Euler Approximation of an Initial Value Problem Is Not Always Obvious

John H. Hubbard, Samer S. Habre, and Beverly H. West

1. INTRODUCTION. Consider the initial value problem

$$\frac{dx}{dt} = \sqrt{|x(t)|}, \quad \text{with } x(-2) = -1. \tag{1}$$

Some solutions are shown in Figure 1, and some Euler approximate solutions are shown in Figure 2.

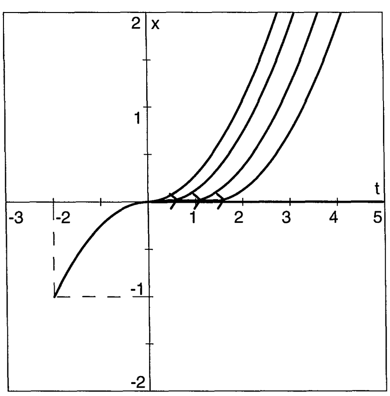

Figure 1. Five correct solutions for $x' = \sqrt{|x|}$ starting at $t_0 = -2$, $x_0 = -1$.

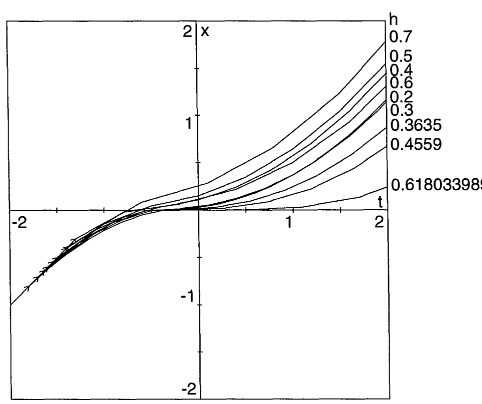

Figure 2. Euler approximate solutions to (1) for various stepsizes.

Analytically, the solutions of (1) are exactly the functions (shown in Figure 3)

$$v_b(t) = \begin{cases} -\dfrac{t^2}{4}, & \text{for } t \le 0 \\ 0, & \text{for } 0 \le t \le b \\ \dfrac{(t-b)^2}{4}, & \text{for } b \le t. \end{cases} \tag{2}$$

There is one solution for each $b \in [0, \infty)$. In particular, there are infinitely many solutions v_b with the same initial condition $x(-2) = -1$.

Since $\partial\sqrt{|x|}/\partial x$ is unbounded in any region containing the t–axis, it is not surprising that uniqueness of the initial value problem (1) does not hold after the solution hits the t–axis.

Let $u_h(t)$ denote the Euler approximation of (1) with stepsize h satisfying $u_h(-2) = -1$. We investigate what happens as $h \to 0$. Our main theorem is:

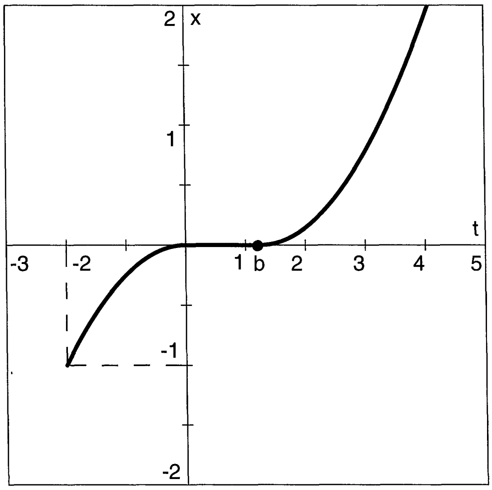

Figure 3. The solution v_b leaves the t–axis at b. This solution is a C^1 function, even at b.

Theorem 1.1. *For every $b \geq 0$, there exists a sequence of stepsizes $\{h_n > 0\}$ tending to 0 with the property that $u_{h_n}(t)$ approaches $v_b(t)$ as $n \to \infty$.*

2. A PARAMETER-SPACE PICTURE. Figure 2 illustrates how, for some values of h, $u_h(2)$ may approach values other than 1. But to produce this picture, we selected our step-lengths very carefully, and used the information in Figure 4 to do so. When the stepsize does *not* land precisely on the terminal value $t = 2$, the final step is shortened appropriately.

Pictures that draw one or several graphs of solutions are inadequate to understand how the solutions vary as a function of h. A movie would be much better, but a substitute is to draw the graph of the function $h \mapsto u_h(2)$, as shown in Figure 4.

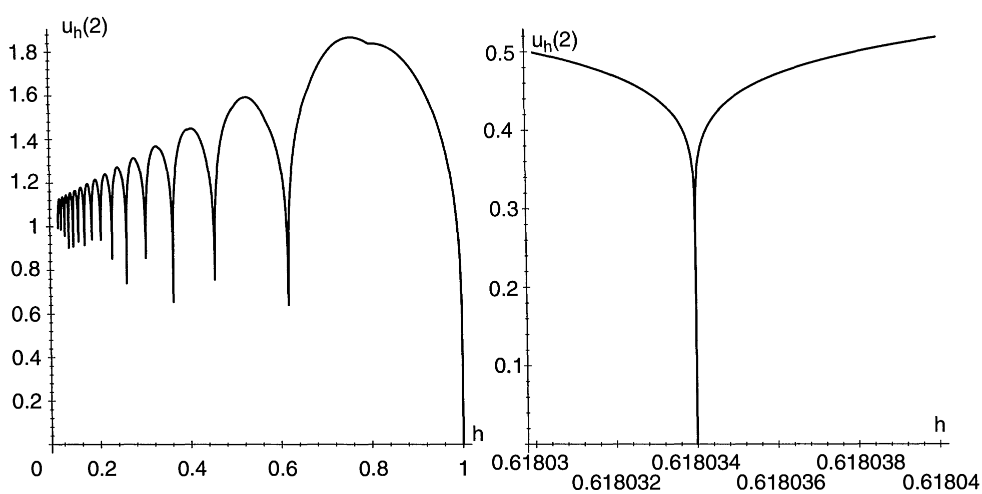

Figure 4. Left: the graph of $f \mapsto u_h(2)$. The cusps do not seem to reach down to zero only because the cusps are *very* sharp and fall between pixels, as illustrated by the blow-up on the right. It is very difficult even with extremely high precision to obtain good approximations to the cusp values.

In fact, a graph of $u_h(2)$ (as a function of the stepsize h) shows that $u_h(2)$ varies continuously with repeated cusp values at zero and peaks greater than one; see Figure 4. The remainder of this article is devoted to proving that these features really occur; they are not artifacts of the computer picture.

3. WHAT REMAINS OF THE CONVERGENCE THEOREM. One (good) way to prove existence and uniqueness for solutions of differential equations is to show that as the step-length tends to 0, Euler approximations to solutions converge. In [**1**, Section 4.5], the authors prove

Theorem 3.1. *If $f(t, x)$ is defined for $t \in [a, b]$, $x \in [c, d]$ and satisfies the Lipschitz condition*

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2| \qquad \textit{for} \quad t \in [a, b],\ x_1, x_2 \in [c, d],$$

and if $u_n : [a, b] \to [c, d]$ is a sequence of Euler approximations for the differential equation $x' = f(t, x)$ with step-lengths tending to 0, and if $\lim_{n\to\infty} u_n(t_0) = c$ exists for some $t_0 \in [a, b]$, then

$$u(t) = \lim_{n\to\infty} u_n(t)$$

exists for $t \in [a, b]$, and is the unique solution to $x' = f(t, x)$ with $u(t_0) = c$.

Of course this uses the Lipschitz nature of the differential equation; Theorem 3.1 is not true for the differential equation (1) when $x = 0$, i.e., when the solution crosses the t-axis. But Euler approximations do converge where the equation is Lipschitz: if a sequence of Euler approximations $u_n(t)$, defined for $t \in [a, b]$, all satisfy $u_n(t) \geq \epsilon$ or $u_n(t) \leq -\epsilon$ for some $\epsilon > 0$, then Theorem 3.1 does apply, and the u_n converge to a solution if they converge at a single point.

We sharpen this in Proposition 3.2.

Proposition 3.2. *Let $h_k > 0$ be a sequence of step-lengths tending to 0, and let u_k be a sequence of Euler approximations with step-length h_k.*

(a) *If there exists t_0 such that $c_k = u_k(t_0)$ converges to c with $c > 0$, then the sequence $u_k(t)$ converges for each $t \geq t_0 - 2\sqrt{c} = b$ to the function $v(t) = (t - b)^2/4$.*
(b) *If there exists t_0 such that $c_k = u_k(t_0)$ converges to c with $c < 0$, then the sequence $u_k(t)$ converges for each $t \leq t_0 + 2\sqrt{-c} = b$ to the function $v(t) = -(t - b)^2/4$.*

The proof depends on the Fundamental Inequality [**1**, Theorem 4.4.1] without which it is essentially impossible to prove anything about differential equations. We restate it here as Theorem 3.3.

Theorem 3.3. *Let $f(t, x)$ be defined for $t \in [a, b]$, $x \in [c, d]$ and satisfy the Lipschitz condition*

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2| \quad \textit{for} \quad t \in [a, b],\ x_1, x_2 \in [c, d].$$

Suppose $u_1, u_2 : [a, b] \to [c, d]$ are continuous, piecewise continuously differentiable functions satisfying

$$|u_1'(t) - f(t, u_1(t)| \le \epsilon_1, \quad |u_2'(t) - f(t, u_2(t)| \le \epsilon_2$$

at all points $t \in [a, b]$ where the functions u_1 and u_2 are differentiable, and

$$|u_1(t_0) - u_2(t_0)| \le \delta,$$

for some $\epsilon_1, \epsilon_2, \delta \ge 0$. Then

$$|u_1(t) - u_2(t)| \le \delta e^{K|t-t_0|} + \frac{\epsilon_1 + \epsilon_2}{K}\left(e^{K|t-t_0|} - 1\right). \tag{3}$$

One may think of the ϵ's as slope errors, which measure to what extent the functions u_1 and u_2 fail to have the proper slope in order to be solutions of $x' = f(t, x)$, and of δ as the error in the initial condition.

Proof of Proposition 3.2. The proofs of the two parts are almost identical; we prove only the first. It is enough to show that u_k converges for $b \le t \le t_0$, and we may assume that all c_k satisfy $c_k \le 2c$, so that $u_k(t) \le 2c$ for all $t \le t_0$.

Choose $\epsilon > 0$, set $\eta = \epsilon^2/16$, and suppose that $c > 2\eta$. Since $v(t) = (t-b)^2/4$, we have $v(b+\epsilon) = 4\eta$. In the region $x > \eta$, the differential equation (1) is Lipschitz with constant

$$K = \sup_{x>\eta} \frac{1}{2\sqrt{|x|}} = \frac{2}{\epsilon}. \tag{4}$$

So long as $u_k(t) > \eta$ we have

$$|u_k'(t) - \sqrt{|u_k(t)|}| \le \frac{2h_k\sqrt{2c}}{\epsilon}, \tag{5}$$

and the fundamental inequality ensures that

$$|u_k(t) - v(t)| \le |c_k - c|e^{2|t-t_0|/\epsilon} + \frac{2h_k\epsilon\sqrt{2c}}{2\epsilon}\left(e^{2|t-t_0|/\epsilon} - 1\right). \tag{6}$$

If we choose k large enough that

$$h_k < \frac{\eta}{\sqrt{2c}(e^{2\sqrt{c}/\epsilon} - 1)} \quad \text{and} \quad |c_k - c| < \frac{\eta}{e^{2\sqrt{c}/\epsilon}}, \tag{7}$$

then

$$|u_k(t) - v(t)| \le \frac{\eta}{4e^{2\sqrt{c}/\epsilon}} e^{2|t-t_0|/\epsilon} + \frac{\eta}{\sqrt{2c}(e^{2\sqrt{c}/\epsilon} - 1)}\left(e^{2|t-t_0|/\epsilon} - 1\right) \le 2\eta. \tag{8}$$

We have to deal with the condition $u_k(t) > \eta$. This is true for $t = t_0$, so if it fails, there is a largest t_1 for which it fails, and (8) is true for $t = t_1$. If $t_1 \ge b + \epsilon$, this gives

$$u_k(t_1) > v(t_1) - 2\eta \ge 4\eta - 2\eta > \eta.$$

Thus (8) is true for $t \in [b+\epsilon, t_0]$ for all sufficiently large k. Since ϵ is arbitrary, this proves the first part of Proposition 3.2. ■

Corollary 3.4. *If u_k is a sequence of Euler approximations for the differential equation (1), with step-lengths $h_k \to 0$, and if there exist t_0, t_1 such that the $u_k(t_0)$ converge* *to* $c_0 < 0$ *and the* $u_k(t_1)$ *converge to* $c_1 > 0$, *then* $u_k(t)$ *converges for every* t *to* $v_{a,b}(t)$, *where*

$$a = t_0 + 2\sqrt{-c_0}, \quad b = t_1 - 2\sqrt{c_1}, \quad \textit{and} \quad v_{a,b}(t) = \begin{cases} -(t-a)^2/4 & t \le a \\ 0 & a \le t \le b \\ (t-b)^2/4 & t \ge b. \end{cases}$$

Proof. Choose $\epsilon > 0$, and break up $[t_0, t_1]$ into the three subintervals

$$[t_0, a-\epsilon], \quad [a-\epsilon, b+\epsilon], \quad \text{and} \quad [b+\epsilon, t_1].$$

We just saw that the sequence u_k converges to $v_{a,b}$ on the first and the last; in fact, for k sufficiently large we have

$$u_k(a-\epsilon) > -\epsilon^2 \quad \text{and} \quad u_k(b+\epsilon) < \epsilon^2. \tag{9}$$

But all the u_k are non-decreasing functions, and thus the middle segment take values in $[-\epsilon^2, \epsilon^2]$ for k sufficiently large, and hence the u_k converge to 0 there. ■

4. SOME RESULTS ABOUT FENCES. Our proofs require *fences*, an elementary way of dealing with differential inequalities, featured in [**1**, Sections 1.3 and 4.4].

Definition 4.1. *Suppose* $x' = f(t, x)$ *is a differential equation defined for* $t \in [a, b]$, $x \in [c, d]$. *A continuous and piecewise continuously differentiable function* $\beta : [a, b] \to [c, d]$ *is called an upper fence if* $f(t, \beta(t)) \le \beta'(t)$ *for all* $t \in [a, b]$.

If $f(t, \beta(t)) < \beta'(t)$, then β is called a *strong upper fence*; otherwise β is called a *weak upper fence*.

Figure 5 shows some examples of upper fences (dashed) in a slope field.

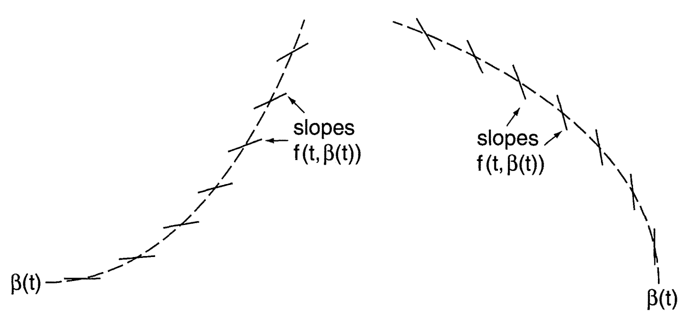

Figure 5. Upper fences $f(t, \beta(t)) \le \beta'(t)$.

The pictures in Figure 5 convey the proper idea that an upper fence pushes solutions down; once a solution crosses a strong upper fence, the solution cannot get back across the fence.

Theorem 4.2. *If* $\beta(t)$ *is a strong upper fence for the differential equation* $x' = f(t, x)$, *then whenever a solution* $x = u(t)$ *satisfies* $u(t_0) < \beta(t_0)$, *then* $u(t) < \beta(t)$ *for all* $t \ge t_0$ *in* $[a, b]$ *where* $u(t)$ *is defined.*

To get a similar result for a weak upper fence, the differential equation must satisfy a Lipschitz condition.

Theorem 4.3. *If $\beta(t)$ is a weak upper fence for the differential equation $x' = f(t, x)$ and the differential equation is locally Lipschitz, then whenever a solution $x = u(t)$ satisfies $u(t_0) \leq \beta(t_0)$, then $u(t) \leq \beta(t)$ for all $t \geq t_0$ in $[a, b]$ where $u(t)$ is defined.*

This article provides excellent examples that this conclusion is not true without the Lipschitz hypothesis.

5. PROOF OF THE MAIN RESULT. For the remainder of the article, $u_h(t)$ denotes the Euler approximation to solutions of (1), with $u_h(-2) = -1$. It is the piecewise linear function joining the "control points" $(-2 + nh, x_n(h))$, where $x_n(h)$ is defined recursively by the formulas

$$x_0 = u_h(-2) = -1, \quad x_{n+1}(h) = x_n(h) + h\sqrt{|x_n(h)|}. \tag{10}$$

Lemma 5.1.

a) *For each integer $n \geq 0$, $x_n(h)$ is a continuous function of h for $h \geq 0$.*
b) *For each integer $n \geq 0$, $x_n(0) = -1$, and when $n \geq 1$, $x_n(h) > 0$ for $h > 1$.*

Proof. Lemma 5.1 is immediate by induction on n. ∎

Proposition 5.2. *There exists a sequence of stepsizes $\{\tilde{h}_n\}$ tending to 0 such that $x_n(\tilde{h}_n) = 0$. Furthermore,*

$$\frac{1}{n} \leq \tilde{h}_n \leq \frac{2}{n+1}.$$

Proof. By the intermediate value theorem, there exists for each n a smallest $\tilde{h}_n > 0$ such that $x_n(\tilde{h}_n) = 0$; see Figure 6 for graphs of x_n versus h.

Next we show that $1/n \leq \tilde{h}_n \leq 2/(n+1)$; it follows that $\tilde{h}_n$ tends to zero as $n \to \infty$. The left-hand inequality is more or less obvious: in $-1 \leq x \leq 0$ we have $\sqrt{|x|} \leq 1$, so the solutions $u_{\tilde{h}_n}$ have slope less than one in this region, and hence take time more than 1 to get from $x = -1$ to $x = 0$. This means that $n\tilde{h}_n > 1$.

The right-hand inequality is less obvious.

Observe that for every $h > 0$, in the interval where $u_h(t) < 0$, $u_h(t)$ is a weak upper fence for the differential equation $x' = \sqrt{|x|}$. In particular, $u_h(t) \geq -t^2/4$ when $t < 0$; see Figure 7.

We must have $x_{n-1}(\tilde{h}_n) = -\tilde{h}_n^2$, since

$$x_{n-1}(\tilde{h}_n) + \tilde{h}_n\sqrt{|x_{n-1}(\tilde{h}_n)|} = 0.$$

The function $-t^2/4$ takes on the value $-\tilde{h}_n^2$ when $t = -2\tilde{h}_n$, and hence $u_{\tilde{h}_n}$ must take on this value earlier. This means that

$$-2 + (n-1)\tilde{h}_n < -2\tilde{h}_n,$$

which gives the desired inequality. ∎

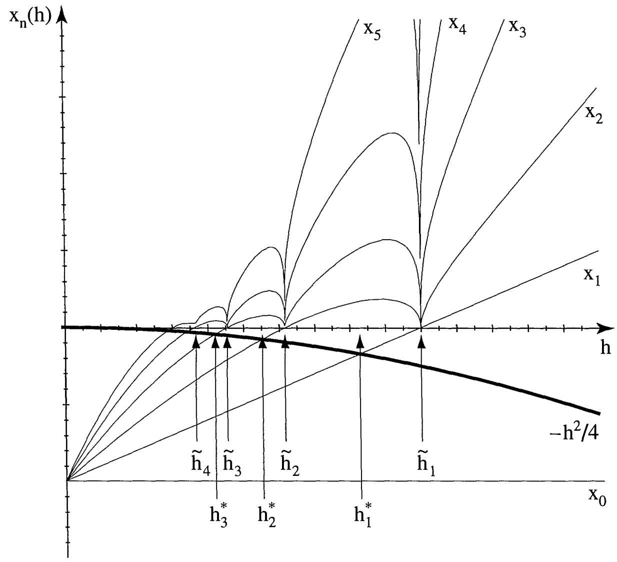

Figure 6. Control points $x_n(h)$ for $n = 0, 1, \ldots, 5$, for Euler approximations. We have marked the points $\tilde{h}_n$ where $x_n(h)$ first vanishes, the points h_n^* where $x_n(h) = -h^2/4$, and the points h_k^*, which come between them, by Lemma 5.3.

Lemma 5.3. *There exists a sequence of stepsizes* $\{h_n^*\}$ *such that* $x_n(h_n^*) = -h_n^{*2}/4$. *These satisfy* $\tilde{h}_{n+1} < h_n^* < \tilde{h}_n$; *in particular, they tend to* 0 *as* n *tends to infinity, and* $-2 + (n+1)h_n^* < 0$.

Proof. As h decreases from $\tilde{h}_n$ to 0, $x_n(h)$ decreases monotonically from 0 to -1, whereas $-h^2/4$ increases from $-\tilde{h}_n^2/4$ to 0; this is illustrated in Figure 6. Again by the intermediate value theorem, for each n there must exist a unique $h_n^* < \tilde{h}_n$ such that

$$x_n(h_n^*) = -\frac{(h_n^*)^2}{4}.$$

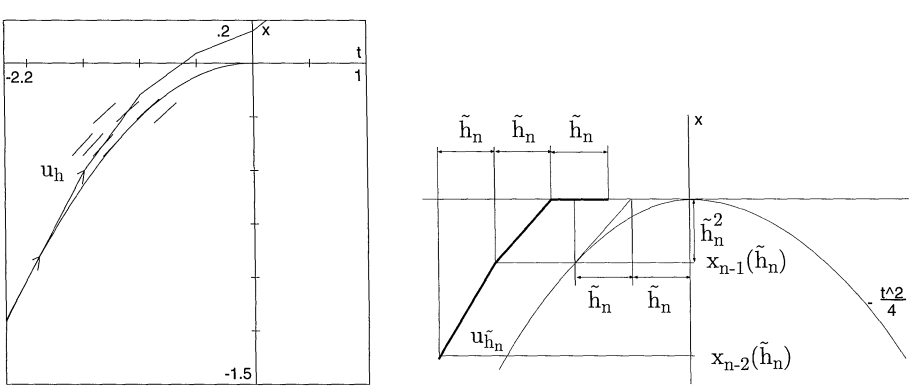

Figure 7. Left: the graph of $u_{\tilde{h}_n}$ is above the graph of $-t^2/4$ in the region $-2 \le t \le 0$, so $n\tilde{h}_n < -h_n$. Right: since $\sqrt{|x|}$ is decreasing in the region $x < 0$, all Euler approximations are upper fences there.

However,

$$x_{n+1}(h_n^*) = -\frac{h_n^*}{4} + h\sqrt{\frac{(h_n^*)^2}{4}} = \frac{h_n^*}{4}, \tag{11}$$

so x_{n+1} vanishes for the first time before h_n^*.

Now $-2+(n+1)h_n^* < 0$ follows from $h_n^* < \tilde{h}_n$, and $-2+(n+1)\tilde{h}_n < 0$. ■

We now aim at Proposition 5.4.

Proposition 5.4. *We have*

$$\lim_{n\to\infty} u_{h_n^*}(t) = v_{0,0}(t).$$

This requires two more intermediate lemmas.

Lemma 5.5. *For $h > 0$, $u_h(t)$ is a strong upper fence for the differential equation*

$$\frac{dx}{dt} = \frac{1}{2}(x(t))^{3/4}$$

in the region $R : h^2/4 \le x \le \frac{1}{2}$.

Proof. We want to show that $\frac{1}{2}(u_h(t))^{3/4} \le u_h'(t)$. Since the slope of an Euler approximation is a constant between any two consecutive points, and since $\frac{1}{2}(x(t))^{3/2}$ increases, it is enough to check the result at the point $(t+h, x+h\sqrt{x})$. Thus it is enough to show that

$$\sqrt{x} > \frac{1}{2}\left(x + h\sqrt{x}\right)^{3/4} \quad \left(x(t) \ge \frac{h^2}{4} > 0\right).$$

Indeed,

$$\left[\frac{1}{2}\left(x+h\sqrt{x}\right)^{3/4}\right]^4 = \frac{1}{16}\left(x+h\sqrt{x}\right)^3 = \frac{x^3}{16}\left(1+\frac{h}{\sqrt{x}}\right)^3.$$

But $x \ge h^2/4 \Rightarrow \sqrt{x} > h/2$ or $h/\sqrt{x} < 2$. Therefore, since $x \le \frac{1}{2}$,

$$\frac{x^3}{16}\left(1+\frac{h}{\sqrt{x}}\right)^3 \le \frac{27}{16}x^3 < \frac{27}{32}x^2 < x^2.$$ ■

Corollary 5.6. *For any n and for $t \ge 0$, we have $u_{h_n^*}(t) \ge \inf\{(t/8)^4, \frac{1}{2}\}$.*

Proof. The function $x(t) = (t/8)^4$ is a solution to $x'(t) = \frac{1}{2}(x(t))^{3/4}$ satisfying $x(0) = 0$. We have

$$u_{h_n^*}(-2+(n+1)h_n^*) = x_{n+1}(h_n^*) = \frac{h_n^{*2}}{4} > 0$$

by (11), and $-2 + (n+1)h_n^* < 0$ by Lemma 5.3. Since $u_{h_n^*}$ is non-decreasing, we have $u_{h_n^*}(0) > (h_n^*)^2/4$. Thus Theorem 4.2 ensures that

$$u_{h_n^*}(t) \geq \left(\frac{t}{8}\right)^4$$

for all $t > 0$ for which $u_{h_n^*}(t) \leq 1/2$. ■

Proof of Proposition 5.4. We can extract a subsequence of $u_{h_n^*}$ such that $u_{h_n^*}(2)$ converges; Corollary 5.6 implies that the limit is greater than $1/256$. Now Corollary 3.4 shows that the subsequence converges to $v_{a,b}$ for some $a \leq b$.

Corollary 5.6 again implies that $v_{a,b}(t) \geq 0$ when $t \geq 0$, so that $b \leq 0$. On the other hand, $v_{a,b}(-2) = -1$, which implies that $a = 0$. Thus $a = b = 0$.

We now invoke the fact that if a sequence and a number a are such that every subsequence has a subsubsequence that converges to a, then the sequence converges to a. This is just what we have proved about $u_{h_n^*}(t)$ for each t. ■

Proof of Theorem 1.1. By Proposition 5.4, there exists a sequence of stepsizes $\{h_n^*\}$ with the property that $\lim_{h_n^* \to 0} u_{h_n^*}(2) = 1$, and by Proposition 5.2, there exists a sequence of stepsizes $\tilde{h}_n$ tending to zero for which $u_{\tilde{h}_n}(2) = 0$.

By the continuity of $u_h(2)$, the intermediate value theorem implies that for every $c \in (0, 1)$, there exists a sequence of stepsizes $h_{c,n}$ satisfying $\lim_{n\to\infty} u_{h_{c,n}}(2) = a$.

Apply Corollary 3.4 to the sequence $u_{h_{c,n}}$; the limit is $v_{0,b}$ where $(2-b)^2/4 = c$, i.e., $b = 2(1-\sqrt{c})$. Thus b can take any value between 0 and 2. ■

6. OTHER NUMERICAL METHODS. Is the pathology described by Theorem 1.1 special to Euler's method, or does it persist for other numerical methods? It appears that it not only persists, but it can get worse. For example, the analog of Figure 6 for mid-point Euler is illustrated in Figure 8.

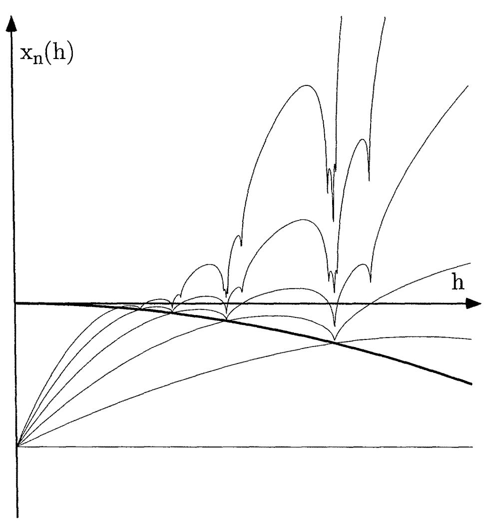

Figure 8. Compared with Euler's method (Figure 6), the control points for mid-point Euler approximations behave in even wilder ways as functions of the step-length.

We observe at least one new phenomenon: there are two kinds of cusps where an approximation to a solution remains bounded: as for Euler's method, this occurs if some control point $x_n(h)$ satisfies $x_n(h) = 0$, but it also happens if some x_n satisfies $x_n(h) = -h^2/4$.

It seems likely, therefore, that for the differential equation (1), all fixed step-length numerical methods fail to converge as the step-length goes to 0. If a variable-step length method is devised to slow down when the Lipschitz constant becomes large, it presumably grinds to a standstill, and never makes it past $t = 0$. This is the case for every such method that we have tested.

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JOHN HUBBARD is a professor of mathematics at Cornell University, and at the Université de Provence in Marseille. He obtained his doctorate in Paris in 1973, then spent three years as a Benjamin Pierce Assistant Professor at Harvard. He has been at Cornell since 1976, with regular visits to universities in France and Germany. Hubbard's research has focussed on complex analysis, more particularly complex analytic dynamics in one and several variables, often in collaboration with A. Douady. He has long been interested in teaching, particularly differential equations and calculus, and his textbooks attempt to revise the curricula in these subjects. For differential equations, this means a focus on the behavior of solutions rather than techniques of solution.
Malott Hall, Cornell University, Ithaca NY 14853
jhh8@cornell.edu

SAMER HABRE received his Ph.D. in mathematics in 1991 from Syracuse University under Tadeusz Iwaniec. He then moved to his home country Lebanon, where he has been since then a professor of mathematics at the Lebanese American University. His research interests have shifted lately to the teaching aspect of mathematics at the college level. In this respect, he spent the year 1998–99 as a Fulbright Scholar at Cornell University doing research on the teaching of differential equations. While at Cornell he collaborated with B. West and J. Hubbard on writing this paper.
Lebanese American University, P.O. Box: 13 - 5053, Beirut, Lebanon;
New York Office Address: 475 Riverside Drive, Suite 1846, New York, NY 10115
shabre@lau.edu.lb

BEVERLY HENDERSON WEST is a Senior Lecturer at Cornell University, where she has been teaching mathematics since 1970, after being educated at Swarthmore College and University of Wisconsin at Madison. Teaching is her special interest, and her favorite subjects are differential equations, multivariable calculus, and complex dynamical systems. She was immediately attracted to the usefulness of John Hubbard's fences and funnels and to the illuminating instructional possibilities of interactive computer graphics, and since the early '80s has been involved with textbook collaborations and software development for her favorite courses.
Malott Hall, Cornell University, Ithaca NY 14853
bhw2@cornell.edu