Exploring the Mandelbrot set.
The Orsay Notes.

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The goal of this work is to present results obtained by A. Douady and J.H. Hubbard in 1981-82. The most important has been stated, with or without proofs, in [**DH1**] or [**Do1**].

The present text is based on notes written weekly by A. Douady for his course "Holomorphic dynamical Systems" during the first semester 1983-1984. There may be repetitions.

The terminology is the one used by A. Douady. Some chapters, especially "Tour de Valse" (part 2), have been prepared with the participation of Pierrette Sentenac.

This text should be developed in a more complete work. We nevertheless think that it is useful to present it to the reader in the current state.

We thank the group of topology in Orsay, who enabled this publication, and Bernadette Barbichon who realized this with task competently with kindness.

Part 1: Chapters 1 to 8.

Part 2: Chapters 9 and next, Appendices.

Contents

Part 1

CHAPTER 1

Goal of the notes.

We will study the family of maps $P_c : z \mapsto z^2+c$, from $\mathbb{C}$ to $\mathbb{C}$, from a dynamical point of view. For each c, we denote by K_c the set of points z such that $P_c^{\circ n}(z)$ does not tends to ∞ (filled-in Julia set of P_c). By a theorem of Fatou and Julia (1919), K_c is connected if $0 \in K_c$; otherwise, it is a Cantor set. We denote by M (Mandelbrot set) the set of parameters c for which K_c is connected, and by M' the set of parameters c for which P_c has an attracting cycle. The set M is compact and connected, M' is open and contained in M.

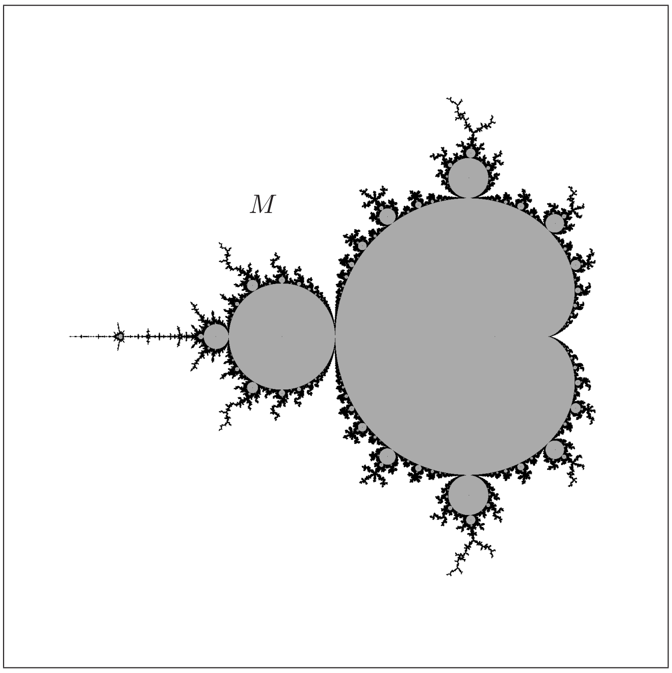


FIGURE 1. The Mandelbrot set M.

The two main conjectures are the following:

(MLC) The set M is locally connected.

(HG2) The interior of M is M'.

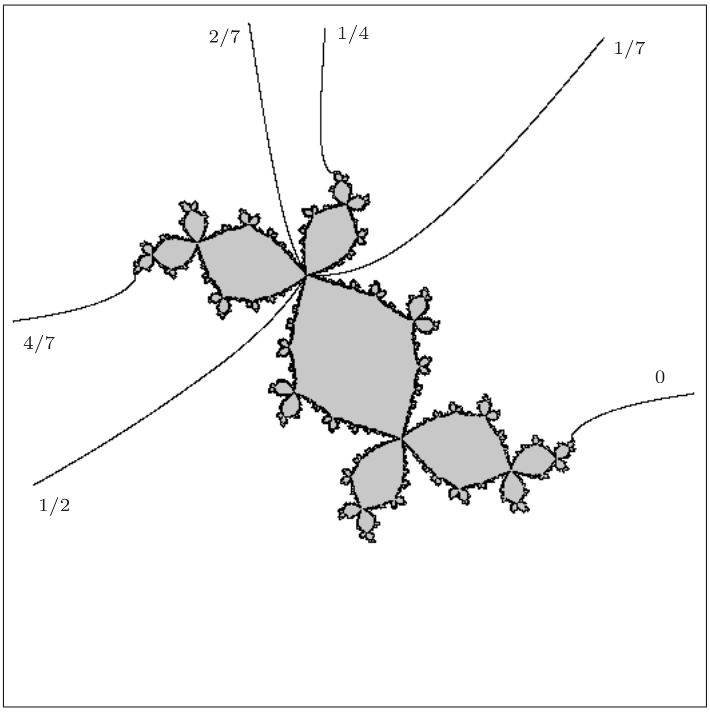
We do not know how to prove either, but we intend to show that (MLC) $\Rightarrow$ (HG2).[1].

[1] Yoccoz proved that the Mandelbrot set is locally connected at every parameter c such that P_c has an indifferent cycle (see [**Hu**]). He also proved that the Mandelbrot set is locally connected

Let $K \subset \mathbb{C}$ be a compact set which is connected and full (i.e., $\mathbb{C} \setminus K$ is connected). Then, the Riemann mapping theorem asserts that there exists a unique pair (r, φ) such that $r \in \mathbb{R}^+$ and $\varphi = \varphi_K$ is a $\mathbb{C}$-analytic homeomorphism between $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus \overline{\mathbb{D}}_r$, with $\varphi(z)/z \to 1$ as $|z| \to \infty$. We say that $r_K = r$ is the *capacity* of K. For $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, the set

$$\mathcal{R}(K, t) = \varphi_K^{-1}\left(\left\{\rho e^{2i\pi t}\right\}_{\rho > r}\right)$$

is the *external ray* of K of argument t (arguments are counted in turns, not modulo 2π). If $\varphi_K^{-1}(\rho e^{2i\pi t})$ has a limit $x \in K$ as $\rho \to r_K$, we say $\mathcal{R}(K, t)$ *lands* at x, or that x has *external argument* t in K. If K is locally connected, a theorem of Carathéodory asserts that every external ray land.



It is a quite remarkable fact that we know φ_M and the maps $\varphi_c = \varphi_{K_c}$ for $c \in M$. [2] But without (MLC), we do not know that every external ray of M lands.

Theorem. *Every external ray of M with rational argument lands.*

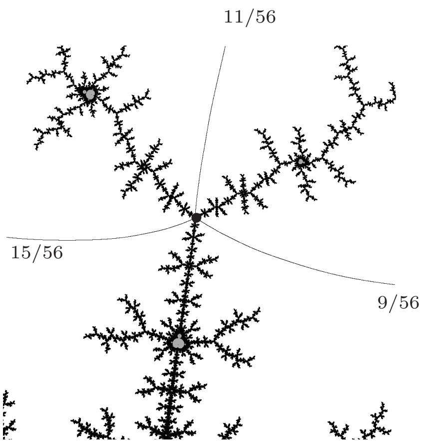
The behavior is different for rational numbers which, in reduced form, have odd denominators and even denominators.

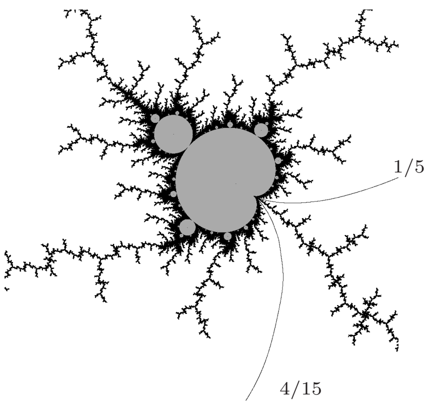
If $t \in [0, 1]$ is a rational number with odd denominator, the external ray $\mathcal{R}(M, t)$ lands at a point c such that P_c has a rationally indifferent cycle. Each such point

at every parameter c such that P_c is finitely renormalizable (see [**Hu**] or [**Ro**]). Also, it is known that $\overset{\circ}{M} \cap \mathbb{R} = M' \cap \mathbb{R}$ (see [**L3**] or [**GS**]) Finally, if there exist a parameter $c \in \overset{\circ}{M} \setminus M'$, then K_c has empty interior, positive Lebesgue measure and carries an invariant Beltrami form (see [**MSS**] for more details).

[2]It is rather easier to find the conformal mapping of $\mathbb{C} \setminus M$ than of the complement of a triangle.

is obtained for 2 values of t (except $c = 1/4$ corresponding to $t = 0$). If t has even denominator, $\mathcal{R}(M, t)$ lands at a point c such that the orbit of 0 under iteration of P_c falls on a repelling cycle. Such values of c are called Misurewicz points. Each Misurewicz point has a finite number of external arguments (each rational with even denominator).





To show those properties, we attack the problem from the other end. We first study the hyperbolic components of $\overset{\circ}{M}$, i.e, the connected components of M'. We will show that for each component U of M' and for each $c \in U$, P_c has a unique attracting cycle $\{z_0(c), \ldots, z_{k-1}(c)\}$ of some period k. The map $U \to \mathbb{D}$ given by $\varphi_U : c \mapsto (P_c^{\circ k})'(z_0(c))$ is a conformal mapping which extends injectively to the boundary. We define the *center* of U to be $\varphi_U^{-1}(0)$ and the *root* of U to be $\varphi_U^{-1}(1)$. We study K_c when c is the center of a hyperbolic component or a Misurewicz point. In particular, we construct in K_c a combinatorial object: *the Hubbard tree.* Thanks to this tree, we determine the arguments of c if c is a Misurewicz point, and if c is the center of a hyperbolic component W, the 2 arguments of the root of W. The analysis, rather elementary in the case of Misurewicz points, is more tricky for the roots of hyperbolic components. It then remains to show that every rational arguments are obtained in such a way. This time, it is much easier for rational numbers with odd denominators.[3]

The technique that leads the implication (MLC) $\Longrightarrow$ (HG2) is the following. Let c_1 and c_2 be two point of M with external arguments θ_1 and θ_2 which can be written as $p/2^k$ (when a point has such an external argument, it has no other external argument).

Assuming M is locally connected, we construct topological arcs Γ_1 and Γ_2 in M, joining 0 to c_1 and c_2 (in fact, we impose certain conditions on those arcs – "allowable arcs"). Let c_3 be the point where Γ_1 and Γ_2 separate. We show that c_3 is a Misurewicz point or the center of a hyperbolic component, and we can construct its tree in terms of those of c_1 and c_2.

If c_3 is a Misiurewicz point, we will show that it has at least 3 external arguments t_1, t_2, t_3 such that $t_1 < \theta_1 < t_2 < \theta_2 < t_3$. If c_3 is the center of a hyperbolic component W, we will call the external arguments of any point of ∂W an *argument*

[3] An alternate approach uses the spider algorithm (see [**HS**]).

associated to c_3. We will show that at least three external arguments t_1, t_2, t_3 such that $t_1 < \theta_1 < t_2 < \theta_2 < t_3$. All the combinatorial part of this study can be performed without assuming M locally connected – the definition of c_3 then seems artificial.

Let us now assume that $\overset{\circ}{M}$ has a ghost (i.e., not hyperbolic) component W. Let w_1, w_2, w_3 be three points of ∂W, and u_1, u_2 and u_3 be external arguments of respectively w_1, w_2 and w_3. Let θ_1 and θ_2 be of the form $p/2^k$ such that $u_1 < \theta_1 < u_2 < \theta_2 < u_3$ and denote by c_1 and c_2 the landing points of $\mathcal{R}(M, \theta_1)$ and $\mathcal{R}(M, \theta_2)$. Then, construct c_3 and t_1, t_2, t_3 as above. Set

$$\begin{aligned} S &= W \cup \{w_1, w_2, w_3\} \cup \mathcal{R}(M, u_1) \cup \mathcal{R}(M, u_2) \cup \mathcal{R}(M, u_3) \\ &= W \cup \overline{\mathcal{R}}(M, u_1) \cup \overline{\mathcal{R}}(M, u_2) \cup \overline{\mathcal{R}}(M, u_3); \\ S' &= \mathcal{R}(M, t_1) \cup \mathcal{R}(M, t_2) \cup \mathcal{R}(M, t_3) \cup c_3 \text{ if } c_3 \text{ is a Misurewicz point and} \\ S' &= \overline{\mathcal{R}}(M, t_1) \cup \overline{\mathcal{R}}(M, t_2) \cup \overline{\mathcal{R}}(M, t_3) \cup W' \text{ if } c_3 \text{ is the center of } W'. \end{aligned}$$

The set S, S' and $\overline{\mathcal{R}}(\theta_1) \cup \overline{\mathcal{R}}(\theta_2)$ must be disjoint which yields a contradiction.

Remark. This technique does not rule out a ghost component whose closure would only meet two external rays. This situation could occur if M were not locally connected.

Prerequisites.

Topology.

Jordan curve theorem. *Let $\varphi : S^1 \to \mathbb{R}^2$ be continuous and injective. Then, there exists a homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi_{|S^1} = \varphi$.*

Additional information. *Assume L is an arc between a and b intersecting $\Gamma = \varphi(S^1)$ at a point c. Assume there exists a homeomorphism ψ between a neighborhood U of c and a neighborhood V of 0 of $\mathbb{R}^2$ such that $\psi(U \cap \Gamma) = V \cap (\mathbb{R} \times 0)$. Then, a and b are in distinct connected components of $\mathbb{R}^2 \setminus \Gamma$.*

We say that a compact set (respectively a bounded open set) $A \subset \mathbb{R}^2$ is *full* if $\mathbb{R}^2 \setminus A$ is connected.

Proposition. *Let $U \subset \mathbb{R}^2$ be a bounded connected open set. The following are equivalent:*

- *U is full;*
- *for any Jordan curve $\Gamma \subset U$, the domain of $\mathbb{R}^2$ bounded by Γ is contained in U;*
- *U is simply connected;*
- *$H^1(U;\mathbb{Z}) = 0$;*
- *$H^1(U;\mathbb{Z}/2) = 0$; $H^1(U;\mathbb{R}) = 0$;*
- *U is homeomorphic to $\mathbb{D}$.*

Proposition. *Let $K \subset \mathbb{R}^2$ be a connected compact set. The following are equivalent:*

- *K is full;*
- *K has a basis of neighborhoods homeomorphic to $\overline{\mathbb{D}}$;*
- *for every $a \in \mathbb{R}^2 \setminus K$, the universal covering (respectively the connected covering of degree 2) of $\mathbb{R}^2 \setminus \{a\}$ induces a trivial covering of K;*
- *every finite covering of K is trivial.*

We will begin those notes with a more detailed study of full, connected and locally connected compact subsets of $\mathbb{R}^2$.[4]

Holomorphic functions.

Uniformization Theorem. *Every simply connected Riemann surface (i.e. $\mathbb{C}$-analytic manifold of dimension 1 over $\mathbb{C}$) is isomorphic to $\mathbb{D}$, $\mathbb{C}$ or $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.*

Poincaré metric. It is the metric defined on $\mathbb{D}$ by

$$\|dz\| = \frac{2|dz|}{1 - |z|^2}.$$

Every automorphism of $\mathbb{D}$ is of the form

$$z \mapsto \lambda \frac{z + a}{1 + \bar{a}z} \quad \text{with } |\lambda| = 1,\ |a| < 1,$$

and is an isometry for the Poincaré metric.

[4] For further developments on the descriptions of compact subsets of $\mathbb{C}$, see for example [**Do5**].

If X is isomorphic to $\mathbb{D}$, we define the Poincaré metric on X by transporting the one on $\mathbb{D}$. If the universal covering $\widetilde{X}$ of X is isomorphic to $\mathbb{D}$, we define the Poincaré metric on X by the condition that $\pi : \widetilde{X} \to X$ is a local isometry.[5]

Let X and Y be two Riemann surfaces such that $\widetilde{X}$ and $\widetilde{Y}$ are isomorphic to $\mathbb{D}$, and let $f : X \to Y$ be an analytic map. Then, f is 1-Lipschitz with respect to the Poincaré metrics. We have $\|T_x f\| < 1$ for all $x \in X$, except if f is a covering. If $f(X)$ is relatively compact in Y, f is k-Lipschitz with $k < 1$ except if T is compact and f is a covering.

Carathéodory Theorem. *Let $U \subset S^2$ be an open set isomorphic to $\mathbb{D}$ and $\psi : \mathbb{D} \to U$ be an isomorphism, i.e., the conformal of U. If ∂U is locally connected, ψ has a continuous extension $\overline{\mathbb{D}} \to \overline{U}$.*

In fact, we will give a proof in theorem 2.1.

Corollary *Assume $U \subset \mathbb{C}$ is a simply connected bounded open set and $\psi : \mathbb{D} \to U$ is an isomorphism. The following are equivalent:*

- *ψ has a continuous extension $\overline{\mathbb{D}} \to \overline{U}$;*
- *∂U is locally connected;*
- *$\mathbb{C} \setminus U$ is locally connected;*
- *there exists L locally connected with $\partial U \subset L \subset \mathbb{C} \setminus U$;*
- *there exists $\gamma : \mathbb{T} \to \partial U$ surjective.*

Morrey-Ahlfors-Bers's Theorem. Let μ be an element of $L^\infty(\mathbb{C})$ satisfying $\|\mu\|_\infty = k < 1$. There then exists a unique homeomorphism $f : \mathbb{C} \to \mathbb{C}$ with distributional partial derivatives locally in L^2 such that $\partial f/\partial \bar{z} = \mu \cdot \partial f/\partial z$ locally in L^2 and such that $f(0) = 0$ and $f(1) = 1$.

[5]This is feasible since the deck transformations are isomorphisms of $\widetilde{X}$, hence isometries for the Poincaré metric.

CHAPTER 2

Compact subsets of $\mathbb{C}$.

1. Paths and arcs.

Let X be a topological space. A *path* in X is a continuous map $\gamma : I = [0,1] \to X$. An *arc* in X is a subset of X homeomorphic to I, in other words, the image of an injective path. It is usual to say that X is *connected by arc* if any pair of points of X can be joined by a path. This terminology makes sense because of the following proposition.

Proposition 2.1. *Let X be a Hausdorff space, a and b be two distinct points in X. If a and b can be joined by a path in X, they can be joined by an arc.*

Idea of the proof. Let γ be a path joining a and b. Let Ω be the set of open subsets $W \subset \overset{\circ}{I} =]0,1[$ such that, for any connected component $]\alpha,\beta[$ of W, we have $\gamma(\alpha) = \gamma(\beta)$.

- For $W \in \Omega$, there exists a unique path γ_W which coincides with γ on $I \setminus W$ and is constant on each connected component of W.
- For any open subset W of $\overset{\circ}{I}$ without adjacent connected components and such that $W \neq \overset{\circ}{I}$, there exists an increasing function $\lambda : I \to I$, constant on each connected component of W and satisfying $\lambda(t) > \lambda(s)$ if $t > s$ and $]s,t[\not\subset W$. If $W \subset \Omega$, the path γ_W is of the form $\tilde{\gamma}_W \circ \lambda$.
- Ω is inductive. If $W \in \Omega$ is maximal, W does not have two adjacent components and $W \neq \overset{\circ}{I}$.
- For W maximal, $\tilde{\gamma}_W$ is injective.

2. Locally connected compact sets.

Let X be a metric space and $h : [0,a[\to \mathbb{R}$ be a continuous and increasing function with $h(0) = 0$. We say that h is a *modulus of local connectivity* for X if for x and y in X such that $d(x,y) = r < a$, there exists a connected subset $L \subset X$ containing x and y, of diameter less than or equal to $h(r)$. Any space having a modulus of local connectivity is locally connected. Any compact metric space X which is locally connected has a modulus of local connectivity (defined on $\mathbb{R}_+$ if X is connected).

Proposition 2.2. *Any compact metric space which is locally connected is connected by arcs.*

Additional information. *Assume h is a modulus of local connectivity for X. Let x and y be two distinct points in X such that $d(x,y) = r$ and suppose $\eta > h(r)$. Then, there exists an arc joining x to y with diameter at most η.*

Proof. We can assume that X is embedded isometrically in a Banach space E, for example, by taking $E = C(X, \mathbb{R})$ and $r(x) = (y \mapsto d(x, y))$. A *polygonal path* γ with vertices in X is a path $\gamma : I \to E$ equipped with a finite set $S = \{s_0, \dots, s_n\} \subset I$ with $s_0 = 0 < s_1 < \dots < s_n = 1$ such that $\gamma(s_i) \subset X$, γ affine on $[s_i, s_{i+1}]$. We say that (γ', S') refines (γ, S) if $S' \supset S$ and $\gamma'_{|S} = \gamma_{|S}$. The *step* of γ is $\sup d(\gamma(s_i), \gamma(s_{i+1}))$.

Lemma 2.1. *Let $\gamma : I \to E$ be a polygonal path of step less than or equal to δ with vertices in X, and $\delta' > 0$. There exists a polygonal path γ' with vertices in X refining γ with step less than or equal to δ', such that $d(\gamma, \gamma') \le h(\delta)$.*

Proof. Let (δ_n) be a sequence of positive numbers tending to 0 such that $\sum h(\delta_n) < \eta - h(r)$. Let γ_1 be a polygonal path joining x to y with vertices in X and step less than or equal to δ_1, diameter less than or equal to $h(r)$, and construct recursively a sequence of paths (γ_n) such that γ_n has step less than or equal to δ_n, $d(\gamma_n, \gamma_{n+1}) \le h(\delta_n)$. This sequence converges uniformly to a path γ_∞ continuous in X joining x to y, of diameter less than or equal to $h(r) + \varepsilon = \eta$. In the image of γ_∞, we can find an arc Γ joining x to y. ■

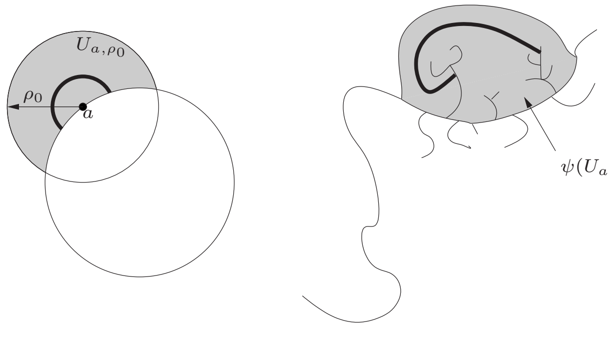
3. Carathéodory Theorem.

Let $K \subset \mathbb{C}$ be a compact set which is full (i.e., such that $\mathbb{C} \setminus K$ is connected). It follows from Riemann's uniformization theorem that there exists a unique pair (r, φ) such that φ is an isomorphism between the Riemann surfaces $\mathbb{C} \setminus K$ and $\mathbb{C} \setminus \overline{\mathbb{D}}_r$, tangent to the identity at ∞, i.e., such that $\varphi(z)/z \to 1$ as $|z| \to \infty$. The number r is the *capacity* of K.

For $z \in \mathbb{C} \setminus K$, $\log |\varphi(z)|$ is the *potential* of z and the argument of $\varphi(z)$ is the *external argument* of z relatively to K. The arguments are counted in turns (not modulo 2π). The set of points $z \in \mathbb{C} \setminus K$ of argument θ is the *external ray* $\mathcal{R}(K, \theta)$.

Theorem 2.1. (Carathéodory) *Let $K \subset \mathbb{C}$ be a full compact set. Assume there exists a locally connected compact set L such that $\partial K \subset L \subset K$. Then, the map $\psi = \varphi^{-1} : \mathbb{C} \setminus \overline{\mathbb{D}}_r \to \mathbb{C} \setminus K$ has a continuous extension $\Psi : \mathbb{C} \setminus \mathbb{D}_r \to \mathbb{C} \setminus \mathring{K}$.*

Proof. For $a \in \partial \mathbb{D}_r$ and $\rho_0 < 2r$, set $U_{a,\rho_0} = \mathbb{D}_{a,\rho_0} \cap \mathbb{C} \setminus \overline{\mathbb{D}}_r$. The open set $\psi(U_{a,\rho_0})$



is bounded, so its area is finite, equal to $\int_0^{\rho_0} A(\rho) d\rho$, where

$$A(\rho) = \int_{\theta_\rho^-}^{\theta_\rho^+} |\psi'(z(\rho, \theta))|^2 \rho d\theta = \frac{1}{\rho} \|\rho\psi'\|^2.$$

Set $\Gamma_\rho = \partial\mathbb{D}_{a,\rho} \cap (\mathbb{C} \setminus \overline{\mathbb{D}}_r)$ and $\lambda(\rho)$ the length of $\psi(\Gamma_\rho)$. We have:

$$\lambda(\rho) = \int_{\theta_\rho^-}^{\theta_\rho^+} |\psi'(z(\rho,\theta))|\rho d\theta = \langle \rho\psi', 1\rangle \, ,$$

and so,

$$\lambda(\rho)^2 \leq \|\rho\psi'\|^2 \cdot \|1\|^2 = \rho A(\rho)(\theta_\rho^+ - \theta_\rho^-) \leq 2\pi\rho A(\rho).$$

Lemma 2.2. *There exists a sequence ρ_n tending to 0, such that $\lambda(\rho_n)$ tends to 0.*

Proof.

$$\int_0^{\rho_0} \frac{\lambda(\rho)^2}{2\pi\rho} \leq \int A(\rho)d\rho < \infty.$$

□

Let h be a modulus of local connectivity for L. Let $a \in \partial\mathbb{D}_r$ and (ρ_n) be as in the lemma; set $U_n = U_{a\rho_n}$. The curve (Γ_{ρ_n}) has finite length, thus has two end points α_n and β_n in ∂K, whose distance is less than or equal to $\lambda_n = \lambda(\rho_n)$. We can join the two points α_n and β_n in L by an arc H_n of diameter less than or equal to $h(\lambda_n)$ and $\varphi(\Gamma_n) \cup H_n$ is a Jordan curve J_n whose diameter is at most $\lambda_n + h(\lambda_n)$. The open set $\psi(U_n)$ is contained in the open set bounded by J_n, thus it also has a diameter less or equal to $\lambda_n + h(\lambda_n)$. It follows that the U_n converge to a point $\psi(a)$. For $h \in \partial\mathbb{D}_r$, with $|b - a| \leq \rho_n$, we have $|\Psi(b) - \Psi(a)| \leq \lambda_n + h(\lambda_n)$, which proves the continuity of Ψ. ■

Remark. The map Ψ induces a surjective map $\gamma_K : \mathbb{T} = \mathbb{R}/\mathbb{Z} \to \partial K$ that we will call the *Carathéodory loop*. For $x \in \partial K$, the elements of $\gamma_K^{-1}(x)$ are called the *external arguments* of x.

4. Components of the interior of K.

Proposition 2.3. *Let $K \subset \mathbb{C}$ be a full, locally connected, compact set and denote by $(U_i)_{i\in I}$ the family of connected components of $\mathring{K}$.*

a) *For all i, $\overline{U}_i$ is homeomorphic to the closed disk.*

b) $\operatorname{diam}(U_i) \to 0$ *(i.e., $\forall\varepsilon > 0$, the set of i such that* $\operatorname{diam}(U_i) > \varepsilon$ *is finite)*

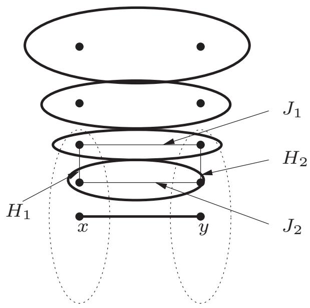
Proof. a) If Γ is a Jordan curve in U_i, the domain bounded by Γ is contained in K, thus in U_i. It follows that U_i is simply connected, thus isomorphic to $\mathbb{D}$ or $\mathbb{C}$. Since U_i is bounded, it is isomorphic to $\mathbb{D}$. Let $\psi : \mathbb{D} \to U_i$ be an isomorphism. We have $\partial U_i \subset \partial K \subset \overline{\mathbb{C}} \setminus U_i$, and ∂K is locally connected. It follows from Carathéodory's theorem that ψ extends to a continuous mapping $\Psi : \overline{\mathbb{D}} \to \overline{U}_i$. We need to show that $\Psi|_{\partial\mathbb{D}}$ is injective.

α) Analytical part. $\forall x \in \partial U_i$, $\Psi^{-1}(x)$ has empty interior in $\mathbb{T}$. This follows from Schwarz reflection principle.

β) Topological part. $\forall x \in \partial U_i$, $\Psi^{-1}(x)$ is connected. If this were not the case, we could find t_1, t_2, u_1, u_2 such that u_1 and u_2 do not belong to the same connected component of $\mathbb{T} \setminus \{t_1, t_2\}$, $\Psi(t_1) = \Psi(t_2) = x$, $\Psi(u_1) \neq x$ and $\Psi(u_2) \neq x$. Let A and B be C^1 arcs with extremities (t_1, t_2) and (u_1, u_2) intersecting transversally at a point. Then, $\Psi(A)$ is a Jordan curve $\Gamma \subset K$ and $\Psi(B)$ intersects it transversally at a point. Thus, one of the two points $\Psi(u_1)$, $\Psi(u_2)$ belongs to the domain bounded by Γ, and the other is outside this domain. But, there cannot be any point of $\partial U_i \subset \partial K$ in the domain bounded by Γ.

a) follows from α) and β).

b) Assume $n > 0$, h is a modulus of local connectivity for K and (U_{i_ν}) is a sequence of connected components of $\overset{\circ}{K}$ such that $\mathrm{diam}(U_{i_\nu}) > m$. In each U_{i_ν}, let (x_ν, y_ν) be a pair of points such that $|y_\nu - x_\nu| \geq m$. Extracting a subsequence if necessary, we may assume that $x_\nu \to x$ and $y_\nu \to y$. We have $x \in K$, $y \in K$, $|y - x| \geq m$. Let A_1 and A_2 be two disjoint connected neighborhoods of x and y in K, k be such that x_k and x_{k+1} are in A_1 and y_k and y_{k+1} are in A_2. Let H_1 be an arc from x_k to x_{k+1} in A_1, H_2 from y_k to y_{k+1} in A_2, J_1 from x_k to y_k in U_{i_k} and J_2 from x_{k+1} to y_{k+1} in $U_{i_{k+1}}$. The Jordan curve $\Gamma \subset H_1 \cup H_2 \cup J_1 \cup J_2$ with $\Gamma \cap J_1$ and $\Gamma \cap J_2$ non empty. Then, $\Gamma \subset K$, the domain bounded by Γ is contained in $\overset{\circ}{K}$, thus in a connected component of $\overset{\circ}{K}$, and intersects both U_{i_k} and $U_{i_{k+1}}$, which is a contradiction. ■



5. Projection to a component.

Proposition 2.4. *Let $K \subset \mathbb{R}^2$ be a full connected and locally connected compact set, U be a connected component of $\overset{\circ}{K}$ and $x \in K$. Let γ_1 and γ_2 be two paths in K, with $\gamma_1(0) = \gamma_2(0) = x$, $\gamma_i(1) \in \overline{U}$; denote by u_i the smallest t such that $\gamma_i(t) \in \overline{U}$. We have $\gamma_1(u_1) = \gamma_2(u_2)$. This point is called the projection of x on $\overline{U}$ and is denoted by $\pi_U(x)$.*

Proof. If $x \in \overline{U}$, we have $u_i = 0$ and $\gamma_i(u_i) = x$. We may therefore assume that $x \notin \overline{U}$. Set $y_i = \gamma_i(u_i)$ and assume $y_1 \neq y_2$. The set $L = \gamma_1([0, u_1]) \cup \gamma_2([0, u_2])$ is a compact set in which y_1 and y_2 are joined by a path, thus also by an arc J, and we have $J \cap \overline{U} = L \cap \overline{U} = \{y_1, y_2\}$. By adding an arc H joining y_1 to y_2 in $\overline{U}$, such that $H \cap \partial U = \{y_1, y_2\}$, we get a Jordan curve Γ. If z_1 and z_2 are two points in ∂U intertwined with (y_1, y_2), one is in the domain V bounded by Γ, but $V \subset K$, and so $V \subset \overset{\circ}{K}$. Since $V \cap U \neq \emptyset$, we have $V \subset U$, which gives a contradiction. ■

Corollary 2.1. *For any arc $\Gamma \subset K$, the set $\Gamma \cap \overline{U}$ is connected.*

Proposition 2.5. *Let $K \subset \mathbb{R}^2$ be a full connected and locally connected compact set and U be a connected component of $\overset{\circ}{K}$. The projection $\pi_U : K \to \overline{U}$ is continuous, locally constant on $K \setminus \overline{U}$.*

Additional information. *Let h be a modulus of local connectivity for K. Then, h is a modulus of continuity for π_U. If $h(d(x,y)) < d(x,\overline{U})$, we have $\pi_U(x) = \pi_U(y)$.*

Proof. Let x and $y \in K$, $\delta > h(|y-x|)$ and γ be a path joining x to y with diameter less than or equal to δ. If $\gamma([0,1]) \cap \overline{U} \neq \emptyset$, let u and v be the smallest and the largest t such that $\gamma(t) \in \overline{U}$. We have $\pi(x) = \gamma(u)$, $\pi(y) = \gamma(v)$, and $d(\pi(x), \pi(y)) \leq \mathrm{diam}\gamma([0,1]) \leq \delta$. If $\gamma([0,1]) \cap \overline{U} = \emptyset$, we can adjoin to γ a path from y to a point of $\overline{U}$. we obtain a path between x and a point of $\overline{U}$, and so, $\pi(x) = \pi(y)$. If $\delta > d(x,\overline{U})$, we are necessarily in the latter case. ■

6. Allowable arcs.

Let $K \subset \mathbb{C}$ be a full connected and locally connected compact set; denote by $(U_i)_{i\in I}$ the family of connected components of $\overset{\circ}{K}$. Choose in each U_i a point w_i. This determines, up to multiplication by λ of modulus 1, a homeomorphism $\varphi_i : \overline{U}_i \to \overline{\mathbb{D}}$ inducing a $\mathbb{C}$-analytic isomorphism between U_i and $\mathbb{D}$ such that $\varphi_i(w_i) = 0$.

Definition 2.1. *An arc $\Gamma \subset K$ will be called allowable if for every $i \in I$, $\varphi_i(\Gamma \cap \overline{U}_i)$ is contained in the union of two rays of $\overline{\mathbb{D}}$.*

Proposition 2.6. *Let x and y be two distinct points of K. There exists a unique allowable arc Γ joining x to y.*

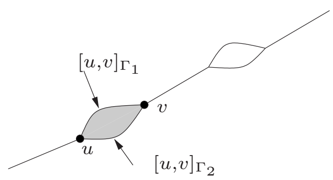
Proof.
a) Existence. If x and y are in the same $\overline{U}_i$, this is clear: if $\varphi_i(x)$ and $\varphi_i(y)$ have the same argument, $\varphi_i(\gamma)$ is the segment $[x,y]$, otherwise it is $[x,0] \cup [0,y]$.

In general, let γ be an injective path between x and y. Order the elements i of I such that $\gamma^{-1}(\overline{U}_i)$ has more than 2 points, in a sequence (i_n), and denote γ_n the path obtained by modifying γ on $\gamma^{-1}(\overline{U}_{i_1})$, $\ldots$, $\gamma^{-1}(\overline{U}_{i_n})$ so that it becomes allowable on those intervals. It follows from proposition 2.3.b) that the γ_n converge uniformly to a path γ^*. We check that γ^* is an injective path and that its image is an allowable arc.

b) Uniqueness. It is a consequence of the following lemma.

Lemma 2.3. *Let Γ_1 and Γ_2 be two allowable arcs. Then, $\Gamma_1 \cap \Gamma_2$ is connected.*

Proof. Assume this is not the case and let $]u,v[_{\Gamma_1}$ be a connected component of $\Gamma_1 \setminus (\Gamma_1 \cap \Gamma_2)$. Then, $[u,v]_{\Gamma_1} \cup [u,v]_{\Gamma_2}$ is a Jordan curve J. Let V be the domain



bounded by J. Then, $V \subset K$, thus $V \subset \overset{\circ}{K}$, so there exists i such that $V \subset U_i$ and $J \subset \overline{U}_i$. The arcs $[u,v]_{\Gamma_1}$ and $[u,v]_{\Gamma_2}$ are distinct allowable arcs between u and v in $\overline{U}_i$, which is not possible. □

■

Notation. We denote by $[x, y]_K$ the allowable arc between x and y. This notation depends on the choice of w_i. If $x = y$, we set $[x, y]_K = \{x\}$.

Properties of allowable arcs. Every sub-arc of an allowable arc is itself allowable.

Let x, y, z be three points of K. Then, $[x, y]_K \cap [y, z]_K$ is of the form $[y, c]_K$ (lemma above). We denote by $c(x, y, z)$ the point c defined in this way.

We have $[x, y]_K = [x, c]_K \cup [c, y]_K$, $[y, z]_K = [y, c]_K \cup [c, z]_K$, $[x, z]_K = [x, c]_K \cup [c, z]_K$. In particular, if $[x, y]_K \cap [y, z]_K = \{y\}$, the arc $[x, y]_K \cup [y, z]_K$ is allowable.

7. Allowable trees.

We will say that a subset $X \subset K$ is *allowably connected* if, for x and y in X, we have $[x, y]_K \subset X$. A union of a family of allowably connected subsets having a common point is allowably connected. The intersection of a family of allowably connected subsets is allowably connected. We define the *allowable hull* $[A]$ of a subset $A \subset K$ as the intersection of all the allowably connected subsets of K containing A.

Proposition 2.7. *Let $x_1, \ldots, x_n$ be points in K. The allowable hull $[x_1, \ldots, x_n]$ of the set $\{x_1, \ldots, x_n\}$ is a topological finite tree.*

Proof. By induction on n, it is clear for $n = 1$ or 2 or for $n = 3$. Assume $[x_1, \ldots x_n]$ is a topological finite tree and let $x_{n+1} \in K$. Let a be an arbitrary point of $[x_1, \ldots x_n]$ and denote by c the first point on the arc $[x_{n+1}, a]$ (starting at x_{n+1}) which belongs to $[x_1, \ldots x_n]$. Then, $[x_1, \ldots x_{n+1}] = [x_1, \ldots x_n] \cup [c, x_{n+1}]$ and $[x_1, \ldots x_n] \cap [c, x_{n+1}] = \{c\}$. ■

Remark. 1) Every end point of $[x_1, \ldots, x_n]$ is one of the points x_i, but there can be points x_i which are not end points.

2) We could define geodesic arcs. But proposition 2.7 would not hold.

CHAPTER 3

Local connectivity of some Julia sets.

1. Julia sets.

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d > 1$. We call *filled-in Julia set* of f the set K_f of points z such that $f^{\circ n}(z) \not\to \infty$. It is a compact set. Indeed, let $f_i : z \mapsto a_d z^d + \ldots + a_0$, set

$$R^* = \sup\left(1, \frac{1 + |a_{d-1}| + \ldots + |a_0|}{|a_d|}\right).$$

For $|z| > R^*$, we have $|f(z)| \geq |z|^d/R^*$. It follows that $K_f = \bigcap f^{-n}(\overline{\mathbb{D}}_{R^*})$.

Set $J_f = \partial(K_f)$; it is the *Julia set*.

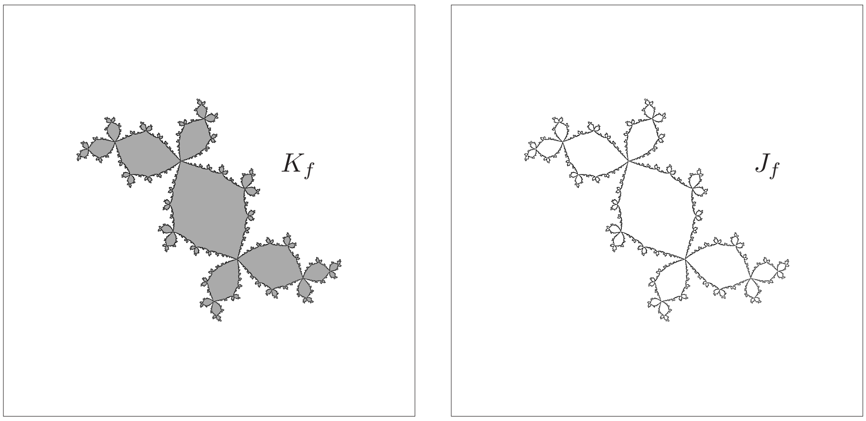


FIGURE 1. The filled-in Julia set and the Julia set of a quadratic polynomial.

We study more specifically the family $(f_c)_{c\in\mathbb{C}}$ defined by $f_c(z) = z^2 + c$. Every polynomial of degree 2 is conjugate to a unique f_c via a unique affine map. For example, $z \mapsto \lambda z + z^2$ is conjugate to f_c for $c = \lambda/2 - \lambda^2/4$. We denote by K_c the filled-in Julia set of f_c.

Proposition 3.1. (JULIA, FATOU)

a) *If $0 \in K_c$, the set K_c is connected.*

b) *If $0 \notin K_c$, the set K_c is homeomorphic to a Cantor set.*

Proof. Choose $R > 1 + |c|$ and set $V_n = f_c^{-n}(\mathbb{D}_R)$ for every n. We have $\overline{V_{n+1}} \subset V_n$ and $K_c = \bigcap V_n$.

a) For every n, $f_c : V_{n+1} \to V_n$ is a covering of degree 2 ramified at one point, V_0 is a disk, and so, V_n is homeomorphic to a disk for every n, and $K_c = \bigcap \overline{V_n}$ is connected.

b) There exists m such that $0 \in V_m$ and $c = f_c(0) \notin V_m$. Then, V_m is homeomorphic to a disk, but for every $n \geq m$, $f_c : V_{n+1} \to V_n$ is a double unramified covering. It follows that for all k, the open set V_{m+k} has 2^k connected components homeomorphic to a disk. Let δ_k be the maximum diameter of those connected components for the Poincaré metric μ on V_m. The map $f_c : V_{m+1} \to V_m$ has 2 sections g_0 and g_1, λ-Lipschitz for μ with $\lambda < 1$, and so, $\delta_k \leq \lambda^{k-1}\delta_1$. In particular, $\delta_k \to 0$, which implies b). ■

For a polynomial f of degree $d > 2$, there are in general several critical points, and so more possibilities. If all the critical points belong to K_f, the set K_f is connected. If no critical point belong to K_f, then K_f is a Cantor set. If there is at least one critical point outside K_f (and possibly others in K_f), the set K_f has an uncountable number of connected components, but some may not be reduced to a point. The proof is analogous.

For every polynomial f, the compact set K_f is full: this follows from the maximum modulus principle. We have $f(K_f) = f^{-1}(K_f) = K_f$. The map f induces a holomorphic (hence open) and proper map from $\mathring{K}_f$ to $\mathring{K}_f$. As a consequence, for each connected component U of $\mathring{K}_f$, its image $f(U)$ is a connected component of $\mathring{K}_f$ and f induces a proper map from U to $f(U)$.

There are polynomials for which K_f is locally connected and others (even in degree 2) for which it is connected but not locally connected.[1] The goal of this chapter is to give sufficient conditions for local connectivity of K_f.

2. Conformal representation of $\mathbb{C} \setminus K_f$.

Proposition 3.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d \geq 2$. Assume K_f is connected. Then, the capacity of K_f is 1 and the conformal representation $\varphi : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \overline{\mathbb{D}}$, tangent to the identity at ∞, conjugates f to $z \mapsto z^d$.*

Proof. Let R be the capacity of K_f; set $r = 1/R$, $\Phi(z) = 1/\varphi(z)$ for $z \in \mathbb{C} \setminus K_f$, and define $g : \mathbb{D}_r \to \mathbb{D}_r$ by $g = \Phi \circ f \circ \Phi^{-1}$ on $\mathbb{D}_r \setminus \{0\}$ and $g(0) = 0$. The map g is holomorphic with a zero of order d at 0, therefore it is of the form $z \mapsto uz^d$ and u does not vanish on $\mathbb{D}_r$. Moreover, g is proper, so $|u(z)|$ tends to $1/r^{d-1}$ as $|z| \to r$; it follows that u is constant. As f is monic and φ is tangent to the identity at ∞, we have $u(0) = 1$, and so, $u(z) = 1$ for all z, $r = 1$, $g(z) = z^d$ for $z \in \mathbb{D}$ and $\varphi \circ f \circ \varphi^{-1}(z) = z^d$. ■

Remark. 1) Assume that $0 \in K_f$, let $z \in \mathbb{C} \setminus K_f$ and set $z_n = f^{\circ n}(z)$. It follows from the functional equation $\varphi(z_{n+1}) = (\varphi(z_n))^d$ that φ is given by the infinite

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[1]The Julia set of a *geometrically finite* polynomial (i.e., the critical points are either in attracting basins, in parabolic basins or are preperiodic) is locally connected (see chapters 3 and 10). The Julia set of a finitely renormalizable quadratic polynomial without indifferent cycle is locally connected (see [**Hu**]). The Julia set of a polynomial having a non linearizable irrationally idifferent fixed point is not locally connected (see [**Do1**]). Finally, there exist examples of infinitely renomalizable quadratic Julia sets with connected but non locally connected Julia sets (see [**Mil**]). For further papers related to local connectivity of Julia sets, see for example [**Ki**], [**Pe**], [**PM**], [**Sø1**], [**Sø2**], [**Z**]

product

$$\varphi(z) = z \prod_{n=0}^{\infty} \left(1 + \frac{a_{d-1}}{z_n} + \ldots + \frac{a_0}{z_n^d}\right)^{1/d^{n+1}},$$

with the notation of section 1. The ambiguity due to the fractional exponent is solved as follows: for n such that $|z_n| > R^*$, take the branch of $(1+\zeta)^{1/d^{n+1}}$ that sends 0 to 1. Also, for each factor, as a function of z, there is a unique choice of continuous branch which tends to 1 as $z \to \infty$. This infinite product converges with a fantastic speed as soon as $|z_n| > R^*$.

2) In the proof (and even in the statement) of proposition 3.2, we assume that Riemann's theorem of existence of a conformal representation is known. We do not really need it since we can effectively construct the conformal representation.

3. The Carathéodory loop.

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d \geq 2$ such that K_f is connected. If K_f is locally connected, the conformal representation $\psi = \varphi^{-1} : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_f$ tangent to the identity at ∞ has a continuous extension to $\mathbb{C} \setminus \mathbb{D}$ (Carathéodory Theorem), and so, there is a surjective and continuous map $\gamma : \mathbb{R} = \mathbb{R}/\mathbb{Z} \to \partial K$ defined by $t \mapsto \psi(e^{2i\pi t})$: it is the *Carathéodory loop* of K_f (or of f). We will give a process of constructing the Carathéodory loop, process that converges *if and only if* K_f is locally connected. In the following, we will use the convergence of this process as a criterion to know whether K_f is locally connected.

Let us consider the external ray $\mathcal{R}(K_f, 0)$ of external argument 0. Let $\gamma_0 : \mathbb{T} \to \mathbb{C}$ be a loop such that $\gamma(\mathbb{T}) \subset \mathbb{C} \setminus K_f$, $\gamma_0(0) \in \mathcal{R}(K_f, 0)$, γ_0 of index 1 relatively to a point (and therefore to all points) of K_f.

Proposition 3.3. a) *We can define by induction a sequence (γ_n) of loops $\mathbb{T} \to \mathbb{C}$ by the conditions $f(\gamma_{n+1}(t)) = \gamma_n(d \cdot t)$, $\gamma_{n+1}(0) \in \mathcal{R}(K_f, 0)$.*

b) *The sequence (γ_n) converges uniformly if and only if K_f is locally connected.*

c) *If K_f is locally connected,* $\lim(\gamma_n)$ *is the Carathéodory loop of K_f.*

Proof. (a) The map $\varphi \circ \gamma_0$ is of the form $t \mapsto \rho(t)e^{2i\pi\theta(t)}$, where $\rho : \mathbb{T} \to]1, +\infty[$ and $\theta : \mathbb{T} \to \mathbb{T}$ are continuous, with θ of degree 1 and $\theta(0) = 0$. The map θ lifts to $\tilde{\theta} : \mathbb{R} \to \mathbb{R}$, continuous with $\tilde{\theta}(0) = 0$ and $\tilde{\theta}(t+1) = \tilde{\theta}(t) + 1$. Then, γ_n is given by $t \mapsto \varphi^{-1}(\rho_n(t)e^{2i\pi\theta_n(t)})$, where

$$\rho_n(t) = \rho(d^n t)^{1/d^n}$$

and $\theta_n : \mathbb{T} \to \mathbb{T}$ comes from

$$\tilde{\theta}_n : t \mapsto \frac{1}{d^n}\tilde{\theta}(d^n t).$$

(b $-$ $\Leftarrow$) and (c). Uniformly on $\mathbb{T}$, $\rho_n \to 1$ and $\theta_n \to \mathrm{Id}$. If K_f is locally connected, $\psi = \varphi^{-1}$ has a continuous extension to $\mathbb{C} \setminus \mathbb{D}$, thus γ_n converges uniformly to $t \mapsto \psi(e^{2i\pi t})$.

(b $-$ $\Rightarrow$) Assume that the γ_n converge uniformly to a loop $\gamma_\infty : \mathbb{T} \to \mathbb{C}$, and let us show that $\gamma_\infty(\mathbb{T}) = \partial K_f$. Every compact subset of $\overline{\mathbb{C}} \setminus K_f$ is contained in a $\varphi^{-1}(\overline{\mathbb{C}} \setminus \mathbb{D}_{1+\varepsilon})$. As $\rho_n \to 1$ uniformly, for every neighborhood V of K_f, we have $\gamma_n(\mathbb{T}) \subset V$ for n sufficiently large. It follows that $\gamma_\infty(\mathbb{T}) \subset \partial K_f$.

Let $x \in \partial K_f$ and $y \in \mathbb{C} \setminus K_f$ be a point close to x. Let L be a path in $\overline{\mathbb{C}} \setminus K_f$ joining y to ∞. For n sufficiently large, $\gamma_n(\mathbb{T}) \cap L = \emptyset$, so γ_n has index 0 with respect to y. Since it has index 1 with respect to x, $\gamma_n(\mathbb{T})$ intersects the segment $[x, y]$, and there exists a t_n such that $\gamma_n(t_n) \in [x, y]$, and so, $|\gamma_n(t_n) - x| < |y - x|$. Since this occurs for every $y \in \mathbb{C} \setminus K_f$, we can find a sequence (n_k) and a sequence $(s_k = t_{n_k})$ such that $\gamma_{n_k}(s_k) \to x$. Extracting a subsequence if necessary, we may assume that (s_k) has a limit s, and then $\gamma_\infty(s) = x$.

This shows that $\gamma_\infty(\mathbb{T}) = \partial K_f$. As the image of a locally connected compact set by a continuous map is locally connected, ∂K_f is locally connected, and it follows that K_f is locally connected. ■

4. Expanding and sub-expanding maps.

Let Ω be an open subset of $\mathbb{C}$. A *Riemannian metric* (compatible with the complex structure, often called a conformal metric) on Ω is the data at every point $z \in \Omega$ of a complex norm on $T_z(\Omega) = \mathbb{C}$; this norm is necessarily of the form $t \mapsto \|t\| = u(z)|t|$, where u is a function on Ω taking its values in $\mathbb{R}_+^*$. We shall write $\|dz\| = u(z)|dz|$. If u is continuous (respectively C^1,C^∞, ...) we shall say that it is a Riemannian metric with continuous coefficients (respectively C^1, C^∞, ...). If Ω is equipped with a Riemannian metric defined by a continuous function u, we define the length $\ell_u(\gamma)$ of a C^1 path γ by

$$\ell_u(\gamma) = \int_0^1 \|d(\gamma(t))\| = \int_0^1 u(\gamma(t))|\gamma'(t)|dt.$$

The *distance* $d_u(x, y)$ for x and y in Ω is the infimum of the length of paths between x and y. Let $f : \Omega \to \Omega_1$ be a holomorphic map, where Ω and Ω_1 are open subsets of $\mathbb{C}$ equipped with Riemannian metrics defined by respectively u and u_1. For $x \in \Omega$, the norm of $T_x f : T_x\Omega \to T_{f(x)}\Omega_1$ (each of those spaces being equipped with its norm) is:

$$\|T_x f\| = \frac{u_1(f(x))}{u(x)}|f'(x)|.$$

Let Ω be an open subset of $\mathbb{C}$, $f : \Omega \to \mathbb{C}$ be a holomorphic map and Λ a subset of Ω such that $f(\Lambda) \subset \Lambda$. Let $u : \Omega \to \mathbb{R}_+^*$ be a continuous function. We say that f is *strongly dilating* on Λ for the Riemannian metric defined by u if

$$(\exists \lambda > 1)\ (\forall x \in \Lambda) \quad \|T_x f\| \geq \lambda.$$

If Λ is compact and u is continuous, it is sufficient that

$$(\forall x \in \Lambda) \quad \|T_x f\| > 1.$$

Definition 3.1. *We say that f is expanding on Λ if there exists a neighborhood V of Λ in Ω and a continuous function $u : V \to \mathbb{R}_+^*$ such that f is strongly dilating on Λ for the Riemannian metric defined by u.*

Exercise. If Λ is a compact set, the following are equivalent:

- f is expanding on Λ;
- $(\exists \lambda > 1)$, $(\exists c > 0)$, $(\forall x \in \Lambda)$, $(\forall n \in \mathbb{N})$, $|(f^{\circ n})'(x)| \geq c\lambda^n$;
- $(\forall x \in \Lambda)$, $(\exists n \in \mathbb{N})$, $|(f^{\circ n})'(x)| > 1$.

We will show that if a polynomial f is expanding on its Julia set J_f, the compact set K_f is locally connected (and also J_f). But this is true under weaker hypothesis. For the formulas, we will introduce the notion of sub-expanding map.

A Riemannian metric $\|dz\| = u(z)|dz|$ is *admissible* on Ω if u is defined, continuous and positive on $\Omega \setminus \{a_1, \ldots, a_k\}$, and if $m_i \leq u(z) \leq \dfrac{c_i}{|z-a_i|^{\beta_i}}$ with $m_i > 0$, $0 < \beta_i < 1$ and $c_i < \infty$, in a neighborhood of each point a_i.

An admissible metric allows us to define a finite length for every arc which is piecewise $\mathbb{R}$-analytic, and a distance which defines the usual topology (in fact $m_i|z-a_i| \leq d(a_i, z) \leq \dfrac{|z-a_i|^{1-\beta_i}}{1-\beta_i}$.) We say that $f : \Omega \to \mathbb{C}$ is strongly dilating on Λ if each $f(\alpha_i)$ is equal to a α_j, and if there exists a neighborhood V of Λ and a $\lambda > 1$ such that for all $x \in V \setminus \big(\{a_i\}_i \cup f^{-1}(\{a_i\}_i)\big)$,

$$\|T_x f\| \geq \lambda.$$

Definition 3.2. *Let $\Lambda \subset \Omega$ be a compact set such that $f(\Lambda) \subset \Lambda$. We will say that f is sub-expanding on Λ if there exists a neighborhood V of Λ in Ω and an admissible Riemannian metric on V such that f is strongly dilating on Λ.*

5. Local connectivity for sub-hyperbolic polynomials.

Definition 3.3. *Let f be a polynomial. We say that f is hyperbolic (respectively sub-hyperbolic) if f is expanding (respectively sub-expanding) on its Julia set J_f.*

Proposition 3.4. *Let f be a polynomial such that K_f is connected. If f is sub-hyperbolic, K_f is locally connected.*

Proof. We will show that the sequence (γ_n) defined in section 3 converges uniformly. Let V be a neighborhood of J_f on which there exists an admissible metric μ for which f is strongly dilating, V_1 be a connected neighborhood of J_f relatively compact in V. For n sufficiently large (let's say $n \geq N$), we have $\gamma_n(\mathbb{T}) \subset V_1$, and γ_n and γ_{n+1} are homotopic in $V_1 \setminus J_f$. Let us denote by $\mathcal{E}$ the set of loops $\eta : \mathbb{T} \to \overline{V}_1$ and $\mathcal{F}$ the set of loops $\eta : \mathbb{T} \to V_1 \setminus J_f$, homotopic to γ_n for $n \geq N$, and such that $\eta(0) \in \mathcal{R}(K_f, 0)$. Equip V with the distance defined by μ, $\overline{V}_1$ with the induced distance, $\mathcal{E}$ with the distance of uniform convergence with respect to this distance, and $\mathcal{F}$ with the following distance:

$$d_{\mathcal{F}}(\eta, \eta') = \inf_{\substack{h \text{ homotopy between} \\ \eta \text{ and } \eta' \\ h(s,0) \in \mathcal{R}(K_f, 0)}} \sup_{t \in \mathbb{T}} \ell_\mu(s \mapsto h(s,t)).$$

We can assume that $\varphi(V_1 \setminus K_f)$ is homeomorphic to an annulus. We have, for $n \geq N$,

$$d_{\mathcal{F}}(\gamma_{n+1}, \gamma_{n+2}) \leq \frac{1}{\lambda} d_{\mathcal{F}}(\gamma_n, \gamma_{n+1}).$$

It follows that (γ_n) is a Cauchy sequence in $\mathcal{F}$, and so also in $\mathcal{E}$ since $d_{\mathcal{E}} \leq d_{\mathcal{F}}$. But $\mathcal{E}$ is complete, thus (γ_n) converges in $\mathcal{E}$ and the topology on $\mathcal{E}$ coincides with the distance of uniform convergence for the usual distance since $\overline{V}_1$ is compact, so (γ_n) converges uniformly for the usual distance. ■

6. Periodic points.

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial (or a holomorphic function).

A *periodic point* for f is a point $x \in \mathbb{C}$ such that there exists a $n > 0$ for which $f^{\circ n}(x) = x$. The smallest such n with this property s called the *period* k of x. The *cycle* of x is then $\{x_0, \ldots, x_{k-1}\}$, where $x_i = f^{\circ i}(x)$, and the *multiplier* of this cycle is $\rho = (f^{\circ k})'(x) = \prod_i f'(x_i)$. We say that x is an *attracting* (respectively *repelling*, respectively *indifferent*) cycle if $|\rho| < 1$ (respectively $|\rho| > 1$, respectively $|\rho| = 1$). A periodic point is *super-attracting* if $\rho = 0$; this is equivalent to the existence of a critical point in the cycle. We say that x is a preperiodic point if there exists an integer ℓ such that $f^{\circ \ell}(x)$ is periodic.

If x is an attracting periodic point of period k, the *basin* of x is the set of points x such that $f^{\circ nk}(z)$ tends to x as n tends to ∞. The immediate basin of x is the connected component of the basin of x that contains x. The basin (respectively immediate basin) of an attracting cycle is the union of the basins (respectively immediate basins) of the points of this cycle.

Let f be a polynomial and x be an attracting periodic point of f. The basin of x is contained in K_f, so $x \in \overset{\circ}{K}_f$.

Lemma 3.1. *The immediate basin of x is the connected component U_x of $\overset{\circ}{K}_f$ that contains x.*

Proof. This immediate basin is clearly contained in U_x.

Let V be a closed disk for the Poincaré metric on U_x centered at x. The map $f^{\circ k}$ induces a holomorphic map from U_x into itself, which is not a isomorphism, so which is λ-Lipschitz on V with $\lambda < 1$; it follows that every point in V is attracted by x. ■

Proposition 3.5. (Fatou, Julia)[2] *The immediate basin of every attracting cycle contains at least one critical point.*

Proof. The immediate basin U_x of a point x of the cycle contains at least one critical point of $f^{\circ k}$; otherwise U_x would be isomorphic to the disk $\mathbb{D}$ and $f^{\circ k}$ would be an automorphism of U_x, the inverse of which would contradict Schwarz's lemma. The proposition follows. ■

Corollary 3.1. *A polynomial of degree d has at most $d-1$ attracting cycles.*

If x is an indifferent periodic point, its multiplier ρ is of the form $e^{2i\pi\theta}$; we say that x is a rationally or Diophantine[3] indifferent periodic point, if θ has those properties. We say that x is linearizable if there exists a diffeomorphism φ from a neighborhood V of x to a disk such that $\varphi \circ f^{\circ k} \circ \varphi^{-1}$ is $z \mapsto \rho z$. The largest possible domain V is the *linearizing domain* of x.[4]

Theorem 3.1. (Siegel 1942) *Every Diophantine indifferent periodic point is linearizable.*

For a proof, see [**Si**]. One can give an simpler proof for θ Diophantine of exponent 2 (Herman).[5]

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[2]This is one of the most important results in dynamics in one complex variable.

[3]A Diophantine number is an irrational number θ satisfying the condition $|\theta - p/q| \geq C/q^\nu$ for some constants $C > 0$ and $\nu \geq 2$ for all rational number p/q. Equivalently, it is an irrational number θ such that $\log q_{n+1} = \mathcal{O}(\log q_n)$, where p_n/q_n are the approximants to θ given by the continued fraction algorithm.

[4]A linearizing domain is also called a Siegel disk.

[5]Assume θ is an irrational number and let p_n/q_n be the approximants to θ given by the continue fraction algorithm. In 1938, Cremer [**Cr**] proved that when $\sup \dfrac{\log q_{n+1}}{q_n} = +\infty$, there

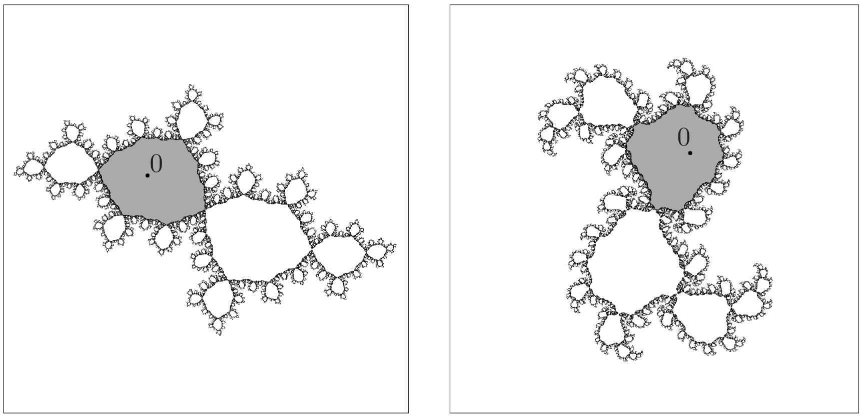

FIGURE 2. Left: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{2}}z + z^2$. Right: the Julia set of the polynomial $z \mapsto e^{2i\pi\sqrt{10}}z + z^2$. In both cases, there is a linearizing domain (light grey).

7. Characterization of hyperbolic or sub-hyperbolic polynomials.

Theorem 3.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial.*

a) *For f to be hyperbolic it is necessary and sufficient that every critical point of f which belongs to K_f is attracted by an attracting cycle*
b) *For f to be sub-hyperbolic) it is necessary and sufficient that every critical point of f which belongs to K_f is preperiodic or attracted by an attracting cycle.*

Proof. a) It is necessary. Let V be a neighborhood of J_f, μ be an admissible Riemannian metric on V, $E \subset V$ a finite set and $\lambda > 1$ such that μ has continuous coefficients on $V \setminus E$ and that, for all $x \in (V \setminus E) \cap f^{-1}(V \cap E)$, we have $\|T_x f\| \geq \lambda$. For $\varepsilon > 0$, denote by V_ε the set of points $x \in V$ such that $d_\mu(x, J_f) \leq \varepsilon$. If ε is small enough, we have $f^{-1}(V_\varepsilon) \subset V$; then $f^{-1}(V_\varepsilon) \subset V_{\varepsilon/\lambda} \subset \overset{\circ}{V}_\varepsilon$. Indeed, for every $x \in V_\varepsilon$ and all $\varepsilon' > \varepsilon$, we can find a path γ whose μ-length is less than ε' joining x to a point of J_f avoiding E; if $y \in f^{-1}(x)$, we can lift γ to a path starting at y, we obtain a path whose μ-length is less than ε'/λ joining y to a point of J_f, and so, $d_\mu(y, J_f) < \varepsilon'/\lambda$.

Let us choose such an ε and set $L = K_f \setminus \overset{\circ}{V}_\varepsilon$. The set L is compact, so, the family $(U_i)_{i\in J}$ of connected components of $\overset{\circ}{K}_f$ that intersect L is finite, and each such component is preperiodic. Also, we have $f(L) \subset L$; thus, if $f^{\circ k}(U_i) \subset U_i$, we have $f^{\circ k}(U_i \cap L) \subset U_i \cap L$ and $f^{\circ k}$ induces a map from $U_i \cap L$ into itself, strongly contracting for the Poincaré metric on U_i. It follows that every point of L is attracted by an attracting cycle.

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exists a germ $f(z) = e^{2i\pi\theta}z + \mathcal{O}(z^2)$ which is not linearizable. Around 1965, Brjuno [**Brj**] refined Siegel's arguments and proved that when the sum $\sum \dfrac{\log q_{n+1}}{q_n}$ converges, every germ $f(z) = e^{2i\pi\theta}z + \mathcal{O}(z^2)$ is linearizable. In the 80's, Yoccoz [**Yo**] proved that when the sum diverges, the quadratic polynomial $e^{2i\pi\theta}z + z^2$ is not linearizable at 0. This proves the optimality of the Brjuno condition.

Let E^* be the set of points $a \in E$ such that the coefficient of μ is bounded in a neighborhood of a. For $a \in E^*$, we have $f(a) \in E^*$ if $f(a) \in V$. Also, for every critical point c of f which belongs to V, we have $f(c) \in E^*$ if $f(c) \in V$. Let c be a critical point of f which belongs to K_f. If for all n, $f^{\circ n}(c) \in V_\varepsilon$, the set of points $f^{\circ n}(c)$ for $n > 0$ is contained in E^*, thus finite, and c is preperiodic. If there exists n such that $f^{\circ n}(c) \notin V_\varepsilon$, for such an n we have $f^{\circ n}(c) \in L$ and c is attracted by an attracting cycle.

If $E = \emptyset$, every critical point is attracted by an attracting cycle.

b) Let $\mathcal{C}_f$ be the set of critical points of f and $\mathcal{P}_f := \bigcup_{n \geq 1} f^{\circ n}(\mathcal{C}_f)$ be the postcritical set of f. Let $\underline{x} = (x_0, \ldots, x_{k-1}, x_k = x_0)$ be an attracting cycle, and let $U_{\underline{x}}$ be the basin of $\underline{x}$.

Lemma 3.2. *There exists an open set $V_{\underline{x}} \subset U_{\underline{x}}$ such that $\overline{f(V_{\underline{x}})} \subset V_{\underline{x}}$ (the closure is in $\mathbb{C}$, not in $U_{\underline{x}}$), and $\mathcal{P}_f \cap U_{\underline{x}} \subset V_{\underline{x}}$.*

Proof. Choose $V_0, \ldots V_{k-1}$ neighborhoods of $x_0, \ldots, x_{k-1}$ so that $\overline{f(V_i)} \subset V_{i+1}$. These exist because $\underline{x}$ is attracting. Choose n sufficiently large so that $f^{\circ n}(\mathcal{C}_f) \subset V := V_0 \cup \ldots \cup V_{k+1}$. Then, the set $V_{\underline{x}} = f^{-n}(V)$ satisfies the requirements. □

Construct for each attracting cycle $\underline{x}$ a subset $V_{\underline{x}} \subset U_{\underline{x}}$ and moreover choose R so that $\overline{f^{-1}(\mathbb{D}_R)} \subset \mathbb{D}_R$. Define

$$U = \mathbb{D}_{\mathbb{R}} \setminus \left(\bigcup_{\underline{x} \textit{ attracting cycle}} \overline{V_{\underline{x}}} \right) \quad \text{and} \quad U' = f^{-1}(U).$$

Lemma 3.3. *U' is relatively compact in U and $f : U' \to U$ is a covering map, ramified at the points of $J_f \cap \mathcal{C}_f$.*

The theorem follows in the case where $J_f \cap \mathcal{C}_f = \emptyset$. Let $\mu = \| \cdot \|_U$ be the Poincaré metric of U and $\| \cdot \|_{U'}$ be the Poincaré metric of U'. Since f is then a covering map $U' \to U$, we have for $x \in U'$

$$\|Df(x)(\xi)\|_U = \|\xi\|_{U'} > \|\xi\|_U.$$

The case where $J_f \cap \mathcal{C}_f \neq \emptyset$ is considerably more elaborate and will require the *orbifold metric* of U.

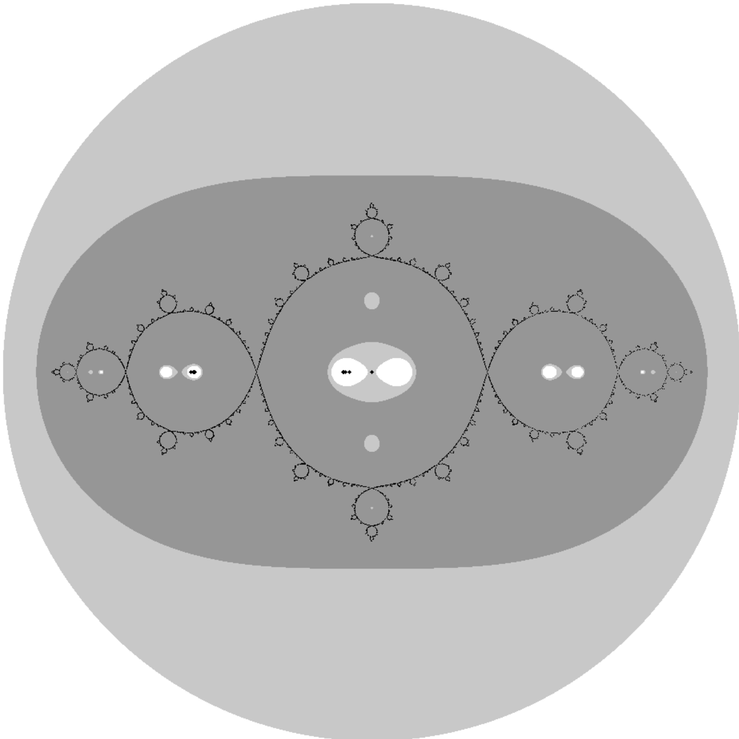
Give each point $x \in \mathcal{P}_f \cap J_f$ the weight

$$\delta_x = \lim_{n \to \infty} \operatorname*{LCM}_{y \in f^{-n}(x)} \deg_y f^{\circ n}.$$

The sequence $n \mapsto \operatorname*{LCM}_{y \in f^{-n}(x)} \deg_y f^{\circ n}$ is eventually constant since the critical points in J_f are assumed preperiodic, so δ_x is well defined.

Let $\pi : X^* \to U \backslash \mathcal{P}_f$ be the covering map corresponding to the normal subgroup $G \subset \pi_1(U \backslash \mathcal{P}_f)$ generated by $\gamma_x^{\delta_x}$, where γ_x is a small loop around $x \in \mathcal{P}_f$. Note that since G is normal, there are no difficulties associated to the choice of base-point.

If D_x is the disk bounded by γ_x, then every component of $\pi^{-1}(D_x \setminus \{x\})$ is a δ_x-fold connected cover. It follows that we can add a point above x to each such component, creating a Riemann surface X and a ramified covering map still denoted $\pi : X \to U$, ramified of order δ_x at every invers image of each $x \in \mathcal{P}_f$.



Lemma 3.4. a) *X is a simply connected Riemann surface isomorphic to $\mathbb{D}$*

b) *π is a Galois cover, so that for all $y_1, y_2 \in \pi^{-1}(y)$, there exists a deck transformation $\sigma : X \to X$ such that $\sigma(y_1) = y_2$.*

c) *The Poincaré metric of X induces en admissible metric on U.*

Proof. a) By definition, $\pi_1(X) = G$. So it is enough to show that for any $\alpha \in \pi_1(X)$ in the conjugacy class of γ_x, the element $\alpha^{\delta_x} \in \pi_1(X^*)$ is contractible in $\pi_1(X)$. This is clear. Moreover, $\pi : X \to \mathbb{C}$ is a bounded analytic function, so $X \simeq \mathbb{D}$.

b) Since G is normal, $\pi : X^* \to U \setminus \mathcal{P}_f$ is Galois. Thus, it is enough to show that the deck transformations of X^* extend analytically to X. Again, this should be clear.

c) By part b), if $x \in U \setminus \mathcal{P}_f$, $y_1, y_2 \in \pi^{-1}(x)$ and $\xi \in T_x U$, then the POincaré metric $\|\cdot\|_X$ of X satisfies

$$\left\|(D\pi(y_1))^{-1}\xi\right\|_X = \left\|(D\pi(y_2))^{-1}\xi\right\|_X,$$

so $\|\cdot\|_X$ does induce a metric μ on U^*. To see how it behaves near a point of $\mathcal{P}_f$, we can choose charts z near x and ζ near $y \in \pi^{-1}(x)$ such that $z = \zeta^\delta$, and the Poincaré metric on X is $u(\zeta)|d\zeta|$ with u continuous and satisfies $u(\zeta) = u(\lambda\zeta)$ if $\lambda^\delta = 1$. Then, the induced metric on U is

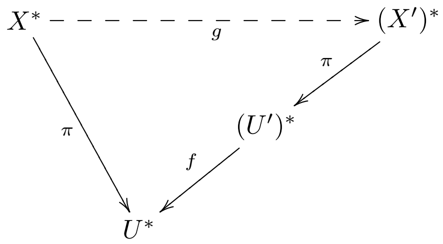
$$v(z)|dz|, \quad \text{where} \quad v(z) = \delta|z|^{\frac{\delta-1}{\delta}} u\left(z^{\frac{\delta-1}{\delta}}\right).$$

□

The following lemma ends the proof of the theorem.

Lemma 3.5. *The map $f : U' \to U$ is strongly dilating on J_f for the orbifold metric μ.*

Proof. Consider the diagram of covering spaces

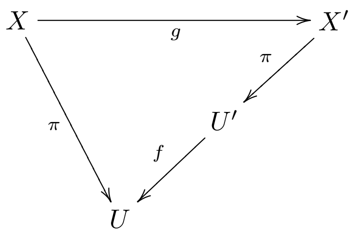


The existence of g making the diagram commute follows from the lifting criterion in covering spaces: g exists if and only if $\pi_*(\pi_1(X^*)) \subset (f \circ \pi)_*(\pi_1(X')^*)$.

Note: this might seem ambiguous because $f \circ \pi$ is not a Galois cover, so $(f \circ \pi)_*(\pi_1(X')^*)$ is not a normal subgroup, and only well defined after choosing base points. But $\pi_*(\pi_1(X^*))$ is a normal subgroup, so if it is contained in so;e $(f \circ \pi)_*(\pi_1(X')^*)$, it is contained in all its conjugates, and there is no ambiguity.

Thus we see that the condition for the existence of g is that for every γ_x, $y \in (f \circ \pi)^{-1}\gamma_x$ (read "every path connecting $(f \circ \pi)y$ to a base point of U"), $\gamma_x^{\delta_x}$ can be lifted to $(X')^*$. This requires that δ_x be a multiple of $\delta_z(\deg_z f)$ for all $z \in f^{-1}(x)$, and the weights δ_x were chosen so that this is true.

The same argument implies that g can be uniquely extended to an analytic mapping $X \to X'$ making the diagram



commute.

Since g is analytic, it is contracting for the Poincaré metrics of X and X', and even more contracting if we use the Poincaré metric of X for both X and X'. More precisely, there is a continuous function $k : X' \to [0, 1[$ such that for every $x \in X'$ and $\xi \in T_x X'$, we have $\|\xi\|_{X'} = k(x)\|\xi\|_X$. If $\pi(x_1) = \pi(x_2)$ then $k(x_1) = k(x_2)$ and since $J_f \subset U' = \pi(X')$ is compact, we see that there exists $k_0 < 1$ such that $k(x) \leq k_0$ for all $x \in \pi^{-1}(J_f)$.

Now, assume $x \in J_f$ and let $\xi \in T_x U'$ be a tangent vector to U'. We can write (choosing $\tilde{x} \in X$ such that $\pi \circ g(\tilde{x}) = x$, and suppressing the points where the derivatives are evaluated)

$$\begin{aligned} \|T_f(\xi)\|_\mu &= \|(T\pi)^{-1}Tf(\xi)\|_X \geq \|Tg(T\pi)^{-1}Tf(\xi)\|_{X'} \\ &\geq \frac{1}{k_0}\|Tg(T\pi)^{-1}Tf(\xi)\|_X = \frac{1}{k_0}\|T\pi Tg(T\pi)^{-1}Tf(\xi)\|_\mu = \frac{1}{k_0}\|\xi\|_\mu. \end{aligned}$$

Therefore f is strongly dilating on J_f, for the metric μ. □

■

CHAPTER 4

Hubbard trees.

1. Action on $\pi_0(\overset{\circ}{K}_f)$.

Let f be a polynomial of degree $d \geq 2$, and denote by $(U_i)_{i \in I}$ the family of connected components of $\overset{\circ}{K}_f$, so that $I = \pi_0(\overset{\circ}{K}_f)$. Remember that for all $i \in I$, $f(U_i)$ is one of the U_j, and we note $j = f_*(i)$. Also, $f : U_i \to U_j$ is proper and holomorphic of degree d_i, where $d_i - 1$ is the number of critical points counted with multiplicities in U_i. In particular, $\sum(d_i - 1) \leq d - 1$, with equality if and only if f is hyperbolic and K_f is connected.

Proposition 4.1. *Assume f is sub-hyperbolic.*

a) *Every element of I is preperiodic for f_*.*
b) *For every periodic i, U_i contains a periodic attracting point and is its immediate basin.*
c) *Every cycle of connected components of $\overset{\circ}{K}_f$ contains at least a critical point.*[1]

Remark. a) is true without the hypothesis of sub-hyperbolicity (Sullivan)[2] but the proof is much more difficult.

Proof. Let V be a neighborhood of J_f, μ be an admissible Riemannian metric on V and $\lambda > 1$ such that $\forall x \in V' = f^{-1}(V)$, $\|T_x f\|_\mu \geq \lambda$. Let $\varepsilon > 0$ be such that $V_1 = \{x \in V \mid d_\mu(x, J_f) < \varepsilon\}$ is relatively compact in V; set $V_1' = f^{-1}(V)$ and $L = K_f \setminus V_1'$. The set L is compact and we have $f(L) = K_f \setminus V_1 \subset L$. Denote by I_L the set of $i \in I$ such that $U_i \cap L \neq \emptyset$. Since the $U_i \cap L$ form a covering of L by disjoint open sets, I_L is finite.

(a) We have $f_*(I_L) = I_L$, thus, every element of I_L is preperiodic. Let $i \in I$, $x \in U_i$ and $n \geq \dfrac{\log \varepsilon - \log d(x, J_f)}{\log \lambda}$. Then, $f^{\circ n}(x) \in L$, and so, $f_*^{\circ n}(i) \in I_L$, thus i is preperiodic.

(b) Let i be such that $f_*^{\circ k}(i) = i$ with $k \geq i$. We then have $f^{\circ k}(L \cap U_i) \subset L' \cap U_i \subset L \cap U_i$. It follows that $\operatorname{diam}(f^{\circ k}(L \cap U_i)) < \operatorname{diam}(L \cap U_i)$, where the diameter is taken for the Poincaré metric on U_i. It follows that $f^{\circ k} : U_i \to U_i$ is not an isomorphism, $\|T_x f^{\circ k}\| < 1$ for all $x \in L \cap U_i$, $\sup_{L \cap U_i} \|T_x f^{\circ k}\| < 1$, $f^{\circ k} : L \cap U_i \to L \cap U_i$ is strongly contracting, thus has an attracting fixed point α_i.

The periodic point α_i attracts every point in $L \cap U_i$, and varying ε, we see that it attracts every point in U_i, so U_i is contained in the immediate basin of α_i. As

[1] This has already been proved in proposition 3.5.

[2] Sullivan proved that every Fatou component of a polynomial (in fact of a rational map) is eventually periodic (see [**Su1**] or [**Mil**] appendix F for a proof). The proof is based on the use of quasiconformal surgery.

it is connected, it is contained in the immediate basin. This immediate basin is connected, contained in $\overset{\circ}{K}_f$ and contains α_i, so is contained in U_i and is equal to it.

(c) The map $f^{\circ k} : U_i \to U_i$ is holomorphic and proper but is not an isomorphism, so its degree is $\delta > 1$. Observe that $\prod_{j=0}^{k-1} d_{f_*^{\circ j}(i)} = \delta$, so one factor is greater than 1, and the corresponding $U_{f_*^{\circ j}(i)}$ contains a critical point. ■

2. The centers of the U_i's.

We keep the notations of the preceding section.

Proposition 4.2. *Assume every critical point of f is periodic or preperiodic. Then, we can choose, for all $i \in I$, an isomorphism $\varphi_i : U_i \to \mathbb{D}$ so that $\forall i$, $\varphi_{f_*(i)} \circ f \circ \varphi_i^{-1} : \mathbb{D} \to \mathbb{D}$ is the map $z \mapsto z^{d_i}$. If $d_i = 2$, this choice is unique.*

Proof.

Lemma 4.1. *Let $h : \mathbb{D} \to \mathbb{D}$ be a holomorphic and proper map of degree δ, such that $h(0) = 0$. Assume every critical point of f is periodic or preperiodic. Then, h is of the form $z \mapsto \lambda z^\delta$ with $|\lambda| = 1$.*

Proof. If $\delta = 1$, h is an isomorphism, so of the form $z \mapsto \lambda z$. We may therefore assume that $\delta > 1$. Then, 0 attracts $\mathbb{D}$, and every critical point of h falls on 0 in finite time.

Denote by A the union of the forward orbits of the critical points of h. The set A is finite. Let γ be a loop surrounding A such that $\gamma \cap h^{-n}(A) = \emptyset$ for all n, and set $\gamma_n = h^{\circ n}(\gamma)$. For n large enough, γ_n is contained in a small disk centered at 0, which contains no other point of A. Then, γ_n is homotopic in $\mathbb{D} \setminus A$ to a loop η_n whose length for the Poincaré metric on $\mathbb{D} \setminus A$ is arbitrarily small. Since $h^{\circ n} : \mathbb{D} \setminus f^{-n}(A) \to \mathbb{D} \setminus A$ is a covering, γ is homotopic in $\mathbb{D} \setminus h^{-n}(A)$ (and a fortiori in $\mathbb{D} \setminus A$) to a loop η lifting η_n. We then have $\text{length}_{\mathbb{D}}(\eta) \le \text{length}_{\mathbb{D} \setminus h^{-n}(A)}(\eta)$ arbitrarily small, which shows that A is reduced to a point, which is necessarily 0. The multiplicity of 0 as a critical point is $\delta - 1$.

The map h is therefore of the form $z \mapsto u(z) \cdot z^\delta$, where u is holomorphic, does not vanish, and $|u(z)| \to 1$ as $|z| \to 1$ since h is proper. It follows that u is constant of modulus 1. □

Let $i \in I$ be a periodic point of period k and α_i be the periodic attracting point of f which belongs to U_i. Let $\varphi_i : U_i \to \mathbb{D}$ be an isomorphism such that $\varphi_i(\alpha_i) = 0$ and set $h = \varphi_i \circ f^{\circ k} \circ \varphi_i^{-1}$. It follows from the preceding lemma that h is of the form $z \mapsto \lambda z^\delta$, and we have $\delta > 1$ according to proposition 4.1 (c). If we replace φ_i by $\mu \varphi_i$ with $|\mu| = 1$, we replace λ by $\mu^{\delta - 1}\lambda$. It follows that we can choose φ_i so that $\lambda = 1$. This choice can be made in $\delta - 1$ ways.

For $0 \le \ell \le k - 1$, the equivalence relation defined on U_i by $f^{\circ \ell}$ is finer than the one defined by $f^{\circ k}$; carried to $\mathbb{D}$ via φ_i, it becomes of the form $z \sim z_1 \Leftrightarrow z^{\delta'} \sim z_1^{\delta'}$, where δ' divides δ. We can therefore choose in a unique way $\varphi_{f_*(i)^{\circ \ell}} : U_{f_*^{\circ \ell}(i)} \xrightarrow{\simeq} \mathbb{D}$, such that $\varphi_{f_*(i)^{\circ \ell}} \circ f^{\circ \ell} \circ \varphi_i^{-1}$ is $z \mapsto z^{\delta'}$. We then have $\varphi_{f_*(j)} \circ f \circ \varphi_j = (z \mapsto z^{d_j})$ for all j in the cycle $\{f_*^{\circ \ell}(i)\}_{\ell = 0, \dots, k-1}$. We proceed in the same way for each of the cycles of f_*. We then construct recursively on ν the φ_i for the i such that $f_*^{\circ \nu}$ is periodic. The point $\varphi_i^{-1}(0)$ is called the *center* of U_i. The inductive step is done by observing that, if there is a critical point in U_i, its image is necessarily the center of $U_{f_*(i)}$ since in this open set, the center is the only point which is preperiodic. We then have d_i possible choices for φ_i. Finally, the number of possible choices for the whole family (φ_i) is

$$\prod_{\zeta \text{ cycle}} \left(\left(\prod_{i\in\zeta} d_i \right) - 1 \right) \cdot \prod_{i \text{ non periodic}} d_i.$$

In particular, if $d = 2$, there is one simple critical point (so with $d_i = 2$) in the unique cycle, so one choice for the family. ∎

Remark. Even if there are choices for the φ_i, for each i the center $\varphi_i^{-1}(0)$ of U_i is uniquely determined.

3. The Hubbard tree.

In the following, f is a polynomial of degree $d \geq 2$ such that every critical point is periodic or preperiodic. Remember that this implies that f is sub-hyperbolic and that K_f is connected and locally connected. We keep the notations of the two preceding sections. In particular, each U_i is equipped with a center, which enables us to define allowable arcs.

Remember that if x and y are two points in K_f, there exists a unique allowable arc $[x, y]_f$ with extremities x and y, and that if (x_s) is a family of points in K_f, the set $\bigcup [x_{s_1}, x_{s_i}]_f$ is a finite topological tree, called the allowable hull of the x_s.

Definition 4.1. *We will call the Hubbard tree of f the allowable hull H_f of the union of the forward orbits of the critical points.*

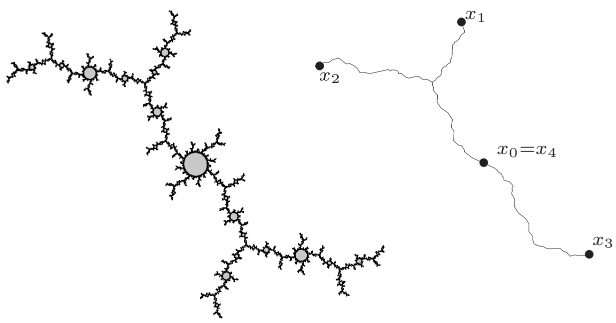


FIGURE 1. An example of Hubbard tree.

Denote by $\mathcal{C}$ the set of critical points of f and (H_σ) the closures of the connected components of $H_f \setminus \mathcal{C}$.

Proposition 4.3. *The map f induces a continuous map from H_f to itself whose restriction to each H_σ is injective.*

Proof.

Lemma 4.2. *Let $\Gamma \subset K_f$ be an allowable arc which does not contain any critical point of f, except possibly its extremities. Then, $f_{|\Gamma}$ is injective and $f(\Gamma)$ is an allowable arc.*

Proof. Let $\gamma : I \to K_f$ be a path with image Γ. If $f \circ \gamma$ is injective, its image is an allowable arc because f maps an internal ray of $\overline{U}_i$ to an internal arc of $\overline{U}_{f_*(i)}$. Let us show that $\eta = f \circ \gamma$ is necessarily injective. It is clear that γ is locally injective, so $S = \{(t_1, t_2) \mid t_1 < t_2 \text{ and } \eta(t_1) = \eta(t_2)\}$ is compact. Assume $S \neq \emptyset$ and let $(t_1, t_2) \in S^2$ with $t_2 - t_1$ minimum and $t_3 \in]t_1, t_2[$. Then, $\eta([t_1, t_3])$ and $\eta([t_3, t_2])$ are allowable arcs with the same extremities; they coincide, which contradicts the injectivity of η on $]t_1, t_2[$. So, we have $S = \emptyset$, η injective and $f(\Gamma) = \eta(I)$ is an allowable arc. □

Lemma 4.3. *Let (x_s) be a finite family of points in K_f and H be the allowable hull of the (x_s). Then, $f(H)$ is the allowable hull of the $f(x_s)$ and the $f(w)$ for $w \in H \cap \mathcal{C}$.*

Proof. The set H is a union of allowable arcs of the form $[x_{s_1}, x_{s_2}]_f$, $[x_s, w]_f$, $[w_1, w_2]_f$ not containing elements of $\mathcal{C}$ except possibly their extremities. Then, $f(H)$ is the union of the corresponding $[f(x_{s_1}), f(x_{s_2})]_f$, $[f(x_s), f(w)]_f$, $[f(w_1), f(w_2)]_f$, thus is contained in the allowable hull of the $f(x_s)$ and the $f(w)$. Since it is connected and contains the $f(x_s)$ and the $f(w)$, $f(H)$ is equal to this hull. □

We now come to the proof of the proposition. The first assertion follows from lemma 4.3. If x and y are two distinct points in the same H_σ, the arc $\Gamma = [x, y]_f$ does not contain any critical point, except possibly x or y, so $f_{|\Gamma}$ is injective and $f(x) \neq f(y)$. ■

4. The case of quadratic polynomials.

Assume now that $d = 2$ and f is of the form $z \mapsto z^2 + c$. The critical point is 0. We set $a_n = f^{\circ n}(0)$ and we denote by A the forward orbit of 0 (which is finite by hypothesis). Two cases are possible.

Periodic case. 0 is periodic, we denote by k its period; so we have $\#A = k$, $A = \{a_0, \dots, a_{k-1}$. The points a_i are super-attracting, so in $\overset{\circ}{K}_f$; we denote by U_i the connected component of $\overset{\circ}{K}_f$ containing a_i (which is its immediate basin). Hence, we have $d_0 = 2$, $d_i = 1$ for $i = 1, \dots, k-1$, and $f^{\circ k} : U_0 \to U_0$ is of degree $\delta = 2$:

$$U_0 \underset{2 \to 1}{\overset{f}{\longrightarrow}} U_1 \underset{1 \to 1}{\overset{f}{\longrightarrow}} U_2 \underset{1 \to 1}{\overset{f}{\longrightarrow}} \dots \underset{1 \to 1}{\overset{f}{\longrightarrow}} U_{k-1} \underset{1 \to 1}{\overset{f}{\longrightarrow}} U_0.$$

Every other connected component of $\overset{\circ}{K}_f$ is mapped by a finite number of iterates, homeomorphically, to one of the U_i.

Strictly preperiodic (Misurewicz) case. 0 is mapped after ℓ iterates in a cycle of order k: we have $a_\ell = a_{k+\ell}$, $a_{\ell-1} \neq a_{k+\ell-1}$, and so $\ell \geq 2$ and $a_{\ell-1} = -a_{\ell+k-1}$.

The set K has empty interior and the cycle $\{a_\ell, \ldots, a_{\ell+k-1}$ is repelling. Denote by $\nu(i)$ the number of branches of the tree H_f at a_i.

Proposition 4.4. *Assume $d = 2$.*

a) *Periodic case. If $k = 1$, we have $c = 0$, $\nu(0) = 0$. If $k > 1$, there exists r, $2 \leq r \leq k$, such that $\nu(i) = 1$ for $1 \leq i \leq r$ and $\nu(i) = 2$ for $r < i \leq k$. The internal arguments of the branches at a_i are: 0 if $\nu(i) = 1$, 0 and $1/2$ if $\nu(i) = 2$.*

b) *Strictly preperiodic case. We have $\nu(0) = 2$, $\nu(1) = \nu(2) = 1 \leq \ldots \leq \nu(\ell) = \ldots = \nu(\ell + k - 1)$.*

Proof. a) Since $f(H_f) \subset H_f$, we have $\nu(1) \geq 1/2\nu(0)$, and $\nu(1) \leq \nu(2) \leq \nu(k) = \nu(0)$. Also, if $H_f \neq \{a_0\}$, this tree has at least two extremities, so there exist at least 2 values of i such that $\nu(i) = 1$, and so $\nu(1) = \nu(2) = 1 \leq \ldots \leq \nu(k) = \nu(0) = 2$. Denote by A_i the set of internal arguments of the branches at a_i (so that $\nu_i = \#A_i$), and $q : \mathbb{T} \to \mathbb{T}$ the map $t \mapsto 2t$. We have $q(A_1) \subset A_1$ since $f^{\circ k}(H_f) \subset H_f$, and so, $a_1 = \{0\}$. We have $q(A_0) \subset A_1 \subset A_2 \subset \ldots \subset A_k = A_0$, and so $A_0 \subset q^{-1}(0) = \{0, 1/2\}$ and $\{0\} \subset A_i \subset \{0, 1/2\}$.

b) We still have

$$\frac{1}{2}\nu(0) \leq \nu(1) \leq \ldots \leq \nu(\ell) \leq \nu(\ell + k + 1) \leq \nu(\ell + k) = \nu(\ell).$$

Let us show that $\nu(0) > 1$. If we had $\nu(0) = 1$, the map $f : H_f \to H_f$ would be injective, contradicting $f(a_{\ell-1}) = f(a_{\ell+k-1})$. Since H_f must have at least two extremities, we have $\nu(1) = \nu(2) = 1$. ■

CHAPTER 5

Julia sets with zero Lebesgue measure.

The results in this chapter have been obtained independently by M. Yu Lyubich. A short proof is given in [**L1**].[1]

The problem of knowing whether J_f has zero Lebesgue measure for every polynomial f is still open (even for the polynomials $z \mapsto z^2 + c$). We show that it is the case if f is hyperbolic, or only sub-hyperbolic (in this last case, the proof is only sketched).

1. Distortion.

Definition 5.1. *Let U be a connected open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a holomorphic function. We call distortion of f on U the quantity:*

$$\mathrm{dist}_U(f) = \sup_{x,y\in U} \left| \log \frac{f'(y)}{f'(x)} \right| .$$

Comment. If f is affine, we have $\mathrm{dist}_U(f) = 0$. We have $\mathrm{dist}_U(f) = \infty$ if f has a critical point or if f' is a map $U \to \mathbb{C}^*$ not homotopic to a constant. In other cases, one must take the determination of log that is 0 when $x = y$.

If $f : U \to V$ and $g : V \to \mathbb{C}$ are holomorphic maps, we have

$$\mathrm{dist}_U(g \circ f) \leq \mathrm{dist}_U(f) + \mathrm{dist}_V(g).$$

If $f : U \xrightarrow{\simeq} V$ is an isomorphism, we have $\mathrm{dist}_U(f) = \mathrm{dist}_V(f^{-1})$. We have

$$\mathrm{dist}_U(f) \leq \overline{\mathrm{diam}}(U) \cdot \sup \left| \frac{f''}{f'} \right| ,$$

where

$$\overline{\mathrm{diam}}(U) = \sup_{x,y\in U} \left(\inf_{\substack{\gamma \text{ path between} \\ x \text{ and } y}} (\mathrm{length}(\gamma)) \right) .$$

Theorem 5.1. (Quasi-self-similarity). *Let U be an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ be a holomorphic map, $\Lambda \subset U$ be a compact set such that $f(\Lambda) \subset \Lambda$ and such that f is expanding on Λ. Then,*

$$(\forall m > 0)\ (\exists b > a > 0)\ (\forall \varepsilon \in]0, a[)\ (\forall x \in \Lambda)\ (\exists n \in \mathbb{N})$$
$$B(f^{\circ n}(x), a) \subset f^{\circ n}(B(x, \varepsilon)) \subset B(f^{\circ n}(x), b) \quad \text{and} \quad \mathrm{dist}_{B(x,\varepsilon)}(f^{\circ n}) \leq m.$$

[1]In all the cases where it is known, the area of the Julia sets of polynomials is zero. For example, this is known for all sub-hyperbolic polynomials (see this chapter), for geometrically finite polynomials (see [**L1**]), for finitely renormalizable quadratic polynomials (see [**L2**] or [**Sh**]), for some cubic polynomials (see [**BH**]) and for some infinitely renormalizable polynomials (see [**Ya**]).

Proof. Let U_1 be an open neighborhood of Λ with U_1 relatively compact in U, $u : \overline{U}_1 \to \mathbb{R}_+^*$ a continuous map defining a Riemannian metric μ and $\lambda > 1$, such that $\|T_z f\|_\mu \geq \lambda$ for all $z \in U_1 \cap f^{-1}(U_1)$.

Set

$$M_1 = \inf_{U_1} u, \quad M_2 = \sup_{U_1} u, \quad M_3 = \sup_{U_1} \left| \frac{f''}{f'} \right|, \quad M_4 = \sup_{U_1} |f'| > 1,$$

$$M = \frac{M_2 M_3}{M_1}, \quad \text{and} \quad b_0 = d(\Lambda, \mathbb{C} \setminus U_1).$$

Let m be such that $0 < m \leq 1$; set $b = \inf \left(b_0 \dfrac{M_1}{M_2}, \dfrac{m(\lambda - 1)}{2M} \right)$ and $a = be^{-2m}/M_4$.

Let $x \in \Lambda$; set $x_k = f^{\circ k}(x)$ and $\rho_k = |(f^{\circ k})'(x)|$ for all k. Choose $n \in \mathbb{N}$ arbitrarily, set $V_n = B(x_n, b)$ and, for $0 \leq k \leq n$, denote by V_k the connected component of $f^{-(n-k)}(V_n)$ which contains x_k, so that $f^{\circ(n-k)}$ induces a homeomorphism from V_k to V_n. We have

$$\overline{\operatorname{diam}}(V_k) \leq \frac{2bM_2}{M_1 \lambda^{n-k}},$$

and so,

$$\operatorname{dist}_{V_k}(f) \leq \frac{2bM}{\lambda^{n-k}} \quad \text{and} \quad \operatorname{dist}_{V_0}(f^{\circ n}) \leq \sum_{k=0}^{n-1} \operatorname{dist}_{V_k}(f) < \frac{2bM}{\lambda - 1} \leq m.$$

It follows that

$$V_0 \supset B\left(x_0, \frac{b}{\rho_n} e^{-m}\right) \quad \text{and} \quad f^{-n}(B(x_n, a)) \cap V_0 \subset B\left(x_0, \frac{a}{\rho_n} e^m\right).$$

Set $\varepsilon_0 = be^{-m}$ and let ε be such that $0 < \varepsilon \leq \varepsilon_0$.

We may now come back on the choice of n. Since we have $\rho_k < \rho_{k+1} \leq M_4 \rho_k$ for all k, and $M_4 e^m a = e^{-m} b$, if n is the largest value of k such that $\rho_k \varepsilon \leq e^{-m} b$, we have: $\rho_n \varepsilon \geq e^m a$, and so

$$f^{-n}(B(x_n, a)) \cap V_0 \subset B(x_0, \varepsilon) \subset V_0, \quad B(x_n, a) \subset f^{\circ n}(B(x_0, \varepsilon)) \subset B(x_n, b)$$

and $\operatorname{dist}_{B(x_0,\varepsilon)}(f^{\circ n}) \leq m$. ■

2. Density.

In $\mathbb{R}^N$ (here $N = 2$), let Λ and V be two measurable sets, with $0 < \operatorname{mes}(V) < \infty$ ($\operatorname{mes}(V)$ stands for the Lebesgue measure on V). We call *density* of Λ in V the quantity

$$d_V(\Lambda) = \frac{\operatorname{mes}(\Lambda \cap V)}{\operatorname{mes}(V)}.$$

If $V \subset V'$ with $\operatorname{mes}(V') < \infty$, we have

$$d_{V'}(\Lambda) \geq \frac{\operatorname{mes}(V)}{\operatorname{mes}(V')} d_V(\Lambda) \quad \text{and} \quad 1 - d_{V'}(\Lambda) \geq \frac{\operatorname{mes}(V)}{\operatorname{mes}(V')}(1 - d_V(\Lambda)).$$

Recall the Lebesgue density theorem: for almost every $x \in \Lambda$, the density $d_{B(x,r)}(\Lambda)$ tends to 1 as $r \to 0$. We shall use the following weaker result.

Proposition 5.1. *Let $\Lambda \subset \mathbb{R}^N$ be a compact set. If $\operatorname{mes}(\Lambda) > 0$, we can choose for every $\rho > 0$ a point $x_\rho \in \Lambda$ such that $d_{B(x_\rho,\rho)}(\Lambda) \to 1$ when $\rho \to 0$.*

Proof. For $\rho > 0$, denote by $\mathcal{P}_\rho$ a tiling of $\mathbb{R}^N$ by cubes of side-length 4ρ, and $\Lambda(\rho)$ the union of the tiles P of $\mathcal{P}_\rho$ that intersect Λ. We have $\text{mes}(\Lambda(\rho)) \to \text{mes}(\Lambda)$, and so; $d_{\Lambda(\rho)}(\Lambda) \to 1$. But $d_{\Lambda(\rho)}\Lambda$ is the mean value of the $d_{P_\rho}(\Lambda)$, so we can choose for each ρ a tile P_ρ in such a way that $d_{P_\rho}(\Lambda) \to 1$. Let P'_ρ be the cube centered at the same point with side-length 2ρ. For ρ small enough, we have $d_{P_\rho}(\Lambda) > 1 - 1/2^N$, hence $\Lambda \cap P_\rho \neq \emptyset$ and we can choose $x_\rho \in \Lambda \cap P'_\rho$. Then, $B(x, \rho) \subset P_\rho$, and

$$\frac{\text{mes}(B(x,\rho))}{\text{mes}(P_\rho)} \geq \frac{\text{mes}(B)}{4^N},$$

and so

$$1 - d_{B(x,\rho)}(\Lambda) \leq \frac{4^N}{\text{mes}(B)}(1 - d_{P_\rho}(\Lambda)) \to 0.$$

■

Proposition 5.2. *Let $f : U \to \mathbb{C}$ be a holomorphic map, $\Lambda \subset U$ be a compact set and $V \subset U$ be an open set such that $f_{|V}$ is injective. If $\text{dist}_V(f) \leq m$, we have*

$$1 - d_{f(V)}f(\Lambda) \leq e^{2m}(1 - d_V(\Lambda)).$$

Proof. Let $h = \inf_V |f'|$. We have $\text{mes}(f(V)) \geq h^2\text{mes}(V)$ and

$$\text{mes}(f(V) \setminus f(\Lambda)) \leq \text{mes}(f(V \setminus \Lambda)) \leq h^2 e^{2m}\text{mes}(V \setminus \Lambda) = h^2 e^{2m}(1 - d_V(\Lambda))mes(V),$$

and so

$$1 - d_{f(V)}f(\Lambda) = \frac{\text{mes}(f(V) \setminus f(\Lambda))}{\text{mes}(f(V))} \leq e^{2m}(1 - d_V(\Lambda)).$$

■

3. The hyperbolic case.

Theorem 5.2. *Let $f : U \to \mathbb{C}$ be a holomorphic map and $\Lambda \subset U$ be a compact set such that $f(\Lambda) \subset \Lambda$ and f is expanding on Λ. Then, Λ has Lebesgue measure zero.*

Corollary 5.1. *For any hyperbolic polynomial f, the Julia set J_f has Lebesgue measure zero.*

Lemma 5.1. *The set Λ has empty interior.*

Proof. Assume W is a connected component of $\mathring{\Lambda}$, and let $x \in W$. The family $f^{\circ n}_{|W}$ is bounded, so the $|(f^{\circ n})'(x)|$ form a bounded sequence on each compact subset of W. However, since f is expanding on Λ, this sequence tends to $+\infty$. Contradiction. ■

Proof of theorem 5.2. Assume $\text{mes}(\Lambda) > 0$. Let us choose $m > 0$ and let (ρ_n) be a sequence of positive numbers tending to 0. By proposition 5.2, we can find for each ν a point $x_\nu \in \Lambda$ such that $d_{B(x_\nu,\rho_\nu)}(\Lambda) \to 1$. By theorem 5.1, we can find numbers a and b (independent of ν), and for each ν an integer $n_\nu \in \mathbb{N}$ such that, setting $y_\nu = f^{\circ n_\nu}(x_\nu)$, we have

$$B(y_\nu, a) \subset f^{\circ n_\nu}B(x_\nu, \rho_\nu) \subset B(y_\nu, b) \quad \text{and} \quad \text{dist}_{B(x_\nu,\rho_\nu)}(f^{\circ n_\nu}) \leq m.$$

So,

$$1 - d_{B(y_\nu,a)}(\Lambda) \leq \frac{b^2}{a^2}\left(1 - d_{f^{\circ n_\nu}B(x_\nu,\rho_\nu)}(\Lambda)\right) \leq \frac{b^2}{a^2}e^{2m}\left(1 - d_{B(x_\nu,\rho_\nu)}(\Lambda)\right) \to 0.$$

Extracting a subsequence if necessary, we can assume that (y_ν) has a limit y, and that $|y - y_\nu| < a/2$ for all ν. Then, $B(y, a/2) \subset B(y_\nu, a)$ and

$$1 - d_{B(y,a/2)}(\Lambda) \leq 4 \cdot \big(1 - d_{B(y_\nu,a)}(\Lambda)\big) \to 0.$$

But this does not depend on ν, and so, $d_{B(y,a/2)}(\Lambda) = 1$ and $\Lambda \supset B(y, a/2)$ since Λ is compact. This contradicts the preceding lemma. ■

4. The sub-hyperbolic case: construction of a covering.

We will pattern the above construction, modifying it in order to take into account the presence of critical points.

Let f be a sub-hyperbolic polynomial. Denote by A (respectively A^*) the union of the forward orbits (respectively of strict forward orbits) of the critical points of f that are in J_f. It is a finite set. For any critical point α, we denote by $\deg_\alpha f$ the degree of ramification of f at α (the value of d such that $f(\alpha+z) = f(\alpha)+cz^d+\ldots$ with $c \neq 0$). For $\alpha \in A$, denote by $\nu(\alpha)$ the product of $\deg_\beta(f)$ for critical points which are in the strict backward orbit of α.

Let U be a relatively compact neighborhood of J_f such that $U' = f^{-1}(U) \Subset U$, $u : \overline{U} \to]0, +\infty]$ a continuous function, such that $u^{-1}(\infty) = A^*$, defining an admissible Riemannian metric μ, and $\lambda > 1$ such that $\|T_x f|_\mu \geq \lambda$ for all $x \in U' \setminus A$. For all $\alpha \in A$, choose three disks Δ_α, Δ'_α and Δ''_α centered at α, with radii r_α, r'_α, r''_α, with $r''_\alpha < r'_\alpha < r_\alpha$, so that

$$\Delta_{f(\alpha)} \Subset f(\Delta_\alpha), \quad \Delta'_{f(\alpha)} \Subset f(\Delta'_\alpha), \quad \Delta''_{f(\alpha)} \Subset f(\Delta''_\alpha),$$

and so that the $\overline{\Delta}_\alpha$ are disjoint.

We will now construct a ramified covering (which is usually not Galois) Y of U, which is ramified only above A^*. Let β be a repelling periodic point of f which does not belong to A, with period k. Denote by Y the set of sequences $\underline{x} = (x_n)_{n\in\mathbb{N}}$ in U such that $f(x_n) = x_{n-1}$ for $n \geq 1$ and such that there exists r with $x_{kp+r} \to \beta$ as $p \to \infty$. We equip Y with the topology of uniform convergence.

Let $\underline{x} \in Y$. There exists an n such that $x_n \notin A^*$. Then, for all $p \geq n$, we have $x_p \notin A^*$, since $f(A^*) \subset A^*$. If we set $\rho = d(x_n, A^*)$, for all $q \geq 0$, the map $f^{\circ q}$ has a continuous inverse branch σ_{n+q} defined on $\mathbb{D}(x_n, \rho)$, such that $\sigma_q(x_n) = x_{n+q}$. Setting $\sigma_{n-q}(z) = f^{\circ q}(z)$, we obtain a section σ of $\pi_n : (z_\nu) \to z_n$, and so a neighborhood $\underline{x}$ homeomorphic to the disk $\mathbb{D}(x_n, \rho)$, and equipped with a chart in this disk. These charts give Y the structure of a manifold. The map $\pi : Y \setminus \pi^{-1}(A^*) \to U \setminus A^*$ is a covering. For $\alpha \in A^*$, the preimage of Δ_α is the union of analytic disks with ramification degree dividing $\nu(\alpha)$ (but usually not the same for two different disks, this is why the covering is usually not Galois).

The map $\tilde{f} : \underline{x} \to (f(x_n)) = (f(x_0), x_0, x_1, \ldots)$ is an isomorphism between $Y' = \pi^{-1}(U)$ and Y: its inverse is $(x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots)$.

We equip Y with the following charts. On $\pi^{-1}(U \setminus \bigcup \overline{\Delta}''_\alpha)$, we take the charts induced by π. If $\tilde{\alpha} \in \pi^{-1}(\alpha)$ with $\alpha \in A$, the connected component $\Delta_{\tilde{\alpha}}$ of $\pi^{-1}(\Delta_\alpha)$ containing $\tilde{\alpha}$ projects to Δ_α by a map which is ramified at $\tilde{\alpha}$ with degree $d_{\tilde{\alpha}}$. We equip $\Delta_{\tilde{\alpha}}$ with a coordinate $w = w_{\tilde{\alpha}}(\underline{x})$ such that

$$(w_{\tilde{\alpha}}(\underline{x})) = \pi(x) - \alpha = x_0 - \alpha.$$

These charts define an atlas $\mathcal{A}$.

We will now equip Y with a Riemannian metric $\tilde{\mu}$. On $Y \setminus \pi^{-1}(A^*)$, consider the Riemannian metric $\pi^*(\mu)$. For each $\alpha \in A$, denote by $\ell(\alpha)$ the smallest i such that $f^{\circ i}(\alpha)$ is periodic.

On each cycle, we can define a family (ν_α) of positive numbers such that $|f'(\alpha)|\nu_{f(\alpha)}/\nu_\alpha > 1$, since in A, every cycle is repelling. We define $\hat{\mu}_{\tilde{\alpha}}$ for periodic α by $\nu_\alpha^{1/d_{\tilde{\alpha}}}|dw_{\tilde{\alpha}}|$. We then have $\|T_{\tilde{\alpha}}\tilde{f}\|_{\hat{\mu}} > 1$ for every $\tilde{\alpha}$ above a periodic point. We can then, by induction on $\ell(\alpha)$, define for all $\alpha \in A$ a ν_α so that, defining $\hat{\mu}_{\tilde{\alpha}}$ by the same formula, we still have $\|T_{\tilde{\alpha}}\tilde{f}\|_{\hat{\mu}} > 1$: it is enough to take ν_α sufficiently small.

We can then find for each α a disk $\Delta'''_\alpha \subset \Delta''_\alpha$ so that $\|T_{\tilde{\alpha}}\tilde{f}\|_{\hat{\mu}} > 1$ for $x \in \Delta'''_{\tilde{\alpha}} = \Delta_{\tilde{\alpha}} \cap \delta'''_\alpha$, $\tilde{\alpha} \in \pi^{-1}(\alpha)$. we set $\tilde{\mu} = \inf(\pi^*\mu, M^*\hat{\mu})$, where M^* is chosen sufficiently large so that $\pi^*\mu < M^*\hat{\mu}$ on $\partial\Delta'''_{\tilde{\alpha}}$ for all $\tilde{\alpha}$ (it is sufficient to check a finite number of $\tilde{\alpha}$, since two point with the same degree of ramification above the same point α give the same thing).

Proposition 5.3. *The metric $\tilde{\mu}$ has the following properties.*

a) *It has continuous coefficients.*
b) *There exists $\lambda > 1$ such that $\|T_{\tilde{x}}\tilde{f}\|_{\tilde{\mu}} \geq \lambda$ for all $\tilde{x} \in Y'$.*
c) *Every point in $U \setminus \bigcup \overline{\Delta}''_\alpha$ has a connected neighborhood above which the change of charts are isometries. For each $\alpha \in A$, we can find a finite number $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_r$ of component of $\pi^{-1}(\Delta_\alpha)$ such that each component of $\pi^{-1}(\Delta_\alpha)$ is isometric above Δ_α to one of the $\Delta_{\tilde{\alpha}_i}$.*

All this follows from the construction of $\tilde{\mu}$.

5. The sub-hyperbolic case.

Theorem 5.3. *If f is a sub-hyperbolic polynomial, J_f has Lebesgue measure zero.*

We only explain the modifications with respect to the proofs of theorems 5.1 and 5.2.

Let $V \subset Y$ be an open set and $g : V \to Y$ be a $\mathbb{C}$-analytic mapping such that V and $g(V)$ are contained in the domain of the charts of $\mathcal{A}$. Denote by $\mathrm{dist}_V(g)$ the distortion of the expression of g in the charts. If there is a choice of charts for V or $g(V)$, we take the supremum of distortions in different expressions.

Let $x \in Y$ and $r > 0$. If there exists a chart $w : \Omega \to \mathbb{C}$ of $\mathcal{A}$ such that $B(w(x), r) \subset w(\Omega)$, we set $B(x, r) = w^{-1}(B(w(x), r)$. If there is a choice, we choose the chart induced by π (or we take the intersection).

Let $\Lambda \subset U$ be a compact set and V be an open subset of U contained in one of the Δ_α. Set
$$\tilde{d}_V(\Lambda) = \inf_{\tilde{\alpha} \in \pi^{-1}(\alpha)} d_{w_{\tilde{\alpha}}(\pi^{-1}(V))}\left(w_{\tilde{\alpha}}(\pi^{-1}(\Lambda))\right).$$
We choose $b_0 > 0$ such that, for all $x \in Y' = \pi^{-1}(U')$, $B(x, b_0)$ is defined and for all $n \geq 0$, $f^{-n}(B(x, b_0))$ is contained in the domain of a chart of $\mathcal{A}$. With those conventions, the proof is analogous.

CHAPTER 6

Constructing a polynomial with a given tree.

This chapter uses ideas and techniques which revealed their power in the work of Thurston (see [**DH3**]). The iteration leading to the Φ_n's is an early example of the iteration in Teichmüller sapce Thurston uses.

Notations and introduction.

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d such that every critical point is preperiodic. Denote by $\mathcal{C}$ the set of critical points of f, A the finite set $\bigcup_{n\geq 0} f^{\circ n}(\mathcal{C})$, J and K the Julia set and filled-in Julia set, H the Hubbard tree, i.e., the allowable hull of A in K. For $\alpha \in A$, denote by $\nu(\alpha)$ the number of branches f H at α, $\tau(\alpha)$ the ramification degree of f at α (the multiplicity of α as a critical point is $\tau(\alpha) - 1$; we have $\tau(\alpha) = 1$ if $\alpha \in A \setminus \mathcal{C}$). The points of A are called *marked points*, adding the branching points of H, we obtain the *remarkable points*.

We will remember on H the structure given by the following data:

- its topology
- the cyclic order of the branches at the branching points (which determines the embedding of H in $\mathbb{C}$ up to isotopy),
- the set A of marked points,
- the dynamics on A, i.e., $f_{|A} : A \to A$,
- the function $\tau : A \to \mathbb{N}$ (if f has degree 2, we have $\tau(\alpha) = 2$ if α is the critical point and $\tau(\alpha) = 1$ otherwise).

Those data are the *primary structure*.

We will show that a polynomial of the form $z \mapsto z^2 + c$, such that 0 is preperiodic, is determined by its tree equipped with its primary structure.

The proof is done in two parts, the first one topological and combinatorial, the second analytic. The second part can be performed as well in degree d arbitrary. To extend the first part, one must define on the tree a *complementary structure*. We then obtain the following result: a polynomial such that every critical point is preperiodic is determined, up to affine conjugacy, by its tree, equipped with its primary and complementary structures.

1. Combinatorial part.

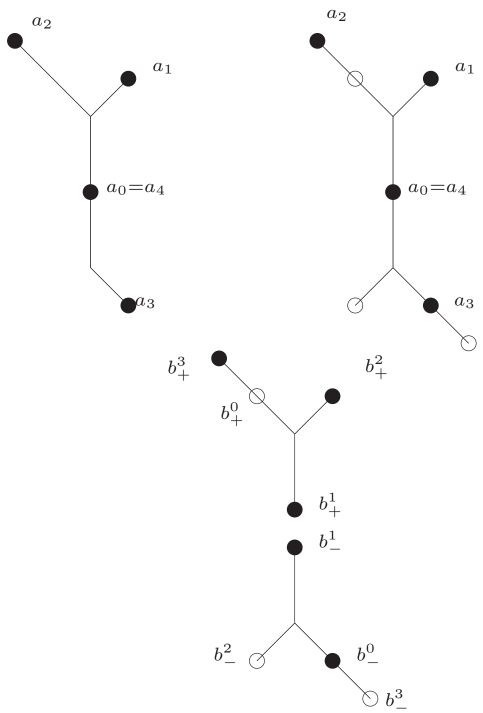
1.1. The tree H^1. We set $H^1 = f^{-1}(H)$. This set is also the allowable hull of $A^{-1} = f^{-1}(A)$ in K_f. The *marked points* of H^1 are the points of A^{-1}. A point of H can be marked (or remarkable) in H^1 without being marked (or remarkable) in H. We define in the same way a primary structure on H^1 (we have $\tau(\alpha) = 0$ if $\alpha \in A^1 \setminus A$). If we denote by $\nu_1(\alpha)$ the number of branches of H^1 at α, we have, for all $\alpha \in A$:

$$\nu(\alpha) \leq \nu_1(\alpha) = \tau(\alpha) \cdot \nu(f(\alpha)).$$

For the tree H of a quadratic polynomial with its primary structure, it is easy to construct H^1 with its primary structure, together with the embedding $H \to H^1$, from H with its primary structure.

If T is a tree and $F \subset T$ is a finite set, the set obtained by *cutting* T at F is the disjoint union of the closures of the connected components of $T \setminus F$.

Assume f is of degree 2, let us say $z \mapsto z^2 + c$. Let $a_0 = 0$ be the critical point, and $a_i = f^{\circ i}(a_0)$. We have $\nu(a_1) = 1$, and so $\nu_1(a_0) = 2$. Cutting H^1 at $\{a_0\}$, we obtain H^1_+ and H^1_- with, let us say, $a_1 \in H^1_+$. The map f induces a homeomorphism from each of the $H^1_\pm$ to H, denote by $b^i_\pm$ the preimage of a_i in $H^1_\pm$. We obtain a homeomorphism $b^i_\pm \mapsto (a_i, \pm)$ from H^1 to $H \times \{+,-\}/(a_1,+) \sim (a_1,-)$. The natural injection $\iota : A \to H^1$ is given by $\iota(a_i) = b^{i+1}_s$ with $s = +$ if a_i is on the same side of a_0 as a_1 in H and $s = -$ otherwise (in H^1, we have $b^1_+ = b^1_- = a_0$). This determines $\iota : H \to H^1$ up to homotopy fixing the remarkable points. The point a_0 is not a branching point in H^1, so every branching point is in $H^1_+ \setminus \{b_1\}$ or $H^1_- \setminus \{b_1\}$ and the cyclic order of the branches is given by the one of the branches of H at the corresponding point.



1.2. The complementary structure. This subsection is relevant only for $d \geq 3$. We will add two complementary pieces of data to the primary structure.

Let $\alpha \in A$. We have $\alpha \in \mathring{K}_f$ if and only if there exist n, k such that $f^{\circ n+k}(\alpha) = f^{\circ n}(\alpha)$ and $\tau(f^{\circ n}(\alpha)) \geq 2$. This can therefore be read on the primary structure of H.

If $\alpha \in \mathring{K}_f$, the branches of H are, in a neighborhood of α, internal rays of the component U_α of $\mathring{K}_f$ containing α. The *first complementary data* is the data, for each $\alpha \in \mathcal{C} \cap \mathring{K}_f$, of the internal angles between the branches of H at α; counted in turns, those are elements of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Those angles determine the angles between the branches at every point in $A \cap \mathring{K}_f$. Indeed, let $\alpha \in A \cap \mathring{K}_f$, and n_0 be the smallest $n \geq 0$ such that $f^{\circ n}(\alpha) \in \mathcal{C}$; if ξ and ξ' are two branches at α, their angle is equal to the one of $f^{\circ n_0}(\xi)$ and $f^{\circ n_0}(\xi')$ at $f^{\circ n_0}(\alpha)$. Those angles are rational, with a denominator that can be computed given the primary structure. The first data therefore contains a finite information.

For $\alpha \in A$, we call *buds* at α the $\tau(\alpha) \cdot \nu(f(\alpha)) - \nu(\alpha)$ branches of H^1 at α which are not branches of H. If $\tau(\alpha) = 1$, the way the buds are inserted in the cyclic order of the branches of H at α is determined by the primary structure. If $\alpha \in \mathcal{C} \cap \mathring{K}_f$, this way is determined by the angles between the branches at α and $f(\alpha)$.

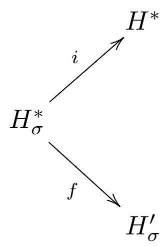
The *second complementary data* is made of the way the buds at α are inserted in the cyclic order of the branches of H for $\alpha \in \mathcal{C} \cap J_f$.

1.3. Rebuilding H^1 (degree $d \geq 3$). Denote by H^* the union of H and the buds (represented by small arcs), and let $\bigsqcup_{\sigma \in S} H^*_\sigma$ be the set obtained by cutting H^* along $\mathcal{C}$. For all $\sigma \in S$, set $\mathcal{C}_\sigma = H^*_\sigma \cap \mathcal{C}$ and denote by H^1_0 the component of H^1 cut along $\mathcal{C}$ that contains H^*_σ. The set H^1 is the union of the H^1_σ.

Lemma 6.1. *The map f induces a homeomorphism from H^1_σ to the component H'_σ of H cut along $f(\mathcal{C}_\sigma)$ that contains $f(H^*_\sigma)$.*

Proof. The map f is injective on H^1_σ, thus is a homeomorphism from H^1_σ to its image which is compact. It therefore induces a homeomorphism from $H^1_\sigma \setminus \mathcal{C}_\sigma$ to a closed subset of $H \setminus f(\mathcal{C}_\sigma)$. Since it is open on $H^1_\sigma \setminus \mathcal{C}_\sigma$ and since $H^1_\sigma \setminus \mathcal{C}_\sigma$ is connected, this closed set is a connected component, and the lemma follows. ■

We give now give the following description of H^1: H^1 is obtained by gluing to H^* the H'_σ via the maps:



The sets H^*, H^*_σ and H'_σ are known once H, its primary structure, its complementary structure and f are known, up to homotopy fixing the remarkable points.

As a consequence, we get the following proposition.

Proposition 6.1. *Let f and g be two polynomials of degree $d \geq 2$, such that every critical point is periodic or preperiodic. Let φ be a homeomorphism from H_f to H_g, respecting the primary and complementary structures. Then there exists a unique homeomorphism $\varphi_1 : H^1_f \to H^1_g$ which coincides with φ on A_f and such that $\varphi \circ \varphi_1 = \varphi \circ f$.*

Additional information. 1) We have $\varphi_1(H_f) = H_g$ and the restriction of φ_1 to H_f is homotopic to φ among the homeomorphisms which coincide with φ on the remarkable points of H_f.

2) The map φ_1 respects the primary structures and the angles at the points of $A_1 \cap \overset{\circ}{K}$.

Remark. 1) In general, we cannot have at the same time $g \circ \varphi_1 = \varphi \circ f$ and $\varphi_{1|H_f} = \varphi$; an homotopy is required on one side or the other. We have chosen the statement that will be useful for us.

2) The proposition is true in degree $d = 2$ without the hypothesis on the complementary structures which is automatic since they do not give any additional information; this follows form subsection 1.1.

1.4. Decorated trees. We will now transform the trees in Christmas trees. Coming back to f, let (U_i) be the family of connected components of $\overset{\circ}{K}_f$, for $\alpha \in A \cap \overset{\circ}{K}_f$, we will denote by U_α the component centered at α. Let (ζ_i) be the family of $\mathbb{C}$-analytic charts $\zeta_i : U_i \xrightarrow{\simeq} \mathbb{D}$ such that the expression of f in those charts is $\zeta_i \mapsto \zeta_{f_*(i)} = \zeta_i^{r_i}$. For $z \in U_i$, we set $\rho(z) = |\zeta_i(z)|$.

The *decorated tree* $\vdash\!\!\circ\!\!\dashv$ is the union of H and the disks $N_\alpha = \{z \in U_\alpha \mid \rho(z) \leq 1/2\}$ for $\alpha \in A \cap \overset{\circ}{K}_f$ (see figure 1).

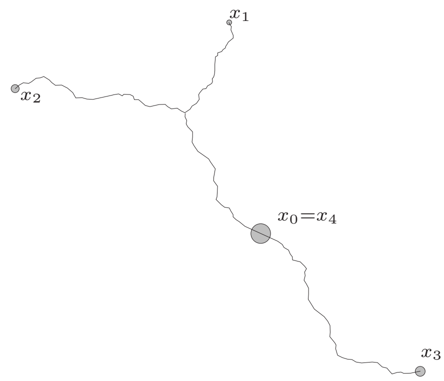


FIGURE 1. A decorated tree.

1.5. Construction of homeomorphisms.

Proposition 6.2. [1] *Under the hypothesis of proposition 6.1, we can find two homeomorphisms $\psi_0, \psi_1 : \mathbb{C} \to \mathbb{C}$ such that:*

a) *$\psi_0(H_f) = H_g$ and $\psi_{0|H_f}$ is homotopic to φ among the homeomorphisms $H_f \to H_g$ which coincide with φ on A_f.*

b) *ψ_0 induces a $\mathbb{C}$-analytic isomorphism from $\overset{\circ}{N}{}^f_\alpha$ to $\overset{\circ}{N}{}^g_{\varphi(\alpha)}$ for $\alpha \in A_f \cap \overset{\circ}{K}_f$.*

c) *$g \circ \psi_1 = \psi_0 \circ f$.*

---

[1] Proposition 6.2 shows that if two postcritically finite polynomials have the same tree, then they are Thurston equivalent (see [**DH3**] for a definition). Proposition 6.6 then shows that the polynomials are affine conjugate, which is the uniqueness part of Thurston's theorem. The proof is in fact the similar.

d) ψ_1 *is homotopic to* ψ_0 *among the homeomorphisms* $\mathbb{C} \to \mathbb{C}$ *inducing a homeomorphism* $H_f \to H_g$ *and coinciding with* ψ_0 *on* $\displaystyle\bigcup_{\alpha \in A_f \cap \mathring{K}_f} N_\alpha \cup A_f$.

Proof.

α) Construction of ψ_0. We can modify φ so that $\rho_g(\varphi(z)) = \rho_f(z)$ for all $z \in H_f \cap N_\alpha$, $\alpha \in A_f \cap \mathring{K}_f$. On each branch $H_f \cap N_\alpha$, the expression of φ in the charts ζ^f_α and $\zeta^g_{\varphi(\alpha)}$ is of the form $\zeta \mapsto \lambda\zeta$ with the condition $|\lambda| = 1$, and the condition of preserving the first complementary data implies that λ is the same λ_α for the different branches issuing from a given $\alpha \in A_f \cap \mathring{K}_f$. We can then define φ_0 on each N_α by $\zeta \mapsto \lambda_\alpha \zeta$. In this way, we obtain a diffeomorphism $\varphi_0 : \vdash\!\!\circ\!\!\dashv_f \to \vdash\!\!\circ\!\!\dashv_g$.

Let τ_f be an isomorphism from $\mathbb{C} \setminus \overline{\mathbb{D}}$ to $\mathbb{C} \setminus \vdash\!\!\circ\!\!\dashv_f$, extending continuously to $\mathbb{C} \setminus \mathbb{D}$, and define in a similar way τ_g. Since φ_0 preserves the cyclic order at the branching points, it follows that there exist a homeomorphism $h : S^1 \to S^1$ such that $\varphi_0(\tau_f(u)) = \tau_g(h(u))$ for all $u \in S^1$. We can then extend φ_0 to a homeomorphism $\psi_0 : \mathbb{C} \to \mathbb{C}$ defined on $\mathbb{C} \setminus \vdash\!\!\circ\!\!\dashv_f$ by $\psi_0(\tau_f(ru)) = \tau_g(rh(u))$.

β) Construction of ψ_1 in a neighborhood of H^1_f. On H_f, ψ_1 is given by proposition 6.1.

For each critical point α of f, let V^f_α and W^f_α be neighborhoods of α and $f(\alpha)$ homeomorphic to $\mathbb{D}$, such that f induces a covering of degree $r_f(\alpha)$ from $V^f_\alpha \setminus \{\alpha\}$ to $W^f_\alpha \setminus \{f(\alpha)\}$. Set $W^g_\alpha = \psi_0(W^f_\alpha)$ and let V^g_α be a neighborhood of $\varphi(\alpha)$ such that g induces a covering $V^g_\alpha \setminus \{\varphi(\alpha)\} \to W^g_\alpha \setminus \{\varphi(f(\alpha))\}$ of degree $r_g(\varphi(\alpha)) = r_f(\alpha)$. We can lift $\psi_0 : W^f_\alpha \setminus \{f(\alpha)\} \xrightarrow{\simeq} W^g_\alpha \setminus \{\varphi(f(\alpha))\}$ to a homeomorphism $\psi^\alpha_1 : V^f_\alpha \setminus \{\alpha\} \to V^g_\alpha \setminus \{\varphi(\alpha)\}$, and thanks to the hypothesis that f preserves the complementary datas, we can do it, in a unique way, by extending ψ_1 already defined on the branches of H^1_f at α.

For each non critical point $x \in H^1_f$, we can find neighborhoods V^f_x, W^f_x, V^g_x, W^g_x of x, $f(x)$, $\psi_1(x)$, $\psi_0(f(x))$ such that we have homeomorphisms

$$V^f_x \xrightarrow{\ f\ } W^f_x \xrightarrow{\ \psi_0\ } W^g_x \xleftarrow{\ g\ } V^g_x\ ,$$

which allows us to define $\psi^x_1 = g^{-1} \circ \psi_0 \circ f : V^f_x \to V^g_x$. All those germs can be glued (it is possible to invoke a lemma by Godement) to obtain a homeomorphism ψ^V_1 from a neighborhood V^f of H^1_f to a neighborhood V^g of H^1_g, such that $g \circ \psi_1 = \psi_0 \circ f$.

γ) Extension of ψ_1 to $\mathbb{C}$. The maps f and g induce coverings of degree d (it is the same because it is $\sum (r(\alpha) - 1) + 1$):

$$f : \mathbb{C} \setminus H^1_f \to \mathbb{C} \setminus H_f, \quad g : \mathbb{C} \setminus H^1_g \to \mathbb{C} \setminus H_g.$$

Those four sets are homeomorphic to an annulus.

Let $x \in V^f \setminus H^1_f$. There exists a unique lift $\psi^\infty_1 : \mathbb{C} \setminus H^1_f \to \mathbb{C} \setminus H^1_g$ of $\psi_0 : \mathbb{C} \setminus H_f \to \mathbb{C} \setminus H_g$ such that $\psi^\infty_1(x) = \psi^V_1(x)$. We can assume that V^f is connected and of the form $f^{-1}(W^f)$, where W^f is a connected neighborhood of H^f. Then, ψ^V_1 and ψ^∞_1 induce two lifts of $\psi_0 : W^{\setminus} H_f \to W^g \setminus H_g$, who coincide at x, so on $V^f \setminus H^1_f$. Then, ψ^V_1 and ψ^∞_1 can be glued to obtain a homeomorphism $\psi_1 : \mathbb{C} \to \mathbb{C}$.

For each $\alpha \in A_f \cap \mathring{K}_f$, the expression of ψ_0 on N_α in the charts ζ^f_α and ζ^g_α is of the form $\zeta \mapsto \lambda_\alpha \zeta$, with $|\lambda_\alpha| = 1$, and the expression of ψ_1 will be of the form

$\zeta \mapsto \lambda_\alpha^1 \zeta$ with $(\lambda_\alpha^1)^{r_\alpha} = \lambda_{f(\alpha)}$. But (except in the trivial case where f would be monomial and $H_f = \{\alpha\}$), there is at least a branch of H_f at α, on which ψ_0 and ψ_1 coincide. So, we necessarily have $\lambda_a^1 = \lambda_\alpha$. It follows that ψ_1 coincide with ψ_0 on $\vdash\!\!\overset{\circ}{\mid}\!\!\dashv_f$.

δ) Homotopy between ψ_0 and ψ_1. Let us consider again τ_f and τ_g which have been used in (α). Let $(\psi_t)_{t\in[0,1]}$ be a homotopy between ψ_0 and ψ_1 among homeomorphisms $\vdash\!\!\overset{\circ}{\mid}\!\!\dashv_f \to \vdash\!\!\overset{\circ}{\mid}\!\!\dashv_g$ which coincide with ψ_0 on $X = A_f \cup \bigcup_\alpha N_\alpha^f$. For all t, there exists a unique homeomorphism $h_t : S^1 \to S^1$ such that $\psi_t \circ \tau_f = \tau_g \circ h_t$, which coincide with $h_0 = h$ on $\tau_f^{-1}(X)$. We can extend this homotopy to $\mathbb{C} \setminus \mathbb{D}$ thanks to that following lemma. We pass from $\mathbb{D}$ to $\mathbb{C} \setminus \mathbb{D}$ by inversion.

Lemma 6.2. *Let φ_0 and φ_1 be two homeomorphisms $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$ and (h_t) be a homotopy between $\varphi_{0|S^1}$ and φ_{1S^1}. There exists a homotopy (φ_t) between φ_0 and φ_1 inducing (h_t) on the boundary.*

Proof. Replacing φ_1 by $\varphi_1 \circ \varphi_0^{-1}$ and h_t by $h_t \circ h_0^{-1}$, we may suppose that $\varphi_0 = \mathrm{Id}$. Set $\tilde{h}_t(r \cdot u) = r \cdot h_t(u)$ for $r \in [0,1]$. Replacing φ_1 by $\varphi_1 \circ \tilde{h}$, we go back to the case where $h_t = \mathrm{Id}$ for all t. We can then define φ_t by $\varphi_t(x) = x$ if $|x| \geq t$ and $\varphi_t(x) = t\varphi_1(x/t)$ for $|x| \leq t$. □

This concludes the proof of proposition 6.2. ■

Remark. ψ_0 is not unique, but once ψ_0 is chosen, then ψ_1 is unique.

1.6. Adjustment at infinity. Choose R and R' such that $R > R' > 1$. Let $\zeta_\infty^f : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be an isomorphism such that the expression of f in this chart is $\zeta \mapsto \zeta^d$, and set

$$N_\infty^f = \{z \in \mathbb{C} \setminus K_f \mid R \leq |\zeta_\infty^f(z)|\}.$$

Define similarly ζ_∞^g and N_∞^g, N'^f_∞ and N'^g_∞.

Proposition 6.3. *In proposition 6.2, we can choose ψ_0 so that $\psi_1 = \psi_0$ on N_∞, and so that ψ_1 is homotopic to ψ_0 among the homeomorphisms $\mathbb{C} \to \mathbb{C}$ which coincide with ψ_0 on*

$$A_f \cap \bigcup_{\alpha \in A_f \cap \overset{\circ}{K}_f \cup \{\infty\}} N_\alpha.$$

Proof. We can modify ψ_0 on N'^f_∞ so that its expression on N_∞^f becomes $\zeta \mapsto \lambda\zeta$ in the charts ζ_∞^f, ζ_∞^g, with $|\lambda| = 1$. The expression of ψ_1 on N_∞^f then becomes $\zeta \mapsto \lambda_1 \zeta$, where $\lambda_1^d = \lambda$.

Let us choose, in $\vdash\!\!\overset{\circ}{\mid}\!\!\dashv_f$, a point x which is a remarkable point of H_f or a point of one of the ∂N_α. Then, we have $\psi_0(x) = \psi_1(x)$, and we even have $\psi_t(x) = \psi_0(x)$ for all $t \in [0,1]$. Let $x' \in H_f$ be another remarkable point, $y \in N_\infty^f$ and η be a path from x to y, homotopic in $\mathbb{C} \setminus \psi_0(x')$ to the path obtained by putting together $\psi_0(\eta)$ with orientation reversed and $\psi_1(\eta)$. Following the argument of ζ along $\tilde{\eta}$, we obtain a $\theta \in \mathbb{R}$ such that $\lambda_1 = \lambda_0 e^{2i\pi\theta}$.

Lemma 6.3. *a) We have $\psi_1 = \psi_0$ on N_∞^f if $\theta \in \mathbb{Z}$.*

b) In order to be able to modify the homotopy from ψ_0 to ψ_1 so that $\psi_t = \psi_0$ for all t, it is necessary and sufficient that $\theta = 0$.

Proof. Part a) is trivial. Part b) follows from the description of the π_0 of the group of homeomorphisms of a closed ring inducing the identity on the boundary. □

Let us now vary ψ_0 with respect to a parameter s so that $\lambda_0(s) = \lambda_0(0)e^{2i\pi s}$. We then have $\theta(s) = \theta(0) + (1/d - 1)s$. For $s = \frac{d}{d-1}\theta(0)$, we have $\theta(s) = 0$ and ψ_0 satisfies the required properties. ■

2. Analytic part.

2.1. Preliminaries on quasiconformal mappings. If U is an open subset of $\mathbb{R}^n$, the Sobolev space $\mathcal{H}^1(U)$ is the space of functions of $L^2(U)$ whose first derivatives in the sense of distributions are in $L^2(U)$. We denote by $\mathcal{H}^1_{\text{loc}}(U)$ the space of functions such that $\forall x \in U$, $\exists V$ neighborhood of x, $f_{|V} \in \mathcal{H}^1(V)$, and $\mathcal{C}H^1_{\text{loc}}(U)$ the space $\mathcal{C}(U) \cap \mathcal{H}^1_{\text{loc}}(U)$. We denote by $\mathcal{C}H^1_{\text{loc}}(U, \mathbb{R}^p)$ the space of functions $f : U \to \mathbb{R}^p$ whose coordinates are in $\mathcal{C}H^1_{\text{loc}}(U)$. If U is an open subset of $\mathbb{C}$, we define $\mathcal{C}H^1_{\text{loc}}(U, \mathbb{C})$ by forgetting the complex structure and identifying $\mathbb{C}$ to $\mathbb{R}^2$.

Let U and V be two open subsets of $\mathbb{C}$ and $f : U \to V$ be a map. We say that f is *quasiconformal* if $f \in \mathcal{C}H^1_{\text{loc}}(U, \mathbb{C})$ and if there exists an $m < 1$ such that for almost every $x \in U$, we have:

$$\left|\frac{\partial f}{\partial \bar{z}}(x)\right| \le m \left|\frac{\partial f}{\partial z}(x)\right|.$$

This inequality means that $T_x f$ (which is defined for almost every x) is orientation preserving and transforms a circle into an ellipse whose ratio of axes is bounded by $M = (1 + m)/(1 - m)$. The smallest such M is the *dilatation ratio* of f.

If f is a quasiconformal homeomorphism from U to V, f^{-1} is quasiconformal with the same dilatation ratio, and we have $\|Df\|^2 \le M\text{Jac}(f)$, hence

$$\|Df\|_2^2 = 2\int_U \left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \le M\text{area}(V).$$

Let us conclude with two remarks that will be useful.

1) Let (f_n) be a sequence in $\mathcal{C}H^1(U)$, $U \subset \mathbb{R}$ open. If $f_n \to f$ uniformly and $\|Df_n\|_2 \le k$ (independently of n), then $f \in \mathcal{C}H^1(U)$ and $\|Df\|_2 \le k$. Indeed,

$$\|Df\|_2 \le k \Longleftrightarrow \forall (u, v) \in \mathcal{C}^\infty_{\text{comp}}(U) \text{ such that } \|u\|_2^2 + \|v\|_2^2 \le 1,$$
$$\left|\int_U f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \le k.$$

2) If $f \in \mathcal{C}H^1(U; \mathbb{C})$ with $U \subset \mathbb{C}$ and $\partial f/\partial \bar{z} = 0$ almost everywhere, f is holomorphic.

2.2. Construction of Φ_0 and Φ_1.

Proposition 6.4. *Under the hypothesis of proposition 6.1, we can find two homeomorphisms Φ_0 and $\Phi_1 : \mathbb{C} \to \mathbb{C}$ such that*

1) *Φ_0 is a diffeomorphism of class $\mathcal{C}^1$.*
2) *Φ_0 induces a $\mathbb{C}$-analytic isomorphism from $\overset{\circ}{N}{}^f_\alpha$ to $\overset{\circ}{N}{}^g_{\varphi(\alpha)}$ for $\alpha \in A_f \cap \overset{\circ}{K}_f \cup \{\infty\}$.*

3) $g \circ \Phi_1 = \Phi_0 \circ f$.
4) Φ_1 *is quasiconformal*
5) Φ_1 *is homotopic to* Φ_0 *among the homeomorphisms* $\mathbb{C} \to \mathbb{C}$ *coinciding with* Φ_0 *on*

$$N_f = A_f \cup \bigcup_{\alpha \in A_f \cap \overset{\circ}{K}_f \cup \{\infty\}} N_\alpha^f.$$

Regarding proposition 6.3, we lost $\psi_0(H_f) = H_g$ but we gained $\Phi_0 \in \mathcal{C}^1$.

Proof. Let ψ_0 and $\psi_1 : \mathbb{C} \to \mathbb{C}$ be homeomorphisms satisfying the conditions of proposition 6.2 and 6.3, Φ_0 a diffeomorphism of class $\mathcal{C}^1$ of $\mathbb{C}$ homotopic to ψ_0 among the homeomorphisms coinciding with ψ_0 on N_f and η_0 a homotopy. The maps $f : \mathbb{C} \setminus A_f^1 \to \mathbb{C} \setminus A_f$ and $g : \mathbb{C} \setminus A_g^1 \to \mathbb{C} \setminus A_g$ are coverings. The map $\psi_1 : \mathbb{C} \setminus A_f^1 \to \mathbb{C} \setminus A_g^1$ is a lift of the map $\psi_0 : \mathbb{C} \setminus A_f \to \mathbb{C} \setminus A_g$; we can therefore lift η_0 to a homotopy η_1 from ψ_1 to a diffeomorphism $\Phi_1 : \mathbb{C} \setminus A_f^1 \to \mathbb{C} \setminus A_g^1$, which extends to a homeomorphism $\mathbb{C} \to \mathbb{C}$.

The diffeomorphism Φ_0 is quasiconformal since C^1 and holomorphic outside a compact set. It follows from (3) that Φ_1 is quasiconformal with the same dilatation ratio. The homotopy η_0 is constant on N_f, so η_1 is constant on $N_f^1 = f^{-1}(N_f)$. We have the homotopies $\Phi_0 \simeq \psi_0 \simeq \psi_1 \simeq \Phi_1$ that are constant on N_f, and so a homotopy h_0 between Φ_0 and Φ_1 constant on N_f. ■

2.3. The sequence (Φ_n)**.** Given Φ_0, Φ_1 and h_0, we construct by induction a sequence of homeomorphisms $\Phi_n : \mathbb{C} \to \mathbb{C}$ coinciding with Φ_0 on N_f, and a homotopy h_n between Φ_n and Φ_{n+1}. The homotopy h_n is obtained by lifting h_{n-1} between Φ_{n-1} and $\Phi_n : \mathbb{C} \setminus A_f \to \mathbb{C} \setminus A_g$ to the coverings $\mathbb{C} \setminus A_f^1$ and $\mathbb{C} \setminus A_g^1$ starting from Φ_n: it determines Φ_{n+1}. We therefore have $g \circ \Phi_{n+1} = \Phi_n \circ f$ for all n, and the homotopy h_n is constant on $N_f^n = f^{-n}(N_f)$.

In particular, Φ_n coincides with Φ_{n+1} on N_f^n, and the sequence (Φ_n) is locally stationary on the open set $\bigcup N_f^n$. This open set is contained in $\mathbb{C} \setminus J_f$, since every point of $\mathbb{C} \setminus J_f$ is attracted by a cycle of $A_f \cap \overset{\circ}{K}_f$ or by ∞.

Proposition 6.5. *The sequence* (Φ_n) *converges uniformly on* $\mathbb{C}$.

Proof. The polynomial g is sub-hyperbolic. Let Ω be an open neighborhood of J_g, μ be an admissible Riemannian metric on Ω and $\lambda > 1$ such that $f^{-1}(\Omega) \subset \Omega$ and $\|T_x g\|_\mu \geq \lambda$ for all $x \in g^{-1}[\Omega)$. Let n_0 be such that $\mathbb{C} \setminus N_g^{n_0} \subset \Omega$. For all $n \geq n_0$, set

$$\rho_n = \sup_{x \in \mathbb{C} \setminus N_f^{n_0}} d_\mu(\Phi_n(x), \Phi_{n+1}(x))$$

(d_μ being the length for μ of the shortest path between $\Phi_n(x)$ and $\Phi_{n+1}(x)$ in Ω, in the class of the path given by h_n). We have $\rho_{n+1} \leq \rho_n/\lambda$. It follows that the sequence (Φ_n) uniformly converges for the distance defined by μ. Since it is constant outside a compact contained in Ω and since the distance d_μ defines the same topology as the usual distance, (Φ_n) converges uniformly on $\mathbb{C}$ for the usual distance. ■

2.4. Holomorphy of Φ**.** Denote by Φ the limit of the (Φ_n). It is a continuous map. For each n, Φ_n is holomorphic on N_f^n. It follows that Φ is holomorphic on $\bigcup N_f^n = \mathbb{C} \setminus J_f$. We know that J_f and J_g have Lebesgue measure zero.

Remark. Let J_1 and J_2 be two closed set of measure zero, $\Psi : \mathbb{C} \to \mathbb{C}$ be a homeomorphism such that $\Psi(J_1) = J_2$ and Ψ holomorphic on $\mathbb{C} \setminus J_1$. This does not imply that Ψ is holomorphic.

Counter example. Let $u : \mathbb{R} \to \mathbb{R}$ be a increasing function which is constant on each connected of the complement of a Cantor set of measure zero, but is however not constant.[2] Then, $\psi : (x + iy) \mapsto x + i(y + u(x))$ provides a counter example.

Proposition 6.6. *The map Φ is holomorphic.*

Proof. [3] Φ_0 et Φ_1 are quasiconformal with dilatation ratio M. It follows that all the Φ_n are quasiconformal with dilatation ratio M, since $g \circ \Phi_{n+1} = \Phi_n \circ f$, with f and g holomorphic.

We have
$$\|D\Phi_n\|_{L^2(\mathbb{C}\setminus N_f^n)} \leq M \mathrm{area}(\mathbb{C} \setminus N_g^n),$$
for all n. It follows, as mentioned at the end of subsection 2.1, that Φ is of class $\mathcal{C}H^1$ on $\mathbb{C} \setminus N_f$. Since $\partial\varphi/\partial\bar{z} = 0$ almost everywhere, the map Φ is holomorphic on $\mathbb{C} \setminus N_f$. It is also holomorphic on $\mathbb{C} \setminus J_f$, and thus on $\mathbb{C}$. ■

Corollary 6.1. *Φ is affine.*

Indeed, Φ is proper with degree 1.

2.5. Conclusion.

Theorem 6.1. *Under the hypothesis of proposition 6.1, f and g are conjugate by an affine map.*

Corollary 6.2. *Let $f : z \mapsto z^2 + c_1$ and $g : z \mapsto s^2 + c_2$ be two polynomials of degree 2. If there exists a homeomorphism from H_f to H_g preserving the primary structure, then we have $c_1 = c_2$.*

Corollary 6.3. *Let c_1 and c_2 be two real numbers such that 0 is periodic with the same period k for $f : z \mapsto z^2 + c_1$ and $g : z \mapsto s^2 + c_2$. Assume that the ordering induced by the one of $\mathbb{R}$ on $\{0, f(0), \ldots, f^{\circ k-1}(0)\}$ and $\{0, g(0), \ldots, g^{\circ k-1}(0)\}$ coincide. Then $c_1 = c_2$.*

This result was known under the name of conjecture of Métropolis-Stein-Stein.

Remark. We can give variants of the condition on the complementary datas. I think that a possible variant would be to complete the tree by adjoining to A the points of external arguments of the form $p/(d_1)$ (fixed points), or p/d (maybe one needs all the $\dfrac{p}{d(d-1)}$).

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[2] Such a function is often called a "devil's staircase".

[3] This is a proof, in this particular case, that a uniform limit of K-quasiconformal mappings is K-quasiconformal.

CHAPTER 7

External arguments in Julia sets.

1. Reminder and introduction.

If $K \subset \mathbb{C}$ is a connected compact set which is full and locally connected, the conformal representation $\varphi_K^{-1} : \mathbb{C} \setminus \overline{\mathbb{D}}_{r(K)} \xrightarrow{\simeq} \mathbb{C} \setminus K$ which is tangent at ∞ to the identity, has a continuous extension to $\mathbb{C} \setminus \mathbb{D}_{r(K)}$ and so we have a continuous map $\gamma_k : \mathbb{T} \to \mathbb{R}/\mathbb{Z} \to \partial K$ that is surjective: the *Carathéodory loop* of K. For $x \in \partial K$, the elements of $\gamma_K^{-1}(x)$ are called the *external arguments* of x.

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d \geq 2$ with every critical point if preperiodic. Then K_f is a connected compact set, full and locally connected; and the Carathéodory loop $\gamma_f : \mathbb{T} \to J_f$ satisfies the functional equation

$$f(\gamma_f(t)) = \gamma_f(d \cdot t).$$

We will now explain how to determine the external arguments of some points in J_f. We are particularly interested in the quadratic polynomials $f_c : z \mapsto z^2 + c$, because we will see that, in the case where 0 is strictly preperiodic for f_c, the external arguments of c in K_c are also (in a certain sense since we do not know that M is locally connected) the external arguments of c in M. There is also a statement about the points c such that 0 is periodic (a bit trickier of course since $c \in \overset{\circ}{K}_c$ and $c \in \overset{\circ}{M}$: there is a game between the center and the root of the components of $\overset{\circ}{K}_c$ and $\overset{\circ}{M}$).

2. Access.

Let K be a connected compact set which is full and locally connected, equipped with a center for each component of $\overset{\circ}{K}$. Let $H \subset K$ be a finite allowable tree, x a point in $H \cap \partial K$ and ν the number of branches of H at x. We call *access* to x (relatively to H) the ends at x of $\mathbb{C} \setminus H$, i.e., the elements of

$$\lim_{U \text{ neighborhood of } x} \pi_0(U \setminus H).$$

In more concrete terms, let Δ be a disk centered at x, containing no other remarkable point of H, and let $[x, y_1]_K, \ldots, [x, y_\nu]_K$ be the branches of H at x, stopped at their first intersection with $\partial\Delta$. The accesses to x are the ν connected components of

$$\Delta \setminus ([x, y_1]_K \cup \ldots \cup [x, y_\nu]_K) .$$

Every external argument t of x determines an access to x: it is the component where $\mathcal{R}(K, t)$ is located in a neighborhood of x.

Proposition 7.1. *Every access to x corresponds to at least one external argument of x.*

Proof. For $r > r(K)$, let us denote by γ_r the loop $t \mapsto \varphi_K^{-1}(re^{2i\pi t})$. Let V be an access to x contained in between two branches $[x, y_i]_K$ and $[x, y_{i+1}]_K$. We have $C \cap \mathbb{C} \setminus K \neq \emptyset$, since otherwise, $[x, y_i]_K$ and $[x, y_{i+1}]_K$ would be in the closure of the same component of $\overset{\circ}{K}$. This property is still true if we replace Δ by a smaller disk. It follows that we can find a decreasing sequence r_n tending to $r(K)$, and for all n a t_n such that $\gamma_{r_n}(t_n) \to x$. Since $\mathbb{T}$ is compact, extracting a subsequence if necessary, we can assume that the sequence (t_n) has a limit θ. Since $(r, t) \to \gamma_r(t)$ is continuous on $[r(K), +\infty[\times\mathbb{T}$, we have $\gamma_{r(K)}(\theta) = x$ and θ is an external argument of x. Let us show that the access to x defined by θ is V. Let ρ be the radius of Δ. If n is large enough, t_n is close enough to θ so that $|\gamma_r(t_n) - \gamma_r(t')| < \rho/2$ for all $r \in [r(K), r_0]$ and all t' between t_n and θ, and moreover, $|\gamma_{r_n}(t_n) - x| < \rho/2$. This implies that $\gamma_{r_n}(\theta)$ is in the same component of $\Delta \setminus ([x, y_i]_K \cup [x, y_{i+1}]_K$ as $\gamma_{r_n}(t_n)$, thus in V. ■

3. Extended tree.

We set $\beta = \gamma_f(0)$. It is a fixed point, repelling since it belongs to J_f and f is sub-hyperbolic. For $i \in \mathbb{Z}/(d)$, we set $\beta_i = \gamma(i/d)$. One can show that $f^{-1}(\beta) = \{\beta_i\}_{i\in\mathbb{Z}/(d)}$ (exercise).

We call *extended tree* the allowable hull $\widehat{H}$ of $\widehat{A} = A \cup \bigcup\{\beta_i\}_{i\in\mathbb{Z}/(d)}$. This tree is equipped with its *primary structure* defined by its topology, the cyclic order of the branches at the branching points, the dynamics on the points of $\widehat{A}$ (marked points) together with the ramification degree at the points of $\widehat{A}$.

In degree 2, we will see that we can reconstruct $\widehat{H}$ given H.

Lemma 7.1. *If* $d = 2$*, the point* β *has no other external argument than* 0.

Proof. We can assume f of the form $z \mapsto z^2 + c$. Then, $\beta_1 = -\beta$ and we have $\beta_1 \neq \beta$, since otherwise we would have $\beta = 0$, so 0 fixed point, $c = 0$ and $\beta = 1$, and so $1 = 0$. Let t be another external argument of β. Conjugating if necessary, we can assume that $t \in]0, 1/2[$ which we lift to $\mathbb{R}$. Let k be such that $2^k t < 1/2 < 2^{k+1}t$. We have: $0 < 2^k t < 1/2 < 2^{k+1}t < 1/2 + 2^k t < 1$. The external rays $\mathcal{R}(K_f, 0)$ and $\mathcal{R}(K_f, 2^{k+1}t)$ land at β, whereas the rays $\mathcal{R}(K_f, 1/2)$ and $\mathcal{R}(K_f, 1/2 + 2^k t)$ land at β_1. Then, $\overline{\mathcal{R}}(K_f, 0) \cup \overline{\mathcal{R}}(K_f, 2^{k+1}t)$ and $\overline{\mathcal{R}}(K_f, 1/2) \cup \overline{\mathcal{R}}(K_f, 1/2 + 2^k t)$ are disjoint curves, which is not compatible with their asymptotes. ■

Corollary 7.1. *β is an extremity of $\widehat{H}$.*

Remark. Lemma 7.1 and its corollary do not extend to $d > 2$. Fig. 1 shows the filled-in Julia set of $z \mapsto z^3 + \frac{3}{2}z$. The external rays of argument 0 and 1/2 both land at the same point.

Let us now assume that $d = 2$, f is of the form $z \mapsto z^2 + c$, and write $\widehat{H}_+ \cup \widehat{H}_-$ with $\widehat{H}_+ \cap \widehat{H}_- = \{0\}$ and $c = f(0) \in \widehat{H}_+$. Denote by α the other fixed point.

Lemma 7.2. *We have* $\beta \in \widehat{H}_-$ *and* $\alpha \in \widehat{H}_+$.

Proof. a) $\beta \in \widehat{H}_-$. Assume first that 0 is strictly preperiodic, so $\overset{\circ}{K}_f = \emptyset$ and $c \in J_f$. Let θ be an external argument of c. Then, $\mathcal{R}(K_f, \theta/2)$ and $\mathcal{R}(K_f, \theta/2 + 1/2)$ both land at 0, because it is the only point in $f^-(c)$. The curves $\mathcal{R}(K_f, 0) \cup [\beta, c]_{K_f} \cup \mathcal{R}(K_f, \theta)$ and $\overline{\mathcal{R}}(K_f, \theta/2) \cup \overline{\mathcal{R}}(K_f, \theta/2 + 1/2)$ must intersect because of the relative position of their asymptotes. They can only cut at 0, and so $0 \in [\beta, c]_{K_f}$.

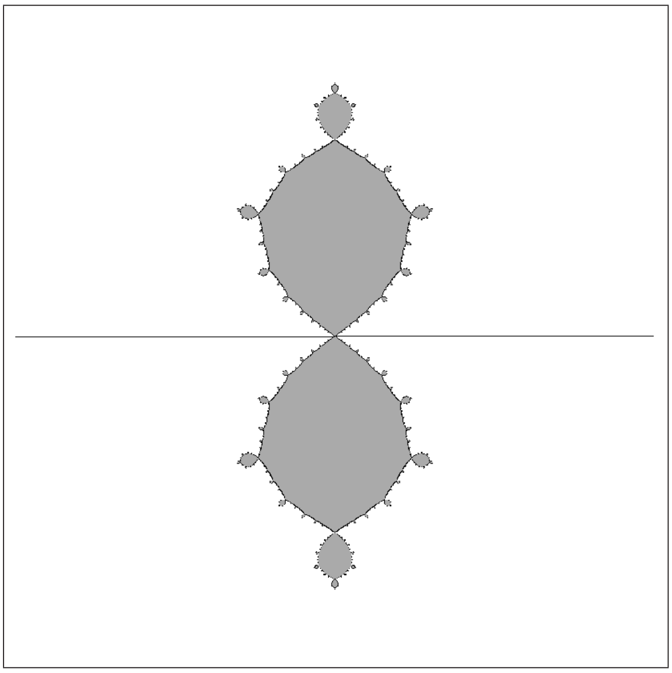


FIGURE 1. The filled-in Julia set of $z \mapsto z^3 + \frac{3}{2}z$ and the two rays $\mathcal{R}(K_f, 0)$ and $\mathcal{R}(K_f, 1/2)$.

If 0 is periodic with $c \neq 0$, then 0 and c are in $\overset{\circ}{K}_f$. Denote by U_0 and U_c the connected components of $\overset{\circ}{K}_f$ containing respectively 0 and c. Let y be a point of ∂U_c which does not belong to $\widehat{H}$ and θ an external argument of y. The landing point y' and y'' of $\mathcal{R}(K_f, \theta/2)$ and $\mathcal{R}(K_f, \theta/2 + 1/2)$ are the preimages of y, they belong to $\partial U_0 \setminus \widehat{H}$. The curves

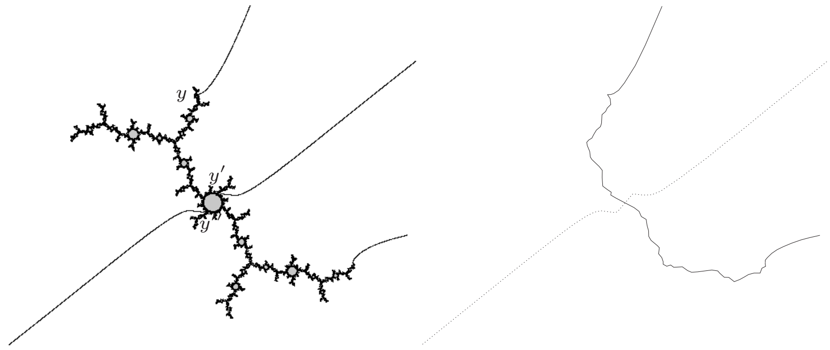
$$\mathcal{R}(K_f, \theta) \cup [y, \beta]_{K_f} \cup \mathcal{R}(K_f, 0) \quad \text{and} \quad \mathcal{R}(K_f, \theta/2) \cup [y', y'']_{K_f} \cup \mathcal{R}(K_f, \theta/2 + 1/2)$$

must intersect because of the relative position of their asymptotes. They can only intersect at 0, and so $0 \in [\beta, y]_{K_f}$ and $0 \in [\beta, c]_{K_f}$.

In the case $c = 0$ where $\alpha = 0$, $\beta = 1$. One must set $\widehat{H}_+ = [-1, 0]$ and $\widehat{H}_- = [0, 1]$. This case is then trivial.

b) Let us define $\pi_+ : \widehat{H} \to \widehat{H}_+$ by $\pi_+(x) = x$ for $x \in \widehat{H}_+$ and 0 for $x \in \widehat{H}_-$. The map $f \circ \pi_+ : \widehat{H}_+ \to \widehat{H}_+$ has a fixed point by Lefschetz's theorem. If $c \neq 0$, it is not 0, and so it is a fixed point of f, which is not β. Thus, it is α and $\alpha \in \widehat{H}_+$. ■

Let us now explain how to reconstruct $\widehat{H}$, knowing H. We first reconstruct $H^1 = f^{-1}(H)$ by gluing two copies H^1_+ and H^1_- of H at their point c, as indicated in chapter 6, section 1.1. The map f induces an injection $[\beta, 0]_{K_f} \to [\beta, c]_{K_f}$; denote by g the inverse map $[\beta, c]_{K_f} \to [\beta, 0]_{K_f}$ and set $z_i = g^{\circ i}(c)$, so that $z_1 = 0$, $z_i \in [\beta, 0]_{K_f}$ for $i \geq 1$. As long as $z_i \in H$, the point z_{i+1} is the preimage of



z_i in H^1_-, so we know its combinatorial position. In this way, we can determine $i^* = \sup\{i \mid z_i \in H\}$, the combinatorial position of z_i in H for $i \leq i^*$, and if $i^* < \infty$, the position of z_{i+1} in H^1. We have $i^* = \infty$ if and only if $\beta \in H$, i.e., if there exists a fixed point in H_+, and in this case, $\widehat{H} = H$. Otherwise, we have a homeomorphism between $\widehat{H}$ and the allowable hull in H^1 of $H \cup \{z_{i^*+1}, -z_{i^*-1}\}$ which coincides with the identity on H, maps β to z_{i^*+1} and $\beta_1 = -\beta$ to $-z_{i^*-1}$. This homeomorphism is compatible with the cyclic ordering of the branches at the branching points.

4. Computation of external arguments.

Let f be a monic polynomial of degree $d \geq 2$, such that every critical point is preperiodic, $X \subset \mathbb{C}$ a finite set such that $f(X) \subset X$, containing all the critical points and the $(\beta_i)_{i \in \mathbb{Z}/(d)}$, and T the allowable hull of X in K_f (for example $X = \widehat{A}$, $T = \widehat{H}$). We equip T with its primary structure: topology, dynamics on marked points (points of X), cyclic order at the branching points and ramification degree at the marked points.

The dynamics of the branching points is determined by the dynamics of the marked points of X, we can therefore add them to X. The dynamics on the branches is also known: if ξ is the germ at x of $[x, y]_T$, with $]x, y[\cap X = \emptyset$, the branch $f(\xi)$ is the germ at $f(x)$ of $[f(x), f(y)]_T$.

For $x \in X$, set $x_n = f^{\circ n}(x)$ and denote by $\nu(x)$ the number of branches of T at x. We have $x \in J_f$ if and only if the ramification degree $r(x_n)$ is 1 for all x_n in the cycle where x ends. We will explain how to determine the external arguments of x in K_f in that case.

If $x \in J_f$, set

$$\tilde{\nu}(x) = \prod_{0 \leq i < n} r(x_i) \cdot \nu(x_n),$$

with n sufficiently large for x_n to be periodic. We can define a tree $\widetilde{T}$ by adding to T, at each point x_i, $\tilde{\nu}(x_i) - \nu(x_i)$ buds, with a cyclic order between branches and buds compatible with the dynamics.

The tree $\widetilde{T}$ can be realized as a subset of $f^{-1}(T)$, and, at each point $x \in X$, there are $\tilde{\nu}(x)$ branches of $\widetilde{T}$ and $\tilde{\nu}(x)$ accesses to x relatively to $\widetilde{T}$. The dynamics on the accesses at points of X is determined by the datas.

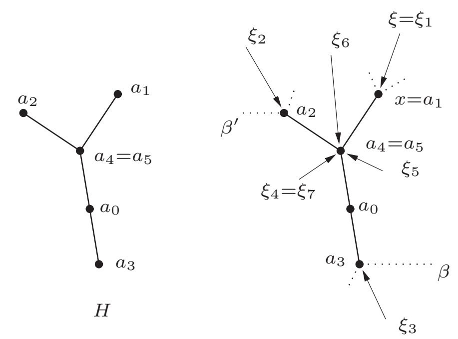
For each $i \in \mathbb{Z}/(d)$, denote by w_i the access to β_i corresponding to the external argument i/d (if $d = 2$, it is the unique access to β_i; if $d > 2$, it is a supplementary data that must be known to do the computation). Set $u(w_i) = i \in \{0, \dots, d-1\}$ and $u(\xi) = i$ if ξ is between w_i and w_{i+1} turning counterclockwise around $\widetilde{T}$.

Theorem 7.1. *Let x be a point of X, θ be an external argument in K_f and ξ the access to x relatively to $\widetilde{T}$ corresponding to θ. Denote by ξ_n the image of ξ by $f^{\circ n-1}$ (so that $\xi_1 = \xi$). We then have*

$$\theta = \sum_{n=1}^{\infty} \frac{u(\xi_n)}{d^n}.$$

In other words, the (ξ_n) are the digits after the "decimal" point of the development of θ in basis d.

Proof. It is immediate for $n = 1$. The access $f^{\circ n}(\xi)$ is the access to $f^{\circ n}(x)$ corresponding to $d^n\theta$, its first digit is the n-th digit of θ. ■



$u(\xi_1) = 0,\ u(\xi_2) = 0,\ u(\xi_3) = 1,\ u(\xi_4) = 1,\ u(\xi_5) = 0,\ u(\xi_6) = 0,\ u(\xi_7) = 1$

$$\theta = \sum \frac{u(\xi_n)}{2^n} = .001\overline{100} = \frac{1}{8}\left(1 + \frac{4}{7}\right) = \frac{11}{56}.$$

Corollary 7.2. *Every access to x relatively to $\widetilde{T}$ corresponds to an external argument of x and only one.*

Corollary 7.3. *Every point x of X has a finite number $\tilde{\nu}(x)$ of external arguments. Those are rational numbers, with denominator coprime with d if and only if x is periodic.*

CHAPTER 8

External arguments in M of Misurewicz points.

1. Conformal representation of $\mathbb{C} \setminus M$.

1.1. Potential for Julia sets. Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree $d \geq 2$. We have seen (Chapter 3 proposition 3.2) that if K_f is connected, there exists a unique isomorphism $\varphi_f : \mathbb{C} \setminus K_f \to \mathbb{C} \setminus \overline{\mathbb{D}}$ tangent to the identity at ∞ (i.e., such that $\varphi(z)/z \to 1$), and which conjugates f to $f_0 : z \mapsto z^d$.

We set $G_f(z) = \log |\varphi_f(z)|$. The function $G = G_f : \mathbb{C} \setminus K_f \to \mathbb{R}_+$ has the following properties:

1) G is harmonic.
2) $G(z) = \log |z| + \mathcal{O}(1)$ as $|z| \to \infty$.
3) $G(z) \to 0$ as $d(z, K_f) \to 0$.
4) $G(f(z)) = d \cdot G(z)$.

Properties 1), 2) and 3), or 2) and 4) are sufficient to characterize G. One can even replace 2) by 2'): $G(z)/\log |z| \to 1$ as $|z| \to \infty$.

In the general case (K_f not necessarily connected), there exists a $\mathbb{C}$-analytic isomorphism φ_f from a neighborhood V of ∞ to a neighborhood V_0 of ∞, tangent to the identity at ∞, such that $f(V) \subset V$, $f_0(V_0) \subset V_0$, $f_0 \circ \varphi_f = \varphi_f \circ f$. For example, if $f = z \mapsto z^d + a_{d-1}z^{d-1} + \ldots + a_0$, one can take $V = \mathbb{C} \setminus \overline{\mathbb{D}}_{R^*}$, where $R^* = 1 + |a_{d-1}| + \ldots + |a_0|$, and define φ_f by

$$\varphi_f(z) = z \cdot \prod_{n=1}^{\infty} \left(1 + \frac{a_{d-1}}{z_n} + \ldots \frac{a_0}{z_n^d}\right)^{1/d^{n+1}}$$

where $z_n = f^{\circ n}(z)$, the fractional exponent being determined by observing that

$$\left| \frac{a_{d-1}}{z_n} + \ldots \frac{a_0}{z_n^d} \right| < 1.$$

We prefer to shrink V so that V_0 is of the form $\mathbb{C} \setminus \overline{\mathbb{D}}_{R_0^*}$. Then, f induces a proper holomorphic map of degree d from V to $f(V)$; in particular, $V = f^{-1}(f(V))$.

The germ φ_f at ∞ is uniquely determined. One can define $G_f : \mathbb{C} \setminus K_f \to \mathbb{R}_+$ by $G_f(z) = \log |\varphi_f(z)|$ for $z \in V$, and in the general case, $z \in \mathbb{C} \setminus K_f$ by $G_f(z) = G(f^{\circ n}(z))/d^n$ where n is sufficiently large so that $f^{\circ n}(z) \in V$ (the result does not depend on the choice of n). The function G_f still has properties 1), 2), 3) and 4) stated above. Properties 2) and 4) are sufficient to characterize G_f, because they imply

$$G_f(z) = \lim_{n \to \infty} \frac{1}{d^n} \log |f^{\circ n}(z)| .$$

It is still true that G_f is characterize by 1), 2) and 3), but this is less obvious.

Denote by $\mathcal{P}_d$ the set of monic polynomials of degree d (that can be identified with $\mathbb{C}^d$), and for all $f \in \mathcal{P}_d$, extend G_f to $\mathbb{C}$ by $G_f(z) = 0$ is $z \in K_f$.

Proposition 8.1. *a) The set $\mathcal{K}$ of pairs (f, z) such that $z \in K_f$ is closed in $\mathcal{P}_d \times \mathbb{C}$.*

b) The map $(f, z) \mapsto G_f(z)$ is a continuous function $\mathcal{P}_d \times \mathbb{C} \to \mathbb{R}_+$.

Proof. For $f : z \mapsto z^d + a_{d-1}z^{d-1} + \ldots + a_0$, set

$$R^*(f) = 1 + |a_{d-1}| + \ldots + |a_0|, \quad R_0^*(f) = R^*(f)^{d/(d-1)}.$$

Set $\mathcal{V}_1 = \{(f, z) \mid R^*(f) < |z|\}$ and $\mathcal{V}_0 = \{(f, z) \mid R_0^*(f) < |z|\}$; define $\Phi : \mathcal{V}_1 \to \mathcal{P}_d \times \mathbb{C}$ by $\Phi(f, z) = (f, \varphi_f(z))$, $F : \mathcal{P}_d \times \mathbb{C} \to \mathcal{P}_d \times \mathbb{C}$ by $(f, z) \mapsto (f, f(z))$, and F_0 by $(f, z) \mapsto (f, z^d)$. One can check that Φ induces an isomorphism between an open subset $\mathcal{V}$ of $\mathcal{V}_1$ and $\mathcal{V}_0$.

We have $\mathcal{P}_d \times \mathbb{C} \setminus \mathcal{K} = \bigcup F^{-n}(\mathcal{V}_1)$, and so a).

The function $(f, z) \mapsto G_f(z)$ is continuous on $\mathcal{P}_d \times \mathbb{C} \setminus \mathcal{K}$, because it is continuous on $\mathcal{V}_1$ where it is given by a series which is locally absolutely convergent, and on each $F^{-n}(\mathcal{V}_1)$, it is given by $G_f(z) = G_f(f^{\circ n}(z))/d^n$. It remains to show that, for all $\varepsilon > 0$, $\mathcal{W}_\varepsilon = \{(f, z) \mid G_f(z) < \varepsilon\}$ is a neighborhood of $\mathcal{K}$. It is enough to show that for every open set Λ relatively compact in $\mathcal{P}_d$, the set $\mathcal{W}_{\varepsilon,\lambda} = \mathcal{W}_\varepsilon \cap \Lambda \times \mathbb{C}$ is open in $\Lambda \times \mathbb{C}$. Set

$$R_0^*(\Lambda) = \sup_{f \in \Lambda} R_0^*(f),$$

and let N be such that $d^N \varepsilon > R_0^*(\Lambda)$. Then,

$$\Lambda \times \mathbb{C} \setminus \mathcal{W}_{\varepsilon,\Lambda} = F^{-N}(\Lambda \times \mathbb{C} \setminus \mathcal{W}_{d^N \varepsilon,\Lambda}) = F^{-N}\left(\Phi^{-1}\left(\{(f, z) \mid d^N \varepsilon \leq |z|\}\right)\right).$$

This is a closed set. ■

1.2. Critical point of G_f.

Proposition 8.2. *The critical points of $G_f : \mathbb{C} \setminus K_f \to \mathbb{R}_+$ are the points which are preimages of the critical points of f in $\mathbb{C} \setminus K_f$.*

Proof. If $\varphi_f : V \to V_0$ is an isomorphism, $G_f = \log |\varphi_f|$ has no critical point in V. The formula $G(z) = G_f(f(z))/d$ shows that z is a critical point of G_f if and only if z is a critical point of f or $f(z)$ is a critical point of G_f.

Let $z \in \mathbb{C} \setminus K_f$ and set $z_n = f^{\circ n}(z)$. For n large enough, $z_n \in V$, so z_n is not a critical point of G_f. It follows that z is a critical point of G_f if and only if one of the z_n is a critical point of f. ■

Corollary 8.1. *If all the critical points of f are in K_f, for all $h > 0$, the set $G_f^{-1}(h)$ is homeomorphic to S^1 and the set of points $z \in \mathbb{C}$ such that $G_f(z) \leq h$ is homeomorphic to a closed disk.*

This gives another proof that in this case K_f is connected. Figure 1 shows some level curves $G_f^{-1}(h)$ for $f : z \mapsto z^2 - 1$.

Corollary 8.2. *Let h_0 be the maximum of $G_f(\alpha)$ for α critical point of f. Then, for all $h > h_0$, the set $G_f^{-1}(h)$ is homeomorphic to S^1 and $\{z \in \mathbb{C} \mid G_f(z) \leq h\}$ is homeomorphic to $\overline{\mathbb{D}}$. The set $L_f = \{z \mid G_f(z) \leq h_0\}$ is a connected compact set. The map φ_f extends to an isomorphism from $\mathbb{C} \setminus L_f$ to $\mathbb{C} \setminus \overline{\mathbb{D}}_R$ with $R = e^{h_0}$.*

Figure 2 shows some level curves $G_f^{-1}(h)$ for a quadratic polynomial with disconnected Julia set.

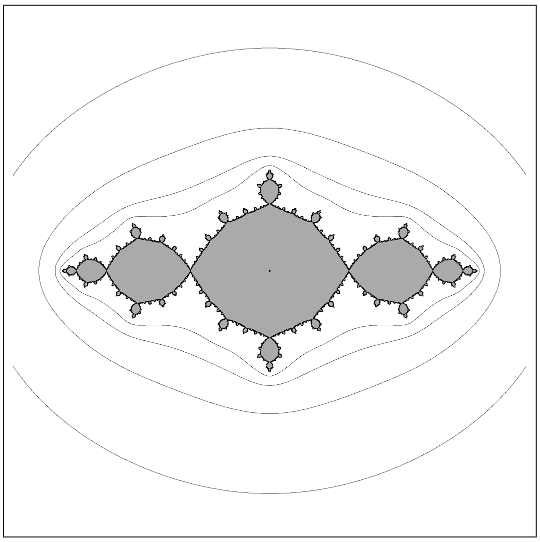


FIGURE 1. Some level curves $G_f^{-1}(h)$ for $f : z \mapsto z^2 - 1$.

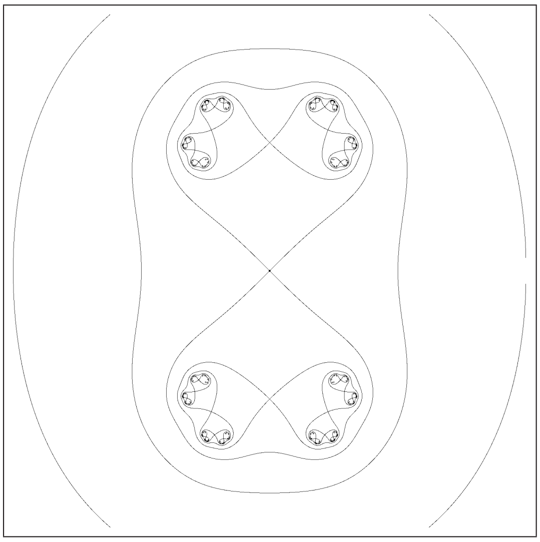


FIGURE 2. Some level curves $G_f^{-1}(h)$ for a quadratic polynomial with disconnected Julia set.

For $z \in \mathbb{C} \setminus L_f$, we define the external argument $\arg_{K_f}(z) = \arg_{L_f}(z) = \arg \varphi_f(z)$. If $0 < G_f(z) \leq h_0$, and if z is not a critical point of G_f, we can define the external ray of f through z as the orthogonal trajectory to the level curves of G_f. This ray, extended on the side of increasing G_f, can go outside L_f, which allows us to define $\arg_{K_f}(z)$, or it can end on a critical point of G_f. A critical point of G_f is a saddle if it has only one critical point, simple, in its forward orbit. In the general case, it is a "monkey saddle" with r rays going up and r rays going down, r being the product of ramification degrees at the critical points in the forward orbit of z. There is a countable family of R-analytic curves on which we cannot define the function "external argument" (see Figure 3).

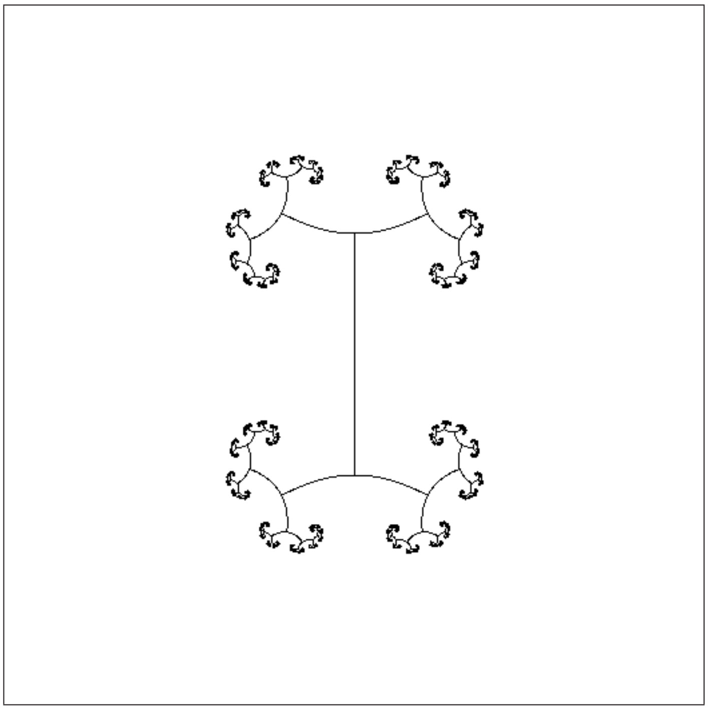


FIGURE 3. The countable family of R-analytic curves on which we cannot define the function "external argument".

1.3. The function Φ. Let us now consider the family of quadratic polynomials $f_c : z \mapsto z^2 + c$. We will write φ_c for φ_{f_c}, and so on...

For $c \in \mathbb{C} \setminus M$, we have $h_0(c) = G_c(0) > 0$ and $G_c(c) = 2G_c(0) > h_0(c)$. We can therefore set

$$\Phi(c) = \varphi_c(c).$$

Theorem 8.1. *We define in this way an isomorphism $\Phi : \mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{\mathbb{D}}$.*

Proof. For $c \in \mathbb{C} \setminus M$, we have $\log |\Phi(c)| = G_c(c) > 0$, so $\Phi(c) \in \mathbb{C} \setminus \overline{\mathbb{D}}$. The map Φ is holomorphic. Indeed,

$$\mathcal{L} = \{(c, z) \mid z \in L_c\} = \{(c, z) \mid G_c(z) \leq G_c(0)\}$$

is closed, and, on $\mathbb{C}^2 \setminus \mathcal{L}$, the map $(c, z) \mapsto \varphi_c(z)$ which is a determination of $(\varphi_c(f_c^{\circ n}(z)))^{1/2^n}$, is holomorphic.

We can write:

$$\frac{\Phi(c)}{c} = \left(1 + \frac{c}{c^2}\right)^{1/2} \cdot \left(1 + \frac{c}{(c^2 + c)^2}\right)^{1/4} \cdots \left(1 + \frac{c}{(f^{\circ n}(c))^2}\right)^{1/2^{n+1}} \cdots$$

This infinite product converges uniformly for $|c| \geq 4$ and all the factors tend to 1 as $c \to +\infty$. It follows that $\Phi(c)/c \to 1$, as $c \to \infty$. We can therefore extend Φ to a holomorphic map $\overline{\mathbb{C}} \setminus M \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, where $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere, by setting $\Phi(\infty) = \infty$. This extension of Φ is proper: indeed, if $c \to c_0 \in \partial M$, $G_c(c) \to G_{c_0}(c_0) = 0$ and $\Phi(c) \to \partial\mathbb{D}$.

As a proper holomorphic map, it has a degree. Since $\Phi^{-1}(\infty) = \{\infty\}$ with multiplicity 1, this degree is 1 and Φ is an isomorphism $\mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{\mathbb{D}}$. ■

Corollary 8.3. *a) The set M is connected.*
b) Its capacity is 1.

Corollary 8.4. *For all $c \in \mathbb{C} \setminus M$, we have:*
a) $G_M(c) = G_c(c)$.
b) $\arg_M(c) = \arg_{K_c}(c)$.

Theorem 8.2 below asserts that in some way, formula b) of corollary 8.4 extends to some points in the boundary of M.

2. External rays for Julia sets.

2.1. Possible behaviors. Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial. As we follow an external ray $\mathcal{R}(K_f, \theta)$ of K_f in the direction of decreasing G_f, this ray may either *bifurcate* on a critical point of G_f, or extend until $G_f \to 0$, i.e., until it tends to K_f.

In that case, it can either tend to a point of K_f – we say that it *lands* at this point – or it may have an accumulation set in K_f which is not reduced to a point – we will say that it *oscillates*.

Figure 4 shows two possible behaviors of external rays for a cubic polynomial with disconnected Julia set. The rays $\mathcal{R}(K_f, 0)$ and $\mathcal{R}(K_f, 1/2)$ both land at α, whereas the rays $\mathcal{R}(K_f, -1/12)$ and $\mathcal{R}(K_f, 7/12)$ bifurcate on the critical point ω. The set $U = \{z \in \mathbb{C} \mid G_f(z) < 1\}$ is a topological disk, whereas the set $\{z \in \mathbb{C} \mid G_f(z) < 1/3\}$ has two connected components U' and U''.

If $\mathcal{R}(K_f, \theta)$ does not bifurcate, we have $f(\mathcal{R}(K_f, \theta)) = \mathcal{R}(K_f, d \cdot \theta)$. This ray lands at $f(x)$ if $\mathcal{R}(K_f, \theta)$ lands at x, oscillates if $\mathcal{R}(K_f, \theta)$ oscillates. It may happen that $\mathcal{R}(K_f, \theta)$ bifurcates, but that $\mathcal{R}(K_f, d\theta)$ does not oscillate.

Proposition 8.3. *If f is sub-hyperbolic, every external ray of K_f either bifurcates or lands.*

Proof. Let V be a neighborhood of J_f, μ be an admissible metric on V and $\lambda > 1$ such that $\|T_x f\|_\mu \geq \lambda$ for all $x \in f^{-1}(V)$. Let $h > 0$ be such that $\{z \mid G(z) \leq h\} \subset V \setminus \mathring{K}_f$; set $Q = \{z \mid h/d \leq G(z) \leq h\}$ and denote by M the supremum of the μ-lengths of external rays in between the levels h and h/d (the fact that there may be critical points of G_f in Q does not forbid M to be finite). The μ-length of an external ray below the level h is bounded from above by $M/(\lambda - 1)$, which proves the proposition. ■

2.2. External rays with rational arguments.

Proposition 8.4. *Assume θ is rational. Then, if $\mathcal{R}(K_f, \theta)$ does not bifurcate, it lands at a point $\alpha \in K_f$. This point is preperiodic (periodic if θ has denominator coprime with d), repelling or rationally indifferent.*

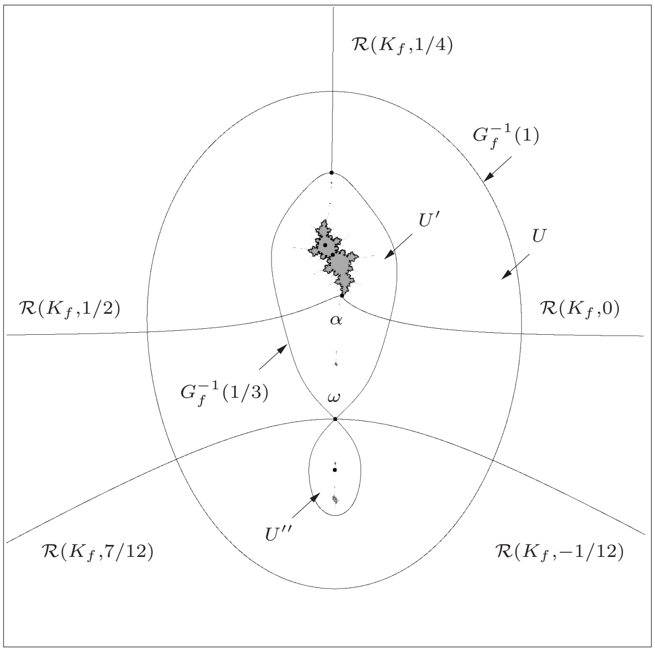


FIGURE 4. The rays $\mathcal{R}(K_f, 0)$ and $\mathcal{R}(K_f, 1/2)$ are defined both land at α, whereas the rays $\mathcal{R}(K_f, -1/2)$ and $\mathcal{R}(K_f, 7/2)$ bifurcate on the critical point ω.

Remark. This is written without the hypotheses that f is hyperbolic or subhyperbolic. Here, f is any monic polynomial.

Proof. Let us first assume that θ has denominator coprime with d. If $\theta = p_0/q$, then d is invertible modulo q, so there exists k such that $d^k \equiv 1 \bmod q$, i.e., q divides $d^k - 1$. We can then write θ, maybe not in an irreducible way but with a minimal k, in the form $p/(d^k - 1)$.[1] We assume that $\mathcal{R}(K_f, \theta)$ does not bifurcate; it is therefore invariant by $f^{\circ k}$.

Let h_0 be the infimum of the $G_f(\omega)$ for ω critical point of f in $\mathbb{C} \setminus K_f$ (if there are none, $h_0 = \infty$) and $h < h_0$. Let $U = \{z \mid 0 < G(z) < h\}$ and $U' = f^{-k}(U) = \{z \mid 0 < G(z) < h/d^k\}$. Denote by $\widetilde{U}$ the universal covering of the connected component of U intersecting $\mathcal{R}(K_f, \theta)$ and $\widetilde{R}$ a lift of $\mathcal{R}(K_f, \theta) \cap U$ in $\widetilde{U}$. There exists a lift $g : \widetilde{U} \to \widetilde{U}$ of f^{-k} such that $g(\widetilde{R}) \subset \widetilde{R}$. Let us choose $x_0 \in \mathcal{R}(K_f, \theta)$ and define $x_n \in \mathcal{R}(K_f, \theta)$ by $G_f(x_n) = G_f(x_0)/d^{kn}$, so that $f^{\circ k}(x_{n+1}) = x_n$. Let L be the Poincaré length in U of $[x_0, x_1]_{\mathcal{R}(K_f,\theta)}$. Since g is contracting for the Poincaré metric $d_{\widetilde{U}}$ on $\widetilde{U}$, we have $d_U(x_n, x_{n+1}) \leq L$ for all n. Since x_n tends to $\partial K \subset \partial U$, the Euclidean distance $|x_{n+1} - x_n|$ tends to 0. If a subsequence $(x_n)_{n \in I}$ tends to a point $\alpha \in \partial K$, the sequence (x_{n-1}) also tends to α, so $f^{\circ k}(\alpha) = \alpha$.

Let $\alpha_1, \ldots, \alpha_r$ be the points such that $f^{\circ k}(\alpha) = \alpha$, $W_1, \ldots, W_r$ be neighborhoods of $\alpha_1, \ldots, \alpha_r$ such that $d_U(W_i \cap U, W_j \cap U) > L$ for $i \neq j$. There exists an n_0 such that $x_n \in \bigcup W_i$ for $n \geq n_0$, since otherwise we could extract from (x_n)

[1] For example with $d = 2$, $\dfrac{1}{5} = \dfrac{3}{15} = \dfrac{3}{2^4 - 1}$.

a sequence tending to $\alpha \notin \{\alpha_1, \dots, \alpha_r\}$. But since $d_U(x_n, x_{n+1}) \leq L$, the x_n for $n \geq n_0$ all belong to the same W_i, let us say W_1. Then $x_n \to \alpha_1$, because for all extracted subsequence converging to a point α we have $\alpha = \alpha_1$. Every $y \in \mathcal{R}(K_f, \theta)$ such that $G_f(y) \leq G_f(x_0)$ belongs to a segment $[x_{n(y)}, x_{n(y)+1}]$ of $\mathcal{R}(K_f, \theta)$, and we have $d_U(y, x_{n(y)}) \leq L$. It follows that $|y - x_{n(y)}| \to 0$, so $y \to \alpha = \alpha_1$ as $G(y) \to 0$. In other words, $\mathcal{R}(K_f, \theta)$ lands at α, which is a periodic point of f, with period k' dividing k.

The point α belongs to ∂K, so it is not attracting. As a consequence, it is repelling or indifferent.

Lemma 8.1. *If α is an indifferent periodic point, we have $(f^{\circ k})'(\alpha) = 1$.*

Proof. Assume $(f^{\circ k})'(\alpha) = e^{2i\pi t}$. We have $t = \lim t_n$, where

$$t_n = \arg\left(\frac{x_{n-1} - \alpha}{x_n - \alpha}\right).$$

Let $\tilde{t}_n$ be the lift of t_n to $\mathbb{R}$ defined by the path from x_n to x_{n-1} following $\mathcal{R}(K_f, \theta)$. The sequence $(\tilde{t}_n)$ tends to a lift $\tilde{t}$ of t. We will show that $\tilde{t} = 0$.

We define a holomorphic function $F : \{z \mid \mathrm{Re}(z) < m\} \to \mathbb{C}$ by the formula $F(\log(z - \alpha)) = \log(f^{\circ k}(z) - \alpha)$, normalized by the convention that $F(\zeta) - \zeta$ tends to $2i\pi\tilde{t}$ as $\mathrm{Re}(z) \to 0$.

We define a parametrization $\gamma : \mathbb{R} \to \mathcal{R}(K_f, \theta)$ by $G_f(\gamma(s)) = G_f(x_0)/d^s$, so that $x_n = \gamma(n)$. Let $\tilde{\gamma}$ be a continuous branch of $s \mapsto \log(\gamma(s) - \alpha)$. Denote by $\widetilde{R}_\alpha$ the image of $\tilde{\gamma}$ and $\tilde{x}_n = \tilde{\gamma}(n)$. The sequence $\mathrm{Re}(\tilde{x}_n)$ tends to $-\infty$ and we have $\tilde{x}_{n-1} = F(\tilde{x}_n)$ for n large enough.

Assume $\tilde{t} > 0$. Taking a smaller m if necessary, we may assume that

$$\inf_{\mathrm{Re}(\zeta) < m} \mathrm{Im}(F(\zeta) - \zeta) = \mu > 0,$$

and that F defines an isomorphism between the half-plane $\{\zeta \mid \mathrm{Re}(\zeta) < m\}$ and an open set containing the half-plane $P_1 = \{\zeta \mid \mathrm{Re}(\zeta) < m_1\}$. Then, $\mathrm{Re}\gamma(s) \to -\infty$ and $\mathrm{Im}\gamma(s) \to -\infty$ as $s \to +\infty$. For all $\eta \in \mathbb{R}$, denote by N_η the connected component of $\{\zeta \mid \mathrm{Im}\zeta < \eta\} \setminus \widetilde{\mathcal{R}}_\alpha$ containing the $u + i(\eta - 1)$ for $u \to -\infty$. If η is sufficiently small, $N_\eta \subset P_1$ and $F^{-1}(N_\eta) \subset N_{\eta - \mu}$.

The image Ω_η of N_η by $\zeta \mapsto \alpha + e^\zeta$ is a neighborhood of α, and the image of $F^{-p}(N_\eta)$ is the image $\Omega_{\eta,p}$ of Ω_η by the branch of f^{-pk} which fixes α. We have $N_{\eta,p} \subset N_{\eta - p\mu}$, which is in a half-plane $\{z \mid \mathrm{Re}(z) < m_2\}$ with m_2 arbitrarily negative if p is sufficiently large, so $\Omega_{\eta,p}$ is arbitrarily small. In particular, we can have $\Omega_{\eta,p} \Subset \Omega_\eta$. Then, Schwarz's lemma implies that $|(f^{-pk})'(\alpha)| < 1$, so α is repelling, which gives a contradiction. □

Let us now complete the proof of proposition 8.4. If θ has a denominator coprime with d, $\theta = p/(2^k - 1)$, $\mathcal{R}(K_f, \theta)$ lands at a point $\alpha \in K_f$, periodic for f of period k' dividing k. We have $(f^{\circ k})'(\alpha) = 1$, so $(f^{\circ k'})'(\alpha)$ is a k/k'-th root of 1.

If θ does not have a denominator coprime with d, we can write θ as $p/(d^l q)$ with q coprime with d. Then, if $\mathcal{R}(K_f, \theta)$ does not bifurcate, $f^{\circ l}(\mathcal{R}(K_f, \theta)) = \mathcal{R}(K_f, \theta_1)$ where $\theta_1 = d^l\theta = p/q$. The ray $\mathcal{R}(K_f\theta)$ cannot oscillate, because $\mathcal{R}(K_f\theta_1)$ would oscillate. Therefore it lands at a point α, so $\mathcal{R}(K_f\theta_1)$ lands at $\alpha_1 = f^{\circ l}(\alpha)$. The above study shows that α_1 is periodic, repelling or rationally indifferent. ■

2.3. A property of stability. Denote by $\mathcal{P}_d$ the space of monic polynomials of degree d. For $f \in \mathcal{P}_d$ and $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, we define $\psi_{f,\theta} : \mathbb{R}_+^* \to \mathcal{R}(K_f, \theta)$ by $G_f(\psi_{f,\theta}(s)) = s$ if $\mathcal{R}(K_f, \theta)$ does not bifurcate; if it bifurcates on a critical point ω of G_f, the funciton $\psi_{f,\theta}$ is only defined on $[G_f(\omega), +\infty[$. If $\mathcal{R}(K_f, \theta)$ lands at a point α, we extend $\psi_{f,\theta}$ to $\mathbb{R}_+$ by setting $\psi_{f,\theta}(0) = \alpha$.

Proposition 8.5. *Let $f_0 \in \mathcal{P}_d$ and $\theta \in \mathbb{Q}/\mathbb{Z}$. Assume $\mathcal{R}(K_{f_0}, \theta)$ lands at a periodic or preperiodic repelling point $\alpha_0 \in J_{f_0}$. Moreover, assume $f^{\circ i}(\alpha_0)$ is not a critical point of f_0 for any value $i \geq 0$. Then, there exists a neighborhood Λ of f_0 in $\mathcal{P}_d$, such that , for all $f \in \Lambda$, the ray $\mathcal{R}(K_f, \theta)$ lands at a periodic or preperiodic repelling point α_f. The map $(f, s) \mapsto \psi_{f,\theta}$ from $\Lambda \times \mathbb{R}_+$ to $\mathbb{C}$ is continuous, holomorphic with respect to f.*

Proof. Assume θ has a denominator coprime with d, so of the form $p/(d^k - 1)$. We can find a neighborhood Λ_1 of f_0 in $\mathcal{P}_d$, neighborhoods $\mathcal{V}$ and $\mathcal{V}'$ of (f_0, α_0) in $\mathcal{P}_d \times \mathbb{C}$, such that $\mathcal{V}' \subset \mathcal{V}$ and an isomorphism $(f, z) \mapsto (f, \zeta_f(z))$ from $\mathcal{V}$ to $\Lambda_1 \times \mathbb{D}$, and an analytic map $\rho : \Lambda_1 \to \mathbb{C} \setminus \overline{\mathbb{D}}$, so that $\zeta_f(f^{\circ k}(z)) = \rho(f) \cdot \zeta_f(z)$ for $(f, z) \in \mathcal{V}'$. We set $\alpha_f = \zeta_f^{-1}(0)$; we then have $f^{\circ k}(\alpha_f) = \alpha_f$ and $(f^{\circ k})'(\alpha_f) = \rho(f)$.

Let $s_0 \in \mathbb{R}_+^*$ be such that $(f_0, \psi_{f_0,\theta}(s_0)) \in \mathcal{V}'$. By lower semi-continuity of the domain of solutions of a differential equation with respect to the initial condition and the parameters, there exists a neighborhood Λ of f in Λ_1 such that, for $f \in \Lambda$, $\psi_{f,\theta}$ is defined on $[s_0, +\infty[$ with $(f, \psi_{f,\theta}(s_0)) \in \mathcal{V}'$, and $\psi_{f,\theta}(s)$ depending continuously on (f, s) and holomorphic with respect to f for $f \in \Lambda$, $s \geq s_0$. For each $f \in \Lambda$, we can extend $\psi_{f,\theta}$ to $\mathbb{R}_+^*$ by setting

$$\psi_{f,\theta}\left(\frac{s}{d^{kn}}\right) = \zeta_f^{-1}\left(\frac{\zeta_f(\psi_{f,\theta}(s))}{[\rho(f)]^n}\right)$$

for $s \geq s_0$, $\psi_{f,\theta}(s) \in \mathcal{V}$. We obtain in this way a map $(f, s) \mapsto \psi_{f,\theta}(s)$ defined on $\Lambda \times \mathbb{R}_+^*$, continuous and holomorphic with respect to f. For each f, the image of $\psi_{f,\theta}$ is $\mathcal{R}(K_f, \theta)$. Finally, $\psi_{f,\theta}(s) \to \alpha_f$ uniformly on every compact subset of Λ as $s \to 0$. We can therefore extend $(f, s) \mapsto \psi_{f,\theta}(s)$ continuously to $\Lambda \times \mathbb{R}^+$.

This proves the proposition in the case where θ has a denominator coprime with d (which implies α_0 periodic). In the general case, there exists $l \geq 0$ such that $\theta^* = d^l\theta$ has a denominator coprime with d. For all $i \geq 0$, the external ray $\mathcal{R}(K_f, d^i\theta)$ lands at $f^{\circ i}(\alpha_0)$. We prove the required property for θ with a reverse induction on i, starting with $i = l$. For $i = l$, it is the case which has already been studied. For $i < l$, the map $F : (f, z) \mapsto (f, f(z))$ has a holomorphic inverse g_i defined in a neighborhood of $\{f_0\} \times \overline{\mathcal{R}}(f_0, d^{i+1}\theta)$ with

$$g_i\left(\{f_0\} \times \overline{\mathcal{R}}(f_0, d^{i+1}\theta)\right) = \{f_0\} \times \overline{\mathcal{R}}(f_0, d^i\theta)$$

since F has no critical point on $\{f_0\} \times \overline{\mathcal{R}}(f_0, d^i\theta)$. We can therefore define $\psi_{f,d^i\theta}(s)$ for $s \leq s_0$ and f sufficiently close to f_0 by

$$\left(f, \psi_{f,d^i\theta}(s)\right) = g_i\left(f, \psi_{f,d^{i+1}\theta}(ds)\right).$$

■

3. Harvesting in parameter space.

We consider the family of quadratic polynomials $(f_c : z \mapsto z^2 + c)_{c \in \mathbb{C}}$.

Theorem 8.2. *Let $c \in M$ be a points such that 0 is strictly preperiodic for f_c (Misurewicz point).*

a) *The point c has a finite number of external rays in K_c, which are all rational with even denominator.*
b) *For each external argument θ of c in K_c, the external ray $\mathcal{R}(M,\theta)$ lands at c.*

Proof. Statement (a) is a particular case of corollary 8.4 in Chapter 7.

Let us prove statement (b). The point c is a repelling preperiodic point of f_c and we have $\psi_{c,\theta}(0) = c$, keeping the notations of the previous subsection. The point c does not have any critical point in its forward orbit, since $c = f_c(0)$ and 0 is not periodic. For λ close to c and $s \in \mathbb{R}^+$, set $H_s(\lambda) = \psi_{\lambda,s}(s) - \lambda$. Denote by ν the order of the zero of H_0 at c. We have $\nu < \infty$ since otherwise we would hae $f_\lambda^{k+l+1}(0) = f_\lambda^{l+1}(0)$ for all λ close to c, and so for all $\lambda \in \mathbb{C}$.

For $s > 0$ close to 0, the equation $H_s(\lambda) = 0$ has ν solutions close to c, counting multiplicities. For such a root λ, we have $\lambda = \psi_{\lambda,\theta}(s)$, and so $\lambda \notin K_\lambda$, i.e., $\lambda \notin M$, and $\Phi(\lambda) = \varphi_\lambda(\lambda) = e^{s+2i\pi\theta}$. Hence, we see that $\Phi^{-1}(e^{s+2i\pi\theta}) \to c$ as $s \to 1$. ■

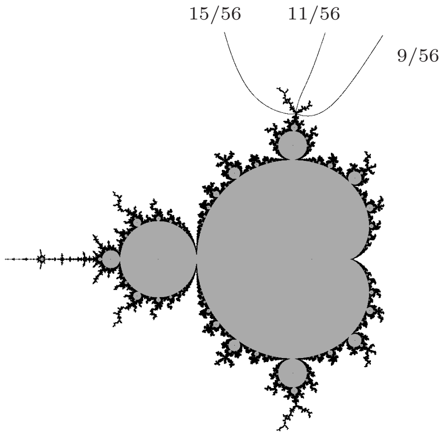
Corollary 8.5. *The equation $f_\lambda^{l+1+k}(0) - f_\lambda^{l+1}(0) = 0$ has a simple root at $\lambda = c$.*

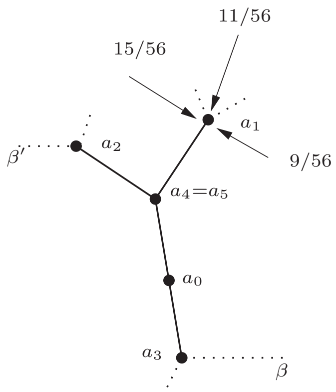
Proof. The multiplicity of c as a root of this equation is equal to the ν introduced in the proof of the theorem.

For $s > 0$, the equation $H_s(\lambda) = 0$ has only one solution, since it necessarily is $\Phi^{-1}(e^{s+2i\pi\theta})$.

We go from the equation $\lambda = \psi_{\lambda,\theta}(s)$, which gives the intersection of the diagonal with the graph of $\lambda \mapsto \psi_{\lambda,\theta}(s)$, to the equation $\varphi_\lambda(\lambda) = e^{s+2i\pi\lambda}$ by transforming those two curves via the diffeomorphism $(\lambda, z) \mapsto (\lambda, \varphi_\lambda(z))$. The multiplicity of the solution of $\lambda = \psi_{\lambda,\theta}(s)$ is equal to the one of $\Phi(\lambda) = e^{s+2i\pi\lambda}$, which is 1 since Φ is an isomorphism. So, we have $\nu = 1$. ■

Remark. There exist other proofs of this corollary, for example an arithmetical proof consisting in counting the 2-adic valuations.





Part 2

CHAPTER 9

Multiple fixed points and rationally indifferent periodic points.

1. Multiple fixed points.

If f is a polynomial having a periodic point of period k at α, with $(f^{\circ k})'(\alpha) = e^{2i\pi p/q}$, the polynomial $f^{\circ kq}$ has a fixed point at α with derivative 1. Because of this, we will first study fixed points with derivative 1.

We can assume that the fixed point is 0. The majojrpart of the study can be performed for maps which are holomorphic in a neighborhood of 0.

1.1. Order of a fixed point. Let f be a holomorphic map in a neighborhood of 0, wuth $f(0) = 0$. The *order* of 0 as a fixed point of f is the order r of vanishing at 0 of $z \mapsto f(z) - z$. We say that 0 is a multiple fixed point if $r \geq 2$. We can then write $f(z) = z + bz^r + \mathcal{O}(z^{r+1})$ with $b \neq 0$. This can also be written as $f(z) = z(1 + bz^{r-1} + \mathcal{O}(z^r))$. The z such that $bz^{r-1} \in \mathbb{R}_+$ (respectively $\mathbb{R}_-$) form $r-1$ hal lines, with angle $1/(r-1)$ turns between each of them. Wee will call them *repelling axes* (respectively *attracting axes*) of 0 for f.

Remark. 1) If φ is a holomorphic map in a neighborhood of 0 with $\varphi(0) = 0$ and $\varphi'(0) \neq 0$, which conjugates f to g (i.e., $g = \varphi \circ f \circ \varphi^{-1}$), the differential $T_0\varphi : z \mapsto \varphi'(0) \cdot z$ sends the repelling (respectively attracting) axes of f to the ones of g.

2) We can conjugate f, via a holomorphic map tangent to the identity at the origin, to a map g of the form $z \mapsto z + bz^r + \mathcal{O}(z^{2r-1})$, or also of the form $z \mapsto z + bz^r + cz^{2r-1} + \mathcal{O}(z^\nu)$ with ν arbitrary. For the obstructions to conjugate $z \mapsto z + bz^r + cz^{2r-1}$, see Ecalle's course.

1.2. A change of variable. In order to study f, we would like to make the change of variable

$$z \mapsto \frac{1}{(r-1)bz^{r-1}}.$$

But this map is not injective in a neighborhood of 0, and this leads us to introduce some conventions.

Let Ω be an open subset of $\mathbb{C}$, $\widetilde{\Omega}$ be a covering of Ω and $\pi : \widetilde{\Omega} \to \Omega$ be the projection. For $z \in \widetilde{\Omega}$, weset $|Z| = |\pi(Z)|$. Let $Z \in \widetilde{\Omega}$ and $u \in \mathbb{C}$ be sufficiently close to 0 so that, for all $t \in [0,1]$, $\pi(Z) + tu \in \Omega$. The path $\gamma : t \mapsto \pi(Z) + tu$ in Ω has a unique lift $\tilde{\gamma}$ in $\widetilde{\Omega}$ starting at Z; we will then denote by $Z + u$ the point $\tilde{\gamma}(1) \in \widetilde{\Omega}$. For λ close to 1, we define λZ by $\lambda Z = Z + (\lambda - 1)\pi(Z)$.

Let $\mathbb{D}_\rho$ be a disk contained in the domain of definition of f. The map

$$z \mapsto \frac{1}{(r-1)bz^{r-1}}$$

defines an isomorphism $h : z \mapsto Z$ from $\mathbb{D}_\rho \setminus \{0\}$ to a covering $\widetilde{\Omega}$ of degree $r-1$ of $\Omega = \mathbb{C} \setminus \overline{\mathbb{D}}_R$, where $R = \dfrac{1}{(r-1)|b|\rho^{r-1}}$. Denote by $\pi : \widetilde{\Omega} \to \Omega$ the projection and by F the map $h \circ f \circ h^{-1}$, defined on an open subset $\widetilde{\Omega}'$ of $\widetilde{\Omega}$ containing $\pi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}_{R'})$ for R' sufficiently large.

Proposition 9.1. *The map F is of the form $Z \mapsto Z - 1 + \mathcal{O}(|Z|^{-1/r-1})$.*

Proof. Let $Z \in \widetilde{\Omega}'$, $z = h^{-1}(Z)$, $z_1 = f(z)$ and $Z_1 = h(z_1) = F(Z)$. We have:

$$z_1 = z + bz^r + \mathcal{O}(|z|^{r+1}) = z\left(1 + bz^{r-1} + \mathcal{O}(|z|^r)\right).$$

Thus

$$\begin{aligned}
\pi(Z_1) &= \frac{1}{(r-1)bz_1^{r-1}} \\
&= \frac{\pi(Z)}{(1 + bz^{r-1} + \mathcal{O}(|z|^r))^{r-1}} \\
&= \pi(Z)\left(1 - \frac{1}{\pi(Z)} + \mathcal{O}(|Z|^{-r/(r-1)})\right)
\end{aligned}$$

Since z and z_1 are close, Z and Z_1 are on the same leaf and we have:

$$Z_1 = Z\left(1 - \frac{1}{\pi(Z)} + \mathcal{O}(|Z|^{-r/(r-1)})\right) = Z - 1 + \mathcal{O}(|Z|^{-1/r-1}).$$

■

Remark. The $r-1$ repelling axes (respectively attracting axes) correspond to the lift in $\widetilde{\Omega}$ of $\mathbb{R}_+ \cap \Omega$ (respectively $\mathbb{R}_- \cap \Omega$).

1.3. Petals. Let R_1 and M be such that, for all $|Z| \geq R_1$, $F(Z)$ is defined and of the form $F(Z) = Z - 1 + \eta(Z)$, with

$$|\eta(Z)| \leq \frac{M}{|Z|^{1/(r-1)}} \leq \frac{1}{2}.$$

Let $\Gamma \subset \mathbb{C}$ be a curve of the form $\{x + iy \mid x = H(y)\}$, where $H : \mathbb{R} \to \mathbb{R}$ is a function with the following properties:

(i) H is converx and even;
(ii) $H(0) < -R_1$;
(iii) $y \mapsto |H(y) + iy|$ is incerasing on $\mathbb{R}_+$;
(iv) $|H'(y)| < 1/(ty^\theta)$ where $\sin\theta = 2M/|Z|^{1/(r-1)}$, $Z = H(y) + iy$;
(v) $H'(y) \to +\infty$ when $y \to \infty$.

Conditions (i) and (v) imply that Γ has a parabolic branch in the direciton of $\mathbb{R}_+$. Conditions (ii) and (iii) imply that $\Gamma \cap \overline{\mathbb{D}}_{R_1} = \emptyset$.

Let $\Gamma_1, \ldots, \Gamma_{r-1}$ be the lifts of Γ in $\widetilde{\Omega}$ and set $F_i(\Gamma) = \pi(F(\Gamma_i))$. Condition (iv) guaranties that each $F_i(\Gamma)$ is strictly to the left of $\tau_{-1/2}(\Gamma)$, where $\tau_{-1/2}$ is the translation $Z \mapsto Z - 1/2$.

Let G be the set of points $Z = x + iy \in \mathbb{C}$ such that $x \leq H(y)$ (region to the left of Γ). The preimage of G in $\widetilde{\Omega}$ is composed of $r-1$ copies $G_1, \ldots, G_{r-1}$ of G and we have $F(G_i) \subset \tau_{-1/2} \overset{\circ}{G_i}$.

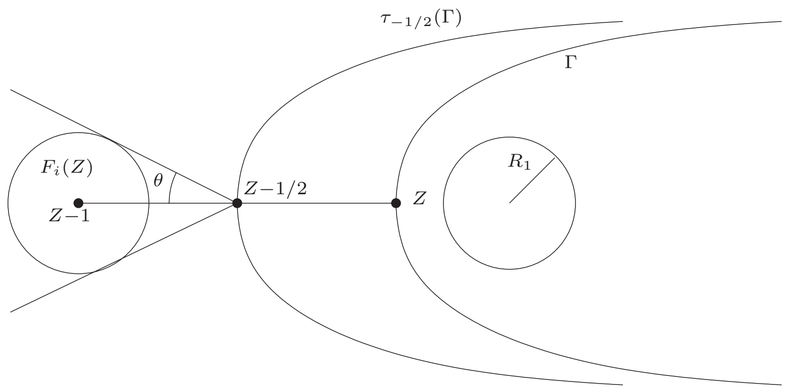


FIGURE 1. A possible curve Γ. The curve $F_i(\Gamma)$ is to the left of $\tau_{-1/2}(\Gamma)$.

For each i, we set $P_i = h^{-1}(G_i) \cup \{0\}$. The P_i are compact sets that we cal the *petals* of f at 0. They depend on the choice of Γ. We have $f(P_i) = \overset{\circ}{P_i} \cup \{0\}$. Each P_i is bounded by a curve $\gamma_i = h^{-1}(\Gamma_i)$, image of a path $[0,1] \to \mathbb{C}$, injective on $]0,1[$, which starts at 0 tangentially to a repelling acis, crosses an attracting axis (called the *axis* of the petal) and comes back at 0 tangentially to the next repelling axis.

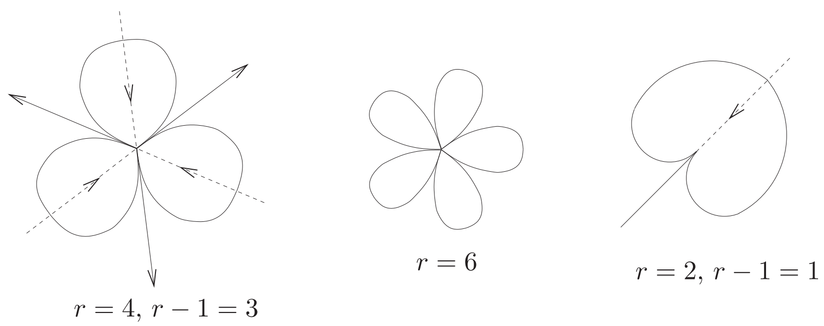


FIGURE 2. The petals P_i.

The flower $\bigcup P_i$ is contained in the disk $\mathbb{D}_{\rho_1} = \{0\} \cup h^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}_{R_1})$. For $\rho' < 1/|H(0)|^{r-1}$, the open set $\mathbb{D}_{\rho'} \setminus \bigcup P_i$ has $r-1$ connected components called the interpetals.

Proposition 9.2. *Let P_i be one of the petals of f at 0 and $z \in P_i \setminus \{0\}$.*

a) $f^{\circ n}(z) \to 0$ *tangentially to the axis of P_i.*

b) $f^{\circ n} \to 0$ *uniformly on P_i.*

Proof. a) Set $z_n = f^{\circ n}(z)$, $Z = h(z)$, $Z_n = h(z_n) = F^{\circ n}(Z)$, and let us write $\mathrm{Re}(Z)$ for $\mathrm{Re}(\pi(Z))$, and so on.... We have $\mathrm{Re}(Z_{n+1}) \leq \mathrm{Re}(Z_n - 1/2$, thus $\mathrm{Re}(Z_n) \to -\infty$, and so $|Z_n| \to \infty$ and $|z_n| \to 0$. We have $Z_{n+1} - Z_n \to -1$, and so $\arg Z_n \to 1/2$, and the angle of z_n with the axis of P_i tends to 0.

b) We have $F^{\circ n}(G_i) \subset \tau_{-n/2}(G_i)$, thus, for all $R'' > 0$, there exists an n_0 such that $\pi(F^{\circ n}(G_i)) \cap \mathbb{D}_{R''} = \emptyset$ for $n \geq n_0$. It follows that $\forall \rho'' > 0$, $(\exists n_0)$ $(\forall n \geq n_0)$ $f^{\circ n}(P_i) \subset \mathbb{D}_{\rho''}$. ∎

2. The case of polynomials.

We assume inthis section that f is a monic polynomial of degree d, having a fixed point of order $r \geq 2$ at 0. Let $P_1, \ldots, P_{r-1}$ be the petals of f at 0.

2.1. Component of $\overset{\circ}{K_f}$ containing a petal. The point 0 belongs to $J_f = \partial K_f$. Indeed, if $f(z) = z + bz^r + \mathcal{O}(|z|^{r+1})$, we have $f^{\circ n}(z) = z + nbz^r + \mathcal{O}(|z|^{r+1})$, and the sequence $((f^{\circ n})^{(r)}(0))_{n \in \mathbb{N}}$ is not bounded.

For each i, $P_i \setminus \{0\} \subset \overset{\circ}{K_f}$. Indeed, $P_i \subset K_f$ and $f(P_i \setminus \{0\} \subset \overset{\circ}{P_i} \subset \overset{\circ}{K_f}$. Since the set $P_i \setminus \{0\}$ is connected, it is contained in a component U_i of $\overset{\circ}{K_f}$.

Proposition 9.3. *For all $x \in U_i$, there exists an n such that $f^{\circ n}(x) \in P_i \setminus \{0\}$.*

Proof. Let $x_0 \in U_i$ and $y_0 \in P_i$. Set $x_n = f^{\circ n}(x)$ and $y_n = f^{\circ n}(y)$. We have, for the Poincar' distances, $d_{U_i}(x_n, y_n) \leq d_{U_i}(x_0, y_0)$, and the Euclidean distances $d(y_n, \partial U_i) \leq |y_n| \to 0$. It follows that $|x_n - y_n| \to 0$ and $|x_n| \to 0$.

If ρ' is sufficiently small, we have $|f(z)| > |z|$ if $z \in \mathbb{D}_{\rho'} \setminus \bigcup P_i$. hence, the x_n cannot all belong to the interpetals, and $\exists n_0$, $\exists j$, $x_{n_0} \in P_j \setminus \{0\}$. We then have $x_n \in \overset{\circ}{P_j} \setminus \{0\}$ for $n \geq n_0$.

Denote by V_j the set of $x \in U_i$ such that $(\exists n)$ $x_n \in \overset{\circ}{P_j} \setminus \{0\}$. The v_j form a partition of U_i in open sets. SinceU_i is connetced, only on is non empty, and since $P_i \setminus \{0\} \subset V_i$, we have $U_i = V_i$. ■

Corollary 9.1. *The U_i are pairwise distinct.*

Proposition 9.4. a) *U_i contains at least a critical point of f.*

b) *f induces a proper holomorphic map $f_i : U_i \to U_i$ of degree $d_i \geq 2$.*

c) *Let $\varphi : U_i \underset{\simeq}{\longrightarrow} \mathbb{D}$ be an isomorphism and set $g = \varphi \circ f_i \circ \varphi^{-1}$. Then, g is the restriction to $\mathbb{D}$ of a rational map $g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ having on S^1 a triple fixed point α. For all $x \in \overline{\mathbb{C}} \setminus S^1$, the sequence $g^{\circ n}(x)$ tends to α.*

Proof.

Lemma 9.1. *Let $x_0 \in U_i$ and set $x_n = f^{\circ n}(x_0)$. Then, $d_{U_i}(x_n, x_{n+1}) \to 0$.*

Proof. $x_n \to 0$ tangentially to the axis of P_i, so $\exists m > 0$, $\exists n_0$, $\forall n \geq n_0$, $d(x_n, \partial U_i) \geq m|x_n|$ (in fact, we can take m arbitrarily close to $\sin(\pi/(r-1))$ if $r \geq 3$, $m = 1$ if $r = 2$).

For $m' > |b|$, we have $|x_{n+1} - x_n| \leq m'|x_n|^r$ for m sufficiently large. As a consequence,

$$\frac{|x_{n+1} - x_n|}{d(x_n, \partial U_i)} \to 0.$$

But,

$$d_{U_i}(x_n, x_{n+1}) \leq d_{\mathbb{D}(x_n, d(x_n, \partial U_i))}(x_n, x_{n+1}) = d_{\mathbb{D}}\left(0, \frac{x_{n+1} - x_n}{d(x_n, \partial U_i)}\right),$$

and so $d_{U_i}(x_n, x_{n+1}) \to 0$. □

We can now prove the proposition. b) The map f induces a proper holomorphic map $f^{-1}(U_i) \to U_i$. But U_i is a connected component of $f^{-1}(U_i)$, so f induces $f_i : U_i \to U_i$ holomorphic and proper. Let d_i be its degree. We have $d_i > 1$

since otherwise f_i would be an isometry for the Poincaré distance on U_i, which contradicts the lemma.

a) follows from b) and the Riemann-Hurwitz formula. More elmetary, if there were no critical point, $f_i : U_i \to U_i$ would be a covering, necessarily trivial since U_i is simply connected, thus of degree 1.

c) The map $g : \mathbb{D} \to \mathbb{D}$ is holomorphic and proper. By Schwarz's reflection principle, it extends to a map $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$, which is a rational map cmmuting with $z \mapsto 1/\bar{z}$. Let us show that there is a triple fixed point on S^1. Let $x_0 \in \mathbb{D}$ and set $x_n = g^{\circ n}(x_0)$. Avery accumulation point α of the sequence (x_n) belongs to S^1 and is a fixed point of g. Indeed, if $x_{n_k} \to \alpha$, we have $|x_{n_k+1} - x_{n_k}| \leq d_{\mathbb{D}}(x_{n_k}, x_{n_k+1}) \to 0$, and $g(x_{n_k}) \to \alpha$, and so $\alpha = g(\alpha)$. Let $\alpha_1, \ldots, \alpha_\nu$ be the fixed points of g on S^1, $W_1, \ldots, W_\nu$ be neighborhoods of respectively $\alpha_1, \ldots, \alpha_\nu$ such taht $d_{\mathbb{D}}(W_i \cap \mathbb{D}, W_j \cap \mathbb{D}) > d_{\mathbb{D}}(x_0, x_1)$. For n sufficiently large, all the x_n are in the union of the W_i, but since $d_{\mathbb{D}}(x_n, x_{n+1}) \leq d_{\mathbb{D}}(x_0, x_1)$, they are all in the same W_i, and the sequence has only one accumulation point $\alpha \in S^1$. Since $\overline{\mathbb{D}}$ is compact, $x_n \to \alpha$.

Since $d_{\mathbb{D}}(x_n, x_{n+1}) \to 0$, we have $(x_{n+1} - x_n)/(x_n - \alpha) \to 0$ and α is a multiple fixed point. Let $y_0 \in \mathbb{D}$ and set $y_n = g^{\circ n}(y_0)$. We have $d_{\mathbb{D}}(x_n, y_n) < d_{\mathbb{D}}(x_0, y_0)$; since $|x_n - \alpha| \to 0$, we have $|y_n - x_n| \to 0$ and $y_n \to \alpha$.

Let s be the order of α as a fixed point of g; let us show that $s = 3$. Let $Q_1, \ldots, Q_{s-1}$ be the petals of g at α. At least half of them meet $\mathbb{D}$. But the $V_i = \{x \in \mathbb{D} \mid g^{\circ n}(x) \in Q_i \setminus \{\alpha\}\}$ form a partition of $\mathbb{D}$ in open sets, so there is at most one which is not empty and $s - 1 \leq 2$, and so $s = 2$ or 3.

Set $z_n = \varphi^{-1}(x_n) \in U_i$. The sequence $z_n \to 0$ tangentially to the axis of P_i. If $\eta : \overline{U}_i \to [0, 1]$ is a harmonic function on U_i, continuous on $\overline{U}_i \setminus \{0\}$, which is equal 0 on ∂U_i in an interpetal adjacent to P_i and 1 is the other, $(\exists m > 0)$, $(\forall n)$, $m \leq \eta(z_n) \leq 1 - m$. It follows that $x_n \to \alpha$ non tangentially to S^1, which excludes the case $s = 2$. We therefore have $s = 3$. ■

Corollary 9.2. *If $d_i = 2$, there is only one critical point $\omega \in U_i$. If we have chosen φ so that $\varphi(\omega) = 0$ and $\alpha = 1$, then g is given by*

$$g(z) = \frac{3z^2 + 1}{3 + z^2}.$$

2.2. External arguments of 0.

Proposition 9.5. *Assume K_f is connected and locally connected.*

a) *Every external argument of 0 is of the form $p/(d-1)$.*
b) *In each interpetal, arrives at least one ray landing at 0.*
c) *In between two external rays landing at 0, there is a critical point and a critical value of f.*

Fig. 3 shows the filled-in Julia set of a cubic polynomial. There is a multiple fixed point with 1 petal. There are two rays (of angle 0 and 1/2) landing at 0. The critical points are plotted.

Proof.

Lemma 9.2. *Let $A \subset \mathbb{R}/\mathbb{Z}$ be a set containing at least one point α of the form $p/(d-1)$, $p \in \mathbb{Z}$. Assume $t \mapsto dt$ induces a bijection from A to A, preserving the cyclic order. Then, every point of A is of the form $p/(d-1)$, $p \in \mathbb{Z}$.*

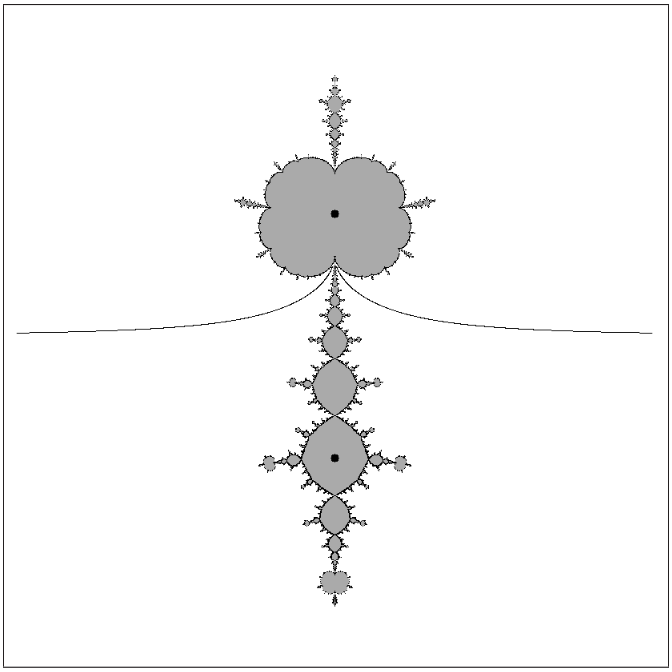


FIGURE 3. The filled-in Julia set of a cubic polynomial having a multiple fixed point.

Proof. We have $(d-1)\alpha = 0$, so $d\alpha = \alpha$, and $t \mapsto t - \alpha$ commutes with $t \mapsto dt$. We can therefore choose α as origin, i.e., assume that $\alpha = 0$. Let $t \in A$ and let $(\varepsilon_1, \ldots, \varepsilon_k, \ldots)$ be the development of t in base d. If the sequence (ε_i) is stationnary, $\varepsilon_i = \varepsilon$ for $i \geq k_0$, we have $d^k t = \varepsilon/(d-1) \in A$, $d^k \varepsilon/(d-1) = \varepsilon/(d-1) = d^k t$, and so $t = \varepsilon/(d-1)$. Otherwise, $\exists i, j$ such that $\varepsilon_i < \varepsilon_{i+1}$ and $\varepsilon_j > \varepsilon_{j+1}$. Fr each element of A, let us choose a representant in $[0,1[$. We have $0 < d^i t < d^{i+1} t$ and $0 < d^{j+1} t < d^j t$, which contradicts the fact that $t \mapsto d^{j-i} t$ preserves the cyclic order. □

We can now prove the proposition. a) Denote by A the set of ecternal arguments of 0. Assume $A \neq \emptyset$ and et $t \in A$ and α and α be the smallest element of A corresponding to a ray $\mathcal{R}(K_f, \alpha)$ which lands in through the same interpetal as $\mathcal{R}(K_f, t)$. The ray $\mathcal{R}(K_f, \alpha)$ is fixed by f, so $\alpha = d\alpha$ and α is of the form $p/(d-1)$. The map f induces a permutation of the rays $(\mathcal{R}(K_f, t))_{t \in A}$ preserving the cyclic order. Part a) then follows form the lemma.

b) Let us choose in each interpetal P_i a center c_i for U_i and let H be the allowable hull of $\{0, c_1, \ldots, c_{r-1}\}$. The interpetals are the access to 0 relatively to H (cf chapter 7). Part b) then follows from proposition 7.1 in the same chapter.

c) Let $R = \overline{\mathcal{R}}(K_f, \theta)$ and $R' = \overline{\mathcal{R}}(K_f, \theta')$ be two external rays of K_f landing at 0, and V a connected component of $\mathbb{C} \setminus (R \cup R')$. According to part a), we may ssume that $\theta = 0$, $\theta' = p/(d-1)$, $p \in \{1, \ldots, d-2\}$, and that V contains the points of $\mathbb{C} \setminus K_f$ of external arguments $t \in]0, p/(d-1)[$. Set $W = f^{-1}(V)$. The boundary ∂W is the union of the rays $\mathcal{R}(K_f, t)$ for $t \in \{i/d, p/(d-1) + i/d\}_{i=0,\ldots,d-1}$. Let W_1 be the connected component of W such that $\partial W_1 \supset R$. We have $W_1 \subset V$

and $\partial W \supset R \cup \mathcal{R}(K_f, t_1)$ where $t_1 = p/(d-1) - p/d = p/(d(d-1))$. The ray $R_1 = \mathcal{R}(K_f, t_1)$ lands at a point $c_1 \neq 0$ since $t_1 \notin \mathbb{Z}(d-1)$. The map f induces a proper holomorphic map $f_1 : W_1 \to V$, denote by d_1 its degree. Let U be a neighborhood of 0, then $f^{-1}(U \cap W) \supset (U' \cap W) \cup (U'' \cap W)$, where U' is a neighborhood of 0 and U'' is a neighborhood of c_1, that we may assume disjoint. It follows that $d_1 \geq 2$, so $(\exists \omega \in W_1)$, $f'(\omega) = 0$ and $f(\omega) \in W$. ■

3. Irrationally indifferent periodic points.

3.1. Number of petals. Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree d, α be a periodic point of period k such hat $\lambda = (f^{\circ k})'(\alpha) = e^{2i\pi p/q}$, with p and q coprime.

Proposition 9.6. *The multiplicity r of α as a fixed point of $f^{\circ qk}$ is of the form $\nu q + 1$, where $\nu \in \{1, \ldots, d-1\}$.*

Proof. The map $T_\alpha f^{\circ k}$, which is the multiplication by λ, acts freely on the repelling axes of α. Hence, their number, which is $r-1$, is of the form $q\nu$. Thee are ν disjoint orbits in $\pi_0(\overset{\circ}{K_f})$, and each one contains at least a critical point of $f^{\circ kq}$, and so a critical point of f. Since f has at most $d-1$ critical points, we have $\nu \leq d-1$. ■

We could also deduce proposition 9.6 from the following lemma, that we give because we will use it later.

Lemma 9.3. *We can find a holomorphic coordinate centered at α such that the expressions of $f^{\circ k}$ in this coordinate is of the form $z \mapsto \lambda(z + z^r + \mathcal{O}(z^{r+1}))$, with $r = \nu q + 1$, $\nu \in \mathbb{N}^*$.*

In other words, we can find a $\mathbb{C}$-analytic diffeomorphism ψ between a neighborhood of α and a neighborhood of 0, with $\psi(\alpha) = 0$, such that $\varphi \circ f^{\circ k} \circ \psi^{-1}$ is of the prescribed form.

Proof. We will show that, if we have a coordinate ζ_j where the expression g_j of $f^{\circ k}$ is $\zeta \mapsto \lambda\zeta + b_j \zeta^j + \mathcal{O}(\zeta^{j+1})$, and if j is not of the form $\nu q + 1$, then we can find a coordinate ζ_{j+1}, tnagent to ζ_j at the order j, such that the expression of g_{j+1} of $f^{\circ k}$ in ζ_{j+1} is $\zeta \mapsto \lambda\zeta + b_{j+1}\zeta^{j+1} + \mathcal{O}(\zeta^{j+2})$. Let us take $\zeta_{j+1} = \zeta_j + c\zeta_j^j$. The map g_{j+1} is given by :

$$\begin{aligned}
\zeta_{j+1} \mapsto \zeta_j \quad &= \quad \zeta_{j+1} - c\zeta_{j+1}^j + \ldots + \\
&\overset{g_j}{\mapsto} \quad \lambda\zeta_{j+1} + (b_j - \lambda c)\zeta_{j+1}^j + \ldots \\
&\mapsto \quad g_{j+1}(\zeta_{j+1}) = \zeta_{j+1} + (b_j - \lambda c + c\lambda^j)\zeta_{j+1}^j + \mathcal{O}(\zeta_{j+1}^{j+1}).
\end{aligned}$$

If j is not of the form $\nu q + 1$, we have $\lambda^j - \lambda \neq 0$, and we can choose c so that $b_j + c(\lambda^j - \lambda) = 0$.

In this way, we can kill each term of $g_j(\zeta) - \lambda\zeta$ until we end up with a term of the form $b\zeta^{\nu q+1}$ with $b \neq 0$. If the process could be always continued, $f^{\circ kq}$ would have a contact of order ∞ with the identity, so $f^{\circ kq} = \mathrm{Id}$, which is not possible if f is a polynomial of degree $d > 1$. Thus, we can find a coordinate ζ such that the expression of $f^{\circ k}$ is $\zeta \mapsto \lambda(\zeta + b\zeta^{\nu q+1} + \mathcal{O}(\zeta^{\nu q+1})$ with $b \neq 0$, and conjugating with a scaling map, we can get $b = \lambda$. ■

Let ψ be a diffeomorphism of a neighborhood of α to a neighborhood of 0. A *flower* F of $f^{\circ kq}$ relatively to ψ is $F = \psi^{-1}(F')$, where F' is a flower (union of petals) of $\psi \circ f^{\circ kq} \circ \psi^{-1}$.

Proposition 9.7. *Let ζ be a coordinate centered at α satisfying the conditions of lemma 9.3 and F the flower of $f^{\circ kq}$ relatively to ζ defined via a curve Γ as in subsection 1.3. For a well chosen curve Γ, we have $f^{\circ k}(F) \subset \overset{\circ}{F} \cup \{\alpha\}$.*

Proof. Let us consider again the change of variable $h : \mathbb{D}_\rho \underset{\simeq}{\longrightarrow} \widetilde{\Omega}$ defined in subsection 1.2, with $b = 1$. Denote by σ the deck transformation $\widetilde{\Omega} \to \widetilde{\Omega}$ conjugate by h to $z \mapsto \lambda z$. The expression $F = h \circ f^{\circ k} \circ h^{-1}$ is of the form

$$Z \mapsto \sigma\left(Z - 1 + \mathcal{O}\left(\frac{1}{|Z|^{1/r-1}}\right)\right).$$

If Γ satifies to the condition (i) to (v), we have $F(G_i) \subset \tau_{-1/2}\ \overset{\circ}{G}_{\sigma(i)}$, and so $f(P_i \setminus \{\alpha\}) \subset \overset{\circ}{P}_{\sigma(i)}$. ■

Remark. Condition (iv) is stronger than the condition required to define the flower of $f^{\circ kq}$, because $f^{\circ kq}$ corresponds to $b = q$ and not $b = 1$.

CHAPTER 10

Local connectivity of some other Julia sets.

1. Results.

Theorem 10.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of dergee $d > 1$. Assume that for each critical point ω of f, one of the following three possibilities occur:*

a) *ω is attracted by an attracting cycle;*
b) *ω ends in finite time on a repelling cycle;*
c) *ω is attracted by a rationally indifferent cycle.*

Then K_f is connected and locally connected.

Corollary 10.1. *Let f be a polynomial of degree 2 having a rationally indifferent cycle. Then, K_f is connected and locally connected.*

This chapter is a complement to chapter 3 "Local connectivity of some Julia sets". We will use results of chpater 9 "Multiple fixed points and rationally indifferent periodic points".

By a theorem of Fatou and/or Julia, since every critical point of f belongs to K_f, the set K_f is connected (chapter 3, proposition 3.1 extended). The case where only possibilities (a) or (b) occur is covered by proposition 3.4 in chapter 3 + the characterization of sub-hyperbolic polynomials. When (c) actually occurs, f is not sub-hyperbolic. In order to show that K_f is locally connected, we will show that the sequence of loops (γ_n) defined in proposition 3.3 in chapter 3 converges uniformly. For this purpose, we will construct a metric on an open subset Ω of $\mathbb{C}$ for which f is strictly (i.e. strictly increases the length of every non trivial rectifiable curve) but not strongly expanding in general. The argument we will have to use in order to conclude the proof will therefore be more delicate than the one of proposition 3.4 in chapter 3: we will have to do something like a Markov partition.

In the following, f is a polynomial satisfying the hypothesis of the theorem.

2. Construction of Ω.

Denote by A_- the set of attracting periodic points of f, A_0 the set of rationally indifferent periodic points and C the union of forward orbits of the critical points. Let us write $C = C_- \cup C_0 \cup C_+$, where C_- (respectively C_0) corresponds to the critical points attracted by an attracting cycle (respectively rationally indifferent) and C_+ to the critical points ending on a repelling cycle.

Proposition 10.1. *We can find a compact set Ω such that*

a) *$\partial\Omega \supset A_0$, $\Omega \cap A_- = \emptyset$, $C_+ \subset \mathring{\Omega}$, $(C_0 \cup C_-) \cap \Omega = \emptyset$;*
b) *$J_f \subset \mathring{\Omega} \cup A_0$, $\gamma_n \subset \mathring{\Omega}$ for n large enough;*
c) *$f^{-1}(\Omega) \subset \mathring{\Omega} \cup A_0$;*

d) $\overset{\circ}{\Omega}$ *is connected;*

e) $\overset{\circ}{\Omega} \cap \mathcal{R}(K_f, 0)$ *is connected.*

Proof. Let L be a closed topological disk containing K_f, bounded by a level curve of the potential G_f of K_f.

For each $\alpha \in A_-$, let us choose a topological disk Δ_α so that $f(\overline{\Delta}_\alpha) \subset \Delta_{f(\alpha)}$. This implies that $\overline{\Delta}_\alpha \subset \overset{\circ}{K}_f$. Let n_- be such that

$$B_- = f^{-n_-}\left(\bigcup_{\alpha \in A_-} \Delta_\alpha\right) \supset C_-.$$

We still have $\overline{B}_- \subset \overset{\circ}{K}_f$.

For each $\alpha \in A_0$, let us construct a flower F_α (relatively to a coordinate ζ_α centered at α) so that $f(F_a \setminus \{\alpha\}) \subset \overset{\circ}{F}_{f(\alpha)}$. Let n_0 be such that

$$B_0 = f^{-n_0}\left(\bigcup_{\alpha \in A_0} \overset{\circ}{F}_\alpha\right) \supset C_0.$$

This implies that $B_0 \subset \overset{\circ}{K}_f$ and $\overline{B}_0 \subset \overset{\circ}{K}_f \cup A_0$. The set $\Omega = L \setminus (B_- \cup B_0)$ has the required properties. ■

3. Construction of $\widetilde{U}$.

Set $U = \overset{\circ}{\Omega}$. We will construct a ramified covering $\widetilde{U}$ of U. Since the periodic points of C_+ are repelling, and so non critical, we can find a function $\nu : C_+ \to \mathbb{N}^*$ such that $\nu(f(x))$ is a multiple of $r(x)\nu(x)$, where $r(x)$ is the local degree of f at x (for example, $\nu(x) = \prod r(y)$, where the product is taken for y in the strict backward orbit of x).

We set $r(x) = 1$ for $x \notin C_+$. Let U^* be a ramified covering of U, with ramification degree equal to $\nu(x)$ for every point above x, and $\widetilde{U}$ the universal covering of U^*. Then, $\widetilde{U}$ is a Galois ramified covering of U. Denote by π the projection $\widetilde{U} \to U$. Let $\widetilde{\mathcal{R}}_0$ be a lift of the open arc $U \cap \mathcal{R}(K_f, 0)$ in $\widetilde{U}$.

Proposition 10.2. *There exists a holomorphic map $g : \widetilde{U} \to \widetilde{U}$ such that $f \circ \pi \circ g = \pi$ and $g(\widetilde{\mathcal{R}}_0) \subset \widetilde{\mathcal{R}}_0$.*

Remark. The condition $f \circ \pi \circ g = \pi$ says in some way that g is a lift of f^{-1}.

Proof. Let X be the set of pairs $(x, y) \in \widetilde{U} \times \widetilde{U}$ such that $f(\pi(y)) = \pi(x)$. The set X is a $\mathbb{C}$-analytic curve with singularities. Let $(x_0, y_0) \in X$. If $\pi(x_0)$ does not belong to C, $\pi(y_0)$ neither, X is smooth at (x_0, y_0) and $\text{pr}_1 : X \to \widetilde{U}$ is an local isomorphism in a neighborhood of (x_0, y_0). Assume $\pi(x_0) \in C$, denote by r the ramification degree of f at $\pi(y_0)$; set $\nu_{x_0} = \nu(\pi(x_0))$ and $\nu_{y_0} = \nu(\pi(y_0))$, $\deg_{y_0}(f \circ \pi) = r\nu_{y_0}$, which by hypothesis divides $\nu_{x_0} = \deg_{x_0}(\pi)$. We can choose on $\widetilde{U}$ coordinates ξ and η centered respectively at x_0 and y_0, so that the expression of π and $f \circ \pi$ are $\xi \mapsto \xi^{\nu_{x_0}}$ and $\eta \mapsto \eta^{r\nu_{y_0}}$. In a neighborhood of (x_0, y_0), the set X becomes

$$\{(\xi, \eta) | \xi^{\nu_{x_0}} = \eta^{r\nu_{y_0}}\} = \bigcup \{(\xi, \eta) | \eta = \lambda \xi^q\},$$

where $q = \nu_{x_0}/r\nu_{y_0}$ and the union is taken on all the λ such that $\lambda^{r\nu_{y_0}} = 1$. in a neighborhood of (x_0, y_0), the curve X is a union of $r\nu y_0$ smooth curves (branches), which intersect transversally, and for each one, the projection $\mathrm{pr}_1 : X \to \widetilde{U}$ induces a local isomorphism.

Replacing the point (x_0, y_0) by $r\nu_{y_0}$ points, one on each branch, and proceeding in the same way for the other points (x, y) such that $\pi(x) \in C$, we obtain a space $\widetilde{X}$ which is a covering of $\widetilde{U}$. But $\widetilde{U}$ is simply connected, this covering is therefore trivial. Let x_1 and y_1 be points in $\mathcal{R}(K_f, 0) \cap U$ such that $x_1 = f(y_1)$; denote by $\tilde{x}_1$ and $\tilde{y}_1$ their lift in $\widetilde{\mathcal{R}}_0$. There eixsts a unique section $\sigma : \widetilde{U} \to \widetilde{X}$ such that $\sigma(\tilde{x}_1) = (\tilde{x}_1, \tilde{y}_1)$. The map $g = \mathrm{pr}_2 \circ \pi_{\widetilde{X}} \circ \sigma : \widetilde{U} \to \widetilde{U}$, where $\pi_{\widetilde{X}} : \widetilde{X} \to X$ is the canonical projection, has the required properties. ■

4. Construction of a metric.

Denote by $\mu_{\widetilde{U}}$ the Poincaré metric on $\widetilde{U}$ and μ_U the admissible Riemannian metric on U such that the projection $\pi : \widetilde{U} \to U$ is a local isometry.

Remark. 1) If $A_0 = \emptyset$, $\pi(g(\widetilde{U}))$ is relatively compact in U, and it follows that $g : \widetilde{U} \to \widetilde{U}$ is strongly contracting for $\mu_{\widetilde{U}}$. As a consequence, f is strongly expanding for μ_U on J_f, in other words, f is sub-hyperbolic. The theorem, in that case, follows from proposition 3.4 in chapter 3.

2) If $A_0 \neq \emptyset$, the map $g : \widetilde{U} \to \widetilde{U}$ is strictly but not strongly contracting. Let $\alpha \in A_0$ and θ be an external argument of α in K_f. The open arc $U \cap \mathcal{R}(K_f, \theta)$ has an infiinte length on the side of α for μ_U, and the sequence $(\gamma_n(\theta))_{n\in\mathbb{N}}$ is not a Cauchy sequence for μ_U. It is for those reasons that we will change the metric μ_U.

For each point $\alpha \in A_0$, we can find a topological disk Δ_α and an isomorphism $\zeta_\alpha : \Delta_\alpha \to \mathbb{D}_r$ so that the expression of $f : \Delta_\alpha \to \Delta_{f(\alpha)}$ is of the form $\zeta \mapsto \lambda(\zeta + b_\alpha \zeta^{q_\alpha+1} + \ldots)$, q_α =number of petals of the flower at α, $\lambda^{q_\alpha} = 1$.

If the disk Δ_α are chjosen sufficiently small, $U \cap \Delta_\alpha$ is contained in the union of the interpetals, and the expression of f has a derivative of modulus > 1. Also, we have: $f(U \cap \Delta_\alpha) \supset U \cap \Delta_{f(\alpha)}$, $f(\overline{\Delta}_\alpha \cap \overline{\Delta}_\beta = \emptyset$ for $\beta \neq f(\alpha)$ and $\overline{\Delta}_\alpha \cap C_+ = \emptyset$.

Denote by μ_α the metric $|d\zeta_\alpha|$ on Δ_α. Choose $M \in \mathbb{R}_+$ large and define on $U \cup \bigcup_{\alpha\in A_0} \delta_\alpha$ a Riemannian metric μ (with discontinuous coefficients) by $\mu = \inf(\mu_U, M\mu_\alpha)$, the infimum being taken at each point z on the metrics defined at this point.

Consider the compact set $\Omega' = f^{-1}(\Omega) \subset U \cup A_0$.

Proposition 10.3. *If we have chosen M sufficiently large, f is strictly expanding for μ on Ω'.*

By "strictly expanding" we mean that for every non trivial rectifiable path $\gamma : I \to \Omega'$, we have $\mathrm{length}_\mu(f \circ \gamma) > \mathrm{length}_\mu(\gamma)$.

Proof. For each $\alpha \in A_0$, $f^{-1}(\Omega \cap \overline{\Delta}_\alpha)$ is a compact set of the form $L'_\alpha \cup L''_\alpha$, where $L'_\alpha \subset \Omega' \cap \Delta_\alpha$ and L''_α is a compact set contained in $U \setminus C_+ = U \setminus C$. Set

$$m_\alpha = \inf_{z\in L''_\alpha} \|T_z f\|_{\mu_U, \mu_\alpha}.$$

We have $m_\alpha > 0$. We choose $M > 1/\inf(m_\alpha)$. Let us show that, for $z \in \Omega' \setminus (A_0 \cup C)$, we have:

$$\|T_z f\|_\mu > 1 \tag{1}$$

We must consider 4 cases:

a) $\mu_z = \mu_{U,z}$, $\mu_{f(z)} = \mu_{U,f(z)}$: the map $f : f^{-1}(U) \to U$ is strictly expanding for μ_U since $g : \widetilde{U} \to \widetilde{U}$ is strictly contracting, and we have (1) in that case.

b) $\mu_z = M\mu_{\alpha,z}$, $\mu_{f(z)} = \mu_{U,f(z)}$: we have $\mu_z \leq \mu_{U,z}$ and so (1) follows as in case (a).

c) $\mu_z = M\mu_{\alpha,z}$, $\mu_{f(z)} = M\mu_{f(\alpha),f(z)}$: inequality (1) follows from the fact that the expression of f in the coordinates $\zeta_\alpha, \zeta_{f(\alpha)}$ has derivative with modulus greater than 1.

d) $\mu_z = \mu_{U,z}$, $\mu_{f(z)} = M\mu_{f(\alpha),f(z)}$: is $z \in L'_\alpha$, we have $\mu_z \leq \mu_{\alpha,z}$ and we conclude as in case (c); if $z \in L''_\alpha$, inequality (1) follows from the condition $M > 1/m_\alpha$ imposed on M.

■

5. A module of continuity.

Set $\Omega'^* = \Omega' \cap U = \Omega' \setminus A_0$, and let $\widetilde{\Omega'}^*$ be the preimage of Ω'^* in $\widetilde{U}$. Let $\tilde{\mu}$ be the metric on $\widetilde{\Omega'}^*$ lifting the metric μ on Ω'^*. denote by $\widetilde{\Omega'}$ the compactification of $\widetilde{\Omega'}^*$ for $\tilde{\mu}$: it is obtained by adding a point at the end of each lift in $\widetilde{\Omega'}^*$ of an interpetal at a point in A_0.

The map $g : \widetilde{U} \to \widetilde{U}$ of proposition 10.2 induces a map $\widetilde{\Omega'}^* \to \widetilde{\Omega'}^*$ which extends to a continuous and strictly contracting map $\hat{g} : \widetilde{\Omega'} \to \widetilde{\Omega'}$.

Proposition 10.4. *There exists an increasing map $h : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $h(s) < s$ for $s > 0$ and $s - h(s) \to +\infty$, such that:*

$$(\forall x, y \in \widetilde{\Omega'}) \quad d_{\tilde{\mu}}(g(x), g(y)) \leq h(d(x, y)).$$

Proof. The group $\Gamma = \mathrm{Aut}_U(\widetilde{U})$ acts on $\widetilde{\Omega'}$ by isometries for $\tilde{\mu}$. Let $e = \gamma_0(0) \in \mathcal{R}(K_f, 0) \cap U$ (cf. chapter 3, proposition 3.3); denote by $\tilde{e}$ the lift of e in $\widetilde{\mathcal{R}}_0$, $\tilde{\gamma}_0$ the lift of γ_0 with origin $\tilde{e}$, and set $\tilde{e}_1 = \tilde{\gamma}_0(1)$. Denote by σ the element of Γ such that $\sigma(\tilde{e}) = \tilde{e}_1$. There exists an element $\sigma_1 \in \Gamma$ such that $g \circ \sigma^d = \sigma_1 \circ g$. Let $F \subset \widetilde{\Omega'}$ bea compactset such that $\Gamma \cdot F = \widetilde{\Omega'}$ and set $F_1 = \bigcup_{0 \leq i \leq d-1} \sigma^i F$. For $s \geq 0$, denote by $B(F_1, s) = \{x \in \widetilde{\Omega'} \mid d_{\tilde{\mu}}(x, F_1) \leq s\}$, it is a compact set. Denote by $H(s)$ the supremum of $d_{\tilde{\mu}}(g(x), g(y))$ for $(x, y) \in \widetilde{\Omega}$, $d_{\tilde{\mu}}(x, y) \leq s$. We have:

$$h(s) = \sup_{\substack{(x,y) \in F_1 \times B(F_1, s) \\ d_{\tilde{\mu}}(x,y) \leq s}} d_{\tilde{\mu}}(g(x), g(y)) < s.$$

It is clear that the funciton h is increasing. et us choose $s_0 > 0$; We have $h(ks_0) \leq kh(s_0)$ and it follows that $s - h(s) \to +\infty$ as $s \to +\infty$. ■

Denote by E the set of loops $\gamma : \mathbb{T} \to \Omega' \setminus (C_+ \cup A_0)$ such that $\gamma(0) = \mathcal{R}(K_f, 0)$, homotopic to γ_0. We define $G : E \to E$ by mapping each γ to the unique loop $\gamma' \in E$ such that $\gamma'(t) \in f^{-1}(\gamma(dt))$, $\gamma'(0) \in \mathcal{R}(K_f, 0)$. In particular, $\gamma_{n+1} = G(\gamma_n)$.

By mapping each loop $\gamma \in E$ to the path $\tilde{\gamma} : [0,1] \to \widetilde{\Omega'}$ which lifts γ, we identify E to a subset $\widetilde{E}$ of $\mathcal{C}(I; \widetilde{\Omega'}$, where $I = [0,1]$. To the map G corresponds a map $\widetilde{G} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{G}(\tilde{\gamma}) = \tilde{\gamma}'$, with $\gamma^{-1}(t) = g(\sigma^i \gamma(s))$ if $t = (i+s)/d$, $s \in [0,1]$. We equip $\widetilde{E}$ with the distance $d(\gamma, \eta) = \sup_{t \in I} d_{\tilde{\mu}}(\gamma(t), \eta(t))$.

Corollary 10.2. *The map $\widetilde{G}$ has h as module of continuity.*

6. The convergence.

Proposition 10.5. *The sequence $(\tilde{\gamma}_n)$ is a Cauchy sequence in $\mathcal{C}(I, \widetilde{\Omega'})$.*

Proof. Set $\ell = d(\tilde{\gamma}_0, \tilde{\gamma}_1)$ and let $L \geq \ell$ be such that $L - h - L) \geq \ell$. Let us show by induction that for all n, we have $d(\tilde{\gamma}_0, \tilde{\gamma}_n) \leq L$. It is clear for $n = 0$ or 1. If $d(\tilde{\gamma}_0, \tilde{\gamma}_n) \leq L$, we have $d(\tilde{\gamma}_1, \tilde{\gamma}_{n+1}) \leq h(L)$, and so $d(\tilde{\gamma}_0, \tilde{\gamma}_{n+1}) \leq \ell + h(L) \leq L$.

For $p \in \mathbb{N}$ and $q = p + n$, we have

$$d(\tilde{\gamma}_p, \tilde{\gamma}_q) = d(G^{\circ p}(\tilde{\gamma}_0), G^{\circ p}(\tilde{\gamma}_n)) \leq h^{\circ p}(L).$$

The sequence $(h^{\circ p}(L))_{p \in \mathbb{N}}$ is strictly decreasing. It has a limit which is a fixed point of h, thus 0. It follows that $(\tilde{\gamma}_n)$ is a Cauchy sequence. ∎

Proof of the theorem. The sequence (γ_n) is a Cauchy sequence in $\mathcal{C}(\mathbb{T}; \Omega')$, equipped with the distance of uifrm convergence for the distance d_μ on Ω'. Hence, it converges uniformly for d_μ, and also for the Euclidean distance d_0 which defines the same topology, since Ω' is compact. The theorem then follows from proposition 3.3 of chapter 3. ∎

CHAPTER 11

A walz.

By Adrien Douady and Pierrette Sentenac

1. Introduction.

Let $c_0 \in M$ be such that the polynomial $f_{c_0} : z \mapsto z^2 + c_0$ ha a rationally indifferent cycle $\{\alpha_1, \ldots, \alpha_k\}$, with eigenvalue $\rho = e^{2i\pi p/q}$ with p and q coprime. According to the preceding chapter, K_{c_0} is locally connected, and according to chapter 9, each α_i has a flower with q petals, and through each interpetal, there are at least one and finitely many external rays of K_{c_0} landing; those rays are fixed by $f^{\circ kq}$ and so have arguments of the form $p/(2^{kq} - 1)$ (we will see later that the number of rays is 2 if $q = 1$ and 1 otherwise).

The point c_0, critical value, is attracted under iteration of $f^{\circ kq}$ by one of the points of the cycle $\{\alpha_1, \ldots, \alpha_k\}$. We may assume that it is α_1. The connected component U_1 of $\overset{\circ}{K}_f$ which contains c_0 contains a petal P_1 for α_1.

Our goal is to prove the following theorem, which will be proved in the next chapter.

Theorem 11.1. *Let θ be the argument of an external ray of K_{c_0} which lands to α_1 in an interpetal adjacent to P_1. Then, the external ray $\mathcal{R}(M, \theta)$ lands at c_0.*

Remark. This theorem is analog to theorem 8.2 in chapter 8. However, this theorem is obtained by passing to the limit on corollary 8.4 of theorem 8.1 in chapter 8. On the contrary, theorem 11.1 above is based on a discontinuous behavior of $\mathcal{R}(K_c, \theta)$ at $c = c_0$.

In this chapter, we will prove theorem 11.2, one particular case of which (corollary 11.1) is a consequence of theorem 11.1. In the next chapter, we will use corollary 11.1 in order to prove theorem 11.1.

Assume $q \neq 1$. Then, α_1 is a simple periodic point of f_{c_0}. For each c close enough to c_0, we can find a $\alpha(c)$ such that $f_c^k(\alpha(c)) = \alpha(c)$, with $c \mapsto \alpha(c)$ analytic and $\alpha(c_0) = \alpha_1$.

Let Δ be a sufficiently small disk centered at $\alpha(c_0)$ (we will be more precise later). Let n_0 be sufficiently large and $r^* > 1$ be sufficiently close to 1 so that $x(c_0) = f_{c_0}^{n_0 kq}(c_0)$ and $y(c_0) = \varphi_{c_0}^{-1}(r^* e^{2i\pi\theta})$ belongs to Δ (where $\varphi_{c_0} : \mathbb{C} \setminus K_{c_0} \xrightarrow{\simeq} \mathbb{C} \setminus \overline{\mathbb{D}}$ is the conformal representation).

For c sufficiently close to c_0, set $x(c) = f_c^{n_0 kq}(c)$ and $y(c) = \varphi_c^{-1}(r^* e^{2i\pi\theta})$. The intermediate statement is the following.

Corollary 11.1. *For any neighborhood W of c_0 there exists an interger $N_0 \geq 0$ such that for all $N \geq N_0$, there exists a $c \in W$ such that $f^{Nkq}(x(c)) = y(c)$.*

Proof of the implication: theorem 11.1 ⇒ corollary 11.1. Let r_N be such that $r_n^{2^{(n_0+N)kq}} = r^*$ and $c = \varphi_M^{-1}(re^{2i\pi\theta})$. We have $\varphi_c(c) = \varphi_M(c) = r_N e^{2i\pi\theta}$, and since $2^{kq}\theta = \theta$, $\varphi_c(x(c)) = r_N^{2^{n_0 kq}} e^{2i\pi\theta}$, $\varphi_c(f^{\circ Nkq}(x(c))) = r^* e^{2i\pi\theta}$, and so $f^{\circ Nkq}(x(c)) = y(c)$. When $N \to \infty$, $r_N \to 1$, thus theorem 11.1 yields $c \to c_0$, and $(\forall W)$, $(\exists N_0)$, $(\forall N \geq N_0)$, $c \in W$. ■

2. Results.

Let us now state the theorem whose proof will be the subject of this chapter.

Let Λ and V be neighborhoods of 0 in $\mathbb{C}$, and $(\lambda, z) \mapsto g_\lambda(z)$ a $\mathbb{C}$-analytic map from $\Lambda \times V$ to $\mathbb{C}$. Assume $g_\lambda(0) = 0$ for all $\lambda \in \Lambda$, $g_0'(0) = \rho_0 = e^{2i\pi p/q}$, $p/q \in \mathbb{Q}$, and that $\lambda \mapsto \rho(\lambda) = g_\lambda'(0)$ is not constant. Assume $g_0^{\circ q}$ is of the form $z \mapsto z + b_0 z^{q+1} + \mathcal{O}(z^{q+2})$ with $b_0 \neq 0$. Let L_+ and L_- be conscutive attracting and repelling axes of $g^{\circ q}$ at 0 (i.e., the ange between them is $\pm 1/2q$ turns). Denote by S_+ and S_- the open sectors with bisectors L_+ and L_- and angle at the vertex equal to $1/8q$ turns.

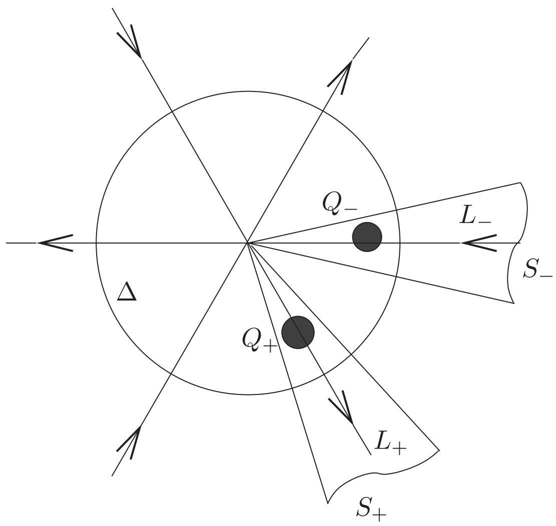


FIGURE 1. The axes $L_\pm$, the sectors $S_\pm$ and the compact sets $Q_\pm$.

Theorem 11.2. *In this situation,*

$$(\exists \Delta, \text{ disk centered at } 0) \quad (\forall Q_+ \subset \Delta \cap S_+) \quad (\forall Q_- \subset \Delta \cap S_-)$$
$$(\forall W \text{ neighborhood of 0 in } \Lambda) \quad (\exists N_0 \in \mathbb{N}) \quad (\forall s_+ : W \to Q_+ \text{ continuous})$$
$$(\forall s_- : W \to Q_- \text{ continuous}) \quad (\forall N \geq N_0) \quad (\exists \lambda \in W)$$
$$g_\lambda^{\circ Nq}(s_-(\lambda)) = s_+(\lambda).$$

Proof of corollary 11.1 knowing theorem 11.2. If $q \neq 1$, α_1 is a simple fixed point of $f_{c_0}^{\circ k}$. Let us set $c(\lambda) = c_0 + \lambda$. By the implicit function theorem, we can find an analytic function $\alpha \mapsto \alpha(\lambda)$ defined in a neighborhood of 0 and such that $\alpha(0) = \alpha_1$, $f_{c(\lambda)}^{\circ k}(\alpha(\lambda)) = \alpha(\lambda)$.

If $q = 1$, i.e., $\rho = 1$, the point α_1 is a double fixed point of $f_{c_0}^{\circ k}$, because there is only one petal. If we set $c(\lambda) = c_0 + \lambda^2$, we can still find a map $\lambda \mapsto \alpha(\lambda)$ which is analytic in a neighborhood of 0 and such that $\alpha(0) = \alpha_1$ and $f_{c(\lambda)}^{\circ k}(\alpha(\lambda) = \alpha(\lambda)$. This follows from Morses' theorem or from Weierstrass preparation theorem. In both cases, we define g_λ by $\alpha(\lambda) + g_\lambda(z) = f_{c(\lambda)}^{\circ k}(\alpha(\lambda) + z)$. We set $s_-(\lambda) = x(c(\lambda) - \alpha(\lambda))$, $s_+(\lambda) = y(c(\lambda) - \alpha(\lambda))$. If we have chosen Δ sufficiently small, we can take n_0 sufficiently large, and r_0 sufficiently close to 1 so that $s_-(0) \in S_-$ and $s_+(0) \in S^+$, because $f_{c_0}^{nkq}$ and $\varphi_{c_0}^{-1}(re^{2i\pi\theta})$ tend to α_1 tangentially to respectively L_- and L_+.

Let Q_- and Q_+ be compact neighborhoods of $S_-(0)$ and $s_+(0)$ in S_- and S_+, we still have $s_-(\lambda) \in Q_-$ and $s_+(\lambda) \in Q_+$ when λ varies in a neighborhood Λ of 0. We can therefore apply theorem 11.2 and we get corollary 11.1. ■

The proof above gives to corollary 11.1 the following additional information which will be useful in the following chapter.

Additional information 1 to corollary 11.1. *We can choose N_0 independent of r^* as long as r^* varies in a compact $J \subset]0, +\infty[$ such that*

$$\varphi_{c_0}^{-1}(J \cdot e^{2i\pi\theta}) \subset \Delta \setminus \{0\}.$$

Also, we will see that, in the situation of theorem 11.2, if $\lambda \mapsto \rho(\lambda) - e^{2i\pi p/q}$ has a zero of order ν at 0, there is at least ν distinct values of λ such that $g_\lambda^{\circ Nq}(s_-(\lambda)) = s_+(\lambda)$ in the conditions of the theorem.

We will use this fact in the next chapter in order to see that, in the situation of theorem 11.1, we necessarily have $\nu = 1$.

3. A change of variable.

We will now work under the hypothesis of theorem 11.2

Proposition 11.1. *We can find a $\mathbb{C}$-analytic diffeomorphism $(\lambda, z) \mapsto (\lambda, \zeta_\lambda(z))$ of a neighborhood $\Lambda' \times V'$ of $(0,0)$ in $\Lambda \times V$ onto an open subset of $\Lambda' \times \mathbb{C}$, such that $\zeta_\lambda(0) = 0$, and for $\lambda \in \Lambda'$, the expression of g_λ in the coordinate ζ_λ is of the form $\zeta \mapsto \rho(\lambda)\zeta + \beta(\lambda)\zeta^{q+1} + \mathcal{O}(\zeta^{q+2})$, with $\beta(0) \neq 0$.*

Proof. (cf chapter 9, lemma 9.3). For $j \in \{2, \dots, q\}$, we have a $\mathbb{C}$-analytic diffeomorphism $(\lambda, z) \mapsto (\lambda, \zeta_{\lambda,j}(z))$ such that the expression of g_λ is $\zeta \mapsto \rho(\lambda)\zeta + \beta_j(\lambda)\zeta^j + \mathcal{O}(\zeta^{j+1})$; setting

$$\zeta_{\lambda,j+1} = \zeta_{\lambda,j} + \frac{\beta_j(\lambda)}{\rho(\lambda) - \rho(\lambda)^j}\zeta_{\lambda,j}^j$$

(the denominator does not vanish in a neighborhood of 0), we obtain a new coordinate where the expression of g_λ is of the form:

$$\zeta \mapsto \rho(\lambda) + \beta_{j+1}(\lambda)\zeta^{j+1} + \mathcal{O}(\zeta^{j+2}).$$

The diffeomorphism $(\lambda, z) \mapsto (\lambda, \zeta_{\lambda,q+1})$ has the required properties; we have $\beta_{q+1}(0) \neq 0$ since otherwise, the flower would have at least $2q$ petals. ■

Corollary 11.2. *The expression of $g_\lambda^{\circ q}$ in the coordinate ζ_λ is of the form $\zeta \mapsto \rho(\lambda)^q\zeta + b(\lambda)\zeta^{q+1} + \mathcal{O}(\zeta^{q+2})$ with $b(0) \neq 0$.*

In fact, $b = (\rho^q + \rho^{2q} + \ldots + \rho^{q^2})\beta$, $b(0) = q\beta(0)$.

We will now make the change of variable defined by the map

$$(\lambda, z) \mapsto \left(\lambda, \frac{\rho(\lambda)^{q(q+1)}}{qb(\lambda)\zeta^q}\right),$$

using the conventions of chapter 9, subsection 1.2.

If Δ is a disk centered at 0, this map defines an isomorphism $h : \Lambda\times\Delta\setminus\{0\} \to \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a covering of degree q of an open set Ω of $\Lambda \times \mathbb{C}$ of the form $\Omega = \{(\lambda, z) \mid R(\lambda) < |z|\}$. We will write $h : (\lambda, z) \mapsto (\lambda, Z)$.

Proposition 11.2. *The expression G_λ of $g_\lambda^{\circ q}$ in the coordinate Z is of the form $Z \mapsto (1 + U_\lambda)Z + 1 + \eta_*(\lambda, Z)$, where $\eta(\lambda, Z)$ is in $\mathcal{O}(1/|Z|^{1/q})$ when $|Z| \to \infty$, uniformly with respect to $\lambda \in \Lambda$.*

Additional information. *$\lambda \to U_\lambda$ is an analytic map given by $1 + U_\lambda = 1/\rho(\lambda)^{q^2}$. We have $U_0 = 0$ and the multiplicity ν of 0 as a zero of $\lambda \mapsto U_\lambda$ is equal to its multiplicity as a zero of $\lambda \mapsto \rho(\lambda) - e^{2i\pi p/q}$.*

Proof of the proposition. Set $\zeta_1 = \tilde{g}_\lambda^{\circ q}(\zeta)$, where $\tilde{g}_\lambda$ is the expression of g_λ in the coordinate ζ_λ, et let Z and Z_1 correpond to ζ and ζ_1. Wehave

$$\zeta_1 = \rho(\lambda)^q\zeta + b(\lambda)\zeta^{q+1} + \ldots = \rho(\lambda)^q\zeta(1 + b(\lambda)\rho(\lambda)^{-q}\zeta^q + \ldots);$$

and so

$$\begin{aligned} Z_1 &= \rho(\lambda)^{-q^2} Z(1 - qb(\lambda)\rho(\lambda)^{-q}\zeta^q + \ldots) \\ &= \rho(\lambda)^{-q^2} Z\left(1 - \frac{\rho(\lambda)q^2}{Z} + \ldots\right) \\ &= \rho(\lambda)^{-q^2} Z - 1 + \ldots \quad . \end{aligned}$$

■

4. Walz of compact sets.

Let K and K' be two compact subsets of $\mathbb{R}^2 = \mathbb{C}$, T be a set homeomorphic to S^1 with orientation (I;E;, equiped with a homotopy class of homeomorphisms $S^1 \to T$), $(\varphi_t)_{t\in T}$ and $(\varphi'_t)_{t\in T}$ two families of embeddings of K and K' in $\mathbb{R}^2$. We set $K(t) = \varphi_t(K)$ and $K'(t) = \varphi'_t(K')$.

Definition 11.1. *We say that $K(t)$ and $K'(t)$ walz d turns when t ranges in T if*

a) *$\forall t \in T$, $K(t) \cap K'(t) = \emptyset$;*
b) *for any map $t \mapsto z(t) \in K(t)$ and $t \mapsto z'(t) \in K'(t)$, the map $t \mapsto$ $\arg(z'(t) - z(t))$ from T to $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has degree d.*

Remark. 1) If K and K' are simply connected and if condition (a) is satisfied, it is sufficient to check (b) for a pair of continuous maps $t \mapsto (z(t), z'(t))$. In particular, there always exists a d such that $K(t)$ and $K'(t)$ walz d turns.

2) Assume $T = \partial\Sigma$, where Σ is a topological disk in $\mathbb{R}^2$, and extend $t \mapsto z(t)$ and $t \mapsto z'(t)$ to continuous mappings $\Sigma \to \mathbb{R}^2$. If $d \neq 0$, there exists at least a $t \in \mathring{\Sigma}$ such that $z'(t) = z(t)$. Indeed, otherwise, the map $t \mapsto \arg(z'(t) - z(t))$ would extend to a continuous mapping $\Sigma \to \mathbb{T}$.

Identifying $\mathbb{R}^2$ and $\mathbb{C}$, if the extensions $t \mapsto z(t)$ and $t \mapsto z'(t)$ are holomorphic in $\overset{\circ}{\Sigma}$, the number d is the number of zeroes of $t \mapsto z'(t) - z(t)$ in $\overset{\circ}{\Sigma}$, counted with multiplicities.

3) Let Ω and Ω' be two simply connected open subsets of $\mathbb{R}^2$ and $\Phi : \Omega \to \Omega'$ be a homeomorphism preserving the orientation.

Assume $K(t) \subset \Omega$ for all t, and that $K(t)$ and $K'(t)$ walz d turns as t ranges in T. Then, $\Phi(K(t))$ and $\Phi(K'(t))$ also walz d turns. Indeed, we can assume that $T = \partial\Sigma$, where Σ is a topological disk in $\mathbb{R}^2$; if $t \mapsto z(t) \in K(t)$ and $t \mapsto z'(t) \in K'(t)$ are continuous, we can extend them to continuous mappings $\Sigma \to \Omega$. The number d is the intersection number in $\Sigma \times \Omega$ of the graphs of $t \mapsto z(t)$ and $t \mapsto z't)$. It is preserved by $\mathrm{Id} \times \Phi : \Sigma \times \Omega \to \Sigma \times \Omega'$.

Identifying $\mathbb{R}^2$ and $\mathbb{C}$, we will say that $K(t)$ is to the left (respectively to the right, above or below) of $K'(t)$ if, for $z \in K(t)$ and $z' \in K'(t)$, we have $\mathrm{Re}(z'-z) > 0$ (repectively $\mathrm{Re}(z'-z) < 0$, $\mathrm{Im}(z'-z) < 0$ or $im(z'-z) > 0$).

Proposition 11.3. *Assume $T = \partial\Sigma$, where $\Sigma = [-1,1]^2$, and assume that for all*

$t \in \{-1\} \times [-1,1]$, *$K(t)$ is to the left of $K'(t)$,*
$t \in [-1,1] \times \{-1\}$, *$K(t)$ is above $K'(t)$,*
$t \in \{1\} \times [-1,1]$, *$K(t)$ is to the right of $K'(t)$,*
$t \in [-1,1] \times \{1\}$, *$K(t)$ is below $K'(t)$,*

Then, $K(t)$ and $K'(t)$ walz 1 turn.

Proof. The map $t \mapsto t/|t|$ and $t \mapsto (z'(t) - z(t))/|z'(t) - z(t)|$ from T to S^1 never take opposite values. Therefore, they are homotopic. ∎

In order to prove theorem 11.2, we set $N = N' + N''$, where $N' = \lfloor N/2 \rfloor$ and $N'' = N'$ or $N' + 1$; then, we construct in Λ topological disks $\sigma_1, \ldots, \sigma_\nu$, where ν is the order of 0 as a zero of $\lambda \mapsto \rho(\lambda) - e^{2i\pi p/q}$, such that, when λ ranges in $\partial\sigma_i$, $g_\lambda^{\circ N'q}(Q_-)$ and $g_\lambda^{\circ -N''q}(Q_+)$ walz 1 turn.

in oreder to check this property, thanks to the third remark, we can work in the coordinate Z.

In order to make the comprehension easier, we will first do the computations without the term η in proposition 11.2, i.e., replacing G_λ by $H_U : Z \mapsto (1+U)Z - 1$.

5. Study of the family (H_U).

We set $H_U(Z) = (1 + U)Z - 1$. The map H_U is a scaling map with center $A = 1/U$ which maps 0 to -1.

Let us fix $a \in]0, 1/2[$; denote by P_a the square $[-a, a]^2$, i.e., $P_a = \{z \mid a \geq |\mathrm{Re}(z)|, a \geq |\mathrm{Im}(z)|\}$ and $\Sigma_N = \{U \mid N \log(1 + U) + 2i\pi \in P_a\}$. The boudary of Σ_N is of the form $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ (see Fig. 2 below).

For $N \geq 8$, we have $\Sigma_N \subset \mathbb{D}_{8/N} \setminus \overline{\mathbb{D}}_{4/N}$ and $|\arg(iU)| < 1/12\text{turn}= 30°$ if $U \in \Sigma_N$.

Proposition 11.4. *Let Q_+ and Q_- be two compact subsets of $\mathbb{C}$, and denote by δ the diamter of $Q_+ \cup Q_- \cup \{0\}$. Let $N > \sup(8, 12\delta/a)$, N' and N'' such that $N' + N'' = N$. Then, $Q-(U) = H_U^{\circ N'}(Q_-)$ and $Q+(U) = H_U^{\circ -N''}(Q_+)$ walz one turn when U ranges in $\partial\Sigma_N$.*

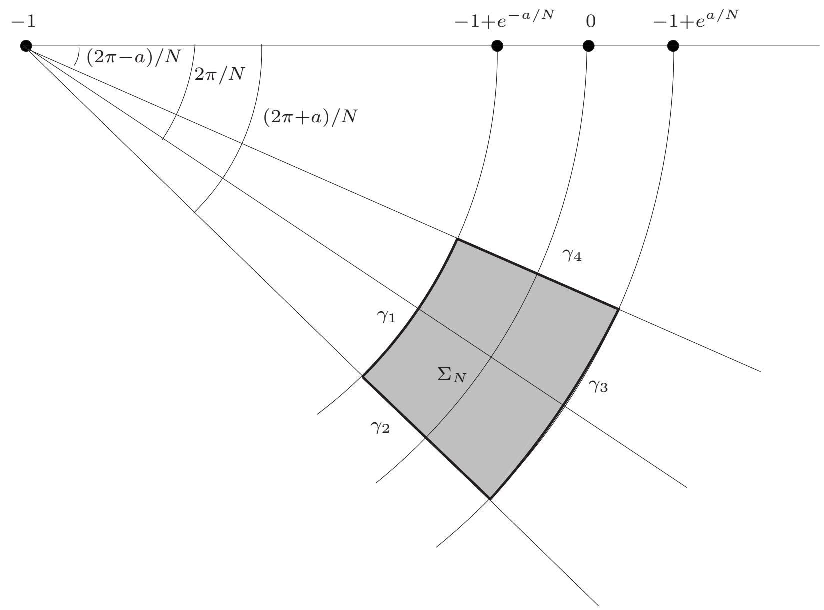


Figure 2. The set Σ_N.

Proof. For $A \subset \Sigma_N$, $z_- \in Q_-$ and $z_+ \in Q_+$, we have

$$\left|\log \frac{z_- - A}{z_+ - A}\right| = \left|\int_{z_-}^{z_+} \frac{dz}{z - A}\right| \le \frac{\delta}{A - \delta} \le \frac{\delta}{N/8 - \delta} < a.$$

Denote by τ_{-A} the translation $Z \mapsto Z - A$, and Φ the map $Z \mapsto \log Z$ on $\mathbb{C} \setminus e^{2i\pi(-N'/N+1/4)}\mathbb{R}_+$. We set $\Phi_A = \Phi_{\tau_{-A}}$. The fact that $Q_-(U)$ and $Q_+(U)$ walz d turns is equivalent to the fact that $\tau_{-A}(Q_-(U))$ and $\tau_{-A}(Q_+(U))$ walz d truns according to the definition and the third remark of section 4. It is also equivalent to $\Phi_A(Q_-(U))$ and $\Phi_A(Q_+(U))$ walz d turns. But for U in γ_1, $\Phi_A(Q_-(U))$ is to the left of $\Phi_A(Q_+(U))$, it is below for $U \in \gamma_2$, to the right in γ_3 and above in γ_4. According to proposition 11.3, $\Phi_A(Q_-(U))$ and $\Phi_A(Q_+(U))$ walz 1 turn.

6. Perturbation.

In this section, we choose a neighborhood Λ of 0 in $\mathbb{C}$, an open subset Ω in $\mathbb{C}$ of the form $\mathbb{C} \setminus (\overline{\mathbb{D}}_R \cup -i\mathbb{R}_+)$, an $a \in]0, 1/2]$ and a family $(G_\lambda)_{\lambda\in\Lambda}$ of holomorphic maps $\Omega \to \mathbb{C}$, such that $G_\lambda(Z) = (1 + U_\lambda)Z - 1 + \eta(\lambda, Z)$ with $|\eta(\lambda, Z)| < a/100$ for $(\lambda, Z) \in \Lambda \times \Omega$.

We assume that $\lambda \mapsto U_\lambda$ is a holomorphic map from Λ to $\mathbb{C}$, with $U_0 = 0$, and that G_λ is injective on every half-plane avoiding $-i\mathbb{R}_+$.

We denote by S_+ and S_- the sectors with angle at the vertex 1/6 turn, centered on respectively $\mathbb{R}_+$ and $\mathbb{R}_-$. Let $Q_+ \subset S_+ \cap (\mathbb{C} \setminus \overline{\mathbb{D}}_{2R})$ and $Q_- \subset S_- \cap (\mathbb{C} \setminus \overline{\mathbb{D}}_{2R})$ be compact sets. Denote by Δ the diameter of $Q_+ \cup Q_- \cup \{0\}$. For $\lambda \in \Lambda$, we set $A_\lambda = 1/U_\lambda$ and $H_\lambda(Z) = (1 + U_\lambda)Z - 1$. We define Σ_N as in the previous section.

Proposition 11.5. *Let $N \geq \sup(8, 24\delta/a)$, $N' = \lfloor N/2 \rfloor$ and $N'' = N - N'$. Let $\sigma \subset \Lambda$ be a compact set such that $\lambda \mapsto U_\lambda$ induces a homeomorphism from σ to Σ_N. Then,*

a) *For all $\lambda \in \sigma$, $Q_-(\lambda) = G_\lambda^{\circ N'}(Q_-)$ and $Q_+(\lambda) = G_l^{\circ -N''}(Q_+)$ are defined;*
b) *When λ ranges in $\partial\sigma$, the compact sets $Q_-(\lambda)$ and $Q_+(\lambda)$ walz 1 turn.*

Proof. In order to facilitate the presentation of the proof of this proposition, we extend G_λ by setting $G_\lambda(Z) = H_\lambda(Z)$ for $z \in \mathbb{C}\backslash\Omega$. This introduces discontinuities, but they will not bother us.

Let us fix $\lambda \in \sigma$, set $A = A_\lambda$, $U = U_\lambda$, $G = G_\lambda$, $H = H_\lambda$. Let $Z_0^+ \in Q - +$ and $Z_0^- \in Q_-$, set $Z_i^- = G^{\circ i}(Z_0^-)$ and choose $Z_i^+ \in G^{-i}(Z_0^+)$.

Lemma 11.1. *For $j \leq N''$, we have $|Z_j^+ - A| > |A|/2$ and $|Z_j^- - A| > |A|/2$.*

Proof. Let us write Z_j for Z_j^+ or Z_j^-. We have $|Z - 0 - A| \geq |A| - \delta$ and $\delta \leq Na/24$; we also have $N/8 < |A|$ because $U \in \Sigma_N \subset \mathbb{D}_{8/N}$. As a consequence,

$$|Z_0 - A| > (1 - a/3)|A|e^{-a/2}|A|.$$

We have

$$|Z_{j+1} - A| > e^{-a/N}|Z_j - A| - \frac{a}{100}.$$

If $|Z_j - A| > |A|/2$, we have $a/100 < 0.16a/N|Z_i - A|$, and so

$$|Z_{j+1} - A| > (e^{-A/N} - .16a/N)[Z_i - A| > e^{-(1.02)a/N}|Z_j - A|.$$

We therefore have

$$\frac{Z_j - A|}{|A|} \geq e^{-(1/2 + 1.02j/N)a}$$

as long as this quantity is greater than 1/2. But $N''/N < .5625$ for $N \geq 8$, and for $a = 1/2$, we obtain $1.71\ldots < 2$. We therefore have $|Z_j - A|/|A| > \frac{1}{2}$ for all $j < N''$. $\square$

Lemma 11.2. *For $j \leq N''$, we have*

a) $\left|\log\dfrac{Z_j^- - A}{Z_0^- - A} - j\log U\right| \leq j\dfrac{a}{3N}$;

b) $\left|\log\dfrac{Z_j^+ - A}{Z_0^+ - A} + j\log U\right| \leq j\dfrac{a}{3N}$;

c) $\left|\log\dfrac{Z_{N'}^- - A}{Z_{N''}^+ - A} - N\log U\right| \leq \dfrac{3}{4}a.$

Proof. a) and b). Let us write Z_j for Z_j^+ or Z_j^-. We have $Z_{j+1} - A = U^\varepsilon(Z_j - A) + \eta_j$, where $\varepsilon = \pm 1$ and $|\eta_j| < a/100$. Hence,

$$\log(Z_{j+1} - A) = \log(Z_j - A) + \pm\log U + \log\left(1 + \frac{\eta_j}{U^\varepsilon(Z_j - A)}\right).$$

We have

$$\left|\frac{\eta}{U^\varepsilon(Z_j - A)}\right| \leq \frac{a/100}{N/12} = .32\frac{a}{N}.$$

But $|\log(1 + .32t)| \leq t/3$ for $|t| \leq 1/16$. As a consequence,

$$|\log(Z_{j+1} - A) - \log(Z_j - A) - \varepsilon\log U| \leq \frac{a}{3N}$$

which gives a) and b).

Inequality c) follows from a) with $j = N'$, b) with $j = N''$ and

$$\left|\log \frac{Z_0^+ - A}{Z_0^- - A}\right| \leq \frac{\delta}{A - \delta} < \frac{Na/24}{N/8 - Na/24} < \frac{a/24}{1/8 - 1/48} = \frac{2}{5}a,$$

by observing that $1/3 + 2/5 < 3/4$. □

Let us now come back to the proof of the proposition.

a) Let ℓ_0 be the half-angle under which we see D_R from the point A (the angles are counted in turns). Set $\ell_+ = \arg A + 1/2 + \ell_0$, $\ell_- = \arg A + 1/2 - \ell_0$, $E_+ = \{Z \mid \arg(Z-A) \in [\ell_+, \ell_+ + 3/4]\}$ and $E_- = \{Z \mid \arg(Z-A) \in [\ell_-, \ell_- - 3/4]\}$. We have $E_- \subset \Omega$ and $G^{\circ j}(Q_-) \subset E_-$ for $j \leq N'$, thus $G^{\circ N'}$ is defined and continuous on Q_-. We have $E_+ \subset G(\Omega)$ and G^{-1} has a holomorphic branch defined on E_+. It follows from the second lemma taht $G^{-j}(Q_+) \subset E_+$ for $j \leq N''$, so $G^{-N''}$ has a continuous branch defined on Q_+.

b) The complex number $L(\lambda) = \log(Z_{N'}^-(\lambda) - A_\lambda)/(Z_{N''}^+(\lambda) - A_\lambda)$ stays at a distance less than $3a/4$ of $-N \log U_\lambda$, or changing of branch, of $-N \log U_\lambda + 2i\pi$, which makes one turn around 0 as λ ranges in $\partial\sigma$, staying at a distance greater or equal to a from 0. It follows that $L(\lambda)$ makes one turn around 0. ■

7. Proof of theorem 11.2.

Let us work under the hypothesis of theorem 11.2, with $g_\lambda^{\circ q}$ of the form $\zeta \mapsto \rho(\lambda)^q \zeta + b(\lambda)\zeta^{q+1} + \mathcal{O}(\zeta^{q+2})$, which we may assume to be true given corollary 11.2. We assume that $g_\lambda^{\circ q}$ is defined on a disk Δ for all $\lambda \in \Lambda$. We have $\arg L_- = \arg L_+ \pm 1/(2q)$; we assume that $\arg L_- = \arg L_+ + 1/(2q)$ (the other case is deduced from this one by conjugating via $z \mapsto \overline{z}$). Let

$$\Theta = \{z \in \Delta \setminus \{0\} \mid \arg z \in (\arg L_+ - 1/(4q), \mathrm{agr} L_+ + 3/(4q)\}.$$

The change of variable $z \mapsto Z$ induces an isomorphism from Θ to an open set Ω as in section 6, and $g_\lambda^{\circ q}$ becomes a ffunction $G_\lambda : \Omega \to \mathbb{C}$ which satisfies the condition of this section, shrinking Δ is necessary, with $U_\lambda = 1/\rho(\lambda)^{2q}$.

If $\lambda \mapsto \rho(\lambda) - e^{2i\pi p/q}$ has a zero of multiplicity ν at 0, it is the same for $\lambda \mapsto U_\lambda$, and if N is sufficiently large, we can find ν disjoint compact sets $\sigma_1, \ldots, \sigma_\nu$ such that $\lambda \mapsto U_\lambda$ induces for each $i \in \{1, \ldots, \nu\}$ a homeomorphism from σ_i to Σ_N.

To the compact sets Q_- and Q_+ of the statement of the theorem correspond compact sets $\mathcal{Q}_-$ and $\mathcal{Q}_+$ in the coordinate Z. Let us fix $i \in \{1, \ldots, \nu\}$. If N is sufficiently large, when λ ranges in $\partial\sigma_i$, $G_\lambda^{\circ N'}(\mathcal{Q}_-)$ and $G_\lambda^{\circ -N''}(\mathcal{Q}_+)$ walz 1 turn by proposition 11.5; thus, $g_\lambda^{\circ N'q}(Q_-)$ and $g_\lambda^{\circ -N''q}(Q_+)$ walz 1 turn by remark 3 in section 4, and there exists a $\lambda \in \sigma_i$ such that:

$$g_\lambda^{\circ N'q}(s_-(\lambda)) = g_\lambda^{\circ -N''q}(s_+(\lambda)).$$

■

8. Additional informations.

The proof of the previous section give the following additional informations.

Additional information 1 to theorem 11.2. *Let ν be the multiplicity of* 0 *as a solution of $\rho(\lambda) = e^{2i\pi p/q}$. In the conditions of theorem 11.2, there exists at least ν distinct values λ such that $g_\lambda^{\circ Nq}(s_-(\lambda)) = s_+(\lambda)$.*

Additional information 2. *The values* $\lambda_{N,1}, \ldots, \lambda_{N,\nu}$ *tend to* 0 *as* $N \to \infty$, *uniformly with respect to* s_+ *and* s_-.

Additional information 3. *We have* $|\arg U_{\lambda_{N,i}} - \varepsilon/4| < 1/12$ *if* $\arg L_- = \arg L_+ + \varepsilon/(2q)$, $\varepsilon = \pm 1$.

(arguments are counted in turns).

This follows form $U_{\lambda_{N,i}} \in \Sigma_N$. In fact, we easily see that $|\arg U_{\lambda_{N,i}} - \varepsilon/4| \to 0$ as $N \to \infty$.

In the introduction, we have given a first additional information to corollary 11.1. We now give a second which follows from the additional information 1 of theorem 11.2. If $q \neq 1$, α_1 is a simple periodic point of period k, for c close to c_0, there is a periodic point $\alpha(c)$ for f_c of period k, which depends analytically on c, we denote by $\rho(c)$ its eigenvalue and ν the order of c_0 as a solution of $\rho(c) = e^{2i\pi p/q}$. If $q = 1$, the point α_1 is double as a periodic point of period k, for c close to d_0, there are 2 points $\alpha(c)$ and $\beta(c)$ periodic of period k close to α_1, and we define ν by $(\alpha(c) - \beta(c))^2 \sim a(c - c_0)^\nu$, $a \neq 0$.

Additional information 2 to corollary 11.1. *In the conditions of corollary 11.1, there are at least* ν *distinct values of* c *in* W *such that* $f_c^{\circ Nkq}(x(c)) = y(c)$.

Proof. In the case $q \neq 1$, this follows immediately from the additional information 1 of theorem 11.2.

In the case $q = 1$, we make the change of parameter $c_\lambda = c_0 + \lambda^2$, which enables us to choose an analytic determination for $\lambda \mapsto \alpha(\lambda)$. The map $\lambda \mapsto \rho(\lambda) - 1$ has a zero of order ν at 0. For N sufficiently large, we find at least ν values of λ such that $f_{c(\lambda)}^{\circ Nkq}(x(\lambda)) = y(\lambda)$ by considering that $\arg L_- = \arg L_+ + 1/2$, and ν other values by considering that $\arg L_- = \arg L_+ - 1/2$, and so 2ν values in total. Those 2ν values of λ correspond to ν distinct values of c. ■

CHAPTER 12

Landing at the right place.

1. Introduction.

Let $c_0 \in M$ be such that the polynomials $f_{c_0} : z \mapsto z^2 + c_0$ has an rationally indifferent cycle $\{\alpha_1, \ldots, \alpha_k\}$ with eigenvalue $\rho = e^{2i\pi p/q}$. We assume that the connected component U_1 of $\overset{\circ}{K}_{c_0}$ which contains c_0 is atracted by α_1 and we denote by P_1 the petal of α_1 contained in U_1. Let θ be the argument of an external ray of K_{c_0} which lands at α_1 through an interpetal adjacent to P_1. We necessarily have $2^{kq}\theta = \theta$ (chapter 9, proposition 9.5 a)).

For $c \in \mathbb{C}$, denote by G_c the potential function $\mathbb{C} \setminus K_c \to \mathbb{R}_+$ (extended to $\mathbb{C}$ by 0 on K_c) and denote by G_M the potential function of M, defined by $G_N(c) = G_c(c)$.

Let us choose Δ, n_0, r^* and define $x(c)$ as in chapter 11, section 2. Set $I^* = [s^*/2^{kq}, s^*]$, where $s^* = \log r^*$. For $s \in I^*$ and c such that $s > G_c(0)$, define $y(c,s)$ by $\arg_{K_c}(y(c,s)) = \theta$; $G_c(y(c,s)) = s$. Let W be a disk centered at c_0 such that, for all $c \in W$, we have $x(c) \in \Delta$, $G_c(0) < s^*/2^{kq}$ and $(\forall s \in I^*)$, $y(c,s) \in \Delta$. Define ν as for the additional information 2 of corollary 11.1 in chapter 11, section 8.

In the preceding chapter, we have defined an $N_0 \in \mathbb{N}$ and constructed, for all $N \geq N_0$ and all $s \in I^*$, ν distinct values of c in W such that

$$f_c^{\circ Nkq}(x(c)) = y(c,s). \tag{2}$$

More precisely, the condition $|\arg U - \varepsilon/4| < 1/12$ (cf chapter 11, additional information 3 to theorem 11.2) defines ν sectors $S_1, \ldots, S_\nu$ in W. In each one, we find a value of c satisfying (2).

Once we have chosen such a sector S, denote by c_t the value of c found in S satisfying (2) for N and s such that $t = t_{N,s} = s/2^{(n_0+N)kq}$. This enables us to define c_t for $0 < t \leq t^* = s^*/2^{(n_0+N_0)kq}$. Observe that if $t = t_{N,s^*/2^{kq}} = t_{N+1,s^*}$, condition (2) for $(N, s^*/2^{kq})$ implies (2) for $(N=1, s^*)$. By the additional information 2 to theorem 11.2, $c_t \to c_0$ as $t \to 0$.

For all polnomial $f : \mathbb{C} \to \mathbb{C}$ and for x and $y \in \mathbb{C}$, we will write $x \sim_f y$ if $(\exists N)$, $f^{\circ n}(x) = f^{\circ n}(y)$. We will specify $x \sim_{f,n} y$ if $f^{\circ n}(x) = f^{\circ n}(y)$.

For $c \in \mathbb{C} \setminus M$, let us define $omega(c) \in \mathbb{C} \setminus K_c$ by:

$$\begin{cases} G_c(\omega(c)) = G_c(c) \\ \arg_{K_c}(\omega(c)) = \theta. \end{cases}$$

For $c = c_t$, we have $c \sim_{f_c} \omega(c)$. Indeed, the points $f_c^{\circ(n_0+N)kq}(c)$ and $f_c^{\circ(n_0+N)kq}(\omega(c))$ have the same potential $2^{(n_0+N)kq}G_c(c)$ and the same external argument θ with respect to K_c, therefore are equal.

In this chapter, we will show the following theorem.

Theorem 12.1. *For $c = c_t$, with $t \in]0, t^*]$ sufficiently close to 0, we have $\omega(c) = c$.*

Let us give consequences immediately.

Corollary 12.1. *For $c = c_t$ with $t > 0$ sufficiently close to 0, we have $c \in \mathbb{C}\setminus M$, $\arg_M(c) = \theta$ and $G_M(c) = t$.*

Corollary 12.2. *We have $\nu = 1$.*

Otherwise, there woulb be several points in $\mathbb{C} \setminus M$ with the same potential and the same external argument with respect to M.

Corollary 12.3. *The external ray $\mathcal{R}(M, \theta)$ lands at c_0.*

This is theorem 11.1 annouced in the previous chapter.

Let us give the sketch of the proof of theorem 12.1. As in chapter 11 for the proof of corollary 11.1, we set $c(\lambda) = c_0 + \lambda$ if $q \neq 1$ and $c(\lambda) = c_0 + \lambda^2$ if $q = 1$, which enables us to define $\alpha(\lambda)$ depending analytically on λ when λ ranges in a disk Λ centered at 0. We assume that $\lambda \in \Lambda \Longrightarrow c(\lambda) \in W$. To the sector $S \subset W$ corresponds a sector $\widetilde{S} \subset \Lambda$.

We will construct for each $c \in S$ a point $\tilde{\omega}(c)$ such that $\tilde{\omega}(c) \sim_{f_c} c$. Then, we will show that on theone hand, $\omega(c_t) = \tilde{\omega}(c_t)$ for $t > 0$ sufficiently close to 0, on the other hand, $\tilde{\omega}(c) = c$ for $c \in S$ sufficiently close to c_0.

Remark. 1) For the proof, we will use not only theorem 11.2, its corollary and their additional informations, but also the inequalities proved in chapter 11 sections 6–7, which have been used in the construction of c_t.

2) We will possibly have to increase n_0, to decrease r^* (and so s^*), which will have the effect to shrikening W, increasing N_0 and decreasing t^*.

2. Definition of $\omega(c, \gamma)$.

Let W and n_0 be as in the previous section. For $c \in W$, we set $x_n(c) = f_c^{\circ(n_0+n)kq}(c) = f_c^{\circ nkq}(x(c))$, and we denote by $\mathcal{C}(c)$ the set of points $f^{\circ i}(c)$ for $0 \leq i < n_0kq$. Let $\gamma : I = [0,1] \to \mathbb{C}$ be a path from $x_1(c)$ to $x_0(c)$ such that $\gamma(I) \cap \mathcal{C}(c) = \emptyset$. Then, there exists a unique path $\tilde{\gamma} : [0, n_0 + 1] \to \mathbb{C}$ extending γ and such that $\tilde{\gamma}(t+1) \in f_c^{-kq}(\tilde{\gamma}(t))$ for $t \in [0, n_0]$. Indeed, we can define $\tilde{\gamma}_{\|[j,j+1]}$ by induction on j: the condition $\gamma(I) \cap \mathcal{C}(c)$ guaranties that $\tilde{\gamma}([j-1, j])$ does not contain critical values of $f_c^{\circ kq}$ for $j \leq n_0$. We then set $\omega(c, \gamma) = \tilde{\gamma}(n_0 + 1)$. We have $\omega(c, \gamma) \sim_{f_c, n_0kq} c$. Indeed, $f_c^{\circ n_0kq}(\omega(c, \gamma)) = x(c) = f_c^{\circ n_0kq}(c)$. If γ is homotopic to γ' among the paths from $x_1(c)$ to $x_0(c)$ avoiding $\mathcal{C}(c)$, we have $\omega(c, \gamma') = \omega(c, \gamma)$.

3. The Fatou-Ecalle cylinder.

We will now work under the conditions of section 7 of chapter 11: the change of variable $z \mapsto Z$ defines an isomorphism from a sector Θ_λ of Δ to $\Omega = \mathbb{C}\setminus(\mathbb{D}_R \cup -i\mathbb{R}_+)$ and, to g_λ deduced from $f_{c(\lambda)}^{\circ kq}$ by the change of variable $z \mapsto \zeta$ corresponds for the variable Z, a map $G_\lambda : \Omega \to \mathbb{C}$ of the form $G_\lambda = H_\lambda + \eta_\lambda$, where $H_\lambda(Z) = (1 + U_\lambda)Z - 1$ and $(\forall Z \in \Omega)$ $|\eta_\lambda(Z)| \leq a/100$. We denote by A_λ the fixed point of H_λ, let $A_\lambda = 1/U_\lambda$. We assume that $R > 1$ and we denote by Λ a disk such that, for $\lambda \in \Lambda$, we have $c(\lambda) \in W$ and $|A_\lambda| > 4R$. The map G_λ has a fixed point A'_λ such that $|A'_\lambda - A_\lambda| \leq |A_\lambda|/100$. Sometimes, we will write A for A_λ, and so on ...

Lemma 12.1. *For $Z \in \Omega$, we have:*

$$\left| \log \frac{G(Z) - A'}{(1+U)(Z - A')} \right| \leq \frac{1}{10|A|}.$$

Proof. a) Case where $|Z-A| \geq |A|/2$. We have

$$\begin{aligned}\log \frac{H(Z)-A'}{(1+U)(Z-A')} &= \log \frac{H(z)-A'}{Z-A'} - \log \frac{H(Z)-A}{Z-A} \\ &= \int_A^{A'} \frac{dt}{t-H(Z)} - \frac{dt}{t-Z} = \int_A^{A'} \frac{(Z-A)U}{(t-H(Z))(t-Z)} dt\end{aligned}$$

So,

$$\begin{aligned}\left|\log \frac{H(Z)-A'}{(1+U)(Z-A')}\right| &\leq |A-A'| \cdot |U| \cdot \left|\frac{Z-A}{t-Z}\right| \frac{1}{|t-H(Z)|} \\ &\leq \frac{|A|}{100} \cdot \frac{1}{|A|} \cdot 1.03 \cdot \frac{1}{.24|A|} \leq \frac{1}{20|A|}.\end{aligned}$$

On the other hand,

$$\log \left|\frac{G(Z)-A'}{H(Z)-A'}\right| \leq \frac{|\eta(Z)|}{|H(Z)-A'|-|\eta(Z)|} \leq \frac{.01}{.24|A|} \leq \frac{1}{20|A|}.$$

The inequality follows in that case.

b) Case where $|Z-A| \leq |A|/2$. On the disk $\mathcal{D} = \mathbb{D}_{A,|A|/2}$, we have $|\eta'| \leq 2/(100|A|)$, because each point of this disk is the center of a disk of radius $|A|/4$ contained in Ω on which η is bounded by $a/100 \leq 1/200$. For $Z \in \mathcal{D}$, we have:

$$|G(Z)-A'-(1+U)(Z-A')| = \left|\int_{A'}^{Z} \eta'(t)dt\right| \leq \frac{1}{50|A|}|Z-A'|,$$

and so, we also have the inequality in this case. ■

We now assume that λ satisfies $|\arg A_\lambda - 1/4| \leq 1/12$, which implies $\arg(1+U_\lambda) \leq -1/(2A_\lambda)$. We define an open set $\Omega'_\lambda \subset \Omega$ in the following way: if V is the largest open sector avoiding $\overline{\mathbb{D}}_R$ with vertex at A'_λ, the open set Ω'_λ is the set of points $Z \in V$ such that $[A'_\lambda, Z] \cap -i\mathbb{R}_+ = \emptyset$.

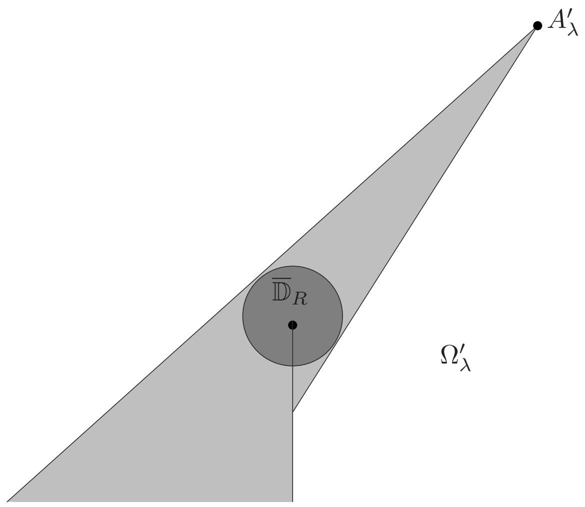


FIGURE 1. The set Ω'_λ.

Denote by E_λ the quotient of Ω'_λ by the equivalence relation identifying Z to $G_\lambda(Z)$ if the segment $[Z, G_\lambda(Z)]$ is contained in Ω'_λ.

Proposition 12.1. Definition. *The space E_λ is a Riemann surface isomorphic to $\mathbb{C}/Z$, that we will call the Fatou-Ecalle cylinder of G_λ.*

Proof. By the change of variable $Z \mapsto \log(Z - A'_\lambda)$ (we choose a branch defined on Ω'_λ), the open set Ω'_λ becomes a strip $\widetilde{\Omega}'_\lambda$ bounded by cruves which make an angle with the horizontals bounded by 1/12 turns (in fact, one of those two curves is a horizontal).

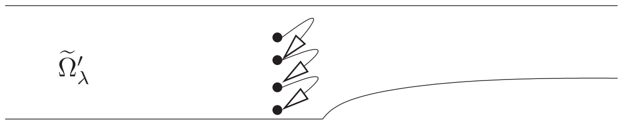


FIGURE 2. The strip $\widetilde{\Omega}'_\lambda$.

The map G_λ is conjugate to a map $\widetilde{G}_\lambda : \widetilde{\Omega'_\lambda} \to \mathbb{C}$ such hat $\widetilde{G}_\lambda(x+iy) = x_1 + iy_1$ with $y_1 - y < .45|U|$ and $|x_1 - x| < |y_1 - y| < 1/2$. As the strip $\widetilde{\Omega}'_\lambda$ has everywhere a width greater than π, $E_\lambda = \Omega'_\lambda/G_\lambda \simeq \widetilde{\Omega}'_\lambda/\widetilde{G}_\lambda$ is isomorphic to a cylinder. ■

Remark. Let Θ'_λ be the subset of Θ_λ which corresponds to Ω'_λ by $z \mapsto Z$. The quotient $\Theta'_\lambda/f_c^{\circ kq}$, where $c = c(\lambda)$, can be identified to $E_\lambda = \Omega'_\lambda/G_\lambda$, we will say that it is the *Fatou-Ecalle cylinder* relatively to the point $\alpha(\lambda)$. The one which interests us here is the one containing the axis of P_1 and the end of $\mathcal{R}(c_0, \theta)$.

4. Defintion of $\tilde{\omega}(c)$.

Let $\lambda \in \widetilde{S}$ and $c = c(\lambda) \in S$. Denote by $E_\lambda = \Theta'_\lambda/f_c^{\circ kq}$ be the Fatou-Ecalle cylinder of $f^{\circ kq}$, $\chi : \Theta'_\lambda \to E_\lambda$ the canonical map and $\xi = \chi(x_0(\lambda))$. A path γ from $x_1(\lambda)$ to $x_0(\lambda)$ in Θ'_λ gives a loop $\chi \circ \gamma$ in E_λ, based at ξ. Abusively, we will say that it is an *injective loop* if it defines an injective map $\mathbb{T} = [0,1]/0 \sim 1 \to E_\lambda$.

Proposition 12.2. Definition

a) *There exists a path γ from $x_1(\lambda)$ to $x_0(\lambda)$ in Θ'_λ giving an injective loop in E_λ.*

b) *Two such loops γ and γ' are homotopic among the loops avoiding $\mathcal{C}(c)$ and we have $\omega(c,\gamma) = \omega(c,\gamma')$.*

We denote be $\tilde{\omega}(c)$ the point $\omega(c,\gamma)$ for γ an arbitrary loop from $x_1(\lambda)$ to $x_0(\lambda)$ giving an injective loop in E_λ.

Proof. If we have chosen n_0 large enough and Δ small enough, we have $\mathcal{C}(c) \cap \Theta_\lambda \subset \{f^{\circ mkq}(c)\}_{m<n_0}$, thus $\chi(z) = \xi$ for all $z \in \mathcal{C}(c) \cap \Theta'_\lambda$.

Let us work in the coordinate $\log(Z - A'_\lambda)$ so that Θ' becomes $\widetilde{\Omega}'$.

a) The affine loop answers the problem.

b) The loops $\eta = \chi(\gamma)$ and $\eta' = \chi(\gamma')$ are homotopic among the injectie loops based at ξ, because E_λ is a cylinder. The open set $\widetilde{\Omega}'$ can be embedded in the universal covering $\widetilde{E}$ of E; denote by π the projection $\widetilde{E} \to E$. Lifting, we obtain a homotopy between γ and γ' among the paths from x_1 to x_0 in $\widetilde{E}$ avoiding

$\pi^{-1}(\xi) \setminus x_1, x_0\}$, and in particular, the image $\widetilde{\mathcal{C}}$ of $\mathcal{C} \cap \Theta'$ by the identification $\Theta' \xrightarrow{\simeq} \widetilde{\Omega}' \longrightarrow \widetilde{E}$. But we can retract $\widetilde{E} \setminus \widetilde{\mathcal{C}}$ on a compact subset of $\widetilde{\Omega}' \setminus \widetilde{\mathcal{C}}$ containing the images of γ and γ'. Hence, we obtain a homotopy in between γ and γ' in $\widetilde{\Omega}' \setminus \widetilde{\mathcal{C}}$, i.e., in $\Theta' \setminus \mathcal{C}$. This establishes the first assertion of b). The second follows (cf section 2). ■

5. Case of c_t.

For all $t \in]0, t^*]$, let us consider the point c_t defined in section 1.

Proposition 12.3. *For all $t > 0$, we have $\omega(c_t) = \tilde{\omega}(c_t)$.*

Proof. Let us set $\mathcal{C}_j(c) = \{f^{\circ m}(c)\}_{0 \leq m < (n_0+j)kq}$. If γ is a path from $x_j(c)$ to $x_{j-1}(c)$ avoiding $\mathcal{C}_{j-1}$, there exists a unique path $\tilde{\gamma} : [0, n_0+j] \to \mathbb{C}$ extending γ and such that $\tilde{\gamma}(t+1) \in f^{-kq}(\tilde{\gamma}(t))$ for $t \in [0, n_0+j-1]$. We then set $\tilde{\omega}(c, \gamma) = \gamma(n_0+j)$ and we have $\omega(c, \gamma) \sim_{f_c} c$.

Lemma 12.2. *Let $c = c(\lambda) \in S$. Let n be such that $x_j(c) \in \Theta'_\lambda$ for $0 \leq j \leq n$, and let γ be a path from $x_n(c)$ to $x_{n-1}(c)$ in Θ'_λ giving an injective path in E_λ. We have*

a) $\gamma(I) \cap \mathcal{C}_{n-1}(c) = \emptyset$;
b) $\omega(c, \gamma) = \tilde{\omega}(c)$.

Proof. a) We have $\mathcal{C}_{n-1} \cap \Theta' = \mathcal{C} \cap \Theta' \cup \{x_0, \ldots, x_{n-2}\}$, and a) follows.

b) For $j = 1, \ldots, n$, denote by γ'_j the path from x_k to x_{j-1} which becomes affine in the coordinate $\log(Z - A')$. The path γ'_j gives an injective path in E, as $f^{\circ kq}(\gamma'_{j-1})$ if $j \geq 2$. The same proof as part (b) in proposition 12.2 shows that:

$$\omega(c, \gamma) = \omega(c, \gamma' n) = \omega(c, f^{\circ kq}(\gamma_{n-1})) = \ldots = \omega(c, \gamma'_1) = \tilde{\omega}(c).$$

□

Let $c = c(\lambda)$, $c \in S$ and $s \in I^*$ (cf section 1). Define the path $\gamma_{c,\mathcal{R},s}$ by $\arg_{K_c} \gamma_{c,\mathcal{R},s}(f) = \theta$ and $G_c(\gamma_{c,\mathcal{R},s}(t)) = 2^{-tkq}s$. This path parametrizes $\mathcal{R}(K_c, \theta)$ from $y(c, s)$ to a point $y_1(c, s) = y(c, s/2^{kq})^i n f^{-kq}(y(c, s))$.

If we have chosen r^* sufficiently close to 1 and W, so S and Λ sufficiently small, the image of $\gamma_{c,\mathcal{R},s}$ is contained in Θ'_λ for all $s \in I^*$ and all $\lambda \in \Lambda$.

Lemma 12.3. *If $\gamma_{c,\mathcal{R},s}$ is a path in Θ'_λ, it defines an injective loop in E_λ.*

Proof. Otherwise, we would find a pair $(t, t') \in I^2$, different from $(1, 0)$, and a $i \geq 1$ such that $\gamma_{c,\mathcal{R},c}(t') = f_c^{\circ ikq}(\gamma_{c,\mathcal{R},s}(t))$, and so $G_c(\gamma_{c,\mathcal{R},c}(t')) = 2^{ikq} G_c(\gamma_{c,\mathcal{R},c}(t))$, which contradicts the definition of $\gamma_{c,\mathcal{R},c}$. □

We now come back to the proof of the proposition. For $c = c_t$, $t = t_{N,s}$, we have $\omega(c) = \omega(c, \gamma_{c,\mathcal{R},c})$ by definition. We have $\omega(c, \gamma_{c,\mathcal{R},c}) = \tilde{\omega}(c)$ thanks to lemma 12.2 and 12.3, and so, $\omega(c) = \tilde{\omega}(c)$. ■

6. Definition of $\tilde{\omega}(c_0)$.

Denote by Ω'_0 the largest open set contained in Ω'_λ for all $\lambda \in \widetilde{S}$. The open set Ω'_0 is bounded by 4 half lines, and has 2 connected components ${\Omega'_0}^+$ and ${\Omega'_0}^-$.

Once a sector Θ has been chosen, to $f_{c_0}^{\circ kq}$ corresponds a map $G_0 : \Omega \to \mathbb{C}$ of the form $Z \mapsto Z - 1\eta$, $\|\eta\| \leq a/100$. The quotients $E_0^+ = {\Omega'_0}^+/G_0$ and $E_0^- = {\Omega'_0}^-/G_0$

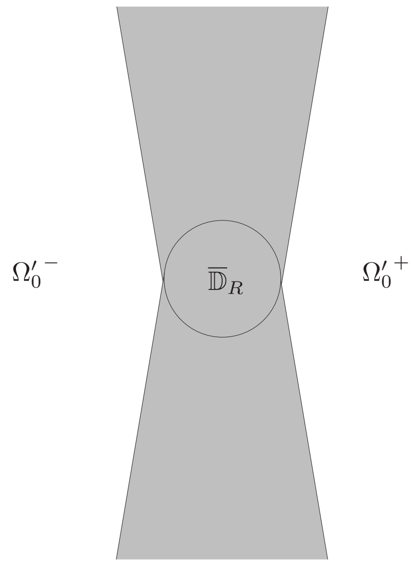


FIGURE 3. The set Ω_0'.

are isomorphic to those cylinders. Those are the two *Fatou-Ecalle cylinders* of G_0. Changing the choice of sector Θ, we obtain in this way $2q$ cylinders attached to the point α_1. The one that interests us is the cylinder E_0^- corresponding to the sector Θ containing the axis of the petal P_1 in a neighborhood of α_1. We assume that this choice was made for Θ. Denote by $\Theta_0'^-$ the preimage by $z \mapsto Z$ of $\Omega_0'^-$ in Θ. If we have chosen n_0 sufficiently large, we have $x_n(c_0) \in \Theta_0'^-$ for all $n \geq 0$, and there is a path from $x_1(c_0)$ to $x_0(c_0)$ in $\Theta_0'^-$ giving an injective loop in E_0^-. This enables us to define $\tilde{\omega}(c_0)$ (we see as in proposition 12.2 that the resuult does not depend on the choice of γ). Of course, we have $\tilde{\omega}(c_0) \sim_{f_{c_0}, n_0 kq} c_0$.

Proposition 12.4. *The point $\tilde{\omega}(c)$ tends to $\tilde{\omega}(c_0)$ as $c \to c_0$ in S.*

Proof. Let γ_0 be a path from $x_1(c_0)$ to $x_0(c_0)$ giving an injective loop – for example the affine path in the coordinate Z. For λ close to 0, we have a path γ_λ close to γ_0 from $x_1'(c(\lambda))$ to $x_0(c(\lambda))$ giving an injective loop –for example again the affine loop in the coordinate Z, or in the coordinate $\log(Z - A')$. Lifting n_0 times, we obtain $\tilde{\omega}(c)$ close to $\tilde{\omega}(c_0)$. ■

7. Identification of $\tilde{\omega}(c_0)$.

Proposition 12.5. *We have $\tilde{\omega}(c_0) = c_0$.*

Proof. Let U_1 be the connected component of $\overset{\circ}{K}_{c_0}$ containing the petal P_1. We have $\Theta_0'^- \subset P_1$, or at least $f_{c_0}^{\circ nkq}(\Theta_0'^-) \subset P_1$ for n sufficiently large, so $\Theta_0'^- \subset U_1$. On the other hand, for all $z \in U_1$, the sequence $f_{c_0}^{\circ nkq}(z) \to \alpha_1$ tangentially to the axis of P_1 (chpater 9, propositions 9.3 and 9.2); thus $(\exists n)$ $f_{c_0}^{\circ nkq}(z) \in \Theta_0'$. As a consequence, $E_{c_0}^- = \Theta_0'/f_{c_0}^{\circ kq}$ is also $U_1/f_{c_0}^{\circ kq}$. This gives a definition of $E_{c_0}^-$ independent of all choices, depending in fact only on the dynamics on U_1.

The map $f^{\circ kq} : U_1 \to U_1$ is holomorphic and proper of degree 2. It has acritical point u, the unique point of $f_{c_0}^{-(kq-1)}(0) \cap U_1$, and its critical value is c_0. Let $\varphi : U_1 \to \mathbb{D}$ be the isomorphism such that $\varphi(u) = 0$ and $\varphi(c_0) \in]0,1[$; we have

$$\varphi \circ f^{\circ kq} \circ \varphi^{-1} = h : z \mapsto \frac{3z^2+1}{z^2+3}$$

(chapter 9, corollary 9.2).

Denote by E_h^- the Fatou-Ecalle cylinder of h at the point 1, relatively to a sector centered on the axis directed by $\mathbb{R}_-$. Similarly, we have $E_h^- = \mathbb{D}/h$, and passing to the quotient, φ gives an isomorphism $\Phi : E_{c_0}^- \to E_h^-$. Let us set $x_n(h) = h^{n_0+n}(1/3) = \varphi(x_n(c_0))$. We can define $\tilde{\omega}(h)$ in the following way: we take a path γ from $x_1(h)$ to $x_0(h)$ which gives an injective loop in $E^-(h)$, we extend it to $[0, n_0+1]$ to a loop $\tilde{\gamma}$ such that $\tilde{\gamma}(t+1) \in h^{-1}(\gamma(t))$ for $t \in [0, n_0]$, and we set $\omega(h) = \tilde{\gamma}(n_0+1)$. We clearly have $\tilde{\omega}(h) \sim_{h,n_0} 1/3$ and $\varphi(\tilde{\omega}(c_0)) = \tilde{\omega}(h)$. Proposition 12.5 is a consequence of the following lemma.

Lemma 12.4. *We have $\tilde{\omega}(h) = 1/3$.*

Proof. We have $h^{\circ n}(1/3) \in]0,1[$ for all $n > 0$; we can take for γ the affine path from $x_1(h)$ to $x_0(h)$. Then, $\tilde{\gamma}$ is an injective path whose image is contained in $]0,1[$ and $\tilde{\gamma}(n_0+1) = 1/3$. □

This completes the proof of proposition 12.5. ■

8. Proof of theorem 12.1.

Proposition 12.5 has the following corollary.

Corollary 12.4. *For $c \in S$ sufficiently close to c_0, we have $\tilde{\omega}(c) = c$.*

Proof. There are 2^{n_0kq} distinct points $v_i(c_0)$, $i = 1, \dots, 2^{n_0kq}$ such that $v_i(c_0) \sim_{f_{c_0}, n_0kq} c_0$. Indeed, $f_{c_0}^{\circ n_0kq}(c_0)$ is not a critical value of $f_{c_0}^{\circ n_0kq}$, because its critical values are the points $f_{c_0}^{\circ m}(c_0)$ for $0 \le m \le n_0kq - 1$, and because c_0 is not preperiodic. We can assume $v_1(c_0) = c_0$. let v_i, $i = 1, \dots, 2^{n_0kq}$ be pairwise disjoint neighborhoods of the $v_i(c_0)$. For c sufficienlty close to c_0, there is in each V_i, a unique $v_i(c)$ such that $v_i(c) \sim_{f_c, n_0kq} c$, and $v_1(c) = c$. For c sufficiently close to c_0, we have $\tilde{\omega}(c) \in V_1$ (cf proposition 12.4), so $\tilde{\omega}(c) = v_1(c) = c$. ■

Theorem 12.1 follows from this corollary and proposition 12.3.

CHAPTER 13

Landing of external rays of M with rational argument.

1. Results.

Theorem 13.1. *Let $\theta \in \mathbb{Q}/\mathbb{Z}$. Then, the external ray $\mathcal{R}(M,\theta)$ lands at a point $c \in M$ qhich is either the root of a hyperbolic component, or a Misurewicz point.*

Remark. The root of the hyperbolic components of $\overset{\circ}{M}$ are the c such that $f_c : z \mapsto z^2 + c$ has a rationally indifferent cycle. The Misurewicz points are the c such that 0 is strictly preperiodic for f_c.

Additional information 1. *If θ has odd denominator, c is the root of a hyperbolic component.*

The point c belongs to a component U_1 of $\overset{\circ}{K}_c$ which is attracted by a point α_1. There are 2 external rays of K_c landing at α_1 through an interpetal adjacent to U_1 (except if $\theta = 0$, i.e., $v = 1/4$, a unique ray), and θ is the argument of one of those.

Additional information 2. *If θ has even denominator, c is a Misurewicz point, and θ is one of the external arguments of c in K_c.*

In this chapter, we will show theorem 13.1. The additional information 1 will be proved in the next chpater. We will show that, if c is a Misurewicz point, then θ is an external argument of c in K_c, which is part of the additional information 2. The additional information 2 will be proved in chapter 17. Theorem 13.1 and its additional informations are a converse statement to theorem 11.1 in chpater 11 and to theorem 8.2 in chapter 8.

2. Accumulation points of $\mathcal{R}(M,\theta)$.

Lemma 13.1. *Let $\theta \in \mathbb{Q}/\mathbb{Z}$ and let c_0 be an accumulation point of $\mathcal{R}(M,\theta)$. Then either f_{c_0} has a rationally indifferent cycle, or c_0 is a Misurewicz point.*

Proof. Let us write θ as $p/(2^l(2^k-1))$, with l and k minimum, and set $\theta' = 2^l\theta$. By chapter 8, proposition 8.4, the ray $\mathcal{R}(K_{c_0},\theta)$ lands at a preperiodic point α_0, $\mathcal{R}(K_{c_0},\theta')$ lands at $\alpha'_0 = f_{c_0}^{\circ l}(\alpha_0)$ which is periodic of period k' dividing k, repelling or rationally indifferent. If α'_0 is periodic and rationally indifferent, we have won. In the following, we assume it is repelling. We can therefore apply proposition 8.5 in chapter 8. In virtue of this proposition, we can find a neighborhood W of c_0 in M, an analytic map $c \mapsto \alpha'(c)$ from W to $\mathbb{C}$ such that $f_c^{\circ k'}(\alpha'(c)) = \alpha'(c)$ and $\alpha'(c_0) = \alpha'_0$, and a continuous map $(c,s) \mapsto \psi_{c,\theta'}(s)$ from $W \times \mathbb{R}_+$ to $\mathbb{C}$ such that $\psi_{c,\theta'}(0) = \alpha'(c)$ and $\psi_{c,\theta'}(s) = \varphi_c^{-1}(e^{s+2i\pi\theta'})$.

Let c_n be a sequence of points in $\mathcal{R}(M,\theta)$ tending to c_0, and set $s_n = G_M(c_n)$ (potential). We have $s_n \to 0$ and $\psi_{c_n,\theta'}(2^l s_n) = f_{c_n}^{\circ l}(c_n)$, and so $f_{c_0}^{\circ l}(c_0) = \psi_{c_0,\theta'}(0) = \alpha'_0$. As α'_0 is periodic and belongs to ∂K_{c_0}, the point c_0 is a Misurewicz point. ■

Additional information 1. *If c_0 is a Misurewicz point, θ is an external argumet of c_0 in K_{c_0}.*

Proof. Let us keep the notations of the previous proof. Let us first show that $f_{c_0}^{\circ i}(\alpha_0) \neq 0$ for all $i \geq 0$ (cf. chapter 8 proposition 8.5). If we had $f_{c_0}^{\circ i}(\alpha_0) = 0$, we would have $\psi_{c_0,2^{i+1}\theta}(0) = f_{c_0}^{\circ i+1}(\alpha_0) = c_0$. However, $\psi_{c_n,2^{i+1}\theta}(2^{i+1}s_) = f_{c_n}^{\circ i+1}(c_n)$ tends to $f_{c_0}^{i+1}(c_0)$. But c_0 has no critical point of f_{c_0} in its forward orbit, and we can apply proposition 8.5 of chapter 8, which gives $\psi_{c_n,2^{i+1}\theta}(2^{i+1}s_n) \to c_0$, and so $f_{c_0}^{\circ i+1}(c_0) = c_0$. As 0 is the only point in $f_{c_0}^{-1}[c_0)$, we deduce that $f_{c_0}^{\circ i+1}(0) = 0$, which contradicts the fact that c_0 is a Misurewicz point.

We can now apply proposition 8.5 of chapter 8 to $\alpha_0 = \psi_{c_0,\theta}(0)$. We have $\psi_{c_n,\theta}(s_n) = c_n$, and so by passing to the limit, $\psi_{c_0,\theta}(0) = c_0$ and $c_0 = \alpha_0$. ■

3. Proof of theorem 13.1.

The set of accumulation points in $\mathcal{R}(M,\theta)$ is a connected compact set. By the previous lemma, it is contained in the union of Misurewicz points wichi is countable andthe set of c such that f_c has a rationally indifferent cycle, which is also countable.

But every countable connected compact set is reduced to a point. Thus, there is a unique acculmulation point c, and since everything occurs in a compact set, $\mathcal{R}(M,\theta)$ lands at c.

We have proved theorem 13.1 and its additional information 2.

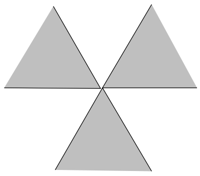
CHAPTER 14

Hyperbolic components.

1. Hyperbolic components.

Let us denote X_k the set of pairs (c,z) such that $f_c^{\circ k}(z)=z$, π the projection $(c,z)\mapsto c$ from X_k to $\mathbb{C}$, and ρ_k or simply ρ the function $(c,z)\mapsto (f_c^{\circ k})'(z)$ on X_k. The set X_k is an algebraic curve over $\mathbb{C}$ and $\pi : X_k \to \mathbb{C}$ is proper of degree 2^k. In virtue of the implicit function theorem, at each point of X_k where $\rho_k \neq 1$, the curve X_k is smooth and π is a local isomorphism.

Denote by A_k the set of pairs $(c,z)\in X_k$ such that $|\rho(c,z)|<1$. As $\rho : X_k \to \mathbb{C}$ is analytic, it is open (even if X_k has singular points). It follows that $\overline{A}_k = \rho_k^{-1}(\overline{\mathbb{D}})$, $\partial A_k = \rho_k^{-1}(S^1)$ and the set of pairs (c,z) such that z is a rationally indifferent periodic point of f_c is dense in ∂A_k. The set ∂A_k is an R-algebraic set, of dimension 1 over $\mathbb{R}$, and its only singularities, outside the singular points of X_k, are of the form



(intersection of ν smooth branches), at the critical points of ρ_k.

Set $M'_k = \pi(A_k)$. It is an open subset of $\mathbb{C}$, and the set of $c \in \partial M'_k$ such that f_c has a rationally indifferent cycle of period dividing k is dense in $\partial M'_k$. But those points belong to ∂M (we even know that they are the landing point of an external ray of M). It follows that $\partial M'_k \subset \partial M$, and every connected component of M'_k is a connected component of $\mathring{M}$.

The set $M' = \bigcup_k M'_k$ is the set of points c such that f_c has an attracting cycle. Each component of M' is a component of $\mathring{M}$. The connected components of $\mathring{M}$ obtained in this way are the hyperbolic components. The question of knowing whether there exist non-hyperblic components (ghost components) is open. For $c \in M'$, there is only one attracting cycle, and the period of this cycle remains constant on each connected component of M'.

If W is a connected component of M', W is simply connected, because it is a connected component of $\mathring{M}$, ∂W is the union of arcs of algebraic curves, so is locally connected, so $\overline{W}$ is homeomorphic to $\overline{\mathbb{D}}$. Let k be the period of W, and let W' be a conected component of A_k above W, we define $\rho_W : W \to \mathbb{D}$ by $\rho_W(c) = \rho_k(c,z)$ for $(c,z)\in W'$ (independant on the choice of W'. The map π induces a homeomorphism from $\overline{W}'$ to $\overline{W}$, so ρ_W extends to a continuous map (still denote ρ_W) from $\overline{W}$ ro $\overline{\mathbb{D}}$ with $\rho'_W \partial W) \subset S^1$. The holomorphic map $\rho_W : W \to \mathbb{D}$ is proper. Every point $c \in W$ (respectively $c \in \partial W$) such that $\rho_W(c) = 0$ (repsectively

$\rho_W(c) = 1$) is called a *center* (respectively *root*) of W. We willsee that $\rho_W : \overline{W} \to \overline{\mathbb{D}}$ is a homeomorphism. It will follow that W has 1 center and 1 root.

2. Deformation of a rationally indifferent cycle, case of $q \neq 1$.

Let $c_0 \in M$ be such that f_{c_0} has a periodic point $\alpha(c_0)$ od period k, with eigenvalue $\rho_0 = e^{2i\pi p/q}$ with p and q coprime and $q \neq 1$, so $\rho_0 \neq 1$. Set $K = kq$. The algebraic curve X_k is the graph of a holomorphic map $c \mapsto \alpha(c)$ in a neighborhood of $(c_0, \alpha(c_0))$. The holomorphic map $c \mapsto \rho_k(c, \alpha(c))$ has a non vanishing derivative at c_0: this follows from corollary 12.2 in chapter 12.

Proposition 14.1. *in a neighborhood of c_0 the open set M'_k is bounded by a subarc of an R-analytic curve.*

Proof. f_{c_0} has no other indifferent cycle than the one of $\alpha(c_0)$. Indeed, in chapter 10 we have constructed a closeb set B attracted by this cycle and on $\mathbb{C} \setminus B$ a metric for which f_{c_0} is strictly expanding. It follows that X_k induces a trivial covering of degree 2^k of a neighborhood of W of c_0. Among the 2^k leaves, k contain a point of the cycle of $\alpha(c_0)$, and $F : (c, z) \mapsto (c, f_c(z))$ exchanges those leaves; on the others we have $|\rho| > 1$ if we have chosen W sufficiently small. We therefore have $M'_k \cap W = \pi(A_k \cap W')$ where W' is the leaf containing $\alpha(c_0)$. ■

Proposition 14.2. *In a neighborhood of $(c_0, \alpha(c_0))$, we have $X_K = X_k \cup X'_K$, where*

a) $X_k \cap X'_K = \{c_0\}$;
b) X'_K *is smooth at (c_0, α_0) with a vertical tangent;*
c) $\pi_K : X'_K \to \mathbb{C}$ *has local degree q at (c_0, α_0);*
d) $\rho_K : X'_K \to \mathbb{C}$ *also has local degree q at (c_0, α_0);*
e) *in a neighborhood of c_0, $M'_K = M'_k \cup M''_K$ where M''_K is the set of points c such that f_c has an attracting cycle of period exactly K; those two open sets are bounded each by an arc of a $\mathbb{R}$-analytic curve;*
f) *those two arcs only meet at c_0.*

Proof. Let ζ be a coordinate satisfying the conditions of proposition 11.1 in chapter 11: the expression g_c of $f_c^{\circ k}$ in this coordinate is of the form

$$g_c : \zeta \mapsto \rho(c, \alpha(c))\zeta + \beta(c)\zeta^{q+1} + \dots,$$

and $f_c^{\circ kq}$ becomes $g_c^{\circ q} : \zeta \mapsto (1 + u(c))\zeta - b(c)\zeta^{q+1} + \dots$, with $u(c_0) = 0$ and $b(c_0) \neq 0$. The equation of X_K is

$$g_c^{\circ q}(\zeta) - \zeta = \zeta(u(c) - b(c)\zeta^q + \dots) = 0.$$

This set is the union of X_k with equation $\zeta = 0$ and of X'_K with equation $u(c) = b(c)\zeta^q + \dots$. Since $c \mapsto u(c)$ has a simple zero at c_0 (corollary 12.2 in chapter 12), X'_K is smooth.

Since $b(c_0) \neq 0$, we have part (c). For $(c, z) \in X'_K$, we have $\rho_K(c, z) = (g_c^{\circ q})'(\zeta) = 1 + u(c) - (q + 1)b(c)\zeta^q + \dots$ and so part (d) follows. The funciton ρ_K takes the same values at the q points of X'_K above a point c close to c_0. It follows that $\rho_K : X'_k \to \mathbb{C}$ factors as $\rho'' \circ \pi$ where ρ'' is holomorphic on a neighborhood W of c_0. For $c \in W$, we hae $c \in M'_K$ if and only if $|\rho_k(c, \alpha(c))| < 1$ or $|\rho''(c)| < 1$, $2^K - (q + 1)k$ other leaves providing repelling periodic points, at least if wa have chosen W sufficiently small. Since π and ρ have the same local degree, ρ'' has a non vanishing derivative at c_0. This gives (e). Part (f) follows from corollary 12.3 in chapter 12, which gives two external rays of M landing at c_0, one in the sector where $\mathrm{Im}(u) > 0$, the other in the sector where $\mathrm{Im} u < 0$. ■

Remark. 1) We will see in chapter 15 that we have the following picture.

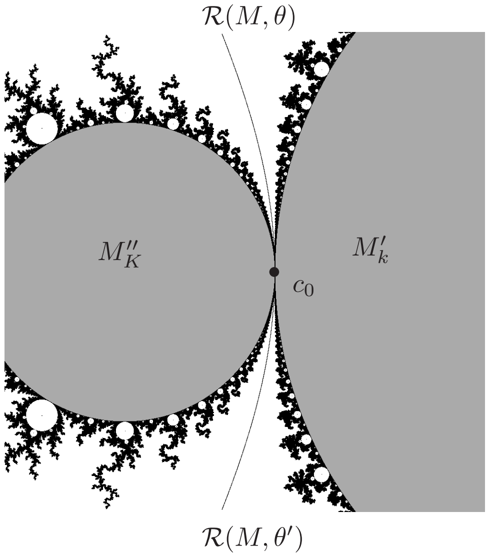


FIGURE 1. The sets M'_k and M''_K.

2) We can also deduce (e) from proposition 9.6 in chapter 9.

3. Case of $q = 1$.

Let $c_0 \in M$ be such that f_{c_0} has a periodic point α_0 of period k, with eigenvalue $\rho_0 = 1$. Then, α_0 is a double fixed point of $f_{c_0}^{\circ k}$ (proposition 9.6, chapter 9).

Proposition 14.3. a) *X_k is smooth at (c_0, α_0) with a vertical tangent;*
b) *$\pi : X_k \to \mathbb{C}$ has degre 2 at this point;*
c) *$\rho_k : X_k \to \mathbb{C}$ has a non-vanishing derivative at this point;*
d) *in a neighborhood of c_0, the open set M'_k is bounded by a $\mathbb{R}$-analytic arc having a cusp at c_0.*

Proof. Part (a) follows from corollary 12.2 in chapter 12. Part (b) follows from the fact that the multiplicity of α_0 as a fixed point of $f_{c_0}^{\circ k}$ is 2. Part (c) comes with (a). In X_k, the open set A_k is bounded by a smooth R-analytic curve in a neighborhood of (c_0, α_0). Since $c_0 \in \partial M'_k$ because it is the landing point of an external ray of M, and since $\pi : A_k \to M'_k$ is injective in a neighborhood of (c_0, α_0), we deduce (d). ■

4. Hubbard tree at a root.

Let $c_0 \in M$ be such that f_{c_0} has a periodic point $\alpha(c_0)$ of period k, with eigenvalue $\rho_0 = e^{2i\pi p/q}$ with p and q coprime; we set $K = kq$.

We will define a tree associated to c_0 as in chapter 4.

Let us first choose a system of centers for the components of $\overset{\circ}{K}_{c_0}$; if $U_0, \dots, U_{K-1}$ is the periodic cycle of components of $\overset{\circ}{K}_{c_0}$, indexed by $\{0, 1, \dots, K-1\}$ with $0 \in U_0$, we will take $f_{c_0}^{\circ i}(0)$ as the center of U_i $(0 \le i < K)$; for the other components V, we take a system of center so that the center of $f_{c_0}(V)$ is the image by f_{c_0} of the center of V. We set $U_K = U_0$.

Definition 14.1. *With those conventions, the Hubbard tree H_{c_0} is the allowable hull of the $f_{c_0}^{\circ i}(0)$, $(0 \le i < K)$.*

Remark. If we do not want to use Sullivan's non wandering domain theorem, the system of centers chosen above is a priori not unique. However, the combinatorial structure (isotopy class of the embedding of a tree in $\mathbb{C}$) is defined without ambiguity.

Proposition 14.4. *If $K \neq 1$, H_{c_0} contains the rationally indifferent cycle and is stable for f_{c_0}.*

Proof. The image by f_{c_0} of the allowable arc Γ between $f_{c_0}^{\circ i}(0)$ to $f_{c_0}^{\circ j}(0)$ (with $0 \le i < K$ and $0 \le j < K$) is, when $0 \notin \Gamma$, the allowable arc from $f_{c_0}^{\circ i+1}(0)$ to $f_{c_0}^{\circ j+1}(0)$ and, when $0 \in \Gamma$ the union of the allowable arc from $f_{c_0}^{\circ i+1}(0)$ to c_0 and the allowable arc from $f_{c_0}^{\circ j+1}(0)$ to c_0. If Γ' is the allowable arc between 0 and $f_{c_0}^{\circ k}(0)$, and $G = H \cup \Gamma'$, we therefore have $f_{c_0}(H) \subset G$. Then, we can apply again the argument of proposition 4.4 in chapter 4 to see that (if $K \neq 1$) $\nu(1)` \dots \le \nu(K-1)$ ($\nu(i)$ is the number of branches of H at $f_{c_0}^{\circ i}(0)$, $0 \le i < K$) (however, at the moment, we do not have $\nu(K-1) \le \nu(K)$); a tree with more than one vertex has at least two extremities, $\nu(1) = 1$, so $H \cap \partial U_1$ contains only one point α_1. G does not intersect ∂U_i $(0 \le i < K)$ so $f_{c_0}(H) \subset G$ implies $f_{c_0}(H \cap \partial U_i) \subset H \cap \partial U_{i+1}$ $(0 \le i < K)$, and so $f_{c_0}^{\circ K}(\alpha_1) = \alpha_1$. Thus, α_1 is a point in the rationally indifferent cycle of f_{c_0}, which is contained in H. In particular, the arc from 0 to α_0, point of the rationally indifferent cycle contained in ∂U_0, is in H. But we know (cf corollary 9.2 in chapter 9) that the dynamics of $f_{c_0}^{\circ k}$ on U_0 is analytically conjugate to the one of $z \mapsto (3z^2+1)/(z^2+3)$ on $\mathbb{D}$, so $f_{c_0}^{\circ K}(0)$ is on the arc between 0 and α_0, and so $\Gamma' \subset H$ and $f_{c_0}(H) \subset H$. ■

We can therefore apply the results of chapters 4 and 7 to c_0.

5. Roots of hyperbolic components; multiplicity.

Let $c_0 \in M$ be such that f_{c_0} has an indifferent cycle of period k, with eigenvalue $e^{2i\pi p/q}$ with p and q coprime, and set $K = kq$.

Proposition 14.5. a) *There exists a unique hyperbolic component W of $\overset{\circ}{M}$ such that c_0 is the root of W. It is a component of period K.*

b) *If $c_0 \neq 1/4$, the point c_0 has at least 2 external arguments in M, of the form $p/(2^K - 1)$.*

Proof. Part (a) follows from propositions 14.2 and 14.3.

(b) Case $q \neq 1$. Let U_1 be the connected component of $\overset{\circ}{K}_{c_0}$ containing c_0, α_1 the point of the indifferent cycle of f_{c_0} attracting U_1. There are q petals, and so q interpetals at α_1, and 2 of those interpetals are adjacent to U_1. landing through each of them there is at least one external ray of K_{c_0} with argument of the form

$p/(2^K-1)$, and the external rays of M with the same argument land at c_0 (corollary 12.2 in chapter 12).

(b) Case $q = 1$. There is only one petal at α_1 and so, only one interpetal. However, there are at least 2 external rays of K_{c_0} landing at α_1: indeed, α_1 is on H_{c_0} without being an extremity, so there are at least 2 accesses to α_1 outside H_{c_0}, and so, by corollary 7.2 two external rays of K_{c_0} landing at α_1.

Those external rays have arguments of the form $p/(2^K - 1)$ (proposition 9.5). The external rays of M with the same arguments land at c_0. ■

Let W be a hyperbolic component of $\overset{\circ}{M}$ of period k. For $c \in W$, denote by $\alpha(c)$ the attracting periodic point of f_c attracting 0.

Proposition 14.6. *The following numbers are equal:*

a) *the degree μ of the proper holomorphic map $\rho_W : W \to \mathbb{D}$;*
b) *the number of zeros in W of $c \mapsto \alpha(c)$, counting multiplicities;*
c) *the number of zeros in W of $c \mapsto f_c^{\circ k}(0)$, counting multiplicities;*
d) *the number of roots of W in W.*

Proof.

Lemma 14.1. *Let $c_0 \in W$ be such that $f_{c_0}^{\circ k}(0) = 0$. The maps $c \mapsto f_c^{\circ k}(0)$, $c \mapsto \alpha(c)$ and $c \mapsto \rho_W(c) = \rho_k(c, \alpha(c))$ have the same order of vanishing at c_0.*

Proof. For $c \in W$, we have $\rho_W(c) = 2^k \alpha(c) \cdot f_c(\alpha(c)) \dots f_c^{\circ k-1}(\alpha(c))$, and in the cycle, $\{\alpha(c), f_c(\alpha(c)), \dots, f_c^{\circ k-1}(\alpha(c))\}$, only $\alpha(c)$ is in the connected component of $\overset{\circ}{K}_c$ that contains 0. Thus, the map $c \mapsto \alpha(c))$ and $c \mapsto \rho_W(c)$ have the same order of vanishing. On the other hand, for c close to c_0, we have

$$\left|f_c^{\circ k}(0) - \alpha(c)\right| < \frac{1}{2}|0 - \alpha(c)| = \frac{1}{2}|\alpha(c)|,$$

so $c \mapsto f_c^{\circ k}(0)$ and $c \mapsto \alpha(c)$ vanish at c_0 with the same multiplicity. □

We now come back to the proof of the proposition. The zeros of the functions $c \mapsto \rho_W(c)$, $c \mapsto \alpha(c)$ and $c \mapsto f_c^{\circ k}(0)$ are the same, and according to the lemma, they have the same multiplicity. The number of zeros of $c \mapsto \rho_W(c)$, counting multiplicities, is the degree μ of ρ_W. The boundary ∂W is homeomorphic to S^1 and $\rho_W : \partial W \to S^1$ is of degree μ. Since it is increasing because ρ_W is holomorphic on W, the number of points in $\rho_W^{-1}(1)$ is also μ. ■

We will call μ the multiplicity of W. We will prove in chapter 19 that $\mu = 1$.

6. Counting.

We will now show the additional information 1 of theorem 13.1 in chapter 13. Let $k \in \mathbb{N}^*$. Denote

$m_1(k)$ the number of values of c such that $f_c^{\circ k}(0) = 0$, counting multiplicities;

$m_2(k)$ the number of hyperbolic components of $\overset{\circ}{M}$ of period dividing k;

$m_3(k)$ the number of roots of hyperbolic components of period dividing k;

$m_4(k)$ the number of $t \in \mathbb{T}$ such that $2^k t = t$, i.e., of the form $p/(2^k - 1)$.

By proposition 14.6 we have $m_1(k) = m_2(k) = m_3(k)$ and by proposition 14.5 we have $m_3(k) \geq 2m_4(k) - 1$. But $f_c(0) = c$, $f_c^{\circ 2}(0) = c^2 + c$, $f_c^{\circ 3}(0) = (c^2 + c)^2 + c$, and so on... $f_c^{\circ k}(0) = P_k(c)$ where P_k is a polynomial of degree 2^{k-1}. On the other hand $m_4(k) = 2^k - 1$. So, we have the equality $m_3(k) = 2m_4(k) - 1$. On the one hand, it follows that each root c_0 of a hyperbolic component (except 1/4) has in M exaclty 2 rational external arguments with odd denominator, on the other hand, we obtain in this way every element in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which is rational with odd denominator. This proves the additional information 1 of theorem 13.1.

CHAPTER 15

The order of contact of hyperbolic components of M.

By Tan Lei

Let c_0 be the intersection of 2 hyperbolic components W and W' of $\overset{\circ}{M}$, one of which is of period k, the other of period kq; then f_{c_0} has a rationally indifferent cycle with eigenvalue $e^{2i\pi p/q}$ of period k.

1. Summary of already known results which are useful for the proof.

Let $\{\alpha_1, \dots, \alpha_k\}$ be the rationally indifferent cycle of f_{c_0}, where α_1 is the point attracting c_0. Let P_1 be the petal of α_1 which contains c_0 and $\mathcal{R}(K_{c_0}, \theta)$ be an external ray which lands at α_1 through an interpetal adjacent to P_1. According to theorem 11.1 of chapter 11, the external ray $\mathcal{R}(M, \theta)$ lands at c_0.

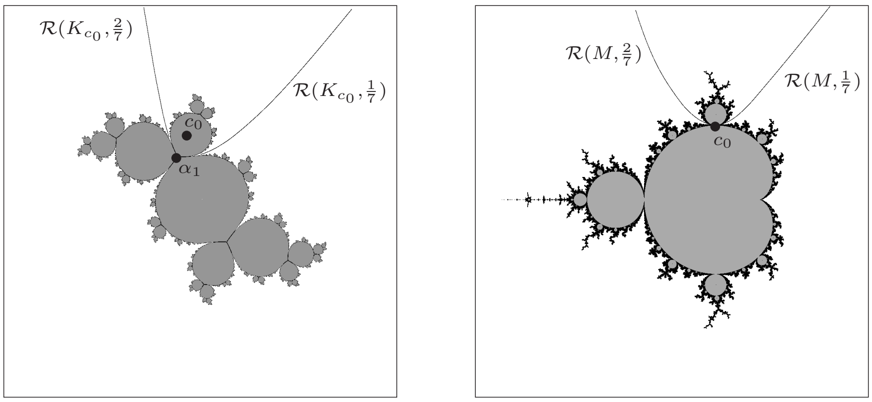


FIGURE 1. The external rays landing at α_1 and at c_0.

As $q \neq 1$, for all c in a neighborhood of c_0, we can find $\alpha(c)$ such that $f_c{}^\circ k(\alpha(c)) = \alpha(c)$ with $c \mapsto \alpha(c)$ analytic and $\alpha(c_0) = \alpha_1$.

Set $c(\lambda) = c_0 + \lambda$,

$$\rho(\lambda) = (f_{c(\lambda)}^{\circ k})'(\alpha(c(\lambda))),$$

$\rho(0) = e^{2i\pi p/q}$ and

$$\tau(\lambda) = \frac{1}{\rho(\lambda)^{q^2}} - 1,$$

$\tau(0) = 0$. Then, $\tau(\lambda)$ has a simple zero at 0 (with multiplicity 1) (cf chapter 12), so $\lambda \mapsto \tau(\lambda)$ is a local homeomorphism in a neighborhood of 0.

If Δ is a sufficiently small disk centered at $\alpha(c_0)$, we take n_0 sufficiently large and $r^* > 1$ sufficiently close to 1 such that

$$x(c_0) = f_{c_0}^{\circ n_0 kq}(c_0)$$

and

$$y(c_0) = \varphi_{c_0}^{-1}(r^* e^{2i\pi\theta})$$

belong to Δ. For c sufiiciently close to c_0, set $x(c) = f_c^{\circ n_0 kq}(c)$ and $y(c) = \varphi_c^{-1}(r^* e^{2i\pi\theta})$. Fix $a \in]0, 1/2]$ and denote by

$$P_a = \{z \mid a \geq |\mathrm{Re}(Z),\ a \geq |\mathrm{Im}(z)\}.$$

If we take

$$\Sigma_N = \{U \mid N \log(1+U) + 2i\pi \in P_a\},$$

we have

$$\Sigma_N \subset \mathbb{D}_{8/N} \setminus \overline{\mathbb{D}}_{4/N}$$

when $N \geq 8$.

2. Proof of the proposition.

Step 1. According to chapter 11, for any neighborhood W_{c_0} of c_0, there exists $N_0 \geq 0$ such that for all $N \geq N_0$, there exists $c_N \in W$ such that $f_{c_N}^{\circ Nkq}(\alpha(c_N)) = y(c_N)$, i.e., $\forall W$ neighborhood of 0, $\exists N_0 \geq 0$, $\forall N \geq N_0$, there exists $\lambda_N \in W$ such that $f_{c(\lambda_N)}^{\circ Nkq}(x(c(\lambda_N))) = y(c(\lambda_N))$ and λ_N is in the piece σ_N of W where $\tau : \sigma_N \to \Sigma_N$ induces a homeomorphism (we can take W sufficiently small so that $\lambda \mapsto \tau(\lambda)$ is a homeomorphism on W).

In a neighborhood of 0, we can write $\tau(\lambda) = a\lambda + \mathcal{O}(\lambda^2)$ with $a \neq 0$ since $\tau(\lambda)$ has a simple zero at 0. Thus, there exists a neighborhood W of 0 such that $\tau(\lambda) - a\lambda| = \mathcal{O}(\lambda^2) < \varepsilon|\lambda|$, $\lambda \in W$, with $0 < \varepsilon < |a|$. For this W, there exists $N_1 \geq 0$ such that $\forall N \geq N_1$, $\sigma_N = \tau^{-1}(\Sigma_N) \subset W$.

For all $\lambda \in \sigma_N \subset W$, we have

$$(|a| - \varepsilon)|\lambda| = |a| \cdot |\lambda| - \varepsilon|\lambda| \leq |\tau(\lambda)| \leq |a| \cdot |\lambda| + \varepsilon|\lambda| = (|a| + \varepsilon)|\lambda|.$$

Since $4/N \leq |\tau(\lambda)| \leq 8/N$, we have:

$$\frac{4}{(|a|+\varepsilon)N} \leq |\lambda| \leq \frac{8}{(|a|+\varepsilon)N}, \qquad \forall \lambda \in \sigma_N.$$

Step 2.

Let U be a simply connected open subset of $\mathbb{C}$. For all $x, y \in U$, $a \in U$, we have the inequality:

$$|y - x| \leq |x - a|(e^{4d_U(x,y)} - 1)$$

where $d_U(x,y)$ stands for the Poincaré distance on U. If $\varphi : U \to V$ is an isomorphism between two simply connected open sets, then $d_U(x,y) = d_V(\varphi(x), \varphi(y))$.

If $\varphi : U \to V$ is an analytic map, then φ is 1-Lipschitz, i.e., $d_V(\varphi(x), \varphi(y)) \leq d_U(x,y)$ for all $x, y \in U$.

The open set $\mathbb{C} \setminus M$ is not simply connected, but if we remove the external ray $\mathcal{R}(M, \theta')$, where

$$\theta' = \begin{cases} \theta + 1/2, & \text{if } \theta < 1/2 \\ \theta - 1/2, & \text{if } \theta > 1/2, \end{cases}$$

$\mathbb{C} \setminus (M \cup \mathcal{R}(M, \theta')) = U$ is open and simply connected. Since $\mathcal{R}(M, \theta')$ lands and avoids $\mathcal{R}(M, \theta)$, U contains $\mathcal{R}(M, \theta)$. If we take $V = \mathbb{C} \setminus (\mathbb{D} \cup \mathcal{R}(\mathbb{D}, \theta'))$, then $\varphi_M : U \to V$ is an isomorphism.

For $N_2 = \max\{N_1, N_0\}$ and $U_2 = \{z \mid \mathrm{Re}(z) > 0, 2\pi\theta - \pi < \mathrm{Im}(z) < 2\pi\theta + \pi\}$, $\forall N \geq N_2$, setting $c_N = c(\lambda_N)$, we have

$$\begin{aligned} d_U(c_N, c_{N+1}) &= d_V(\varphi_M(c_N), \varphi_M(c_{N+1})) \\ &= d_{U_2}(\log \varphi_M(c_N), \log \varphi_M(c_{N+1})) \\ &= d_{U_2}(z_N, z_{N+1}). \end{aligned}$$

We can write $r^* = e^{s_0}$ with $s_0 > 0$. We have

$$f_{c_N}^{\circ Nkq}(x(c_N)) = \varphi_{c_N}^{-1}\left(e^{s_0 + 2i\pi\theta}\right),$$

so

$$\varphi_{c_N}(f_{c_N}^{\circ (N+n_0)kq}(c_N)) = e^{s_0 + 2i\pi\theta},$$

thus

$$[\varphi_{c_N}(c_N)]^{2^{(N+n_0)kq}} = e^{s_0 + 2i\pi\theta}$$

and

$$\varphi_M(c_N) = \varphi_{c_N}(c_N) = e^{\dfrac{s_0}{(N+n_0)kq} + 2i\pi\theta'}$$

with $2^{(N+n_0)kq}\theta' = \theta$.

In K_{c_0}, $\alpha(c_0) = \alpha_1$ has q petals and is fixed by $f_{c_0}^{\circ k}$, so $2^{kq}\theta = \theta$, and we have seen in chapter 12 that in fact, $\theta' = \theta$. It follows that

$$\varphi_M(c_N) = e^{s_N + 2i\pi\theta} \quad \text{where} \quad s_N = \frac{s_0}{(N+n_0)kq},$$

hence c_N is in the external ray $\mathcal{R}(M, \theta)$ and $c_N \to c_0$ as $N \to +\infty$. We deduce that $d_U(c_N, c_{N+1}) = d_{U_2}(s_N + 2i\pi\theta, s_{N+1} + 2i\pi\theta)$.

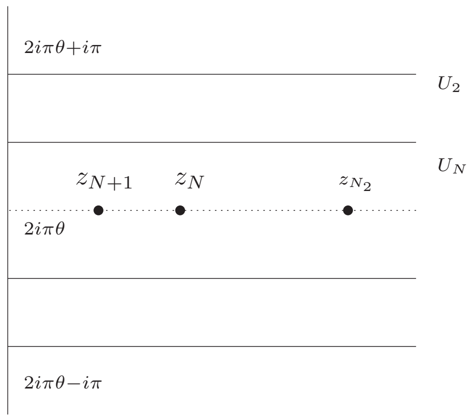


FIGURE 2. The open set U_N.

For each $N \geq N_2$, we take

$$\varphi_N(z) = 2i\pi\theta + \frac{1}{2^{(N-N_2)kq}}(z - 2i\pi\theta),$$

it is an isomorphism which maps z_{N_2} to z_N and which maps U_2 to U_N with U_N of the form

$$U_N = \{z \mid \mathrm{Re}(z) > 0, |\mathrm{Im}(z) - 2\pi\theta| < \varepsilon(N), \quad \varepsilon(N) = \frac{\pi}{2^{(N-N_2)}},$$

$\varepsilon(N) < \varepsilon(N-1) < \ldots < \varepsilon(N_2) = \pi$. The injection $U_n \to U_{N+1}$ is an analytic map and thus it is 1-Lipschitz. We have

$$d_{U_2}(z_N, z_{N+1}) \leq d_{U_N}(z_N, z_{N+1}) = d_{U_2}(z_N, z_{N+1}) \leq A.$$

Finally,

$$d_U(c_N, c_{N+1}) = d_{\mathbb{C}\setminus(M\cup\mathcal{R}(M,\theta'))}(c_N, c_{N+1}) = d_{U_2}(z_N, z_{N+1}) \leq A.$$

For $b_N \in \partial W \cup \partial W'$,

$$|c_N - c_{N+1}| \leq |c_N - b_N| \left(e^{4d_U(c_N, c_{N+1})} + 1\right) \leq B_1 |c_N - b_N|,$$

where B_1 is a constant.

Step 3. Assume the contact between W and W' at the point c_0 is of order greater or equal to 4. Then, for N_3 large enough and $N \geq N_3$, there exists $b_N \in \partial W \cup \partial W'$ such that $|c_N - b_N| \leq B_2 |c_N - c_0|^4$; so, when $N \geq N_3$, we have $|c_N - c_{N+1}| \leq B|c_N - c_0|^4$, where $B = B_1 B_2$.

For $k \geq N_3$, we have

$$|c_k - c_0| \leq \sum_{N=k}^{\infty} |c_k - c_{k+1}| \leq B \sum_{N=k}^{\infty} |c_N - c_0|^4.$$

Since $c_N - c_0 = \lambda_N \in \sigma_N$, according to step 1, there exist two positive constants a_1 and a_2 such that $a_1/N \leq |\lambda_N| \leq a_2/N$, so for all $k \geq N_3$,

$$\frac{a_1}{N} \leq |\lambda_K| = |c_k - c_0| \leq B \sum_{N=k}^{\infty} |\lambda_N|^4 = B a_2^4 \sum_{N_k}^{\infty} \frac{1}{N^4}.$$

This means that for all $k \geq N$,

$$0 < \frac{a_1}{B a_2^4} \leq \sum_{N=k}^{\infty} \frac{k}{N^4} < \sum_{N=k}^{\infty} \frac{N}{N^4} = \sum_{N=k}^{\infty} \frac{1}{N^3}.$$

We have a contradiction because $\sum 1/N^3$ converges.

CHAPTER 16

Identification of cylinders: study of the limiting case.

By Pierre Lavaurs

1. Notations and position of the problem.

The setting is the same as in chapter 12: $f_{c_0} : z \mapsto z^2 + c_0$ has a rationally indifferent cycle with eigenvalue $e^{2i\pi p/q}$ and period k, α_1 is a point in the indifferent cycle.

In chapter 12, we have constructed, for c close to c_0, q Fatou-Ecalle cylinders ($2q$ for $c = c_0$) under the restriction that c tends to c_0 in some given sector.

For technical reasons, we will modify slightly the definition of the cylinders (without changing the cylinders themselves) and the region where c close to c_0 ranges.

Instead of defining Ω'_λ as in chpater 12, we define it by removing from $\mathbb{C}$ only $\overline{\mathbb{D}}_R$, the segment between A'_λ and iR (if $\mathrm{Im}A'_\lambda > 0$; the segment between A'_λ and $-iR$ otherwise) and the half-line $\mathrm{Re}(Z) = 0$, $\mathrm{Im}(Z) \leq -R$ (if $\mathrm{Im}A'_\lambda > 0$; $\mathrm{Im}(Z) > R$ otherwise). By extending the segment from iR (or $-iR$) to A'_λ up to infinity, we get a partition of Ω'_λ into ${\Omega'_\lambda}^+$ and ${\Omega'_\lambda}^-$. Finally, for $c = c_0$, we cut with the half-lines $\mathrm{Re}(Z) = 0$, $\mathrm{Im}(Z) \leq -R$ and $\mathrm{Im}(Z) \geq R$.

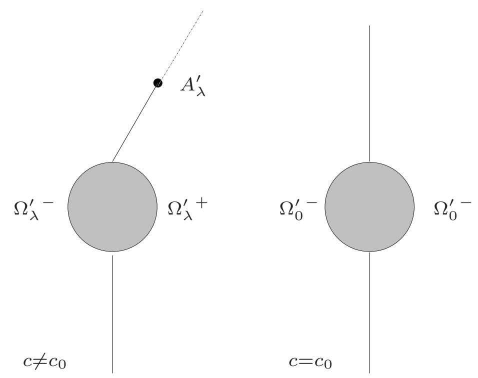


FIGURE 1. The sets ${\Omega'_\lambda}^+$ and ${\Omega'_\lambda}^-$.

The inequalities of chapter 12 show that the points added to Ω'_λ (${\Omega'_0}^+$, ${\Omega'_0}^-$) are equivalent to points of the "old" open sets: this change in the quotiented regions does not modify the quotient cylinders.

In chapter 12, we allowed λ to tend to 0 with $|\arg A'_l \pm 1/4| \leq 1/12$, which is the same, since $1/A'_\lambda$ is holomorphic in λ, to restrict to chosen sectors centered on the 2 half-lines in the λ plane corresponding in the c plane to the half-line which are tangent to the hyperbolic components of $\overset{\circ}{M}$ which are tangent at c_0 (to the half line for $q = 1$). Here, instead of allowing λ to tend to 0 in a sector, we will allow it to tend in between two circles tangent at 0 to the aove half-lines:

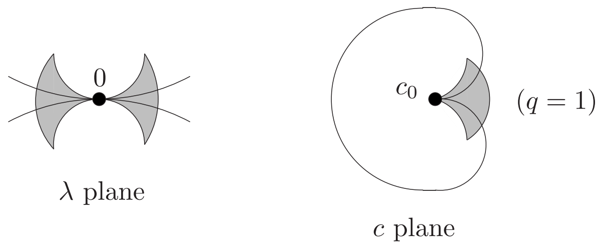


FIGURE 2. The region where λ is allowed to tend to 0.

This restriction is sufficient for applications, because it allows c to take any value outside M sufficiently close to c_0. It has the following advantage: since $1/A_\lambda$ and $1/A'_\lambda$ are analytic with respect to λ, it forces A_λ and A'_λ to stay in between two lines $\mathrm{Re}(Z) = -K$ and $\mathrm{Re}(Z) = +K$. As a consequence, $\{(\lambda, Z) \mid Z \in {\Omega'_\lambda}^-\}$ and $\{(\lambda, Z) \mid Z \in {\Omega'_\lambda}^+\}$ are open subsets of $V \times \mathbb{C}$ (V is the region where λ varies): if a point is in ${\Omega'_0}^-$ (respectively ${\Omega'_0}^+$, it is in ${\Omega'_\lambda}^-$ (respectively in ${\Omega'_\lambda}^+$ for λ sufficiently close to 0.

In order to fix the notations, we assume that we have chosen to stay, in the λ plane, around a half-line corresponding to $\mathrm{Im}A_\lambda > 0$ (and so alos $\mathrm{Im}A'_\lambda > 0$).

Let us look at what this gives in the z plane.

For $c = c_0$, the $2q$ cylinders are the quotient of $2q$ sectors with equal angle in a disk centered at α_1.

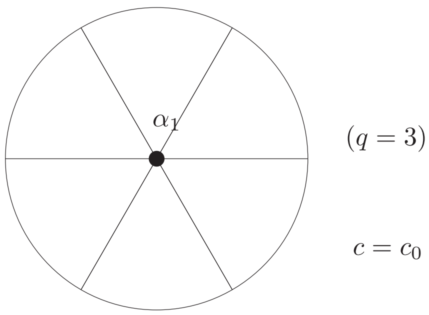


FIGURE 3. The situation for $c = c_0$.

For c close to c_0, $\partial\mathbb{D}_R$ corresponds to a circle "close" to the previous one; α_1 splits into a $\alpha(c)$ fixed by $f_c^{\circ k}$ and a cycle $\beta_1(c), \ldots, \beta_q(c)$ for $f_c^{\circ k}$, α and β_i being continuous functions of c (which varies in the region, that we will denote by Θ, to which we have restricted c). The point α corresponds to ∞ in the Z plane, β_1 (for example) to A'_λ. We have only q cylinders. However, if we also represent (dashed on the figure below) the curves bounding in the z plane the sets ${\Omega'_\lambda}^+$ and ${\Omega'_\lambda}^-$, we see that we may also consider that we have, as for $c = c_0$, $2q$ cylinders which are pairwise canonically identified.

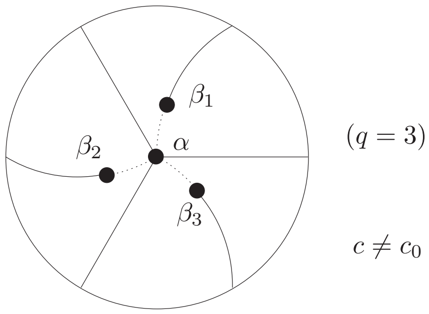


FIGURE 4. The situation for $c \neq c_0$.

We will study the cylinder around $\beta_1(c)$, which becomes 2 when c becomes c_0.

The regions corresponding to Ω'_λ (${\Omega'_\lambda}^+$, ${\Omega'_\lambda}^-$) in the z plane will be denoted by $U(c)$ ($U^+(c)$, $U^-(c)$). The set $\mathcal{U}^+ = \{(c, z) \mid z \in U^+(c)\}$ (respectively $\mathcal{U}^-$) is an open subset of $\Theta \times \mathbb{C}$. The cylinder $U(c)/f_c^{\circ kq}$ will be denoted by $E(c)$ as in chapter 12.

For $c = c_0$, we have one $E^+(c_0)$ and one $E^-(c_0)$; the region providing $E^-(c_0)$ is completely contained in a component of $\overset{\circ}{K}_{c_0}$, the one providing $E^+(c_0)$ contains an interpetal.

The *outgoing curve* will be the curve $\partial U \cap \partial U^+$. The *incoming curve* will be the curve $\partial U \cap \partial U^-$. They both join α and β_1.

The *outgoing fundamental domain* W_0 in U^+ will be bounded by the outgoing curve and its preimage by $f_c^{\circ kq}$ in U^+. Similarly, the *incoming fundamental domain* Y_1 in U^- will be bounded by the incoming curve and its image by $f_c^{\circ kq}$.

The following figure sketches the dynamics of $f_c^{\circ kq}$ in U.

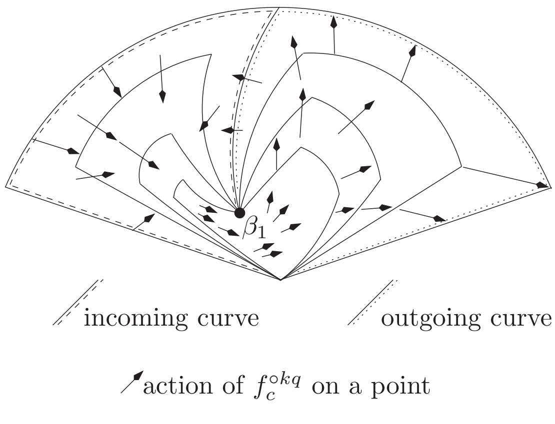


FIGURE 5. The dynamics of $f_c^{\circ kq}$ in U.

W_0 and Y_1 are fundamental domains for the cylinders (identified for $c \neq c_0$) of f_c in a neighborhood of α_1 (by convention, we consider that neither W_0 nor Y_1

contain the points α and β_1 and contain the incoming curve for Y_1 and the outgoing curve for W_0).

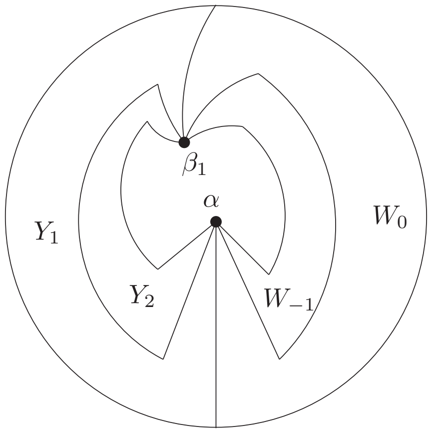
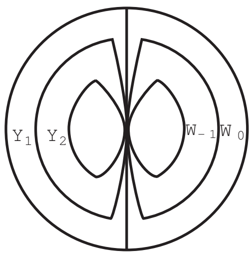


FIGURE 6. The sets Y_i and W_i.

For $i \geq 0$, the region Y_{i+1} will be defined as $(f_c^{\circ kq})^{\circ i}(Y_1)$. For $i < 0$ not too small, we will define W_i inductively. Assume W_{i+1} is defined as a subset of U bounded by curves mapped one onto the other by $f_c^{\circ kq}$; in that case W_{i+1} does not contain critical values of $f_c^{\circ kq}$: we can define the branch of $(f_c^{\circ kq})^{-1}$ on W_{i+1} which maps one of the two bounding curves onto the other; the image of this branch will be W_i under the condition that it is contained in U.

The inequalities of chapter 12 show that W_i is defined for $|i| \leq .8|A|/\pi$, so on a subset of $\mathbb{Z}$ which tends to $\mathbb{Z}^-$ as $c \to c_0$.



Let us choose a marked point P^+ in $U^+(c_0)$ and a point P^- in $U^-(c_0)$: for c close to c_0, they are both in $U(c)$; therefore they provide a point $\widetilde{P}^-(c)$ and a point $\widetilde{P}^+(c)$ in $E(c)$ (in $E^-(c_0)$ and $E^+(c_0)$ for $c = c_0$).

All the cylinders E^+ and E^- can have their extremities labelled in a natural way; one corresponds to a neighborhood of ∞ in the Z plane, the other to A'_λ; on the fundamental domain W_0 or Y_1, this corresponds on the one hand to a neighborhood of α, on the other hand to a neighborhood of β_1. When c becomes c_0, α and β_1 are identified, but in the fundamental domain corresponding to W_0 or Y_1 in the Z plane, the end taht was corresponding to a neighborhood of β_1 is represented by $\text{Im}(Z) > 0$, the other by $\text{Im}(Z) < 0$. Therefore, we can talk, even for $c = c_0$ of the end at β_1 and the end at α of the cylinder.

The marked point and the marking of the ends of the cylinder provide an identification of E^+ with $\mathbb{C}^*$, sending $\widetilde{P}^+$ to 1 and the end at β_1 to the end at 0. Similarly, we will identify E^- to $\mathbb{C}^*$.

It may be convenient to modelize the cylinders by $\mathbb{C}/\mathbb{Z}$ instead of $\mathbb{C}^*$. In this model, $\widetilde{P}$ is identified with 0, the end at β_1 to the end $\mathrm{Im}(Z) < 0$. $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^*$ are indentified by $z \mapsto \exp(-2i\pi Z)$, it is easy to transfer the results that we will obtain for the model $\mathbb{C}^*$ to the model $\mathbb{C}/\mathbb{Z}$.

For $c \neq c_0$, E^+ and E^- are in fact the same cylinder that has been identified via two different isomorphisms φ^+ and φ^- to $\mathbb{C}^*$; $\varphi^+ \circ (\varphi^-)^{-1}$ then provide an isomorphism from $\mathbb{C}^*$ to $\mathbb{C}^*$ which respectes the position of 0 and so, of the form $z \mapsto G(c) \cdot z$, for a $G(c) \in \mathbb{C}^*$.

$G(c)$ contains in a concise way the information on the dynamics of $f_c^{\circ kq}$ between the incoming curve and the outgoing curve: starting with a point in Y_1, iterating a large amot of time, we end in W_0. It may therefore be difficult to control the stability of several phenomenon suting such a large amont of iterations. But, if we look at $E^-(c) = Y_1/f_c^{\circ kq}$ and $E^+(c) = W_0/f_c^{\circ kq}$, the knowledge of $G(c)$ provide a *direct* passage from one to the other.

Since there is no natural identification between $E^-(c_0)$ and $E^+(c_0)$, we may therefore expect that $G(c)$ does not have any limit as c tends to c_0. We will give a Taylor expansion. The purpose of this chapter is to prove the following theorem.

Theorem 16.1. *When c tends to c_0, with the restriction that it belongs to Θ, λ being the coordinate defined in chapter 11 (i.e., $c = c_0 + \lambda$ if $q \neq 1$, $c = c_0 + \lambda^2$ if $q = 1$), we have :*

$$G(c) = \exp\left(g_0 + \frac{k}{\lambda} + o(1)\right).$$

Additional information. *$\rho_1(\lambda)$ being the eigenvalue of the cycle $f_c^{\circ kq}$ containing $\beta_1(c)$, $k = 4\pi^2/\rho_1'(0)$; thus, k/λ is contained in a strip around the postive imaginary axis.*

Remark. We stated here the theorem with the model of $\mathbb{C}^*$. If we prefer to work with $\mathbb{C}/\mathbb{Z}$, multiplication by G is replaced by the translation identified with an element G of $\mathbb{C}/\mathbb{Z}$, and the Taylor expansion becomes: $G(c) = G_0 + k/\lambda + o(1)\ (\mathrm{mod}\mathbb{Z})$, where here, $k = 2i\pi/\rho_1'(0)$ is such that k/λ belongs to a strip around the negative imaginary axis.

Example. $c_0 = 1/4$, the fixed points of f_c for c close to c_0 are $(1 \pm \sqrt{1-4c})/2$. Let us take λ close to the postivie real axis. In order to have $\mathrm{Im}(A'_\lambda) > 0$, we must choose $\alpha = (1+i\lambda)/2$ and $\beta = (1-i\lambda)/2$; so $\rho_1'(0) = -i$ and $G(c) = \exp(g_0 + 4\pi^2 i/\lambda + o(1))$ or, in the model $\mathbb{C}/\mathbb{Z}$, $G(c) = G_0 - 2\pi/\lambda + o(1)$.

2. Continuity of the projection on E.

The choice ot a basepoint enabled us to identify all the E^- (respectively E^+) to $\mathbb{C}^*$. We then have the following proposition.

Proposition 16.1. *The map $\pi^+ : \mathcal{U}^+ \to \mathbb{C}^*$ (respectively $\pi^- : \mathcal{U}^- \to \mathbb{C}^*$) which maps (c, z) to the projection of z on $E^+(c)$ (respectively $E^-(c)$) identified with $\mathbb{C}^*$ is continuous.*

Remark. What this proposition asserts (and it is in those terms that we will recall it when we use it) is that a limiting picture in the plane provides a limiting picture in the cylinders.

Proof.

Lemma 16.1. *Let C_R be a ring contained in between the circles $|s| = 1/R$ and $|z| = R$, with $R > e^{2\pi}$, and let h be a univalent map from C_R to $\mathbb{C}^*$ so that the image of C_R surronds 0, on the ssame side of C_R with $h(1) = 1$.*

Then, for $u \in C_R$, we have an inequality $|h(u) - u| \leq f(u, R)$ where, when u varies in a compact set K of C^* and R tends to infinity (so $K \subset \mathbb{C}_R$ for R sufficiently large), $f(u, R)$ tends to 0 uniformly on K.

Proof. Let first $f : \mathbb{D} \to \mathbb{C}$ be a univalent map with $f(0) = 0$ and $f(a) = a$ (for $a \in]0, 1[$). Then, for $|z| \leq r$, we have a branch of $\log(f(z)/z)$ which satisfies

$$\left|\log \frac{f(z)}{z}\right| \leq 2|\log(1 - r)(1 - a)|.$$

Indeed, for $g : \mathbb{D} \to \mathbb{C}$ univalent with $g(0) = 0$ and $'(0) = 1$, we have for all $z \in \mathbb{D}$ (cf. [**Go**], page 117, inequality (19)):

$$\left|\log \frac{g(z)}{z} + \log(1 - |z|^2)\right| \leq \log \frac{1 + |z|}{1 - |z|}, \tag{3}$$

where $\log(g(z)/z)$ is the continuous branch defined on $\mathbb{D}$ and which maps 0 to 0.

Let us take $g(z) = f(z)/f'(0)$. Applying (3) at $z = a$, we find $|\log f'(0)| \leq 2|\log(1 - a)|$ for a well chosen branch of $\log f'(0)$; applying (3) at z, we then find a branch of $\log(f(z)/z)$ such that

$$\left|\log \frac{f(z)}{z} - \log f'(0)\right| \leq 2|\log(1 - |z|) \leq 2|\log(1 - r)|;$$

and so

$$\left|\log \frac{f(z)}{z}\right| \leq 2|\log(1 - r)(1 - a)|.$$

We will show that for

$$\exp(\sqrt{\log^2 R - \pi^2})^{-1} \leq |u| \leq \exp(\sqrt{\log^2 R - \pi^2}),$$

there is a branch of $\log h(u)$ such that, $\log u$ being the principal branch,

$$\left|\frac{\log h(u)}{\log u}\right| \leq \left(1 - \frac{2\pi}{\log R}\right)^{-2} \left(1 - \frac{\sqrt{\log^2 |u| + \pi^2}}{\log R}\right)^{-2},$$

which provides an inequality of the required form. We can undoubtely refine considerably this inequality: the following proof indeed uses wery few information with respect to what the situation could bring. However, this inequality will be sufficient for our purposes.

h provides by passing to the logarithms a univalent map g of the strip $-\log R \leq \mathrm{Im}Z \leq \log R$ in $\mathbb{C}$; we can assume $h(0) = 0$: given the assumption on the relative position of 0 and $h(C_R)$, we have $g(2i\pi) = 2i\pi$. Let us consider the map $f : z \mapsto g(zi \log R)/\log R$ univalent from $\mathbb{D}$ to $\mathbb{C}$. We have $f(0) = 0$ and $f(2\pi/\log R) = 2\pi/\log R$. The bounds in between u varies have been chosen so that the principal branch of $\log u$ has a sufficiently small modulus so that $|\log u|/\log R$ is bounded from above by $r = \sqrt{\log^2|u| + \pi^2}/\log R \le 1$, and so $\log u/\log R \in \mathbb{D}$. It follows that there exists a branch of

$$\log \frac{f(\log u/\log R)}{\log u/(i\log R)} = \log \frac{g(\log u)}{\log u} = \log \frac{\log h(u)}{\log u}$$

for a branch of $\log u$ which is such that

$$\log \frac{\log h(u)}{\log u} \le 2\left|\log\left(1-\frac{2\pi}{\log R}\right)\left(1-\frac{\sqrt{\log^2|u|+\pi^2}}{\log R}\right)\right|;$$

and so

$$\frac{\log h(u)}{\log u} \le \left(1-\frac{2\pi}{\log R}\right)^{-2}\left(1-\frac{\sqrt{\log^2|u|+\pi^2}}{\log R}\right)^{-2}.$$

□

We now come back to the proof of the propostion. We will show the continuity of π^- at a point (c_1, z_1) of $\mathcal{U}^-$ (for π^+ it is sufficient to replace Y_i by W_i in the proof below; to lighten the notations, the minus sign for π^- and $E(c)^-$ will be omitted in this proof).

We take i such that $z_1 \in Y_i(c_1)$.

If z_1 is not on the bondary of $Y_i(c_1)$, for (c, z) sufficiently close to (c_1, z_1), $z \in Y_i(c)$.

If it is on the boundary of Y_i, this is not a real problem: by increasing slightly R in the proof given in chapter 12, the cylinders are not changed, but Y_i is silghtly shifted, which bring us back in the case $z_1 \in \overset{\circ}{Y}_i(c_1)$ that we will assume from now on.

For $r > 0$, we will define $Y_i^{(r)}(c)$: on the curve bounding $Y_i(c)$ which is mapped to the other by $f_c^{\circ kq}$, we tka the points at distance r from α and β_1 and we truncate $Y_i(c)$ by the segments joining those points to their images.

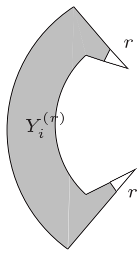


FIGURE 7. The set $Y_i^{(r)}$.

Then, $Y_i^{(r)}(c)/f_c^{\circ kq}$ is a cylinder $E^{(r)}(c)$ of finite modulus. On each $Y_i(c)$, $\widetilde{P}^+(c)$ provides a marked point, which is therefore in $Y_i^{(r)}(c)$ for r sufficiently small (it depends continuously on c since it is the image by $f_c^{\circ kqi}$ of a given point in the z-plane). Thus, there is a unique way to identify conformally $E^{(r)}(c)$ with a ring $C_{R_1R_2}$ of the form $R_1 < |z| < R_2$ in C^*, sending the marked point to 1 and respecting the relative position to 0 of $E^{(r)}(c)$ embedded in $E(c)$ identified with

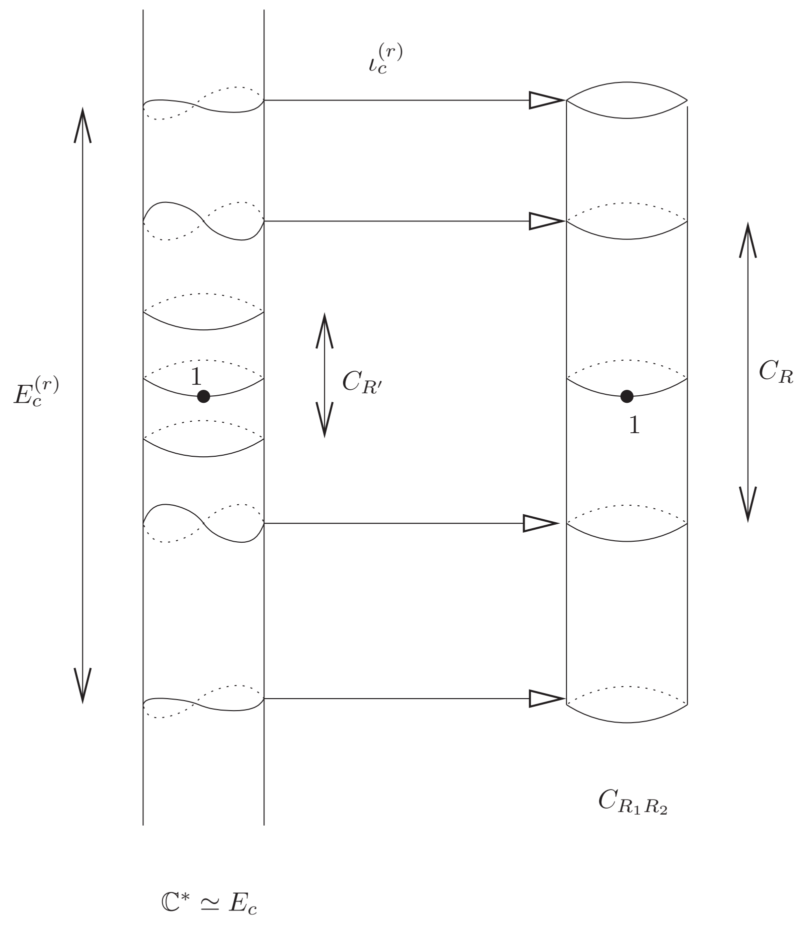
C^*. We will denote by $\iota^{(r)}(c)$ the isomorphism between $E^{(r)}(c)$, considered as a subset of $\mathbb{C}^*$ and $C_{R_1R_2}$.

It is clear that R_2/R_1, the modulus of $E^{(r)}(c)$, tends to ∞ as r tends to 0; it is less clear that R_2 tends to ∞ and R_1 tends to 0.

This is a consequence of Teichmüller extremal problem (cf [**A**], pages 35–37): for r sufficiently small, $E^{(r)}(c)$ contains an annulus C_R symmetric with respect to the unit circle, of modulus $2M$ arbitrarily large.

$\iota_c^{(r)}$ maps the unit circle to a curve containing 1: the annulus bounded by this curve and the circle $|z| = R_2$ therefore has a modulus larger than M. Teichmüller extremal problem guaranties (with the notations of [**A**]) that $M \leq \frac{1}{2\pi} \log \psi(R_2) \leq \frac{1}{2\pi} \log[16(R_2+1)]$ (cf [**A**], page 47); so R_2 tens to infinity as r tends to 0. Similarly, R_1 tends to 0 with r.

We therefore know that if r is sufficiently small, $C_{R_1R_2}$ contains the ring C_R for a fixed R. Applying the same argument to $[\iota_c^{(r)}]^{-1}(C_R)$ for R sufficiently large, $[\iota_c^{(r)}]^{-1}(C_R)$ contains $C_{R'}$ for a fixed R'.



We can map $\overline{Y_i^{(r)}(c)}$ to $\overline{Y_i^{(r)}(c_1)}$ by a C^1 morphism $\Phi_{(c,c_1)}^{(r)}$ which commutes with the dynamics of $f_c^{\circ k}$ on the curves which bound $\overline{Y_i^{(r)}(c)}$ and $\overline{Y_i^{(r)}(c_1)}$, weds the marked point on the marked point and tend to the identity for the C^1 norm as c tends to c_1.

$\Phi^{(r)}_{(c,c_1)}$ induces a map $\widetilde{\Phi}^{(r)}_{(c,c_1)}$ from $E^{(r)}_c$ to $E^{(r)}_{c_1}$ which leaves 1 fixed and is quasiconformal with a dilatation ratio which tends to 1.

$\iota^{(r)}_{c_1} \circ \widetilde{\Phi}^{(r)}_{(c,c_1)} \circ \iota^{(r)}_c$ then is a map between to "true" rings, which fixes 1 and is quasiconformal with a dilatation ratio which tends to 1. It therefore tends to the identity uniformly on every compact set.

As a consequence, if $\pi^{(r)}(c,z)$ stands for $\iota^{(r)}_c \circ \pi(c,z)$ for $z \in Y^{(r)}_i(c)$, $\pi^{(r)}(c,z)$ tends to $\pi^{(r)}(c_1,z)_1)$ as (c,z) tends to (c_1,z_1).

Then, let $\varepsilon > 0$ be fixed. We choose a R' sufficiently large so that $\pi(c_1,z_1) \in C_{R'}$. We then take R_1 sufficiently large so that, if $C_{R_1} \subset \iota^{(r)}_c(E^{(r)}(c))$, we have $C_{R'} \subset \iota^{(r)-1}_c(C_{R_1})$. Then, we choose R_2 sufficiently large so that $\iota^{(r)}_{c_1}(E^{(r)}(c_1))$ (and thus, also $\iota^{(r)}_c(E^{(r)}(c))$ for c sufficiently close to c_1) contains C_{R_2}.

Since $\pi(c_1,z_1) \in C_{R'} \subset \iota^{(r)-1}_c(C_{R_1})$, $\pi^{(r)}(c_1,z_1) \in C_{R_1}$. Let us apply the lemma to $\iota^{(r)-1}_{c_1}$ on C_{R_2}: we obtain

$$|\pi(c_1,z_1) - \pi^{(r)}(c_1,z_1)| < \varepsilon/3.$$

For (c,z) wufficiently close to (c_1,z_1), we then have: $|\pi^{(r)}(c_1,z_1) - \pi^{(r)}(c,z)| < \varepsilon/3$ and $\pi^{(r)}(c,z)$ is still in C_{R_1}.

Applying the lemma to $\iota^{(r)-1}_c$, we have $|\pi(c,z) - \pi^{(r)}(c,z)| < \varepsilon/3$. And so, $|\pi(c,z) - \pi(c_1,z_1)| < \varepsilon$. ■

Corollary 16.1. *G is continuous (on $\Theta \setminus \{c_0\}$).*

Proof. Let c_1 be a point where we want to check the continuity of G. Then, there exists an n such that $f^{\circ n}_{c_1}(P-) \in W_{-1} \subset U^+(c_1)$. For c sufficiently close to c_1 so that $f^{\circ n}_c(P^-) \in U^+(c)$, we then have $G(c) = \pi^+(c, f^{\circ n}_c(P^-))$. ■

3. The germ F.

By construction of the cylinders, a neighborhood of β_1 in W_0 is mapped by $f^{\circ kq}_c$ onto a neighborhood of β_1 in Y_1; this phenomenon remains true for $c = c_0$ (β_1 becomes α_1, and the neighborhood restricts to a "horn" of the region W_0).

Passing to the quotient, we obtain a map $F(c)$ holomorphic and bijective from a neighborhood of the end at 0 of $E^+ = Y_0/f^{\circ kq}_c$ identified with $\mathbb{C}^*$ and the end at 0 of $E = Y_1/f^{\circ kq}_c$ also identified with $\mathbb{C}^*$.

Considered as a map from a neighborhood of 0 in $\mathbb{C}^*$ to a neighborhood of 0 in $\mathbb{C}^*$, $F(c)$ extends continuously by $[F(c)](0) = 0$: extended in this way, it is holomorphic and does not have a critical point at 0.

We will denote $L(c)$ the map from $\mathbb{C}^* \simeq E^+ \to \mathbb{C}^* \simeq E^-$ which maps z to $[F(c)]'(0) \cdot z$. It is the morphism of cylinders tangent to $F(c)$.

In order to compute a Taylor expansion of $G(c)$, we will study $L(c) \circ G(c)$ (we identifie abusuvely the complex number $G(c)$ and the map $z \mapsto G(c) \cdot z$). The advantage is that we have a automorphism of a cylinder, which we can study by looking at its behaviour at an end, forgetting the information contained at the marked point.

$F(c)$ is also defined for c_0: we may expect to have $L(c) \to L(c_0)$ when $c \to c_0$, which will enabe us to pass from the Taylor expansion of $L(c) \circ G(c)$ to a Taylor expansion of $G(c)$.

4. Limiting behavior of $L(c)$.

Proposition 16.2. *When $c \to c_0$, $L(c) \to L(c_0)$.*

Proof. Observe that we can find a fixed r such that, in the z-plane, the intersection of the disk centered at β_1 with radius r and the "end" of W_0 close to β_1 (i.e., the one which to corresponds to $\mathrm{Im}Z > 0$ in the Z-plane) is mapped by $f_c^{\circ kq}$ in Y_1; "a limiting position in the plane corresponding to a limiting position in the cylinder", there is a fixed neighborhood of 0 in $\mathbb{C}^*$ where $F(c)$ is defined.

Let us take a closed curve γ with index 1 around 0 inside this neighborhood. We can develop it in $U^+(c_0)$ as an arc (not closed) Γ_{c_0}; by the proposition in section 2, we can develop it for c close to c_0 as an arc Γ_c close to Γ_{c_0} (in the sense of uniform convergence).

$L(c)$ can then be computed with the help of Cauchy's formula on γ by:

$$\frac{1}{2i\pi}\int_\gamma \frac{[F(c)](u)}{u^2}du = \frac{1}{2i\pi}\int_{\Gamma_c} \frac{\pi_c^-(f_c^{\circ kqN}(z))}{[\pi_c^+(z)]^2}\frac{d\pi_c^+}{dz}dz$$

where π_c stands for the map $z \mapsto \pi(c,z)$ and N is chosen sufficiently large so that $f_c^{\circ kqN}(\Gamma_c) \subset U^-(c)$. In this integral, the path as the function depend continuously on c, which gives the required result. ■

5. Study of $L(c) \circ G(c)$ $(c \neq c_0)$.

$L(c) \circ G(c)$ is the tangent map to $F(c) \circ G(c)$ in a neighborhood of 0 in $E(c) \simeq \mathbb{C}^*$. It is determined by the purely local knowledge of the dynamics ot $f_c^{\circ kq}$ around β_1.

We will see that it can be computed very easily.

As announced in section 1, $\rho_1(\lambda)$ stands for the eigenvalue of the cycle $f_c^{\circ kq}$ containing $\beta_1(c)$; when $|\rho_1(\lambda)| \neq 1$, $f_c^{\circ kq}$ is linearizable in a neighborhood of β_1. We will set $\rho_1 = \rho_1(\lambda) = re^{2i\pi\theta}$. As we are in the case $\mathrm{Im}A'_\lambda > 0$, we have $\mathrm{Im}\rho_1 < 0$.

Let us consider $g : \mathbb{C}^* \to \mathbb{C}^*$ which maps z to $\rho_1 z$ and $p : \mathbb{C} \to \mathbb{C}^*$ the universal covering which maps u to $\exp u$. g induces a map $\tilde{g} : \mathbb{C} \to \mathbb{C}$; we will suppose that it sends 0 to the principal determination $\log \rho_1$ of the logartihm of ρ_1, so $\tilde{g} : u \mapsto u + \log \rho_1$. $\tilde{\tau}$ will stand for the map $u \mapsto u + 2i\pi$ from $\mathbb{C}$ to $\mathbb{C}$ (change of leaf). We have $\tilde{\tau} \circ \tilde{g} = \tilde{g} \circ \tilde{\tau}$.

We can identify the cylinder $\mathbb{C}/\tilde{g}$ to $\mathbb{C}^*$ by sending the "end at $\mathrm{Re}u < 0$ to 0 (ρ_1 is not real); $\tilde{\tau}$, which commutes with $\tilde{g}$, then indces an isomorphism $K(\rho_1)$: $\mathbb{C}^* \to \mathbb{C}^*$.

Lemma 16.2. *If $|\rho_1(c)| \neq 1$, $L(c) \circ G(c) = K(\rho_1)$.*

Proof. Let us take a neighborhood of β_1, sufficiently small so that $f_c^{\circ kq}$ is linearizable on it; we can conjugate analytically $f_c^{\circ kq}$ on this neighborhood to g in a neighborhood of 0. The slit Γ which separates Y_1 from W_0 becomes a curve Γ which has a tangent at 0.

$U \setminus \Gamma / f_c^{\circ kq}$ can be identified with $\mathbb{C}/\tilde{g}$.

Let us examine what $F(c) \circ G(c)$ (defined in the neighborhood of β_1) becomes in this model. We start with a point y in the fundamental domain for g bounded by Γ and $g(\Gamma)$; we consider the successive iterates of y by g until we come back in this domain after iterating n times. In terms of the universal covering, we strat from $\log y$ (principal determination) and we apply $\tilde{g}$ n times: we end on $\log(g^{\circ n}(y)) - 2i\pi$ (log still being the principal branch) which is therefore $\tilde{g}$-equivalent to $\log y$, the map deduced from $F(c) \circ G(c)$ by identifying $U \setminus \Gamma / f_c^{\circ kq}$ to $\mathbb{C}/\tilde{g}$ maps the projection of $\log y$ to the one of $\log(g^{\circ n}(y))$, which is $\tilde{g}$-equivalent to $\tilde{\tau}(\log y)$: it is therefore the restriction of K to a neighborhood of 0, and so $L(c) \circ G(c) = K(\rho_1)$. ■

We can now prove the theorem stated in section 1. We only have to compute $K(\rho_1)$. $\mathbb{C}/\tilde{g}$ can be identified with $\mathbb{C}^*$ via $u \mapsto \exp(-2i\pi u/\log\rho_1)$ (since $\mathrm{Im}(\log\rho_1) < 0$, the minus sign enables to send the end $\mathrm{Re}u < 0$ to the end at 0). $K(\rho_1)$ sends the projection of 0 to the one of $2i\pi$, so 1 to $\exp(4\pi^2/\log\rho_1)$, and so $L(c) \circ G(c) = \exp(4\pi^2/\log\rho_1)$. $\rho_1(\lambda)$ is of degree 1 at 0; for $q = 1$, $\rho_1(\lambda) = \rho(-\lambda)$ is of degree 1 by corollary 12.2 of theorem 12.1 in chapter 12; for $q > 1$, this follows from the analysis of the shape of the hyperbolic components of M tangent at c_0 (and in fact, in an inderect way of this corollary 12.2). We can therefore write $\log\rho_1(\lambda) = \rho_1(0) \cdot \lambda + o(\lambda)$ with $\rho_1(0) \neq 0$; and so

$$L(c) \circ G(c) = \exp\left[\frac{4\pi^2}{\rho_1'(0) \cdot \lambda} + K_0' + o(1)\right].$$

Since $L(c)$ has a limit,

$$G(c) = \exp\left[\frac{4\pi^2}{\rho_1'(0) \cdot \lambda} + C_0 + o(1)\right].$$

A priori, this expansion is only valid when $|\rho_1(\lambda)| \neq 1$, but since $G(c)$ is continuous, it is valid everywhere.

CHAPTER 17

A property of continuity.

By Pierre Lavaurs

1. Bifurcation of external rays with rational arguments.

For a $\theta \in \mathbb{R}/\mathbb{Z}$, we will denote by $\mathcal{R}(M,\theta)$ the external ray of M with argument θ, and by $\mathcal{R}(K_c,\theta)$ the external ray of K_c (filled-in Julia set of $z \mapsto z^2+c$) parametrized by the potential. Tis last ray can bifurcate: in that case, it is not defined on all $\mathbb{R}_+^*$. We will suppose that the rays are oriented in the sense of decreasing potentials: progressing on a ray will mean moving from ∞ towards M or K_c. Finally, note that even when $\mathcal{R}(M,\theta)$ or $\mathcal{R}(K_c,\theta)$ lands, we do not consider the landing point as belonging to the ray.

In the all chapter, we will assume that $\theta \in \mathbb{Q}/\mathbb{Z}$.

Proposition 17.1. *The set of points $c \in \mathbb{C}$ for which $\mathcal{R}(K_c,\theta)$ bifurcates is $\bigcup_{n\geq 1} \mathcal{R}(M,2^n\theta)$ (and since $\theta \in \mathbb{Q}/\mathbb{Z}$ is preperiodic for the multiplication by 2, the union of finitely many external rays of M).*

Proof. If $c \in \mathcal{R}(M,2^n\theta)$ for some $n \geq 1$, the external argument of c for K_c is $2^n\theta$. As aconsequence, the ray $\mathcal{R}_{K_c}(2^{n-1}\theta)$ meets 0, and bifurcates, and so, $\mathcal{R}_{K_c}(\theta)$ also bifurcates since it is a $(n-1)$-th preimage of this ray.

Conversely, if c is not on a ray $\mathcal{R}(M,2^n\theta)$, $n \geq 1$, let $2^l\theta,\dots,2^{l+d-1}\theta$ be the cycle under multiplication by 2 on which θ ends. The rays $\mathcal{R}(K_c,2^j\theta)$ $(l \leq j \leq l+d-1)$ are defined in a neighborhood of infinity, so on an interval $[t,\infty[$ and do not encounter c on this interval. Then, taking for each j the preimage by $z \mapsto z^2+c$ of $\mathcal{R}(K_c,2^{j+1}\theta)$ restricted to $[t,2t]$ with terminal extremity $\mathcal{R}(K_c,2^j\theta)(t)$, we extend all the rays $\mathcal{R}(K_c,2^j\theta,\ l \leq j \leq l+d-1$, to $[t/2,\infty[$, still avoiding c. Iterating the process, we see that the rays $\mathcal{R}(K_c,2^j\theta)$ can be defined on $\mathbb{R}_+^*$ for all integer $l \leq j \leq l+d-1$. The same is true for the rays $\mathcal{R}_{K_c}(2^i\theta)$, $0 \leq i \leq l-1$, that we can construct similarly by induction on $l-i$ since the do not contain c for $i \geq 1$. ■

Thus, for $c \in \mathbb{C} \setminus \bigcup_{n\leq 1} \mathcal{R}_M(2^n\theta)$ (which contains M), the ray $\mathcal{R}(K_c,\theta)$ exists.

By proposition 8.4 in chapter 8, it lands at a point that we will denote by $\gamma_c(\theta)$.

If K_c is locally connected, $\gamma_c(\theta)$ is the value at c of the Carthéodory lopp γ_c defined on $\mathbb{R}/\mathbb{Z}$; otherwise, $\gamma_c(\theta)$ has only a meaning on $\mathbb{Q}/\mathbb{Z}$.

2. Statements and first results.

The goal of this chapter is to prove the following theorem.

Theorem 17.1. *For $\theta \in \mathbb{Q}/\mathbb{Z}$, $\gamma_c(\theta)$ depends continuously on c in* $\mathbb{C} \setminus \bigcup_{n \leq 1} \mathcal{R}_M(2^n\theta)$.

Proposition 8.5 in chapter 8 provides a proof in most cases: indeed, it states that $\gamma_c(\theta)$ in a neighborhood of $c_0 \in \mathbb{C}$, and continuous as a function of c as soon as $\gamma_{c_0}(\theta)$ is preperiodic and repelling and is not on the backward orbit of the critical point (if we denote by $f_c : z \mapsto z^2 + c$, with the notations of this proposition, $\gamma_c(\theta) = \psi_{f_c,\theta}(0)$).

Therefore, there are two remaining cases:

- the one of a Misiurewicz point c_0 when $\gamma_{c_0}(0)$ is on the backward orbit of 0;
- the one of the root c_0 of a hyperbolic component, when $\gamma_{c_0}(\theta)$ is on the backward orbit of the rationally indifferent cycle (which also contains this cycle).

Remark. Using the conclusions of chapters 8 and 12, we can easily see that we already know the continuity of $c \mapsto \gamma_c(\theta)$ on $\mathbb{C} \setminus \bigcup_{n \leq 1} \overline{\mathcal{R}_M(2^n\theta)}$.

3. Case of a Mizurewicz point.

Let c_0 be a Misiurewicz point and $\theta \in \mathbb{Q}/\mathbb{Z}$.

Thus, we assume that there exists $n \in \mathbb{N}$ such that $f_{c_0}^{\circ n}(\gamma_{c_0}(\theta)) = \gamma_{c_0}(2^n\theta) = 0$.

Then, $\mathcal{R}(K_c, 2^{n+1}\theta)$ lands at c_0, and so $\mathcal{R}_M(2^{n+1}\theta)$ also by theorem 8.2 in chapter 8.

By the additional information 1 to lemma 13.1 of chapter 13, none of the rays $\mathcal{R}(M, 2^p\theta)$ for $p \neq n+1$ lands at c_0. Since those are in finite number, there exists a neighborhood Λ of c_0 which meets none of those rays.

We then have the following statement, which is analog to proposition 8.5 in chapter 8.

Proposition 17.2. *The map $(c, s) \mapsto \psi_{f_c,\theta}(s)$ from $[\Lambda \setminus \mathcal{R}_M(2^{n+1}\theta)] \times \mathbb{R}_+^*$ to $\mathbb{C}$ is continuous.*

Proof. By proposition 8.5 in chapter 8, $(c, s) \mapsto \psi_{f_c,2^{n+1}\theta}(s)$ is continuous on $[\Lambda \setminus \mathcal{R}_M(2^{n+1}\theta)] \times \mathbb{R}_+^*$; we will now prove by a decreasing induction on i that $(c, s) \mapsto \psi_{f_c,2^i\theta}(s)$ is continuous on $[\Lambda \setminus \mathcal{R}_M(2^{n+1}\theta)] \times \mathbb{R}_+^*$

Lemma 17.1. *Let $f : E \to F$ be a covering map with F locally ocnnected, A be a topological space and $g : A \times [0, 1] \to F$ (respectively $A \times]0, 1] \to F$) be continuous. Let $h : A \times [0, 1] \to E$ (respectively $A \times]0, 1] \to E$) be a lift of g (i.e., $f \circ h = g$) which is*

- *continuous with respect to $t \in [0, 1]$ (respectively $t \in]0, 1]$)*
- *continuous at points $(a, 1)$, $a \in A$.*

Then, h is continuous on $A \times [0, 1]$ (respectively $A \times]0, 1]$).

Proof. It is enough to show that for $a \in A$,

$$T_a = \{t \mid h \text{ is continuous at the point } (a, t)\}$$

is open and closed in $[0, 1]$ (respectively $]0, 1]$).

• It is open: let (a, t) be a point at which h is continuous; choose a connected neighborhood V of $g(a, t)$ sufficiently small so that f^{-1} is trivial above V, and let

f' be the unique continuous branch of f^{-1} defined on V and which takes the value $h(a,t)$ at $g(a,t)$. Then, by continuity of h at (a,t), for (b,s) sufficiently close to (a,t), $h(b,s) = f'[g(b,s)]$, so h is continuous at (a,s) for s sufficiently close to t.
• It is closed: let $t \in \overline{T}$, V as above and U a neighborhood of (a,t) of the form $W \times I$ with I interval, such that $g(U) \subset V$. By continuity of h in the second variable, for s in I, $h(a,s) = f'[g(a,s)]$; let us choose such an s in T; by continuity of h at the point (a,s), for b sufficiently close to a, $h(b,s) = f'[g(b,s)]$, so, by continuity in the second variable, and since $g(\{b\} \times I) \subset V$, $h(b,u) = f'[g(b,u)]$ for b sufficinetly close to a and u in I, which gives the continuity of h at (a,b). □

Let us now apply this lemma: we will choose for A the set $\Lambda \setminus \mathcal{R}(M, 2^{n+1}\theta)$, we will replace $[0,1]$ (or $]0,1]$) by $[0,\infty]$ (or $]0,\infty]$) in order to work with the parametrization by potentials, E and F will both be $\mathbb{C}^*$ to which we add a point at infinity in each direction of half-line, and f is the map $z \mapsto z^2$ extended in the obvious way at infinity.

The map $(c,s) \mapsto \psi_{f_c,2^i\theta}(s)$ extends to $s = \infty$ by mapping (c,∞) to the point at infinity in F in the direction $2^i\theta$.

<u>Going from $i = n+1$ to $i = n$</u>

We can consider $g : A\times]0,\infty] \to F$

$$(c,s) \mapsto \psi_{f_c,2^{n+1}\theta}(s) - c$$

since $A \cap \mathcal{R}(M, 2^{n+1}\theta) = \emptyset$ this map never takes the values zero and so, takes its values in F.

g satifies the hypothesis of the lemma.

$$h : A\times]0,\infty] \to E$$

$$(c,s) \mapsto \psi_{f_c,2^n\theta}(s)$$

is then a lift of g which satisfies the hypothesis of the lemma: it is therefore continuous on $A\times]0,\infty]$.

Moreover, if we extend g to $[0,\infty]$, which is possible (it no longer takes its values in F but in $F \cup \{0\}$), we know it is continuous. Then, for all $\varepsilon > 0$, for c sufficiently close to c_0 and s sufficiently small, $\left|\psi_{f_c,2^{n+1}\theta}(s) - c\right| < \varepsilon^2$ so, $|\psi_{f_c,2^n\theta}(s)| < \varepsilon$.

<u>Going from $i+1$ to i for $i < n$.</u>

For $i < n$, the step from the proposition for $i+1$ to the proposition for i is then easier: indeed, we can directly apply the le;;a on $[0,\infty]$ for $g : (c,s) \mapsto \psi_{f_c,2^{i+1}\theta}(s) - c$ which takes its values in F, since Λ has been chosen sufficiently small in order to avoid the rays $\mathcal{R}(M, 2^p\theta)$ different from $\mathcal{R}(M, 2^{n+1}\theta)$. ■

4. Case of points with a rationally indifferent cycle.

We will work in the same setting and with the same notations as in chapter 16.

θ is a rational number with odd denominator such that $\mathcal{R}_{K_c}(\theta)$ lands on α_1 in K_{c_0}.

In order to simplify the presentation, the proof will be explained in the case $q = 1$. We will show at the end how to modify it for an arbitrary q.

We will restrict to let c tend to c_0 in a region Θ delimited by two curves tangent with order 2 at c_0, we will assume that Θ entirely contains the region contained in between the two hyperbolic components of M which are tangent at c_0 (for $q = 1$, the region the region contained outside the hyperbolic component whose boundary has a cusp at c_0).

We will show the following three propositions, for c in $\Theta \setminus \bigcup_{n\geq 1} \mathcal{R}(M, 2^n\theta)$.

Proposition 17.3. *The landing point of $\mathcal{R}(K_c, \theta)$ is α or β.*

Proposition 17.4. *For n fixed and c sufficiently close to c_0, $\mathcal{R}(K_c, \theta)$ does not go through $f_c^{\circ n}(0)$.*

Proposition 17.5. *Fr $\theta_1 \in \mathbb{Q}/\mathbb{Z}$ such that there exists $d \in \mathbb{N}$ such that $2^d\theta_1 = \theta$, the landing point of $\mathcal{R}(K_c, \theta_1)$ is continuous as a function of c (and in particular, it is defined for c sufficiently close to c_0).*

Proposition 17.5 implies theo 17.1; indeed, for a θ_c such that $2^d\theta = \theta$, it asserts that $\gamma_c(\theta_c)$ tends to $\gamma_c(\theta_1)$ as c tends to c_0 in $\Theta \setminus \bigcup_{n\geq 1} \mathcal{R}(M, 2^n\theta)$.

Now, if c tends to c_0 in M, the landing point of $\mathcal{R}_{K_c}(\theta_1)$ is a continuous function of c, since it is a repelling preperiodic point. Since c tends to c_0 in $M \cap \Theta$, it tends to $\gamma_{c_0}(\theta_1)$, it also tends to $\gamma_{c_0}(\theta_1)$ as c tends to c_0 in $\mathring{M}$, and so, we get the continuity at c_0.

Proposition 17.3 is a first step to proposition 17.5, proposition 17.4 is the key to the additional information 2 of chapter 13.

4.1. Bounding the distortion of the ray in U. For $c = c_0$, the ray $\mathcal{R}(K_{c_0}, \theta)$ lands at α_1 with a tangent perpendicular to the diameter bouding U^+. Increasing R in the construction of chapter 12 if necessary, we can therefore assume that this ray is transverse to the outgoing curve. (which is the curve by which the ray enters U^+ with our convention of orienting with decresing potentials !); we will take $s(c_0)$ as the smallest parameter where $\mathcal{R}_{K_{c_0}}(\theta)$ intersects the outgoing curve; for c close to c_0, $\mathcal{R}(K_c, \theta)$ intersects the outgoing curve at a point corresponding to a parameter $s(c)$ close to $s(c_0)$, since the restriction of this ray to $[s(c_0)/2, \infty[$ is a continuous function of c.

On $[s(c_0)/2^k, s(c_0)]$, the ray $\mathcal{R}(K_{c_0}, \theta)$ evolve on a finite number of regions W_i $(i \leq 0)$, which we will denote M_1.

The restriction of $\mathcal{R}(K_c, \theta)$ to $[s(c)/2^k, s(c)]$ being continuous as a function of c, and since a limiting position in the plane provides a limiting position on the cylinder (cf. chapter 16 proposition 16.1), for c sufficiently close to c_0, this curve is still on $M_1 + 1$ domains W_i at most.

This arc of ray defines on the cylinder an injective loop; we can therefore rebuild the ray step by step as long as we stay in U: in this way, in between $s(c)$ and $s(c)/2^k$, $\mathcal{R}(K_c, \theta)$ goes along a loop in $W_{-M_1}, \ldots, W_0$; on $[s(c)/(2^k)^2, s(c)/2^k]$, it will go along a loop in $W_{-M_1-1}, \ldots, W_{-1}$; if i_0 is the largest integer such that W_{-i_0} is defined, we will therefore be able to go back to a loop on $W_{-i_0}, \ldots, W_{-i_0+M_1}$.

As $i_0 \geq 0.8|A|/\pi$, for c sufficiently close to c_0, there is at least an arc of $\mathcal{R}(K_c, \theta)$ defined on an interval of the form $[t/2^k, t]$ evolving in W_{j_1} and W_{j_2} for $j_1 = \lfloor -.3|A|/\pi \rfloor$ and $j_2 = \lfloor -.7|A|/\pi \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integer part).

According to the inequality of chapter 12, a W_j for $j_2 \leq j \leq j_1$ is completely covered by finitely many Y_i's. Therefore, the arc $\mathcal{R}_{K_c}(\theta)$ evolves in between t and $t/2^k$ in a finite number $M(c)$ of domains Y_i. Moreover, t can be chosen so that $\mathcal{R}_{K_c}(\theta)(t)$ is on one of the ∂Y_i and so that in between t and $t/2^k$ $\mathcal{R}(K_c, \theta)$ evolves in Y_i's, with $i' \leq i$: it is enough to choose t the least possible so that $\mathcal{R}_{K_c}(\theta)(t)$ is in Y_i among the $t \geq s(c)/(2^k)^{i_0-M_1}$. Lifting the arc sufiiciently many times, we find a w such that $\mathcal{R}(K_c,\theta)$ evolves in between w and $w/2^k$ on $Y_1 \cup \ldots \cup Y_{M(c)}$ and has its extremities on $\partial Y_{M(c)}$ (w depends on c).

A priori, $M(c)$ depends on c. But we have the following lemma.

Lemma 17.2. *We can bound $M(c)$ by a bound M which does not depend on c.*

Proof. Putting a fixed base point on W_0 and one on Y_1, we know (cf chapter 16) that the morphism between marked cylinders identified with $\mathbb{C}/\mathbb{Z}$ defined by the identification of $W_0/f_c^{\circ k}$ and $Y_1/f_C^{\circ k}$ has when c tends to c_0 (in the region to which we retricted ourselves) an asymptotic expansion of the form $G(c) = G_0 + k/\lambda + o(1)$ (mod$\mathbb{Z}$); $G(c)$ evolves in a strip around $\mathbb{R}^-$.

We can then unroll $Y_1/f_c^{\circ k}$ and $W_0/f_c^{\circ k}$ on $\mathbb{C}$, sending the marked points to $\mathbb{Z}$. The restriction of $\mathcal{R}(K_c,\theta)$ to $[s/2^k, s]$ and to $[w/2^k, w]$ each unroll as an arc of curve (compact), γ_s and γ_w where by convention we lift $W_{-M_1} \cup \ldots \cup W_0$ and $Y_1 \cup \ldots \cup Y_{M(c)}$ are lifted by lifting the marked points in W_0 or Y_1 to 0.

The restriction of $\mathcal{R}(K_c,\theta)$ to $[s/2^k, s]$ has a limiting position in the plane as c tends to c_0, so also on the cylinder; thus, γ_1 has a limiting position in $\mathbb{C}$, and therefore stays, for c sufficiently close to c_0, in a rectangle around 0.

γ_w is obtained after γ_s by a translation by $-G'$, where G' is a lift of $G(c)$.

But the above expansion shows that the imaginary part of G, so also of G', is bounded as c tends to c_0: γ_w therefore staysat a finite height, so in a strip B bounded by two horizontals.

As c tends to c_0, the boundary of Y_1 has a limiting position in the plane, so the curve that it defines on the cylinder also has a limiting position, uniformly on every compact; the lift of Y_1 in $\mathbb{C}$ restricted to the strip B therefore has a limiting position; since γ_w intersects this lift, it stays at a bounded distance from zero, so in a rectangle of $\mathbb{C}$.

Passing to the limit, a finite number of lifts of domains Y_i ($i \geq 1$) intersect this bounded rectangle, so $M(c)$ is bounded □

Multiplying W by a power of 2^k for some c if necessary, we will assume that $\mathcal{R}(K_c,\theta)(w)$ and $\mathcal{R}(K_c,\theta)(w/2^k)$ are on ∂Y_M, and so, that in between those two potentials, $\mathcal{R}_{K_c}(\theta)$ evolves in $Y_1 \cup \ldots \cup Y_M$. This arc is obtained by unrolling, with extremities on ∂Y_M, the loop defined on the cylinder by the first entry of the ray, in between s and $s/2^k$.

We will denote by $\omega(c)$ the critical point of $f_c^{\circ k}$ varying continuously with c and equal for c_0 to the critical point of $f_{c_0}^{\circ k}$ located in the component of $\overset{\circ}{K}_{c_0}$ adjacent to α_1: there exists i_0 such that $f_c^{\circ i_0}(\omega) = 0$, with $0 \leq i_0 < k$.

Moreover, there exists a n_0 such that $f_c^{\circ n_0}(\omega)$ is in Y_i for c sufficiently close to c_0; we can finally assume that, looking at c even closer to c_0, that for $j < k(n_0+1)$ and $j \neq kn_0$, $f_c^{\circ k}(\omega) \notin U$. We will denote by $\tilde{\omega}$ the point on the cylinder corresponding to $(f_c^{\circ k})(\omega)$.

Proposition 17.6. *c belongs to $\mathcal{R}(M, 2^{i_0}\theta)$ if and only if the injective loop defined on the cylinder by the "first entry" of $\mathcal{R}(K_c,\theta)$ in U goes through $\tilde{\omega}$.*

Proof.

• On the one hand, if the loop defined on the cylinder by the "first entry" of $\mathcal{R}(K_c,\theta)$ goes through $\tilde{\omega}$, according to what we just said, on $[w/2^n, w]$, the loop goes through a point of U projecting to $\tilde{\omega}$ on the cylinder, and located in a Y_i $(1 \le i \le M)$ so through a $(f_c^{\circ k})^{\circ n_0+i}(\omega)$ $(0 \le i \le M-1)$: the argument of chapter 12 then shows that $\mathcal{R}(K_c,\theta)$ also goes through $f_c^{\circ k}(\omega)$; it follows that $\mathcal{R}_{K_c}(2^{i_0}\theta)$ goes through c, so c belongs to $\mathcal{R}_M(2^{i_0}\theta)$.

• On the other hand, if c belongs to $\mathcal{R}_M(2^{i_0}\theta)$, $\mathcal{R}(K_c,\theta)$ goes through $f_c^{\circ 2k}(\omega) = f_c^{\circ k-i_0}(c)$ so also through all the $f_c^{\circ ki}(\omega)$ $(i \ge 2)$ and the loop defined on the cylinder by the "first entry" of $\mathcal{R}(K_c,\theta)$ goes through $\tilde{\omega}$. ■

Let us now analyze what happens when we are not on $\mathcal{R}_M(2^{i_0}\theta)$: the loop on the cylinder is then, since injective, homotopic on the cyloiner minus $\tilde{\omega}$ to a parallel located on one side or the other of $\tilde{\omega}$; coming back to the unrollments on $Y_1 \cup \ldots \cup Y_M$, we see that this expresses that the arc of loop $\mathcal{R}(K_c,\theta)|[w/2^k, w]$ can be brought in

$$(Y_1 \cup \ldots \cup Y_M) \setminus \left\{(f_c^{\circ k})^{\circ n_0+i}(\omega),\ (0 \le i \le M-1)\right\}$$

on α or β, with the constraint that the extremities stay on ∂Y_M, so that one remains the image of the other by $f_c^{\circ k}$ during the whole homotopy.

Given the setting $(\Theta \setminus \bigcup_{n\ge1} \mathcal{R}_M(2^i\theta))$ in which we stated propositions 17.3 to 17.5, we are outside $\mathcal{R}(M, 2^{i_0}\theta)$, so in the case analized above.

4.2. Return in U.

Proposition 17.7. *There exists an $N \ge 1$ (independent of c) such that on $[w/(2^k)^{N+1}, w/(2^k)^N]$, $\mathcal{R}(K_c,\theta)$ is in $U^+(c)$.*

Remark. This proposition asserts that the ray, which after the interval $[w/2^k, w]$ will soon cross the incoming curve and exit from the region U, is well behaved outside U and comes back in U^+ after a bounded time.

Proof.

α) First control on the position of the ray.

For $n \ge 0$ fixed and all i such that $1 \le i \le M$, $(f_c^{\circ k})^{-n}(Y_i)$ has a finite number of connected components, at most 2^{kn}. They will be called "zones" in the following.

In between $w/(2^k)^n$ and $w/(2^k)^{n+1}$, $\mathcal{R}(K_c,\theta)$ evolves in the $2^{kn}M$ (at most) zones defined in this way.

Lemma 17.3. *Actually, it evolves at most on $M \cdot 2^M$ such zones, for c sufficiently close to c_0.*

Remark. The proximity condition depends on n but it will not be a problem in the following.

Proof. As above we take n_0 such that $(f_c^{\circ k})^{\circ n_0}(\omega) \in Y_1$, and we assume c is sufficiently close to c_0 so that no $(f_c^{\circ k})^{\circ i}(\omega')$ for $0 \le i \le n$ and ω' critical point of $f_c^{\circ k}$ except the $(f_c^{\circ k})^{\circ j}(\omega)$ $(n_0 \le j \le n_0 + M - 1)$ is in $Y_1 \cup \ldots \cup Y_M$.

In between w and $w/2^k$, $\mathcal{R}(K_c,\theta)$ evolves in M zones.

In between $w/2^k$ and $w/(2^k)^2$, it a priori evolves in at most $2^k \cdot M$ zones: the connected components of preimages of $Y_1 \cup \ldots \cup Y_M$ by $F_c^{\circ k}$; but (if $n_0 \ge 1$) $Y_1 \cup \ldots \cup Y_M$ does not contain critical values of $f_c^{\circ k}$: the 2^k preimages of $Y_1 \cup \ldots \cup Y_M$

are therefore pairwise distinct, and since the arc of ray corresponding to the $t \in [w/(2^k)^2, w/2^k]$ is connected, it can only evolve in one of the determinations of $(f_c^{\circ k})^{-1}(Y_1 \cup \ldots \cup Y_M)$, so on M zones at most.

So, each time we take a preimage, we will stay on the same number of zones, except when we take the preimage of a region containing a critical values of $f_c^{\circ k}$. Thanks to the condition imposed on hte position of the points in the critical orbit with respect to $Y_1 \cup \ldots \cup Y_M$, this occurs M times exactly; those times, the region of which we take the preimage contains exactly one critical value which is $f_c^{\circ k}(\omega)$. Its preimages are grouped in $2^k - 1$ connected components, usually composed of M zones, except the one of ω which is composed with $2M - 1$ zones. In any case, this step at most doubles the number of zones on which the ray evolves.

Finally, for the n-th preimages, we have the required upper bound $M \cdot 2^M$. □

The ray, which does not go through $\tilde{\omega}$ when we look at the cylinder, can be brought back, as mentioned above, to α or β (in a sense made precise at the end of 4.1). We will see that the knowledge of the position of the ray with respect to $\tilde{\omega}$ on the cylinder is enough to determine 2^M branches (at most) of $(f_c^{\circ k})^{-n}$ (some being 2-valued) to apply to each Y_i $(1 \leq i \leq M)$ to obtain the $M \cdot 2^M$ (at most) zones defined above.

Indeed, let V be the simply connected region that is covered by the ray when we deform it to α (or β) on $Y_1 \cup \ldots \cup Y_M$ without corssing the points of $(f_c^{\circ k})^{\circ j}(\omega)$ $(n_0 \leq j \leq n_0 + M - 1)$. $(f_c^{\circ k})^{-n}(V)$ then has 2^{kn} connected components, with disjoint closures; only one contains α (respectively β) in its closure. This one is contained in the region where the ray in between $w/(2^k)^n$ and $w/(2^k)^{n+1}$ evolves, because the homotopy that brings the ray toward α (respectively β) can be lifted to a homotopy that brings the ray toward α (respectively β) in $(f_c^{\circ k})^{-i}(Y_1 \cup \ldots \cup Y_M)$ $(0 \leq i \leq n)$. The 2^M branches (at most) of $(f_c^{\circ k})^{-n}$ to be considered are therefore the ones that send $Y_1 \cup \ldots \cup Y_M$ on the connected component of $(f_c^{\circ k})^{-n}(Y_1 \cup \ldots \cup Y_M)$ which contains α (respectively β) in its closure.

β) Remark on the dynamics of $z \mapsto (3z^2+1)/(z^2+3)$.

We will use the conjugacy between the dynamics of $f_{c_0}^{\circ k}$ on the connected component of $\overset{\circ}{K}_{c_0}$ adjacent to α_1 and $F : z \mapsto \dfrac{3z^2+1}{z^2+3}$ on $\mathbb{D}$.

We will denote by $\widehat{U}^-$, $\widehat{Y}_i$ (for $i \geq 0$) the images of $U^-(c_0)$, $Y_i(c_0)$ by this conjugacy; $\widehat{U}^+$ will be the image of the intersection of $U^+(c_0)$ with the component of $\overset{\circ}{K}_{c_0}$ adjacent to α_1.

We will have to define a curve γ in $\mathbb{D}$: γ will be defined on $\mathbb{R}^*$ and will satisfy:

(a) $\mathrm{Im}\gamma \geq 0$
(b) on $[1, 2[$, $\gamma(t) = t/3 - 1/3$
(c) $\gamma(2t) = F[\gamma(t)]$.

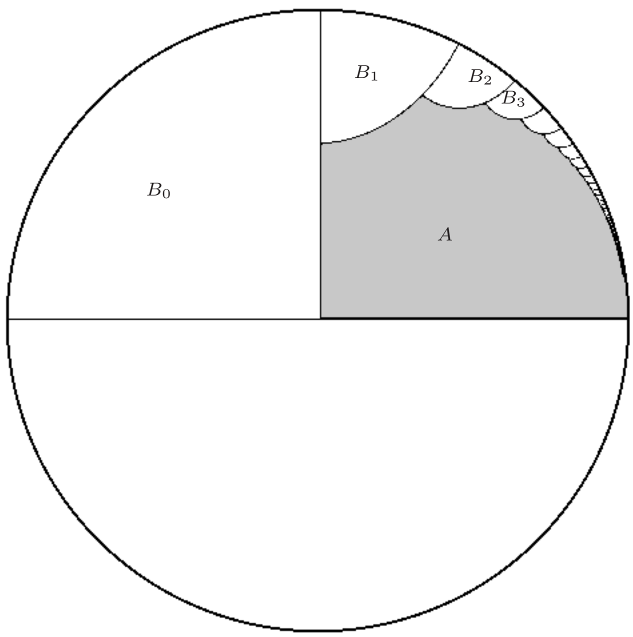
This curve γ is well defined and unique: indeed, (c) enables us to define it on $[1, +\infty[$ where it then covers the segment $[0, 1[$ of the real axis; in between $1/2$ and 1, condition (c) gives two choices for each value of $\gamma(t)$, but (a) determines this value: so, on this interval, γ ranges in the segment joining $i/\sqrt{3}$ to 0; in a similar way, (c) and (a) allows us to determine without ambiguity γ on each $[1/2^{n+1}, 1/2^n[$. The curve obtained in such a way is continuous: indeed, it is continuous on each interval $[2^p, 2^{p+1}[$, $p \in \mathbb{Z}$ and by construction, since $\lim_{t\to 2^-} \gamma(t) = \gamma(2)$, we have continuity at points of the form 2^p.

Lemma 17.4. *As t tends to 0, $\gamma(t)$ tends to 1 with a vertical tangent.*

Proof. It will be convenient to make the change of variable $w = (1+z)/(1-z)$. The map $F : z \mapsto \dfrac{3z^2+1}{z^2+3}$ from $\mathbb{D}$ to $\mathbb{D}$ becomes $F_1 : w \mapsto w + 1/w$ from the half-plane $\mathrm{Re}w > 0$ into itself. The curve $\gamma(t)$ becomes a curve $\eta(t)$ which satisfies: (a) $\mathrm{Im}(\eta) > 0$, (b) on $[1,2[$, $\eta(t) = t$, (c) $\eta(2t) = F_1[\eta(t)]$. It is more convenient to consider η', parametrized by the change of variable $\eta'(u) = \eta(2^u)$: we must show that when $u \to -\infty$, $\eta'(u)$ tends to infinity in the asymptotic direction $y'y$. Given the expression of F_1, a point of $\mathrm{Im}z > 0$ has as preimage in $\mathrm{Im}z > 0$ a point located further to the left and above; so, for $u \le 1$, $\mathrm{Re}\eta'(u) \le 2$: the asymptotic direction will be obtained as soon as $\mathrm{Im}[\eta'(u)]$ will tend to ∞; also, we see that if $\varphi_n(u) = \mathrm{Im}[\eta'(u-n)]$, the sequence $\varphi_n(u)$ is increasing for each $u \in [0,1]$. But for each fixed $u \in [0,1]$, the sequence $\eta'(u-n)$ has a limit (possibly infinity), since its real part decreases and its imaginary part increases. This limit must therefore be a fixed point of F_1, so infinity: so, the sequence φ_n tends to infinity, uniformly by Dini's theorem, which guaranties that $\eta'(u) \to \infty$ as $u \to -\infty$. □

The curve γ therefore bounds a region A in D. We will denote by B_0 the upper left quadrant of $\mathbb{D}$

$$B_{i+1} = [F^{-1}(B_i)] \cap (\mathrm{Im}z > 0).$$



Lemma 17.5. *The sets B_i tend to 1 as $i \to \infty$ (in the sense that $\forall \varepsilon > 0$, $\exists i_0$ such that $\forall i \ge i_0$, $\forall z \in B_i$, $|1-z| < \varepsilon$).*

Proof. If this were not the case, the B_i's would have another accumulation point than 1 in $\overline{\mathbb{D}}$. This point a would not be on $\partial\mathbb{D}$: indeed, the monotonicity of the action of F on $\partial\mathbb{D}$ guaranties that every point in $\partial\mathbb{D}$ eventually lands on ∂B_1; however, every forward image of a must be an accumulation point of the sequence B_i.

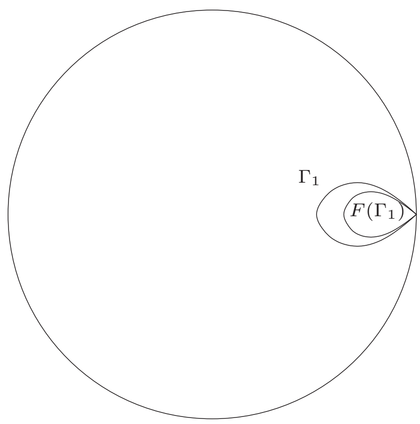
a is therefore in $\mathbb{D}$, and its orbit is "trapped" in $(\mathrm{Im} z > 0) \setminus (B_0 \cup A)$.

But lemma 17.4 then forbid the $F^{\circ n}(a)$ to tend towards 0 tangentially to the ray, thus leading to a contradiction, given proposition 9.2. ■

γ) Choice of N.

We will now apply those considerations on the dynamics of F to obtain informations on $F^{-n}(\widehat{Y}_1 \cup \ldots \cup \widehat{Y}_M)$.

Let us look at the shape of $\widehat{Y}_1$: it is bounded by two curves Γ_1 and $F(\Gamma_1)$ having tangents at 1, which make angles $\pm\pi/4$ with the real axis.



Increasing R if necessary, we can assume that they belong to the union of the closure of A and the rgion conjugate to A (it is better not to try to note this with usual notations !), transverse to the real axis only at one pont.

Motivated by the study done in α), we will only be interested in the branches of F^{-n} that send $\widehat{Y}_1 \cup \ldots \cup \widehat{Y}_M$ in a region containing 1 in its closure.

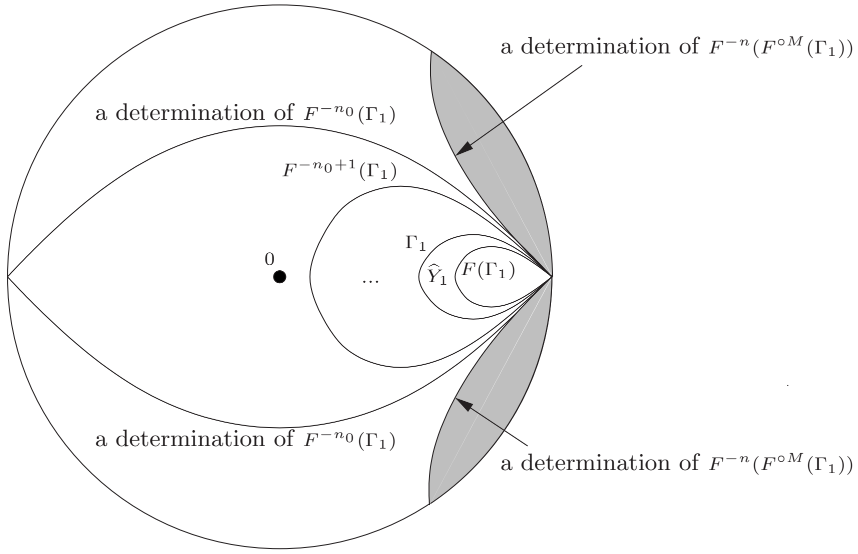
Those connected components of $F^{-n}(\widehat{Y}_1 \cup \ldots \cup \widehat{Y}_M)$ are then in the regions bounded by the boundary of the disk and the two determinations of $F^{-n}(F^{\circ M}(\Gamma_1))$ which start at 1 (cf. shaded zone in the picture below).

Lemma 17.6. *There exists N_0 such thatr for $N \geq N_0$, the two connected components of $F^{-n}(\widehat{Y}_1 \cup \ldots \cup \widehat{Y}_M)$ which contain 1 in their closures are in $\widehat{U}^+$.*

Proof. It is clearly enough to prove it for the connected component of $F^{-n}(\widehat{Y}_1 \cup \ldots \cup \widehat{Y}_M)$ located above the real axis.

This one is in the region bounded by the boundary of $\mathbb{D}$ and a curve, determination of $F^{-N}[F^{\circ M}(\Gamma_1)]$.

The curve Γ_1' which bounds $\widehat{U}^+$ is composed of an arc in A, which coincides with Γ_1 in a neighborhood of 1, then leaves it to go to $\partial\mathbb{D}$, bounding in such a way a region (the upper half of $\widehat{U}^+$) in $\mathbb{D}$.

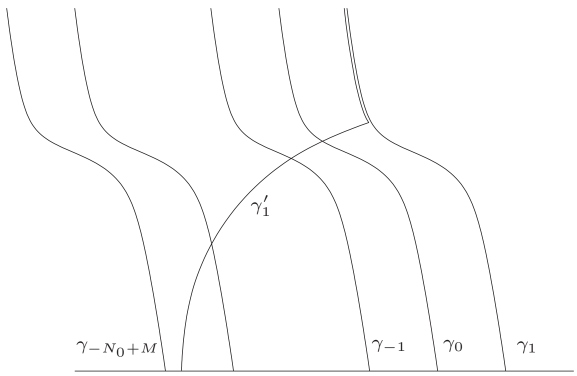


It is enough to show that for N large enough, $F^{-N+M}(\Gamma_1)$ (where we take the determination with positive imaginary part starting at 1) is in this region.

We will distinghuish $F^{-N+M}(\Gamma_1 \cap A)$ and $F^{-N+M}(\Gamma_1 \cap \overline{A})$ (here $\overline{A}$ stands for the conjugate of A).

F restricted to A is bijective: we can therefore conjugate this restriction of F to A to a bijective transformation g of the Poincaré half-plane; $F_{|A}$ has an indifferent fixed point on ∂A: we can therefore choose for g the parabolic transformation $g(Z) = Z + 1$.

The curve $\Gamma_1 \cap A$ is then represented in this model as a curve γ_1 joining a point on the real axis to ∞, with asymptotic direction $y'y$ (corresponding to the tangent to Γ_1 making an angle of $\pi/4$ with the real axis), and $\Gamma_1' \cap A$ is represented as a curve γ_1' coinciding with γ_1 in a neighborhood of infinity, landing on the real axis at a point Choice offurther to the left than the landing point of γ_1; the determination of F^{-1} one must apply to γ_1 in order to find γ_{-N+M} (corresponding to $F^{-N+M}(\gamma_1 \cap A)$) is $Z \mapsto Z - 1$: it is clear that for N large enough, we are to the left of γ_1', and so, $F^{-N+M}(\gamma_1 \cap A)$ is in $\widehat{U}^+$.



For $F^{-N+M}(\Gamma_1 \cap \overline{A})$, we will use lemma 17.5: by assumption made at the beginning of part γ) of the proof on the position of Γ_1, $F^{-N+M}(\Gamma_1 \cap \overline{A})$ evolves in B_{N-n_0-M} (for $N \geq n_0 + M$), so is close to 1 for N large enough, and is also in $\widehat{U}^+$. ■

δ) Such an N is convenient.

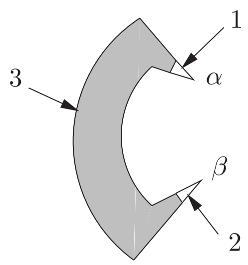
we will choose $N \geq N_0$ (with the additional requirement that $0.3(N-M) \geq 2K$, whose necessity will appear in the following computations), and check that it answers the statement of proposition 17.7.

The branches of $(f_c^{\circ k})^{-N}$ who, applied to $Y_1, \ldots, Y_M$, provide regions in which $\mathcal{R}(K_c, \theta)$ can evolve in between $w/(2^k)^N$ and $w/(2^k)^{N+1}$ are in finite number independent of c; they define in such a way a certain number of "zones", images of Y under iteration of branches of $(f_c^{\circ k})^{-1}$. For $c = c_0$, we have proven in γ) that those zones are all in U^+.

Most of those regions (all except $2M$) do not contain α_1 in their closure (for $c = c_0$): they are therefore relatively compact in $U^+(c_0)$ and stay in $U^+(c)$ for c close to c_0 in Θ.

For the other $2M$, we must study more carefully the behaviour of $f_c^{\circ k}$ in a neighborhood of α or β.

We will cut Y_1 in three pieces: two of them are bounded by small arcs of circles around α or β; the third is the remaining part of Y_1.



This third piece won't be a problem. For $c = c_0$, it does not contain α_1 in its closure.

Let us look at the piece close to α: in the Z-plane, it corresponds to a sector close to $-\infty$ on the side $\mathrm{Im}Z < 0$. It is bounded by a curve which stays at distance R' of zero. Decreasing the size of the three small circles that subdivide Y_1 in three pieces, we may assume that R' is as large as necessary, and in particular $R' > R$.

N being chosen, for R' sufficiently large, the determination of $(f_c^{\circ k})^{-1}$ to be iterated i times ($N - M + 1 \leq i \leq N$) and to be applied to Y_1 to providew the M considered zones close to α, can be written $Z = \left(1 + \dfrac{1}{A_\lambda}\right)^{-1} Z' + 1$ up to an error term, whose modulus is bounded from above by $a/100$ (it is enugh to choose R' sufficiently large in order to make sure to stay outside $\overline{\mathbb{D}}_R$ during N iterations of this $(f_c^{\circ k})^{-1}$). For R' chosen sufficiently large and c sufficiently close to c_0, the real part is then at least increased by .9 each time we apply $(f_c^{\circ k})^{-1}$ and we are sure that after i iterations ($N - M + 1 \leq i \leq N$) in $\mathrm{Re}Z > 0$, $\mathrm{Im}Z < 0$: the small sector around α in Y_1 is mapped in $U^+(c)$ by $(f_c^{\circ k})^{-N+M-1}, \ldots, (f_c^{\circ k})^{-N}$ for c sufficiently close to c_0.

The computation is more delicate close to β. Here, we use the fact that A'_λ, and so also the cut between A'_λ and $i\mathbb{R}$, stays in between two vertical lines $\mathrm{Im}Z = -K$ and $\mathrm{Im}Z = K$.

For $|Z - A| \leq A/2$, the inequalities of chapter 12 show that $(f_c^{\circ k})^{-1}$ looks like the scaling map of center A' and ratio $(1 + 1/A)^{-1}$ up to an error bounded by $\dfrac{2a}{100|A|}|Z - A'_\lambda|$; its i-th iteration ($N - M + 1 \leq i \leq N$) therefore shifts like this scaling map up to an error bounded by $\dfrac{2ai}{100|A|}|Z - A'_\lambda|$; this guaranties that we actually rotated and that we are to the right of the cut between A' and $i\mathbb{R}$.

4.3. Construction of a trap. For c sufficiently close to c_0, we have a $s(c)$ such that $\mathcal{R}_{K_c}(\theta)$ is on the outgoing curve, and that in between s and $s/2^k$, $\mathcal{R}(K_c, \theta)$ evolves in $U^+(c)$.

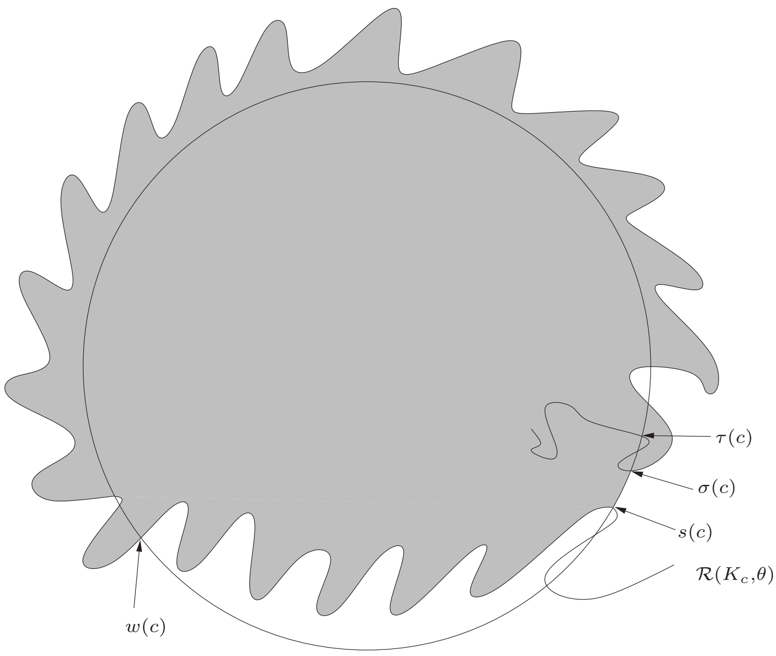
$\mathcal{R}(K_c, \theta)$ eventually intersect the outgoing curve for a potential $w(c)$. Proposition 17.7 then shows that (for c sufficiently close to c_0), there exists $t \leq w(c)$ such that on $[t/2^k, t]$, $\mathcal{R}_{K_c}(\theta)$ again evolves in $U^+(c)$. Let us set

$$\tau(c) = \sup\{t \mid t \leq w \text{ and on } [t/2^k, t],\ \mathcal{R}_{K_c}(\theta)\ |rmevolves\ in\ U^+(c)\}.$$

Now, $\mathcal{R}(K_c, \theta)(\tau(c))$ is on the outgoing curve Γ.

in between $\tau(c)$ and $w(c)$, it is possible that $\mathcal{R}(K_c, \theta)$ intersects Γ at several points; let us call $\sigma(c)$ the parameter contained in between $\tau(c)$ and $w(c)$ corresponding to the intersection of Γ and $\mathcal{R}(K_c, \theta)([\tau(c), w(c)]$ the closets to $\mathcal{R}_{K_c}(\theta)(s(c))$ (on the same side as $\mathcal{R}_{K_c}(\theta)(\tau(c))$.

The capture zone P is then defined as the open set bounded by $\mathcal{R}_{K_c}([\sigma(c), s(c)])$ and Γ in between $\mathcal{R}_{K_c}(\theta)(\sigma(c))$ and $\mathcal{R}(K_c, \theta)(s(c))$.



Let us observe that for c sufficiently close to c_0, $\overline{P}$ contains no critical value of $f_c^{\circ k}$: indeed, regarding $f_c^{\circ k}(\omega')$ for ω' distinct from $\omega(c)$, when $c = c_0$, they are not in the regions covered by the images of the $2 \cdot 2^M \cdot M$ branches of $(f_c^{\circ k})^{-i}$ (for $N - M + 1 \leq i \leq N$) for which the image of Y_1 provide the region where a continuous curve γ, satisfying $\gamma(2t) = f_{c_0}^{\circ k}(\gamma(t))$ and evolving in $Y_1 \cup \ldots \cup Y_M$ in between 1 and 2^k, can evolve in between $1/(2^k)^N$ and $1/(2^k)^{N+1}$; therefore, they are not in those regions for c close to c_0, so they are not in $\overline{P}$.

The same reasonning shows that the ω'' distinct from ω such that $f_c^{\circ k}(\omega'') = f_c^{\circ k}(\omega)$ are not in $\overline{P}$ either.

Moreover, we have a continuous determination of $(f_c)^{-1}$ on ∂P (equal to $\mathcal{R}(K_c, \theta)(u) \mapsto \mathcal{R}(K_c, \theta)(u/2^k)$ on the part of P defined by an arc of the ray $\mathcal{R}(K_c, \theta)$ and by the determination of $(f_c^{\circ k})^{-1}$ which sends W_0 to W_{-1} on the part of ∂P defined by an arc of Γ), which sends ∂P on a simple closed curve $\partial P'$ contained in $\overline{P}$.

If $f_c^{\circ k}(\omega)$ were in $\overline{P}$, one preimage should be in $\overline{P}'$. Since no $\omega'' \neq \omega$ such that $f_c^{\circ k}(\omega'') = f_c^{\circ k}(\omega)$ is in $\overline{P}'$, it would necessarily be ω. But we then have a contradiction, since if $\omega \in P'$, $f - c^{\circ k}$ would map P', simple closed curve turning around a single critical point onto a simple closed curve, and if ω were in $\partial P'$, it would be on $\mathcal{R}(K_c, \theta)$ so c would be on $\mathcal{R}(M, 2^{i_0}(\theta)$, what we excluded.

The continuous determination of $(f_c^{\circ k})^{-1}$ which sends ∂P on $\partial P'$ therefore extends has a univalued holomorphic branch of $(f_c^{\circ k})^{-1}$ which sends P into itself.

Let us finally observe, which will be useful for later purposes, that for $m \geq 0$ fixed, if we take c sufficienlty close to c_0 (the proximity condition depends on m), the same reasonning guaranties that none of the $(f_c^{\circ k})^i(\omega')$ for ω' critical point of $f_c^{\circ k}$ distinct from ω and $1 \leq i \leq m$), and none of the $\omega'' \neq \omega$ such that $(f_c^{\circ k})^{\circ m}(\omega'') = (f_c^{\circ k})^{\circ m}(\omega)$, are in $\overline{P}$; since $f_c^{\circ k}(\omega)$ is not in $\overline{P}$ and since every point in $\overline{P}$ has a preimage in $\overline{P}$, we deduce that the $(f_c^{\circ k})^{\circ i}(\omega)$ $(1 \leq i \leq m)$ are not in $\overline{P}$ either. In terns of f_c, this means that the $f_c^{\circ i}(0)$ are not in $\overline{P}$ for $1 \leq i \leq km$.

We now have the tools to prove the three propositions.

4.4. Proof of proposition 17.3. For $t \leq \sigma(c)$, $\mathcal{R}(K_c, \theta)(t)$ evolves in P: indeed, we see by induction on $n \geq -1$, that the branch of $(f_c^{\circ k})^{-1}$ to be chosen to go from $\mathcal{R}_{K_c}(\theta)\left(\left[\frac{\sigma(c)}{(2^k)^{n+1}}, \frac{\sigma(c)}{(2^k)^n}\right]\right)$ to $\mathcal{R}_{K_c}(\theta)\left(\left[\frac{\sigma(c)}{(2^k)^{n+2}}, \frac{\sigma(c)}{(2^k)^{n+1}}\right]\right)$, maps $\mathcal{R}(K_c, \theta)\left(\frac{\sigma(c)}{(2^k)^n}\right)$ which is in $\overline{P}$ to $\mathcal{R}_{K_c}(\theta)\left(\frac{\sigma(c)}{(2^k)^{n+1}}\right)$ which is also in $\overline{P}$, and so, this branch is the one that sends $\overline{P}$ into itself.

$(f_c^{\circ k})^{-1}$, of degree 1 from an open subset of $\mathbb{C}$ into an open set strictly contained in the first one, is strictly contracting for the Poincaré metric on P; it has a fixed point which is α or β (depending on the position on the cylinder of the loop corresponding to the first entry of the ray); the sequence of $(f_c^{\circ k})^{-n}(u)$ therefore converges to α or β. ■

4.5. Proof of proposition 17.4. We observed that $\overline{P}$ does not contain $f_c^{\circ i}(0)$ $(1 \leq i \leq km)$ for c sufficiently close to c_0; since $\mathcal{R}_{K_c}(\theta)\left[\frac{s(c)}{2^k}, +\infty\right[$ is continuous with respect to c, we see that for c close to c_0, it does not go through $f_c^{\circ i}(0)$ $(1 \leq i \leq km)$ and $\mathcal{R}_{K_c}(\theta)\left]0, \frac{s(c)}{2}\right]$ neither, since it is contained in $\overline{P}$.

4.6. Proof of the additional information 2 of theorem 13.1 in chapter 13. We already know (chapter 13 – additional information 1 to lemma 13.1) that if the additional information 2 were not true, it would be that an external ray of M whose argument θ_1 has even denominator, lands at the root c_0 of a hyperbolic component. Let us consider n such that $2^n\theta_1$ has odd denominator. There exists c's arbitrarily close to c_0 such that $c \in \mathcal{R}(M, \theta_1)$, so $c \in \mathcal{R}(K_c, \theta_1)$ and so, $f_c^{\circ n}(c) = f_c^{\circ n+1}(0) \in \mathcal{R}(K_c, 2^n\theta_1)$.

• If $\mathcal{R}(K_{c_0}, 2^n\theta_1)$ lands on a point in a repelling cycle, $\overline{\mathcal{R}(K_c, \theta_1)}$ moves continuously with respect to c in a neighborhood of c_0, so $\overline{\mathcal{R}_{K_{c_0}}(\theta_1)}$ goes through c_0 which is a contradiction since $c_0 \in \overset{\circ}{K}_{c_0}$.

• If $\mathcal{R}(K_{c_0}, 2^n\theta_1)$ lands on a point in a rationally indifferent cycle, we can (modifying n if necessary) assume that it is on α_1. But then, the fact that we can find c arbitrarily close to c_0 such that $f_c^{\circ n+1}(0) \in \mathcal{R}(K_c, 2^n\theta_1)$ contradicts proposition 17.4. ■

4.7. Proof of proposition 17.5. The position of the ray with respect to $f_c^{\circ i}(0)$ $(1 \leq i \leq km)$ does not change, as long as c is sufficiently close to c_0. In order to give a sense to this remark, let us compactify $\mathbb{C}$ by a point at infinity in each direction of half-lines, et let us exted $\mathcal{R}(K_c, \theta)$ by choosing for $\mathcal{R}(K_c, \theta)(\infty)$ the point at infinity in the direction of θ: the ray is then continuous on $[0, \infty]$.

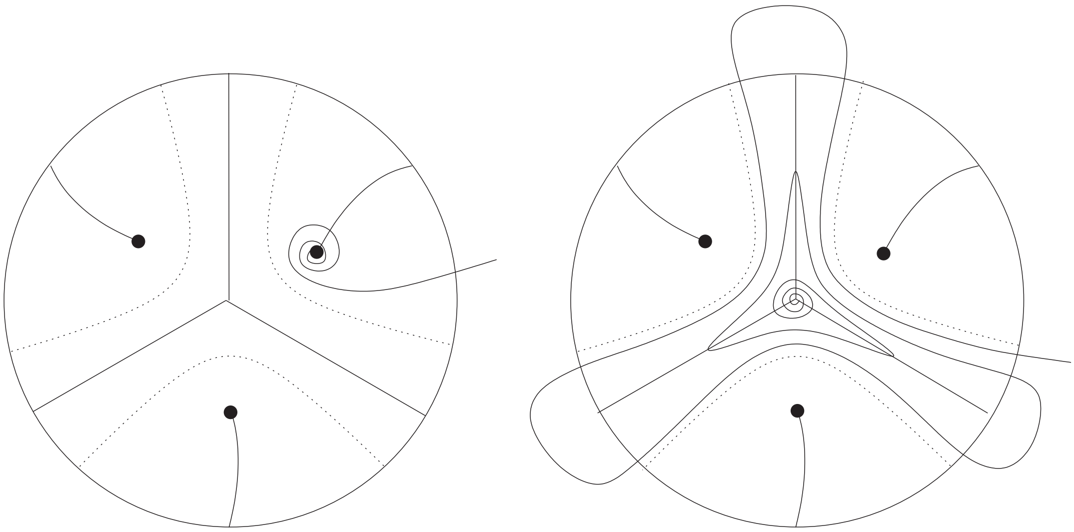
In the space $\widehat{\mathbb{C}}$ constructed above, for c sufficiently close to c_0 and not on $\mathcal{R}(M, 2^{i_0\theta})$, the ray $\mathcal{R}(K_c, \theta)$ is homotopic with fixed extremitites to a curve η_c equal to $\mathcal{R}_{K_c}(\theta)$ on $[s(c), +\infty]$ and to a segment of line between α or β and $\mathcal{R}(K_c, \theta)(s(c))$, without going through $f_c^{\circ i}(0)$ $(1/ \leq i \leq km)$.

So, $\mathcal{R}(K_c, \theta)$ is not continuous with respect to c, but is homotopic to η_c which is continuous with respect to c. The extremity of $\mathcal{R}(K_c, \theta_1)$ for $2^n\theta_1 = \theta$ (with $n \leq km$) is then equal to the extremity of the determination of $(f_c)^{-n}(\eta_c)$ which maps the point at infinity in the direction θ to the point at infinity in the direction of θ_1.

The same reasonning as in the case of Misurewicz points, given the fact that the homotopy occurs at each step in the space F there defined, shows that this extremity is continuous with respect to c. ■

Remark. When $q \neq 1$, we have $2q$ cylinders and no longer 2. We can however use the same argument starting with the entrance of the ray in one of the cylinders, and bound its distortion. If we are on the side of β_1, then the argument used in the case $q = 1$ shows that in bounded time, we come back in the zone defining the first cylinder, and we can construct the same trap around β_1 as in the case $q = 1$.

If we are on the side of α on the cylinder, after crossing the zone corresponding to the first cylinder, we will end on a second in finite time. But we will again be on the side of α: indeed, for c sufficiently close to c_0, the ray evolves in a petal for K_{c_0} during its possible passage outside the zones providing the cylinders: we cross in such a way a second cylinder, with again limitations on the distortion of the ray during this crossing. After crossing the $2q$ zones providing the cylinders, the ray comes back in the one where it did its first entry, and a trap is formed around α.



CHAPTER 18

Additional information on trees.

By Pierre Lavaurs

1. Trees at the centers and the roots.

Let W be a hyperbolic component of $\overset{\circ}{M}$ with multiplicity μ, and $c_1, \ldots, c_\mu$ its roots.

For all point $c \in M$ and $\theta \in \mathbb{Q}/\mathbb{Z}$, the ray $\mathcal{R}(K_c, \theta)$ lands at a point $\gamma_c(\theta) \in K_c$: we can therefore define an equivalence relation $\sim_c$ on $\mathbb{Q}/\mathbb{Z}$ by $\theta \sim_c \theta' \iff \gamma_c(\theta) = \gamma_c(\theta')$.

Proposition 18.1. *$\sim_c$ is constant on $W \cup \{c_1, \ldots, c_\mu\}$.*

Proof.

• The property is true on W:

For θ and θ' fixed, the set of $c \in W$ such that $\gamma_c(\theta) = \gamma_c(\theta')$ is closed in W since those two functions are continuous with respect to c (cf. theorem in chapter 17; in fact, on W, we are in the easy case of this theorem).

Let $c_0 \in W$ be given; for c close to c_0 the functions $c \mapsto \gamma_c(\theta)$ and $c \mapsto \gamma_c(\theta')$ providing two preperiodic points, varying continuously with respect to c, and sent by a constant number of iterations n, without going through 0, on a cycle of length constantly dividing an fixed integer p. Those two functions therefore satisfy the functional equation $f_c^{\circ n+p}(\alpha(c)) = f_c^{\circ p}(\alpha(c))$. If we assume $\theta \sim_{c_0} \theta'$, the implicit function theorem guaranties the equality of $\gamma_c(\theta)$ and $\gamma_c(\theta')$ for c close to c_0: the set of c such that $\theta \sim_c \theta'$ is therefore open in W.

For all pair (θ, θ'), the equivalence $\theta \sim_c \theta'$ is therefore true for all c in W or for none.

• The property is true at the roots:

It is still true, for the same reasons, that for θ and θ' fixed, the set of c in $W \cup \{c_1, \ldots, c_\mu\}$ such that $\gamma_c(\theta) = \gamma_c(\theta')$ is closed in $W \cup \{c_1, \ldots, c_\mu\}$: the equivalence relation is therefore bigger at a root than in W.

Let us take θ and θ' equivalent at c_1 for example.

If $\gamma_{c_1}(\theta) = \gamma_{c_1}(\theta')$ is repelling preperiodic, the argument used on W can be used identically, and we see that θ and θ' are still equivalent in a neighborhood of c_1, so in W.

If $\gamma_{c_1}(\theta) = \gamma_{c_1}(\theta')$ is indifferent preperiodic, it is necessary to analyze more carefully the cycles that "merge" at c_0 to provide the indifferent cycle. This analysis has been made at the beginning of chapter 16; it shows that if $\alpha(c_1)$ is on the indifferent cycle of f_{c_1}, there are $q+1$ ways of defining $\alpha(c)$ continuous in a neighborhood of c_1 in W so that $\alpha(c)$ is a periodic point of f_c (one is of period kq, the q others are of period k). Among those $q+1$ determinationsm, q are attracting in W so only one can be the landing point of the rays. If $\gamma_{c_1}(\theta) = \gamma_{c_1}(\theta')$ is on the rationally indifferent cycle, the continuity of $\gamma_c(\theta)$ or $\gamma_c(\theta')$ with respect to c implies that for c close to c_0, we still have $\theta \sim_c \theta'$; if $\gamma_{c_1}(\theta) = \gamma_{c_1}(\theta')$ is rationally indifferent preperiodic, there exists n such that $\gamma_{c_1}(2^n\theta)$ is rationally indifferent periodic: since $\gamma_c(\theta)$ and $\gamma_c(\theta')$ both satisfy the functional equation

$$f_c^{\circ n}(\gamma_c(\theta)) = f_c^{\circ n}(\gamma_c(\theta')) = \gamma_c(2^n(\theta),$$

they are the same branch of $f_c^{-n}(\gamma_c(\theta))$ and both are equal for c sufficiently close to c_1 in W. ■

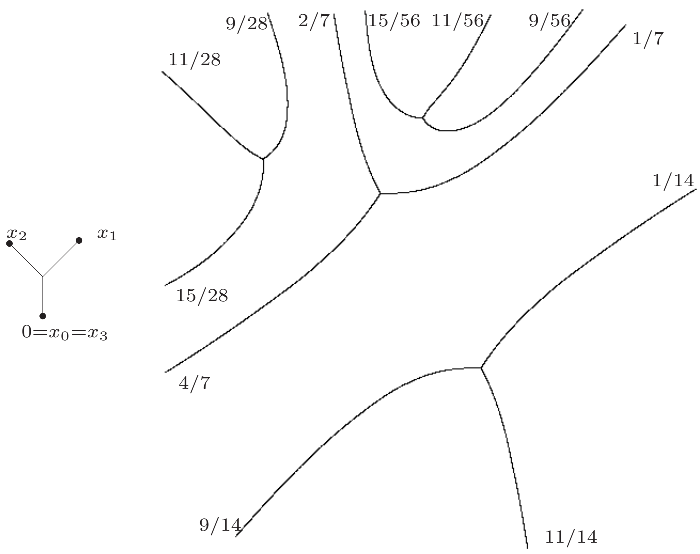
Proposition 18.2. *Let c and c' be centers or roots of hyperbolic components; if $\sim_c = \sim_{c'}$, the Hubbard trees H_c and $H_{c'}$ are isomorphic.*

Proof. We associate to c, center or root of a hyperbolic component, a (finite) subset Θ_c of $\mathbb{Q}/\mathbb{Z}$ composed of:

- external arguments of branching points in the Hubbard tree of f_c
- external arguments of points in $\partial U_0, \dots, \partial U_{n-1}$ (where $U_0, \dots, U_{n-1}$ are the components of the periodic cycle of f_c) which form a cycle of length n
- external arguments of points in $\partial U_0, \dots, \partial U_{n-1}$ with internal argument opposite to the previous ones.

Let us consider $\Theta_c \cup \Theta_{c'}$: the restriction of $\sim$ to this set is the same for c and c'. The external rays having those arguments give the same drawing in the plane (in the sense of the existence of a homeomorphism of the plane respecting the directions at infinity) for K_c and $K_{c'}$.

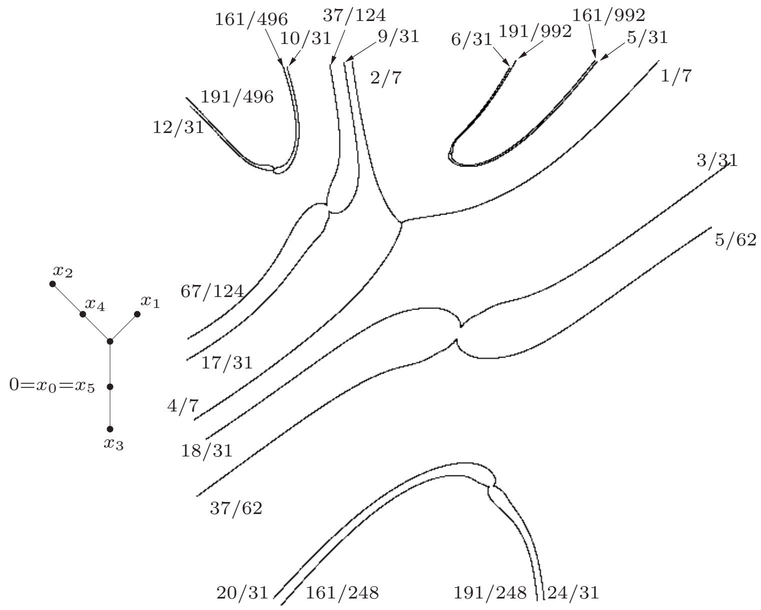
Examples: • c landing point of the external ray of M of argument $1/7$



• c landing point of the external ray of M of argument $5/31$

This drawing subdivides the plane in a certain number of regions.

Lemma 18.1. *For all $i \geq 0$, $f^{\circ i}(0)$ is in the same region for c and for c'.*



Proof. We will show that for $\theta, \theta' \in \mathbb{Q}/\mathbb{Z}$ equivalent for $\sim_c = \sim_{c'}$, and distinct, we know where to put $f^{\circ i}(0)$ with respect to the curve joining ∞ to ∞ through $\overline{\mathcal{R}_{K_c}(\theta)} \cup \overline{\mathcal{R}(K_c, \theta')}$ or $\overline{\mathcal{R}_{K'_c}(\theta)} \cup \overline{\mathcal{R}(K'_c, \theta')}$.

For $i = 0$, it is obvious: K_c (or $K_{c'}$) being symmetric with respect to 0, 0 is on the side that leaves the largest arc in between θ and θ' on the circle $\mathbb{R}/\mathbb{Z}$.

We will then proceed by induction on i: $\mathcal{R}(K_c, \theta)$ and $\mathcal{R}_{K_c}(\theta')$ do not land at 0, so $\theta \neq \theta' + 1/2$: θ and θ' have in total four halves $\theta/2$, $\theta'/2$, $\theta/2 + 1/2$ amd $\theta'/2 + 1/2$. The rays indexed by those halves are grouped on K_c in two points mapped by f_c on the landing point of $\mathcal{R}(K_c, \theta)$. Since $\sim_c = \sim_{c'}$, the rays for $K_{c'}$ are grouped pairwise in the same way, subdividing the plane in three regions; by induction hypothesis, we know where to put $f^{\circ i}(0)$ with respect to those three regions; we deduce where to put $f^{\circ i+1}(0)$ with respect ot $\overline{\mathcal{R}(K, \theta)} \cup \overline{\mathcal{R}(K, \theta')}$ which is therefore the same for c and for c'. □

We will deduce from this lemma that H is the same for c and for c'.

Let us observe that

- there is at most one point of H_c (or $H_{c'}$ since the lemma shows that they are put in the same way) in each region: it is for this reason that we put in Θ_c rays landing at two points in each component of the cycle of components of $\overset{\circ}{K}_c$
- the branching points of H_c or $H_{c'}$ are all points where at least three regions of the drawing formed by those meet.

Now, there is only one way, from the point of view of topology and up to isotopy in $\mathbb{C}$, to join the marked points: on the boundary of each region, composed of one or two curves, we put an exiting point by curve, and the tree must go through those exiting points to join the marked points. ■

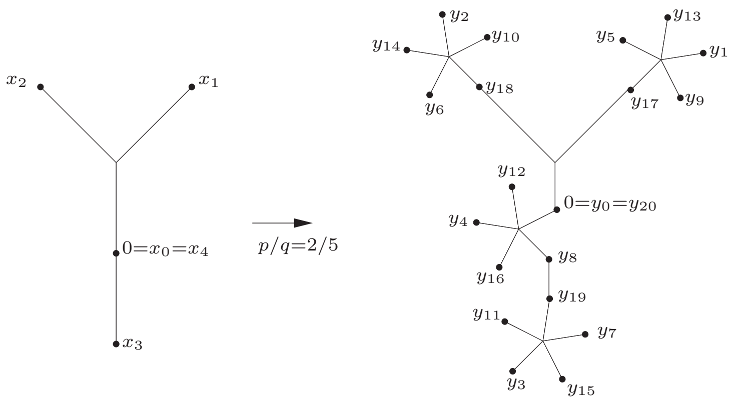
2. Tree at a bifurcation.

Let W be a hyperbolic component of $\overset{\circ}{M}$, $\rho(c)$ the multiplier of the cycle for f_c which is attracting in W. We will describe how to obtain thre tree H' of a point c of ∂W where $\rho = e^{2i\pi p/q}$ with p and q coprime, $q \neq 1$ (the case $q = 1$ has been treated in the previous section), given the Hubbard tree H at the center of W. We will say that H' is obtained from H after *bifurcation* of argument $e^{2i\pi p/q}$. (To be rigorous, it is of course the isotopy class of isomorphism of tree that we construct).

2.1. Description of the construction. Let $(x_i)_{0\leq i\leq k-1}$ be the marked points of H. Let us replace each x_i by a star with q branches, centered at x_i, with extremities $(y_j)_{j=i+lk}$ $0 \leq l \leq q-1$, indexed so that the rotation of angle p/q around x_i sends y_j to y_{j+k} mod kq. We then have points $y_0, \dots y_{kq-1}$.

When a branch of H come to a point x_i, we let it come to a point y_j with $j = i$ mod k. There us a unique way of doing this, so that the tree H' we get, equipped with the y_i, satsify to the Hubbard condition: $\exists F : H' \to H'$, continuous and injective on each component of H' cut along y_0, with $F(y_i) = y_{i+1}$, $F(y_{kq-1}) = y_0$.

Example:



2.2. Justification of the construction. Let c_1 be a center of U, and consider Θ_{c_1} defined as in the beginning of the proof of proposition 18.2. The argument of proposition 18.1 immediately shows that the graph of $\sim_c$ contains the one of $\sim_{c_1}$; moreover, the landing points of the rays indexed by Θ_c land in K_{c_1} on repelling (pre)periodic points which are not backward images of 0, so do not "merge": as in the case of a root, we see that those ray provide the same "drawing" in the plane of K_{c_1} and the plane of K_c (here, we only have to use the "easy" case of chapter 17). Lemma 18.1 can be applied: we know how to put the points $(y_i)_{0\leq i\leq kq-1}$ with respect to the rays of Θ_c. The tree H' is therefore obtained from the tree H by replacing the point $(x_i)_{0\leq i\leq k-1}$ by the subtree generated by the $(y_j)_{j=i+lk}$ $0 \leq l \leq q-1$. However, this subtree is associated to the q components adjacent to a periodic point of period k: it is therefore the star with q barnches described above.

H', isomorphic to the Hubbard tree for a c root of hyperbolic component, satisfy the Hubbard condition.

We still have to show that this construction is not ambiguous, i.e., that there is only one way to put the stars on H. But δ standing for the degree, Hubbard condition implies easily (see chapter 4 proposition 4.4), $\delta(y_0) = 2$,

$$\delta(y_1) = 1 \leq \delta(y_2) \leq \ldots \leq \delta(y_{kq-1}) \leq 2 = \delta(y_{kq})$$

(setting $y_{kq} = y_0$); at x_1 it is therefore necessary to attach the star to H by its vertex having the largest index.

Finally, the injectivity of F on the two components of H' cut at y_0 allows to determine how the stars are mapped one to another (the path going to the star cotaining y_0 being mapped to the path going to the star containing y_1), so to complete the indexing of the points $(y_i)_{0 \leq i \leq kq-1}$ on the stars. ■

3. Computation of external arguments in M.

Let c_1 be a root of a hyperbolic component W of $\overset{\circ}{M}$ and H be its Hubard tree. We will explain how we find its external arguments in M given the tree H. Let $x_0 = 0$, $x_1 = c, \ldots, x_k = x_0$ be the ;arked points of H (where k is the period of x_0). Let us choose a continuous map $F : H_c \to H_c$, such that $F(x_i) = X_{i+1}$, injective on each component of H_c cut at 0. We assume that we have chosen F so that it has a periodic point α_1 inside the edge going to x_1 (let us recall that x_1 is an extremity in H_c). The period of α_1 is k. The arguments $\theta_- = \arg_-(c)$ and $\theta_+ = \arg_+(c)$ are the arguments of α_1, computed with the algorithm desribed in chapter 7 section 4 (which only uses combinatorial datas).

Variation. We can (as in [**DH1**]) put α_1 at x_1, but we only put one bud at the extremities of H_c (instead of an infinity), and we choose the dynamics on the accesses to x_i (which is not determined for $x_0 \mapsto x_1$) so that those accesses are periodic of period k.

Let us show that this algorithm is convenient: it is clear that the result does not depend on the position of the point α_1 on the edge going to x_1, neither on the chosen F; the rationally indifferent periodic point is in the tree H_c precisely on this edge, possibly at its root a (cf chapter 14 proposition 14.4); if it is strictly on the edge, the above computation gives all its external arguments in K_c which are two: we are in the case $q = 1$ (primitive root) and we have found the external arguments of c in M; if it is at a, the computation gives the same result as the one we would obtain by considering the accesse to a adjacent to the edge $[a, x_1]$, i.e., precisely the external arguments of c in M.

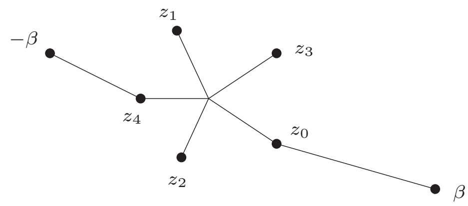
From this algorithm and the one described in section 2, we can deduce an arithmetic algorithm which provides the external arguments $\theta^-_{p/q}$ and $\theta^+_{p/q}$ in M of a point c in ∂W obtained from the bifurcation of argument $e^{2i\pi p/q}$ given the external arguments θ^- and θ^+ in M of the point c_1.

If $c_1 = 1/4$, the Hubbard tree of f_c is a star with q branches indexed so that we go from the point y_i to the point y_{i+1} by a rotation of p/q. We can use the above algorithm to compute the external arguments of c in M; we will denote their representatives in $]0,1[$ by $f^-(p/q)$ and $f^+(p/q)$ with $f^-(p/q) < f^+(p/q)$; in this way, $f^-(1/2) = 1/3$, $f^+(1/2) = 2/3$, $f^-(1/3) = 1/7$, $f^+(1/3) = 2/7, \ldots$. $\overset{\circ}{\theta}{}^-$ and $\overset{\circ}{\theta}{}^+$ stand for the representatives of θ^- and θ^+ in $]0,1[$ and we assume $\overset{\circ}{\theta}{}^- < \overset{\circ}{\theta}{}^+$.

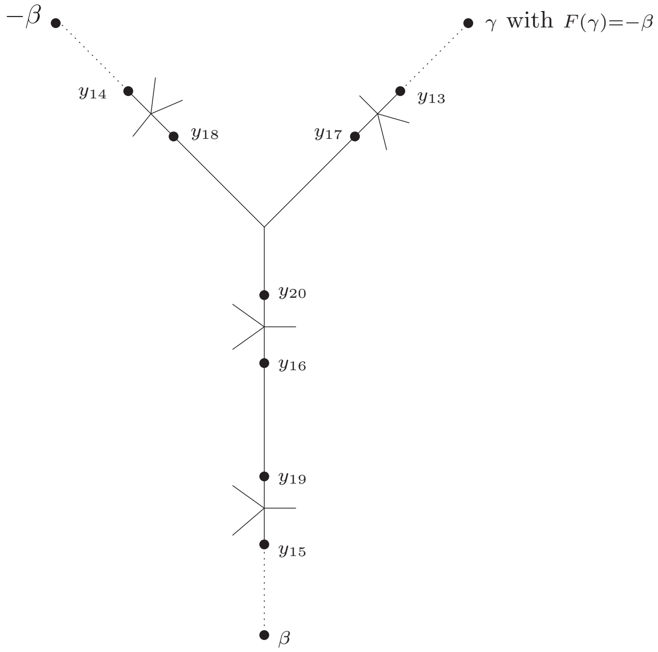
Proposition 18.3. *The representatives of $\theta^-_{p/q}$ and $\theta^+_{p/q}$ in $]0,1[$ are obtained in the following way: we write $\mathring{\theta}^- = .\overline{u_1^- \dots u_k^-}$ and $\mathring{\theta}^+ = .\overline{u_1^+ \dots u_k^+}$ in base 2. Then, in the diadic development of $f^-(p/q)$ (respectively $f^+(p/q)$) in base 2, we replace the zeros by $u_1^- \dots u_k^-$ and the ones by $u_1^+ \dots u_k^+$.*

Proof. It is enough to explicit what happens when we aply the algorithm of section 3 to find the tree associated to c, and then the one of the first part of this section to compute the associated arguments.

Indeed, one can check that the extended star in the sense of chapter 7, the arc $[-\beta, \beta]$ goes through z_{q-1} and z_0 (we denote by z_0 the critical point for this tree, z_{i+1} is the image of z_i for its dynamics)



In the tree H_c, the backward images of $[-\beta, \beta]$ go through the stars at $y_{(l-2)q+l}$ and $y_{(k-1)q+l}$ $(1 \leq l \leq k)$.



Therefore, if we choose a point α_1 on the edge going to y_1, an access to α_1 will evolve during the k first iterations as an access to x_1 "to the right" when we go to x_1 on H_{c_1} if y_1 is "to the right" of the arc y_{kq-1} to y_{kq-q-1}, i.e., if z is "to the right" of the path from z_0 to z_{q-1}, i.e., if $f^-(p/q)$ (or $f^+(p/q)$ depending on the bud considered at α_1) starts with a 0, as an access to x_1 "to the left" if $f^-(p/q)$

starts with a 1: we replace the first digit of $f^{-}(p/q)$ by $u_1^- \dots u_k^-$ if it is a 0 and by $u_1^+ \dots u_k^+$ if it is a 1. We see that it is again what we must do for each sequence of k iterations. ■

Proposition 18.4. *If* $c_1 \neq 1/4$, 0 *and* W *are on both sides of the curve* $\mathcal{L} = \mathcal{R}_M(\theta_+) \cup \mathcal{R}(M, \theta_-) \cup \{c_1\}$.

Proof. Indeed, 0 and W are positionned with respect to $\mathcal{L}$ as 1/4 with respect to a point c obtained from c_1 after bifurcation of argument 1/2. The algorithm of proposition 18.3 shows that if $\overset{\circ}{\theta}{}^-_{1/2}$ and $\overset{\circ}{\theta}{}^+_{1/2}$ are the representatives in $]0,1[$ of $\theta^-_{1/2}$ and $\theta^+_{1/2}$, we have

$$0 < \overset{\circ}{\theta}{}^- < \overset{\circ}{\theta}{}^-_{1/2} < \overset{\circ}{\theta}{}^+_{1/2} < \overset{\circ}{\theta}{}^+ < 1,$$

so $\mathcal{L}$ separates c from 1/4, landing point of $\mathcal{R}(M, 0)$. ■

CHAPTER 19

Simplicity of hyperbolic components.

Theorem 19.1. *For all hyperbolic component W of $\overset{\circ}{M}$, the map $\rho_W : W \to \mathbb{D}$ is an isomorphism*

We can give several proofs of this theorem. Here are the sketches.

1. First proof.

Lemma 19.1 (Gleason). *All root of P_k is simple. (P_k is defined in chapter 14, section 6).*

Proof. Let us denote by A the ring of $z \in \mathbb{C}$ which are algebraic integers on $\mathbb{Z}$. The polynomial P_k is monic, so if $P_k(c) = 0$, we have $c \in A$. But $P_k(2) = (P_{k-1}(2))^2 + 2$, so $P'_k = 2P_{k-1}P'_{k-1} + 1$ and if c is a root of P_k, we have $P'_k(c) \equiv 1 \bmod 2A$, so $P'_k(c) \neq 0$. ■

Proof 1 of Theorem 19.1. A component W only has one center by chapter 18 proposition 18.1 and by chapter 6. This center is simple by lemma 19.1.

2. Second proof.

Using proposition 18.4 in chapter 18, we show the following proposition.

Proposition 19.1. *Let W_1 and W_2 be two hyperbolic components, c_1 a root of W_1 and c_2 a root of W_2. If $c_1 \neq c_2$, we have $\overline{W}_1 \cap \overline{W}_2 = \emptyset$, $\{c_1\}$ or $\{c_2\}$.*

Proof. For $i = 1, 2$, c_i is the landing point of 2 rays $\mathcal{R}(M, \theta_i^+)$ and $\mathcal{R}(M, \theta_i^-)$. We set $\mathcal{L}_i = \mathcal{R}_M(\theta_i^+) \cup \mathcal{R}(M, \theta_i^-) \cup \{c_i\}$. We have $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$, so $\mathbb{C} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2$ has three connected components U_1, U_2, U_3 with $\partial U_1 = \mathcal{L}_1$, $\partial U_2 = \mathcal{L}_1 \cup \mathcal{L}_2$, $\partial U_3 = \mathcal{L}_2$. If $0 \in U_1$, we have $\overline{W}_2 \subset U_3 \cup \{c_2\}$ and $\overline{W}_1 \subset U_2 \cup \{c_1, c_2\}$ so $\overline{W}_1 \cap \overline{W}_2 \subset \{c_2\}$. If $0 \in U_2$, we have $\overline{W}_1 \subset U_1 \cup \{c_1\}$ and $\overline{W}_2 \overset{\circ}{K} \subset U_3 \cup \{c_2\}$, so $\overline{W}_1 \cap \overline{W}_2 = \emptyset$. If $0 \in U_3$ we have $\overline{W} - 1 \subset U_1 \cup \{c_1\}$ and $\overline{W}_2 \subset U_2 \cup \{c_1, c_2\}$, so $\overline{W}_1 \cap \overline{W}_3 = \emptyset$ or $\{c_1\}$. ■

Corollary 19.1. *Theorem 19.1.*

Because we cannot have $W - 1 = W_2$ and $c_1 \neq c_2$.

3. Third proof.

Lemma 19.2 (Sullivan). *Let $c \in W$ be such that $\rho(c) \neq 0$. Then, $\rho'(c) \neq 0$.*

The proof relies on the Measurable Riemann Mapping Theorem of Morrey-Ahlfors-Bers.

Proof. Let $\{\alpha_1, \ldots, \alpha_k\}$ be the attracting cycle of f_c. The basin of this cycle is $\overset{\circ}{K}$, let us set $V = \overset{\circ}{K} \setminus \{z \mid \exists n,\ f^{\circ n}(z) = \alpha_1\}$. Let E be the quotient of V by the equivalence relation $z_1 \sim z_2 \iff \exists(n_1, n_2)\ f_c^{\circ n_1}(z_1) = f_c^{\circ n_2}(z_2)$. Then E is a compact Riemann surface of genus 1. If σ is a continuous Beltrami form of norm < 1 on E, denoting by $\chi : V \to E$ the canonical map, $\tilde{\sigma} = \chi^*(\sigma)$ extended by 0 is a Beltrami form invariant by f_c, measurable and bounded with norm < 1, so integrable. If $\varphi : \mathbb{C} \to \mathbb{C}$ is such that $\bar{\partial}\varphi/\partial\varphi = \tilde{\sigma}$, the map $\varphi \circ f_c \circ \varphi^{-1} : \mathbb{C} \to \mathbb{C}$ is holomoprhic (since $\tilde{\sigma}$ is f_c-invariant), proper of degree 2, it is therefore a polynomial of degree 2, affinely conjugated to one $z \mapsto z^2 + c(\sigma)$. We have $c(0) = c$ and $\log \rho_W(c(\sigma))/2i\pi$ is equal to the ratio between the periods of (E, σ). We can choose a continuous map $\rho \mapsto \sigma(\rho)$ so that $\rho \mapsto c(\sigma(\rho))$ is a continuous section of $c \mapsto \rho_W(c)$. ■

Proof 1 of Theorem 19.1. Lemma 19.2 and 19.1 show that $\rho_W : W \to \mathbb{D}$ is a covering map. Since $\mathbb{D}$ is simply connected, it is trivial. ■

For a fourth proof, variant of the preceding one but avoiding Gleason's lemma, see [**Do1**].

CHAPTER 20

Veins.

We first give a technique to study a full, connected, locally connected compact susbset of $\mathbb{C}$. Le compact set M is full and connected, but we do not know how to show that it is locally connected. We will see that due to its combinatorial aspect, our techniques can be applied to M.

1. Extremal points.

Let $K \subset \mathbb{C}$ be a compact, full, connected and locally connected, and choose a center for each connected component of $\overset{\circ}{K}$. For x and y in K, we denote by $[x, y]_K$ the allowable arc from x to y, and if $x_1, \ldots, x_n$ are elements of K, we denote by $[x_1, \ldots, x_n]_K$ the allowable hull of $\{x_1, \ldots, x_n\}$ (cf chapter 2).

Denote by γ_K the Caratheodory loop $\mathbb{T} \to \partial K$. For $x \in K$, the points of $\gamma_K^{-1}(x)$ are the *external arguments* of x.

Proposition 20.1. and definitions. *Let $x \in K$. The following conditions are equivalent:*

(i) *$x \in \partial K$ and $K \setminus \{x\}$ is connected*
(ii) *x has one external argument and only one*
(iii) *one cannot find two points $y, z \in K \setminus \{x\}$ such that $x \in [y, z]_K$.*

If x satisfies these conditions, we say that x is an extremal point in K.

Proof. a) (not (i)) $\Rightarrow$ (not (iii)): if x belongs to a connected component U of $\overset{\circ}{K}$, we can find y and $z \in \partial U$ such that $x \in [y, z]$. If $K \setminus \{x\}$ is not connected, we take y and z in two different connected components of $K \setminus \{x\}$; then, $x \in [y, z]_K$.

b) (not (iii)) $\Rightarrow$ (not (ii)): If $x \in \overset{\circ}{K}$, it does not have any external argument. If $x \in \partial K$ and $x \in]y, z[_K$, there are two accesses to x relatively to the tree $[y, z]_K$ (cf. chapter 7), so x has at least 2 external arguments.

c) (not (ii)) $\Rightarrow$ (not (i)): Let t and t' be two arguments of x and $\mathcal{L} = \mathcal{R}(K, t) \cup \mathcal{R}(K, t') \cup \{x\}$. We can find $s \in [t, t']$ and $s' \in [t', t]$ such that $y = \gamma(s) \neq x$ and $z = \gamma(s') \neq x$ (a Carathéodory loop is never constant on an interval). Then, y and z are on both sides of $\mathcal{L}$ and $x \in [y, z]_K$. ■

2. Veins in K.

Same assumptions on K. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of extremal points in K. We denote by t_n the external argument of a_n, and we assume that $t_0 = 0$. We set $H_n = [a_0, \ldots, a_n]_K$[1]. For each n, H_n is a finite topological tree, and for $n \geq 1$ we

[1] If $a_0 \in \partial U_0$, where U_0 is a component of $\overset{\circ}{K}$ (necessarily unique since a_0 is extremal), we choose for H_0 the internal ray of U_0 with extremity a_0 and not the set $\{a_0\}$

can write $H_n = H_{n-1} \cup [d_n, a_n]_K$ with $H_{n-1} \cap [d_n, a_n]_K = \{d_n\}$. This determines $d_n \in K$.

We denote $h(t_n)$ and we call *vein* of extremal external argument t_n, or vein with extremity a_n, the allowable arc $[d_n, a_n]_K$. The point d_n is called the *origin* of $h(t_n)$, and $h^*(t_n) =]d_n, a_n]_K = [d_n, a_n]_K \setminus \{d_n\}$ is called the *strict vein* with extremity a_n. For x and $y \in K$, we will say that y is *after* x and we will write $x \le y$ if $x \in [a_0, y]_K$.

We will define, for all $x \in K$, two numbers $\arg_-(x)$ and $\arg_+(x)$ in $[0,1]$. For $t \in \mathbb{T}$, denote $\overset{\circ}{t}$ the representative of t in $[0,1[$.

If $x \in \partial K \setminus \{a_0\}$, we set $\arg_-(x) = \inf_{t \in \gamma_K^{-1}(x)} \overset{\circ}{t}$ and $\arg_+(x) = \sup_{t \in \gamma_K^{-1}(x)} \overset{\circ}{t}$.

If $x = a_0$, we set $\arg_-(x) = 0$ and $\arg_+(x) = 1$.

If x is the center of a component U of $\overset{\circ}{K}$, teh arc $[a_0, x]_K$ cuts ∂U at a unique point $y = \pi_U(a_0)$ (projection of a_0 on $\overline{U}$, and $\arg_-(y)$ and $\arg_+(y)$ correspond to 2 accesses to y relatively to $[a_0, x]_K$. We take for $\arg_-(x)$ the largest $\overset{\circ}{t}$ for t external argument of y in the same access as $\arg_-(y)$, and we define $\arg_+(x)$ symetrically. We therefore have

$$0 \le \arg_-(y) \le \arg_-(x) \le \arg+(x) \le \arg_+(y) \le 1.$$

If x belongs to U without being the center, denote by x_0 the center of U and let $[x_0, x_1]_K$ be the internal ray of $\overline{U}$ going through x. We set:

$$\begin{aligned} \arg_\pm(x) &= \arg_\pm(x_1) \quad \text{if } x_1 \ne \pi_U(a_0) \quad \text{and} \\ &= \arg_\pm(x_0) \quad \text{if } x_1 = \pi_U(a_0). \end{aligned}$$

In any case, we denote by $I(x)$ the interval $[\arg_-(x), \arg_+(x)]$.

We also define the *arguments associated* to x.

If $x \in \partial K \setminus \{a_0\}$, the arguments associated to x are the $\overset{\circ}{t}$, where t is an external argument of x.

If $x = a_0$, they are 0 and 1.

If x is the center of a component U of $\overset{\circ}{K}$, they are $\arg_-(y)$ and $\arg_+(y)$ for $y \in \partial U \setminus \{\pi_U(a_0)\}$, together with $\arg_-(x)$ and $\arg_+(x)$.

If x belongs to U without being the center, the arguments associated to s are $\arg_-(x)$ and $\arg_+(x)$.

For all interval $I \subset [0,1]$ containing one of the $\overset{\circ}{t}_n$, we call *leader* of I the number $t_{n(I)}$ where $n(I)$ is the smallest n such that $\overset{\circ}{t}_n \in I$.

Proposition 20.2. *Let $n \in \mathbb{N}$ and $x \in K$. Assume $x \in \partial K$ or x is the center of a component of $\overset{\circ}{K}$.*

a) *We have $x \in h^*(t_n)$ if and only if t_n is the leader of $I(x)$.*
b) *We have $x \in h(t_n)$ if and only if t_n) is the leader of an interval of the form $[\theta', \theta'']$, where θ' and θ'' are arguments associated to x.*

Proof. Let us set $\theta_\pm = \arg_\pm(x)$; let x_1 be the common landing point of $\mathcal{R}(K, \theta_-)$ and $\mathcal{R}(K, \theta_+)$ (which is x if $x \in \partial K$). If x belongs to a component U of $\overset{\circ}{K}$, we set $x_0 = x$. We set $\mathcal{L} = \nabla_K(\theta_-) \cup \mathcal{R}_K(\theta_+) \cup \{x_1\}$.

(a) Case $x_1 = a_0$. With the chosen conventions, we have $x \in h^*(t_0) = h(t_0)$. However, $I(x) = [0,1]$, and the leader of $[0,1]$ is $0 = t_0$.

(a) $x_1 \neq a_0$, $\Leftarrow$. $\mathcal{L}$ is homeomorphic to $\mathbb{R}$, and $\mathbb{C}\backslash\mathcal{L}$ has 2 connected components V_0 and V_1, $V_0 \supset \mathcal{R}(K,0)$. By definition of the leader, we have: $a_i \in V_0$ for $i = 0, \dots, n-1$ and $a_n \in V_1 \cup \{x_1\}$. Since $d_n \in H_{n-1}$, we have $d_n \in V_0 \cup \{x_1\}$.

If $d_n = x_1$, let t_i be such that $i < n$ and $a_i \geq x_1$; we have $t_i \in I(x_1)$, so $I(x_1) \neq I(x)$, this is only possible if $x = x_0$ is the center of a component U, et in this case, $x \in]d_n, a_n]$.

(a) $x_1 \neq a_0$, $\Rightarrow$. Let us keep the definitions of $\mathcal{L}$, V_0 and V_1. We have $a_n \geq x$. We deduce that $a_n \in V_1 \cup \{x_1\}$, and so $t_n \in I(x)$. For $i < n$, we have $a_i \not\geq x$. We deduce that $a_i \in V_0 \cup \{x_1\}$. We could have $a_i = x_1$ only if $x_0 > x_1$, but in this case x_1 is not extremal, so necessarily $a_i \neq x_1$, $a_i \in V_0$ and $t_i \notin I(x)$. As a consequence, t_n is the leader of $I(x)$.

(b) $\Leftarrow$. Let us set $\mathcal{L}' = \mathcal{R}(K,\theta') \cup \mathcal{R}_K(\theta'') \cup [x_1', x_1']_K$, where x_1' and x_1'' are the landing points of respectively $\mathcal{R}(K,\theta')$ and $\mathcal{R}_K(\theta'')$. Denote by V_0' and V_1' the connected components of $\mathbb{C} \setminus \mathcal{L}'$, with $V_0' \supset \mathcal{R}(K,0)$. We have $t_n \in [\theta', \theta'']$, so $a_n \in V_1' \cup \{x_1', x_1'\}$. For $i < n$, we have $t_i \notin [\theta', \theta'']$, so $a_i \in V_0'$. As a consequence, $d_n \in V_0' \cup \{x_0\}$, and in each case, we deduce that $x \in [d_n, a_n] = h(t_n)$.

(b) $\Rightarrow$. If $x \in \partial K$, let θ' and θ' be external arguments of x in accesses to H_n such that among the points $a_0, \dots, a_n$, only a_n is in between $\mathcal{R}(K,\theta')$ and $\mathcal{R}_K(\theta'')$. Then, $t_n \in [\theta', \theta'']$, $t_i \notin [\theta', \theta'']$ for $i < n$ and t_n is the leader of $[\theta', \theta'']$.

Let us assume that x belongs to a component U of $\mathring{K}$ with center x_0. Denote by y the piont where $[x, a_n]_K$ cuts ∂U. If $x \notin]x_0, \pi_U(a_0)]$, we take $\theta' = \arg_-(y0$ and $\theta'' = \arg_+(y)$. If $x \in]x_0, \pi_U(a_0)]$, we take $\theta' = \arg_-(x_0)$ and $\theta'' = \arg_+(x_0)$. In each case, we have $t_i \notin [\theta', \theta'']$ for $i < n$, and we have $t_n \in [\theta', \theta'']$ since $t_n \geq y$ in one case and $t_n \geq \pi_U(a_0)$ in the other. So, t_n is the leader of $[\theta', \theta']$. ■

3. Combinatorial veins.

In the Mandelbrot set M, let us define a countable subset: we denote $\mathcal{D}_0$ (respectively $\mathcal{D}_1$) the set of centers (respectively of roots) of hyperbolic components of $\mathring{M}$. Denote by $\mathcal{D}_2$ the set of Misurewicz points, and set $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2$.

We will define for each $c \in \mathcal{D}$ the (combinatorial) arguments associated to c.

If $c \in \mathcal{D}_2$, the point c has in K_c a finite nu;ber of external arguments (which are rational with even denominator). Those are the arguments associated to c. The smallest and the largest are denoted by $\arg_-(c)$ and $\arg_+(c)$.

If $c \in \mathcal{D}_1$, let α_1 be the indifferent periodic point attracting the component U_1 of $\mathring{K}_c$ containing c. The argument associated to c are the 2 external arguments of α_1 corresponding to the interpetals adjacent to U_1. The smallest (respectively the largest) is denoted by $\arg_-(c)$ (respectively $\arg_+(c)$).

If c is the center of a hyperbolic component W of $\mathring{M}$, the arguements associated to c are the arguements associated to the points of ∂W of rational internal arguments (which are points in $\mathcal{D}_1$). The smallest and the largest are respectively $\arg_-(c_1)$ and $\arg_+(c_1)$, where c_1 is the root of W. We will set $\arg_-(c) = \arg_-(c_1)$ and $\arg_+(c) = \arg_+(c_1)$.

If $c \in \mathcal{D}_1 \cup \mathcal{D}_2$, for all argument θ associated to c, the external ray $\mathcal{R}(M,\theta)$ lands at c.

We denote by (t_n) the sequence of numbers in $[0,1[$ of the form $p/2^k$, ordered by k increasing and for each k by p increasing, so that:

$$p/2^k = t_n \quad \text{with} \quad 2n+1 = 2^k + p.$$

Remark. The fact of choosing for each k the ordering of increasing p has no relevance for what we want to do, since the order is useful to define the leader of intervals of $[0,1[$. But each interval $I \subset [0,1[$ which contains a point of the form $p/2^k$ (in particular every interval not reduce to a point) contains a unique such point with k minimal. Indeed, if $p/2^k$ and $p'/2^k$ are in I, with p and p' odd and $p' > p$, $p+1/2^k \in I$ but this fraction simplifies.

Remark. For all $n \geq 0$, denote by a_n the landing point of $\mathcal{R}(M, t_n)$. The polynomial f_{a_n} is such that 0 ends in finite time on the fixed point $\beta(a_n)$ with external argument 0. In particular, $a_n \in \mathcal{D}_2$, and there is only one argument associated to a_n, namely t_n. We obtain in such a way a bijection from $\{t_n\}_{n>0}$ to $\{c \mid (\exists k),\ f_c^{\circ k+1}(0) = \beta(c)\}$. (This follows from chapter 13).

We have $a_0 = 1/4 \in calD_1$. We set $I(c) = [\theta_-(c), \theta_+(c)]$. We define the combinatorial vein $N(t_n)$ and the strict combinatorial vein $N^*(t_n)$ by:

$N^*(t_n) = \{c \in \mathcal{D} \mid t_n \text{ is the leader of } I(c)\}$
$N(t_n) = \{c \in \mathcal{D} \mid (\exists \theta', \theta' \text{ associated to } c)\ t_n \text{ is the leader of } [\theta', \theta'']\}$.

Remark. If, as we imagine, M is locally connected and all point $c \in \mathcal{D}_1 \cup \mathcal{D}_2$ has no other external argument in M than its associated arguments, we have $N(t_n) = h(t_n) \cap \mathcal{D}$ and $N^*(t_n) = h^*(t_n) \cap \mathcal{D}$ thanks to proposition 20.2.

4. Ordering on $\mathcal{D}$.

For $c \in \mathcal{D}$, let us set $I(c) = [\arg_-(c), \arg_+(c)]$. We will write $c < c'$ if $I(c)$ contains strictly $I(c')$ or if c is the root and c' is the center of a common hyperbolic component of $\overset{\circ}{M}$. We will write $c \leq c'$ if $c < c'$ or $c = c'$. We define in such a way an order on $\mathcal{D}$.

Proposition 20.3. *Let c and c' be in $\mathcal{D}$. We have $I(c) \supset I(c')$ or $I(c') \subset I(c)$ or $I(c) \cap I(c') = \emptyset$.*

Proof. If c is the center of a hyperbolic component W, let us denote by c_1 the root of W; let us set $c_1 = c$ if $c \in \mathcal{D}_1 \cup \mathcal{D}_2$. If c is not extremal[2], $\mathcal{L}(c) = \mathcal{R}(M, \theta_-(c) \cup \mathcal{R}(M, \theta_+(c)) \cup \{c_1\}$ is homeomorphic to $\mathbb{R}$, and $\mathbb{C} \setminus \mathcal{L}(c)$ has 2 connected components $V_0(c)$ and $V_1(c)$ (by convention, $\mathcal{R}(M,0) \subset V_0(c)$). If $c_1 = c_1'$, we have $I(c) = I(c')$. Otherwise, we have $\mathcal{L}(c) \cap \mathcal{L}(c') = \emptyset$, so $V_1(c)$ is strictly contained in $V_1(c')$ or $V_1(c)$ strictly contains $V_1(c')$ or $V_1(c) \cap V_1(c') = \emptyset$, and the conclusion follows. ■

Corollary 20.1. *In the ordered set $\mathcal{D}$, every subset which has an upper bound is totally ordered.*

Proof. Let X be a subset which admits $\hat{c}$ as upper bound. For c and c' in X, we have $I(c) \supset I(\hat{c})$, $I(c') \supset I(\hat{c})$, so $I(c) \cap I(c') \neq \emptyset$, and $c \leq c'$ or $c' \leq c$. ■

Corollary 20.2. *For all $n \neq 0$, the set $N^*(t_n)$ is a totally ordered set, whose largest element is a_n.*

Remark. $N(a_0) = N^*(0) = \{1/4, 0\}$ with $a_0 = 1/4$, $0 > 1/4$.

Proposition 20.4. *Let c and $\tilde{c}$ be two elements of $N(t_n)$ with $\tilde{c} \in N(t_n) \setminus N^*(t_n)$; then $\tilde{c} \leq c$.*

[2] If c is extremal or $c = 0$, $I(c)$ is reduced to a point or $c = [0,1]$

Proof. If c is extremal, $c = t_n$ and $\tilde{c} \leq c$. We assume that c is not extremal. Let θ' and θ'' be the arguments associated to c, such that t_n is the leader of $[\theta', \theta'']$. We define $\mathcal{L}(\theta', \theta'')$ in the following way:

If $c \in calD_1 \cup \mathcal{D}_2$, we set $\mathcal{L}(\theta', \theta'') = \mathcal{R}(M, \theta') \cup \mathcal{R}(M, \theta'') \cup \{c\}$.

If $c \in calD_0$, denote by c_1' and c_1'' the landing points of $\mathcal{R}(M, \theta')$ and $\mathcal{R}(M, \theta'')$. We have c_1' and $c_1'' \in \partial W$, where W is the component centered at c. If $c_1' = c_1''$, we set $\mathcal{L}(\theta', \theta'') = \mathcal{R}(M, \theta') \cup \mathcal{R}(M, \theta'') \cup \{c_1'\}$. Otherwise we set $\mathcal{L}(\theta', \theta'') = \mathcal{R}(M, \theta') \cup \mathcal{R}(M, \theta'') \cup [c, c_1']_{\overline{W}} \cup [c, c_1'']_{\overline{W}}$.

In both cases, $\mathcal{L}(\theta', \theta'')$ is homeomorphic to $\mathbb{R}$, and $\mathbb{C} \setminus \mathcal{L}(\theta', \theta'')$ has 2 connected components $V_0(\theta', \theta'')$ and $V_1(\theta', \theta'')$, with $\mathcal{R}(M, 0) \subset V_0(\theta', \theta'')$.

We have $V_1(\theta', \theta') \cap V_1(\tilde{c}) = \emptyset$, since those two sets contain $\mathcal{R}(M, t_n)$, and $V_1(\theta', \theta'') \not\supset V_1(\tilde{c})$ since $I(\tilde{c})$ contains a t_n with $n' < n$ and $[\theta', theta'']$ does not. If $\tilde{c}_1 \neq c_1', c_1''$, we have $\mathcal{L}(\theta', \theta'') \cap \mathcal{L}(\tilde{c}) = \emptyset$, so $V_1(\theta', \theta') \subset V_1(\tilde{c})$ and $\tilde{c} \leq c$. If we had $\tilde{c}_1 = c_1' \neq c_1$, we would have $V_1(\tilde{c}) \subset V_1(\theta', \theta'')$ which is not the case. If we had $\tilde{c}_1 = c_1' = c_1'' = c_1$, we would have $V_1(\theta', \theta'') = V_1(\tilde{c})$, which is not possible. We are therefore in the case $\tilde{c}_1 \neq c_1', c_1''$ and we have $\tilde{c} \leq c$. ■

Corollary 20.3. and definition. *The set $N(t_n) \setminus N^*(t_n)$ has at most one point. If it has a point, it is the smallest element of $N(t_n)$, we say it is the origin of the vein $N(t_n)$.*

We will show – it is the main theorem in the theory of veins – that every vein has an origin.

CHAPTER 21

Tree of the origin of a vein of M.

1. Abstract Hubbard trees.

We call an *abstract Hubbard tree* a topologically finite tree H, equipped with a finite preperiodic sequence x_n of points, and of a isotopy class of embeddings $H \to \mathbb{C}$ (or equivalently of a cyclic order at the branching points), satisfying the following conditions:

(i) Every extremity is one of the (x_i);
(ii) $H \setminus \{x_0\}$ has at most 2 components;
(iii) There exists an injective and continuous map $F : H \to H$ preserving the cyclic order at the branching points on each component of $H \setminus \{x_0\}$, and such that $F(x_i) = x_{i+1}$ for all i.

The map F is determined uniquely, up to isotopy fixing the points x_i.

An abstract Hubbard tree is called periodic or preperiodic depending on whether x_0 is periodic or preperiodic for F.

The Hubbard tree of a polynomial $f : z \mapsto z^2 + c$ such that 0 is preperiodic has an underlying abstract tree.

Given an abstract Hubbard tree H, we can define $\arg_- H()$ and $\arg_+(H)$, together with the argument associated to H, using the algorithms described in chapters 7, 20 and 18.

2. Results and notations.

Let $\tau \in \mathbb{Q}/\mathbb{Z}$ be an element of the form $p/2^k$. We want to show that the combinatorial vein $N(\tau)$ of M has an origin in $\mathcal{D}_0 \cup \mathcal{D}_2$.

In this chapter, we will construct an abstract tree $\check{H}$ having the property that if it is the tree of a point $\check{c} \in \mathcal{D}_0 \cup \mathcal{D}_2$, the point $\check{c}$ is at the origin of the vein $N(\tau)$.

Theorem 21.1. *There exists an abstract Hubbard tree $\check{H}$ having the following properties:*

a) *τ belongs to $I(\check{H}) = [\arg_(H), \arg_+(H)]$ without being its leader.*
b) *there exist two arguments θ' and θ'' associated to $\check{H}$ such that τ is the leader of $[\theta', \theta'']$.*

In the remaining, we fix $\tau = p/2^k$ (with $k \geq 1$, p odd, $0 < p < 2^k$). Let $c = a_\tau$ be the landing point of $\mathcal{R}(M, \theta)$. We set $f = f_c : z \mapsto z^2 + c$, $K = K_c$, $H = H_c$ (Hubbard tree of f), $\beta = \beta_c$ (fixed point of f with external argument 0. We have $f^{\circ k}(c) = f^{\circ k+1}(0) = \beta$, $f^{\circ k-1}(c) = -\beta$, the compact set K is full, connected, locally connected, with empty interior. For x and y in K, we denote by $[x, y]$ the arc from x to y in K; more generally, for all finite subset A of K, we denote by $[A]$ the connected hull of A in K. We set $x_n = f^{\circ n}(0)$. If X is a tree and $z \in X$, we denote by $\nu_X(z)$ the number of branches of X at z.

3. Rank of a point in K.

For $z \in K$, we set $\mathrm{rg}(z) = \inf\{r \mid f^{\circ r}(z) \in [\beta, -\beta]\}$. We therefore have $\mathrm{rg}(z) = \infty$ if $(\forall r)$, $f^{\circ r}(z) \notin [\beta, -\beta]$ and $\mathrm{rg}(z) = 0$ if $z \in [\beta, -\beta]$. If $\mathrm{rg}(z) > 0$, we have $\mathrm{rg}(f(z)) = \mathrm{rg}(z) - 1$. We have $\mathrm{rg}(c) = k - 1$. Indeed, $f^{\circ k-1}(c) = -\beta \in [\beta, -\beta]$ and if $k \geq 2$, $f^{\circ k-2}(c) \in f^{-1}(\beta)$; it is an extremal point of K distinct from β and $-\beta$, so it does not belong to $[\beta, -\beta]$[1].

Remark. If $\mathrm{rg}(z) = r > 0$, we have $f^{\circ r}(z) \in [0, -\beta]$. Indeed, $f([beta, 0]) = [\beta, x] \supset [\beta, 0]$. As a consequence, $f^{-1}([\beta, 0]) \subset [0, -\beta] = [\beta, -\beta]$, and if $f^{\circ s}(z) \in [0, \beta]$ with $s > 0$, we have $f^{\circ s-1}(z) \in [\beta, -\beta]$ and $s > \mathrm{rg}(z)$.

Remark. If $\mathrm{rg}(z) = r$, the point z has 2 external arguments t and t' in K, having expansions in base 2 $t = .\varepsilon_1\varepsilon_1 \ldots \varepsilon_n \ldots$, $t' = \varepsilon'_1\varepsilon'_2 \ldots \varepsilon'_n \ldots$ such that $\varepsilon_i = \varepsilon'_i$ for $i \leq r$, $\varepsilon'_{r+1} \neq \varepsilon_{r+1}$ (possibly $t = t'$ with with two different expansions). Indeed, a point in $[\beta, -\beta]$ has 2 external arguments $t = .0\ldots$ and $t' = .1\ldots$, and this characterizes the points in $[\beta, -\beta]$, and the external arguments of $f^{\circ i}(z)$ are obtained from the external arguments of z by multiplication by 2^i, which is interpreted in terms of the expansion in base 2 by the deleting of the first i digits.

4. The tree Z_r.

For $r \in \mathbb{N}$, we denote by Z_r the set of $z \in K$ such taht $\mathrm{rg}(z) \leq r$. We have $Z_{r+1} = f^{-1}(Z_r) \cup [+\beta, -\beta]$.

Proposition 21.1. *The set Z_r is the connected hull in K of $f^{-(r+1)}(\beta)$.*

Proof. (by induction on r). We have $Z_0 = [\beta, -\beta] = [f^{-1}(\beta)]$. Let us write K as $K_+ \cup K_-$, with $K_+ \cap K_- = \{0\}$, $K_- = -K_+$, and for all subset A of K, let us set $A_+ = A \cap K_+$ and $A_- = A \cap K_-$. The map f induces a homeomorphism from $(f^{-1}(Z_r))_+$ to Z_r and similarly for $(f^{-1}(Z_r))_-$. If $Z_r = [f^{-(r+1)}(\beta)]$, we have $(f^{-1}(Z - r))_+ = [(f^{-(r+2)}(\beta))_+]$ and similarly with $-$.

For all finite subset $A \subset K$ containing β and $-\beta$, we have $[A] = [A_+] \cup [A_-] \cup [\beta, -\beta]$. As a consequence,

$$[f^{-(r+2)}(\beta] = [f^{-1}(Z - r)]_+ \cup (f^{-1}(Z_r))_- \cup [\beta, -\beta] = f^{-1}(Z_r) \cup [\beta, -\beta] = Z_{r+1}.$$

■

Corollary 21.1. *For all r, the set Z_r is a finite topological tree.*

Proposition 21.2. *For $r \in \mathbb{N}$, the set $Z_{r+1} \setminus Z_r$ contains no branching point of Z_{r+1}.*

Proof. (by induction on r). Sice $0 \in Z_0$, f induces a locally injective map $Z_1 \setminus Z_0 \to Z_0$. However, $Z_) = [\beta, -\beta]$ has no branching point. So, $Z_1 \setminus Z_0$ has no branching point. For each $r \geq 1$, the map f induces a locally injective map from $Z_{r+1} \setminus Z_r$ to $Z_r \setminus Z_{r-1}$. Hence, we see by induction that $Z_{r+1} \setminus Z_r$ has no branching point. ■

Proposition 21.3. *For $r \geq k - 1$, we have $f(Z_r) \subset Z_r$.*

Proof. For all $r \in \mathbb{N}$, we have $f(Z_r) \subset Z_{r-1} \sup f([\beta, -\beta]) = Z_{r-1} \cup [\beta, x_1]$, and $Z_{r-1} \subset Z_r$. But if $r \geq k_1$, we have $[\beta, x_1] \subset Z_r$, so $f(Z_r) \subset Z_r$. ■

[1] What about $c = -2$?

5. The point y_1.

We have $x_1 \in Z_{k-1} \setminus Z_{k-2}$ since $\mathrm{rg}(x_1) = k-1$. Denote by y_1 the projection of x_1 on Z_{k-2}, i.e., the first point (starting at x_1) where the tree $[x_1, 0]$ meets Z_{k-2}.

We set $y_n = f^{\circ n-1}(y_1)$.

Proposition 21.4. *The point y_1 is a branching point of Z_{k-1}.*

Proof. For $z \in K$, if $z \neq 0$ and if $f(z)$ is extremal in K, z is extremal. Since β is extremal (chapter 7, lemma 7.1), if $f^{\circ n}(z) = \beta$ and $f^{\circ i}(z) \neq 0$ for $0 \leq i \leq n$, the point z is extremal. In particular, if $f^{\circ k}(z) = \beta$, z is extremal in K.

Yet, y_1 is not an extremal point in Z_{k-1} since $y_1 \in x_1[\subset]\beta, x_1[$. So, $f^{\circ k}(y_1) \neq \beta$, and y_1 is neither an extremal point in Z_{k-2}. There are therefore at least two branches of Z_{k-2} at y_1, and $[y_1, x_1]$ is a branch of Z_{k-1} at y_1, distinct from the preceding ones since $[y_1, x_1] \cap Z_{k-2} = \{y_1\}$ by definition of y_1. This makes at least three branches of Z_{k-1} at y_1. ■

Corollary 21.2. *The point y_1 is preperiodic for f.*

Proof. The point $x_1 = f(0)$ is an extremal point of Z_{k-1}. As a consequence, 0 is not a branching point of Z_{k-1}, and if z is a branching point of Z_{k-1} (with $\nu_{k-1}(z)$ branches, $\nu_{k-1}(z) \geq 3$), it is the same for $f(z)$ (with $\nu_{k-1}(f(z) \geq \nu_{k-1}(z))$. Since the number of branching points in Z_{k-1} is finite, every branching point is preperiodic. ■

Since $f^{-1}(x_1) = \{0\}$, the set $f^{-1}([y_1, x_1])$ is an arc $[\delta, -\delta]$. The points δ and $-\delta$ are the points of $f^{-1}(y_1)$. The open arc $]\delta, -\delta[$ contains no branching point of Z_{k-1}, since $f(]-\delta, \delta[) =]y_1, x_1]$, and since this arc, contained in $Z_{k-1} \setminus Z_{k-2}$, contains no branching point of Z_{k-1} (proposition 21.2).

In particular, $(\forall n)$, $y_n \notin]\delta, -\delta[$.

We will now define a point y_0: if y_1 is periodic of period k, we have $y_k = \delta$ or $-\delta$. We then set $y_0 = y_k$. If y_1 is strictly preperiodic, we have $y_n \notin \{\delta, -\delta\}$ for all n, we take for y_1 one of the two points δ or $-\delta$.

6. The tree $\check{H}$.

We denote by $\check{H}$ the connected hull of $(y_n)_{n\geq 0}$ in Z_{k-1} (or in K which is the same).

Proposition 21.5. *We have $f(\check{H}) \subset \check{H} \cup [y_1, x_1]$ and $\check{H} \cap [y_1, x-1] = \{y_1\}$.*

Proof. Let $(Z_{k-1})_+$ and $(Z_{k-1})_-$ be the components of Z_{k-1} cut at 0, and set $\check{H}_+ = \check{H} \cap (Z_{k-1})_+$, and similarly for $\check{H}_-$. The set $\check{H}_+$ is the connected hull of $(y_n)_{n\in\Lambda_+}$, where $\Lambda_+ = \{n \mid y_n \in (Z_{k-1})_+\}$, ad possibly of 0. Then,

$$f(\check{H}_+) \subset [\{y_{n+1}\}_{n\in\Lambda_+}, x_1], \quad f(\check{H}_-) \subset [\{y_{n+1}\}_{n\in\Lambda_-}, x_1]$$

and

$$f(\check{H}) \subset [\{y_n\}, x_1] = \check{H} \cup [y_1, x_1].$$

The second assertion follows from the fact that $]y_1, x_1]$ contains no branching point of Z_{k-1}, so no y_n. ■

Remark. We can show, using the fact that f is sub-hyperbolic, that $f(\check{H}) = \check{H} \cup [y_1, x_1]$, except if $\check{H} = \{y_1\}$, which happens if y_1 is a fixed point of f.

Proposition 21.6. *The point y_1 is an extremal point of $\check{H}$.*

Proof. We will write $\check{\nu}(z)$ for $\nu_{\check{H}}(z)$.

If y_1 is periodic of period K, we have:

$$\check{\nu}(y_1) \leq \check{\nu}(y_2) \leq \ldots \leq \check{\nu}(y_k) \leq \check{\nu}(y_1) + 1$$

according to proposition 21.5 and to the fact that none of the y_i is 0. Yet, $\check{H}$ has at least one extremity. We therefore have $\check{\nu}(y_1) \leq 1$.

If y_1 is strictly preperiodic, we have:

$$\check{\nu}(y_0) - 1 \leq \check{\nu}(y_1) \leq \check{\nu}(y_2) \leq \ldots.$$

Yet, $\check{H}$ is not reduced to a point, there are at least 2 extremities, so $\check{\nu}(y_1) = 1$. ■

7. Hubbard's condition for $\check{H}$.

The tree $\check{H}$ is equipped from the topology and the embedding in $\mathbb{C}$ inherited from those of K (or of Z_{k-1}, or of H), and the points (y_n).

Proposition 21.7. *The tree $\check{H}$ is an abstract Hubbard tree.*

Proof. Condition (i) of the definition of abstract Hubbard trees follows from the fact that $\check{H}$ is the connected hull of the y_i. We have $\check{\nu}(y_0) \leq \check{\nu}(y_1) + 1 \leq 2$, so (ii) follows. We will show condition (iii).

Let us set $\check{f} = \rho \circ f$, where $\rho : Z_{k-1} \to Z_{k-1}$ coincides with the identity on $Z_{k-1} \setminus]y_1, x_1]$, and maps $[y_1, x_1]$ on y_1. We have $\check{f}(\check{H}) \subset \check{H}$ according to proposition 21.5, and $\check{f}(y_n) = y_{n+1}$ for all n. However, f is constant on $[\delta, -\delta] = [y_0, -y_0]$.

If $\check{H} \cap]y_0, -y_0] = \emptyset$ (which only occurs if $H = \{y_1\}$), $\check{f}$ is injective on $\check{H}$ and $\check{H}$ is an abstract Hubbard tree. We now assume that this intersection in not empty, so $[y_0, -y_0] \subset \check{H}$. In the periodic case as in the preperiodic case, the point $-y_0$ is not a marked point of $\check{H}$. We have $\check{\nu}(-y_0) \leq \check{\nu}(y_1) + 1 \leq 2$, so it is an ordinary point or an extremity. But if it were an extremity, it would be a marked point. Thus, it is an ordinary point ($\check{\nu} = 2$), i.e., a non remarkable point. Let α be the first remarkable point after $-y_0$ coming from y_0. The map $\check{f}|_{[\alpha, y_0]}$ is injective on $[\alpha, -y_0]$ and constant on $[-y_0, y_0]$.

We can find a continuous map $F : \check{H} \to \check{H}$ which coincides with $\check{f}$ (so with f) on $\check{H} \setminus]\alpha, y_0[$ and in a neighborhood of α, and which is injective on $[\alpha, y_0]$ with $F([\alpha, y_0]) = \check{f}([\alpha, +y_0]) = \check{f}([\alpha, -y_0])$. Since $\check{f}$ is injective on each of the 2 components of $\check{H} \setminus]-y_0, y_0[$, the map F is injective on each component of $\check{H}$ cut at y_0. Of course, we have $F(y_n) = y_{n+1}$ for all n. As a consequence, $\check{H}$ is an abstract Hubbard tree. ■

8. External arguments of y_1.

Proposition 21.8. a) *If y_1 is strictly preperiodic, the external arguments of y_1 are the same in H and in $\check{H}$.*

b) *If y_1 is periodic, all the external arguments of y_1 in H belong to $I(\check{H}) = [\arg_-(\check{H}), \arg_+(\check{H})]$.*

c) *If y_1 is periodic, the arguments in H of the accesses to y_1 adjacent to $[y_1, x_1]$ are associated to H.*

Proof. Let us set $\widetilde{H} = \check{H} \cup [\beta, -\beta]$. The map $\widetilde{F} : \widetilde{H} \to \widetilde{H}$ which coincides with F on $\check{H}$ and with f (and $\check{f}$) on $[\beta, -\beta] \setminus \check{H}$ is injective on each component of $\widetilde{H}$ cut at

y_0, extend F and maps β and $-\beta$ to β. As a consequence, $\widetilde{H}$ is the tree obtained by extending $\check{H}$ (cf. chapter 7).

a) The forward orbit of y_1 does not meet $[\delta, -\delta]$, so in a neighborhood of each of those points, F coincides with f. The tree we obtain by adding buds to $\widetilde{H}$, as explained in chapter 7 section 4 can be identified with a neighborhood of $\check{H}$ in H. If ξ is an access to y with respect to H, or to $\check{H}$ which is the same, the $f^{\circ n}(\xi)$ and $F^{\circ n}(\xi)$ coincide, so the digits in the expansion in base 2 are the same for the arguments of ξ relatively to H and to $\check{H}$.

b) Denote by Θ the set of external arguments of y_1 in H (i.e., in K). Denote by θ_- and θ_+ the smallest and the largest element of Θ, and let $\theta \in \Theta$. Let k be the period of y_1, and for all $t \in [0, 1[$ equipped with an expansion in base 2, let $\varepsilon_i(t)$ be the i-th digit after the comma of this expansion.

The arguments $\check{\theta}_- = \arg_-(\check{H})$ and $\check{\theta}_+ = \arg_+(\check{H})$ are characterized by the fact that they have a periodic expansion of period k, with $\varepsilon_i(\check{\theta}_-) = \varepsilon_i(\theta_-)$ and $\varepsilon_i(\check{\theta}_+) = \varepsilon_i(\theta_+)$ for $1 \le i \le k$. For each $s \in \mathbb{N}$, we have $2^{sk}\theta \in \Theta$, so $(.\varepsilon_{sk+1}(\theta) \dots \varepsilon_{sk+k}(\theta))$ is contained in between $(.\varepsilon_{sk+1}(\theta_-) \dots \varepsilon_{sk+k}(\theta_-))$ and $(.\varepsilon_{sk+1}(\theta_+) \dots \varepsilon_{sk+k}(\theta_+))$. As a consequence, $\check{\theta}_- \le \theta \le \check{\theta}_+$.

c) Let k be the period of y_1. Set $q = \nu_{Z_{k-1}}(y_1)$ and define p_1 by the condition that the branch $[y_1, x_1]$ is the p_1-th branch after $[y_1, \beta]$ turning counter-clockwise. Let z_1 be a point on $]y_1, x_1[$ next to y_1, set $z_n = f^{\circ n-1}(z_1)$ for $n \le kq + 1$. The point z_{kq_1+1} also belongs to $]y_1, x_1[$.

Denote by $\check{H}_1$ the connected hull of $\{z_1, \dots, z_{kq}\}$ in Z_{k-1}. Define α as in the proof of proposition 21.7. We can construct $F_1 : Z_{k-1} \to Z_{k-1}$ coinciding with f on $Z_{k-1} \backslash]y_0, \alpha[$, and also on each $[y_i, z_{i+sq}]$ for $0 \le i \le k-1$ and q such that $1 \le i + sq \le kq - 1$, with $F(z_{kq}) = z_1$, and F injective on eqch component of $\check{H}_1$ cut qt $z_0 = z_{kq}$.

The tree $\check{H}_1$ is an abstract Hubbard tree, which can be indentified with the tree obtained from $\check{H}$ by bifurcation of argument p/q. As in a), the arguments of $x-1$ are the same in H and in $\check{H}_1$. The arguments of the accesses adjacent to $[x_1, z-1]$ (i.e., to $[x_1, y_1]$) therefore are $\arg_-(\check{H}_1)$ and $\arg_+(\check{H}_1)$, they are associated to $\widehat{H}$. ■

9. Proof of the theorem.

Denote by θ_- and θ_+ the smallest and the largest external argument of y_1 in K (or in Z_{k-1}, it is the same). Denote by θ' and θ'' the external argument of the accesses to y_1 adjacent to $[y_1, x_1]$ with respect to Z_{k-1}, with $\theta' < \theta''$. We therefore have $\theta_- \le \theta' < \theta'' \le \theta_+$. The arguments θ' and θ'' are associated to $\check{H}$.

The external argument of x_1 in K is τ and we have $\theta' < \tau < \theta''$. For all $\tau' = p'/2^{k'}$ with $k' < k$, the point $\gamma(\tau')$ of K, with external argument τ', is an extremity of Z_{k-2}, so $\tau' \notin [\theta', \theta'']$. Thus, the argument τ is the leader of $[\theta', \theta'']$.

The point y_1 in not an extremal point of Z_{k-2}, so there exists an extremal point $\zeta \in Z_{k-2}$ such that $y_1 \in]\beta, \zeta[$. We have $\arg_K(\zeta) \in [\theta_-, \theta_+]$, and $\arg_K(\zeta)$ is of the form $p'/2^{k'}$ with $k' < k$ (proposition 21.1). As a consequence, τ is not the leader of $[\theta_-, \theta_+]$.

We have $I(\check{H}) \supset [\theta_-, \theta_+] \supset [\theta', \theta'']$, so τ belongs to $I(H)$ without being its leader. ■

CHAPTER 22

Addresses.

1. Origin of a vein.

We keep the notations of the two preceding chapters. $\mathcal{H}$ is the set of isomorphisms classes of abstract Hubbard trees. For each $H \in \mathcal{H}$, we have defined the associated arguments and the interval $I(H) = [\arg_-(H), \arg_+(H)]$.

This enables us to define the veins in terms of trees: for $\tau = p/2^k$ with $k > 0$,

$$N_{\mathcal{H}}(\tau) = \{H \in \mathcal{H} \mid (\exists \theta', \theta'' \text{ associated to } H)\ \tau \text{ is the leader of } [\theta', \theta'']\},$$

$$N^*_{\mathcal{H}}(\tau) = \{H \in \mathcal{H} \mid \tau \text{ is the leader of } I(H)\}.$$

In chapter 21, we have proved the following theorem.

Theorem 22.1. *For all τ of the form $p/2^k$ with $k > 0$, there exists a tree $\check{H} \in N_{\mathcal{H}}(\tau) \setminus N^*_{\mathcal{H}}(\tau)$.*

But we do not know whether the tree we constructed is the tree of a point $c \in \mathcal{D}_0 \cup \mathcal{D}_2$.

However, we will prove the following theorem.

Theorem 22.2. *There exists $c \in \mathcal{D}_0 \cup \mathcal{D}_2$ such that every arguemtn associated to $\check{H}$ is associated to c.*

We immediately obtain the following theorem.

Theorem 22.3. *For all $\tau = p/2^k$ with $k > 0$, the vein $N_{\mathcal{D}}(\tau)$ has an origin.*

The vein $N_{\mathcal{D}}(0)$ consists in the points 0 and 1/4. We will declare that 1/4 is the origin of the vein $N_{\mathcal{D}}(0)$, even if it belongs to $N^*_{\mathcal{D}}(0)$. In the following, we will write N for $N_{\mathcal{D}}$.

We will call proper arguments of an abstract tree H its associated arguments if it is preperiodic, but only its arguments $\arg_-(H)$ and $\arg_+(H)$ if it is periodic; for $c \in \mathcal{D}_1 \cup \mathcal{D}_2$, the proper arguments of the tree of f_c are the arguments associated to c, i.e., the external arguments of c in M.

We will fix a $\tau = p/2^k$, $k > 0$; we will denote by c_τ the point of M of external argument τ, H the Hubbard tree of f_{c_τ}.

1.1. Construction of the point anounced in theorem 22.2.

We take the proper arguments $\theta_1 \dots \theta_m$ of $\check{H}$; we consider the points of $\mathcal{D}_0 \cup \mathcal{D}_2$ smaller than c_τ (in the sense of relation $<$ of chapter 20 section 4) whose tree has a $2^i\theta_j$ ($i \geq 0$, $1 \leq j \leq n$) as proper argument. Those are finitely many; since the set of points in $\mathcal{D}_0 \sup \mathcal{D}_2$ smaller than c_τ is totally ordered (chapter 20, corollary 20.1), there is a greatest point c among the points smaller than c_τ, whose tree has a $2^i\theta_j$ as proper argument; we will show that c gives a solution to theorem 22.2.

For $c_1 \in M$, $\sim_{c_1}$ stands for the equivalence relation on $\mathbb{Q}/\mathbb{Z}$ defined by $\theta \sim_{c_1} \theta'$ if and only if θ and θ' are external arguments of a common point of K_{c_1}.

We will now explain how $\check{H}$ enables us to define an equivalence relation $\sim_H$ on

$$A = \{2^i\theta_j \mid i \geq 0,\ 1 \leq j \leq n\} \cup \{1/2 + 2^i\theta_j \mid i \geq 0,\ 1 \leq j \leq n\}.$$

The construction of the extended tree $\check{H}$ at the beginning of proposition 21.8 and the algorithm described in chapter 18 section 3 enables us to define, for all preperiodic point of $\check{H}$, its rational "combinatorial associated arguments".

To the opposite of what happens in Julia sets, it is possible that two distinct points in H have common combinatorial arguments.

For $\theta, \theta' \in A$, we will set $\theta \sim_{\check{H}} \theta'$ if there is a point of $\check{H}$ of the form y_i or $-y_i$ having α and α' as combinatorial arguments. (We do not claim at the moment that $\sim_{\check{H}}$ is an equivalence relation).

We have defined y_i for all $i \geq 0$ with $y_{i+1} = f(y_i)$, f being the dynamics on $\check{H}$; for $i \geq 1$ in the preperiodic case, $i \geq 0$ in the periodic case, y_i is also a well defined point of H.

The following proposition completes proposition 21.8.

Proposition 22.1. a) *All elements of A are external arguments of points y_i or $-y_i$ $(i \geq 1)$ of H.*

b) *For all $i > 0$, the combinatorial arguments of y_i in $\check{H}$ are external arguments of y_i in H.*

Proof. a) θ_j is an external argument of y_1 $(1 \leq j \leq n)$, so for $i \geq 0$, $2^i\theta_j$ is an external argument of y_{i+1}; $1/2 + 2^i\theta_j$ is therefore an external argument of $-y_i$.

b) In the preperiodic case as in the periodic case, with the notations of chapter 21, $y_i \notin [-\delta, \delta]$ and we can reproduce the proof of proposition 21.8 a). ■

Corollary 22.1. $\sim_H = \sim_{c_\tau|A}$.

We will denote by $\gamma_M(\alpha)$ the point of M with external argument α for $\alpha \in \mathbb{Q}/\mathbb{Z}$.

Lemma 22.1. *Let U be the connected component of $M \setminus \{\gamma_M(2^i\theta_j) \mid i \geq 0,\ 1 \leq j \leq n\}$ containing c_τ. On U, $\sim_{c|A}$ is constant.*

Proof. Let $\theta, \theta' \in A$. Let us show that the set of $C_1 \in U$ such that $\theta \sim_{c_1} \theta'$ is open and closed in U.

It is closed because $\theta \sim_{c_1} \theta' \iff \gamma_{c_1}(\theta) = \gamma_{c_1}(\theta')$, where $gamma_{c_1}(\theta)$ (respectively $\gamma_{c_1}(\theta')$) stands for the landing point of the ray of argument θ (respectively θ') in K_{c_1}, and $\gamma_{c_1}(\theta)$ as $\gamma_{c_1}(\theta')$ is a continuous function of c_1 (see chapter 17; here we only use the easy part of the theorem).

It is open since by choice of the points we removed in M, $\gamma_{c_1}(2^i\theta_j)$ or $\gamma_{c_1}(1/2 + 2^i\theta_j)$ (for $i \geq 0$, $1 \leq j \leq n$) is repelling preperiodic and not in the backward orbit of zero for f_{c_1}, so there exists in a neighborhood of c_1 only one preperiodic determination of the same kind for the continuous function $\gamma_{c_1}(2^i\theta_j)$ or $\gamma_{c_1}(1/2 + 2^i\theta_j)$. ■

Lemma 22.2. *If c is periodic, $\sim_{c_\tau|A} = \sim_{c|A}$.*

If c is preperiodic, the graph of $\sim_{c|A}$ contains the one of $\sim_{c_\tau|A}$.

Proof. In the periodic case, U defined in lemma 22.1 contains c, since otherwise one of the points $\gamma_M(2^i\theta_j)$ $(i \geq 0,\ 1 \leq j \leq n)$ would not be in between c and c_τ

in the sense of the order defined on $\mathcal{D}$, which would contradict the definition of c. Thus, lemma 22.1 implies that $\sim_{c_\tau|A}=\sim_{c|A}$.

In the preperiodic case, we can only say that $c \in \overline{U}$ by the same argument; the relations $\alpha \sim \alpha'$ are open at c (i.e., $\alpha \not\sim_c \alpha'$), so they remain open in q neighborhood of c, so in U, which yields the result. ■

Remark. What can happen in that case is, if c is of the form $\gamma_M(2^{i_0}\theta_j)$ with $i_0 > 0$ and $1 \leq j \leq n$, that the $2^{i_0}\theta_j$ and $1/2 + 2^{i_0}\theta_j$ which are not linked for $\sim_{c_0}$ get grouped fr $\sim_c$; for $i < i_0$, the $2^i\theta_j$ and $1/2 + 2^i\theta_j$ land in K_c at opposite points, so are not linked together.

We therefore deduce, given corollary 22.1, the following result.

Corollary 22.2. *If c is in $\mathcal{D}_0$, $\sim_{\check{H}}=\sim_{c|A}$.*

1.2. Proof of theorem 22.2. Case where H is periodic. Here, there are only two θ_j: θ_1 and θ_2, which have odd denominator. By definition of c, at least one proper argument of the tree associated to c is of the form $2^i\theta_j$ ($i \geq 0$, $j = 1, 2$); reindexing $\theta_1 2$ and θ_2 if necessary, we may assume that it is $2^i\theta_1$ which has odd denominator: c is therefore in $\mathcal{D}_0$.

Given the purely algorithmic construction of the arguments associated to a periodic tree provided the proper arguemnt (see chapter 18 section 3), it is enough to show that the tree associated to c and $\check{H}$ have the same proper arguments. Since it is equal to the tree associated to c (see chapter 18 propositions 18.1 and 18.2), we may consider the tree at the root c_1 of the hyperbolic component M of which c is the center.

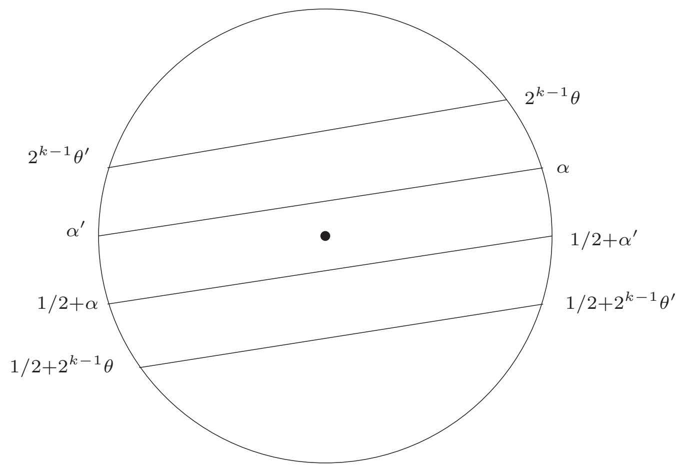
Case where c_1 is a primitive root (i.e., the multiplier of the rationally indifferent cycle is 1). Then, every ray landing at a point of the rationally indifferent cycle which attracts c_1 in K_{c_1} provides a proper argument of c_1: since there are two external rays of M landing at c_1, there are exactly two such rays. We have $2^{i_0}\theta_1 \sim_{\check{H}} 2^{i_0}\theta_2$, so $2^{i_0}\theta_1 \sim_{c_1} 2^{i_0}\theta_2$ and the proper arguments of the tree of c_1 therefore are $2^{i_0}\theta_1$ and $2^{i_0}\theta_2$ for some $i_0 \geq 0$. We still have to show that $i_0 = 1$: on a circle, for each i, let us join with a segment the point of argument $2^i\theta_1$ to the point of argument $2^i\theta_2$ on the one hand, and the point of argument $1/2 + 2^i\theta_1$ to the point of argument $1/2 + 2^i\theta_2$ on the other hand. If k is the period of θ_1 or θ_2 for multiplication by 2, i_0 is determied by the fact that the segments from $2^{i_0+k-1}\theta_1$ to $2^{i_0+k-1}\theta_2$ and from $1/2 + 2^{i_0+k-1}\theta_1$ to $1/2 + 2^{i_0+k-1}\theta_2$ bound a region which contains the center of the circle; but it is precisely the case for $i = 1$ (the rays of arguments $2^k\theta_1$, $2^k\theta_2$, $1/2 + 2^k\theta_1$, $1/2 + 2^k\theta_2$ landing in H at δ and $-\delta$ with the notations of chapter 21).

θ_1 and θ_2 are the proper arguments of c_1.

Case where c_1 is not a primitive root. Here, f_{c_1} acts transitively on the rays landing at the points of the rationally indifferent cycle; since $\theta_1 \sim_{\check{H}} \theta_2$, we also have $\theta_1 \sim_{c_1} \theta_2$, so θ_2 is of the form $2^a\theta_1$ for some $a \geq 0$. As in the case where c_1 is primitive, let us join with a segment $2^i\theta_1$ to $2^i\theta_2$ and $1/2 + 2^i\theta_1$ to $1/2 + 2^i\theta_2$. Here again, the segments from $2^k\theta_1$ to $2^k\theta_2$ and from $1/2 + 2^k\theta_1$ to $1/2 + 2^k\theta_2$ bound a region which contains the center of the circle, which shows that the rays in K_{c_1} which are adjacent to the petal containing c_1 are θ_1 and θ_2, and those are the proper arguments of c_1.

Case where H is preperiodic. As in the previous case, let us join on a circle the points of argument α and α' for all pair α, α' of points of A which are equivalent for $\sim_H$, and which are therefore also equivalent for $\sim_c$ (see lemma 22.2).

Case where $c \in \mathcal{D}_0$. We will show that this case is not possible. It is not enough here to consider $\sim_{\check{H}}$, and we must come back to $\sim_{c_\tau}$, which is equal to $\sim_c$ on A by lemma 22.2. Then, in the auxilliary scheme, we can add the segments between the external arguments of δ (respectively $-\delta$). Since δ and d are on both sides of zero on H (see chapter 21 section 5), there is no branching point of Z_{k-1} on $]-\delta, \delta[$, we can therefore find α and α' such that 2α and $2\alpha'$ are one of the θ_j and one of the θ'_j ($1 \le j \le n$, $1 \le j' \le n$), such that the associated rays land in K_{c_τ} at δ, and such that on the auxilliary scheme, the segments from α to α' and from $1/2+\alpha$ to $1/2+\alpha'$ separate the center of the circle from all the segments corresponding to $\sim_{\check{H}|A}$. We then have the following situation:

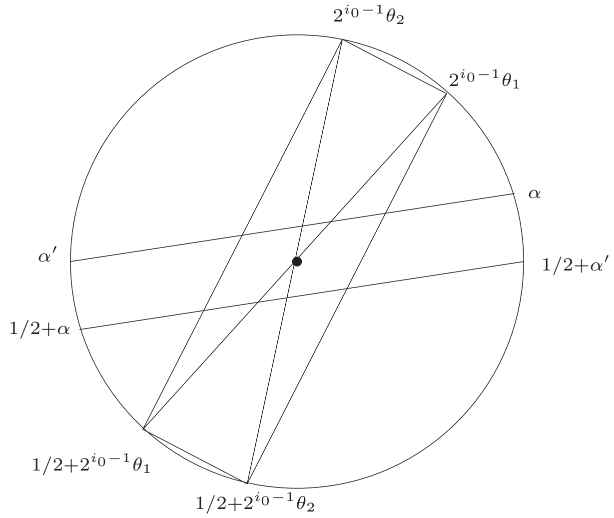


If θ and θ' are the proper arguments of tree of c, and k the periodi of the cycle of f_c, we then have a contradiction, since the reasonning of lemma 22.1 show that $\mathcal{R}(K_c, \alpha)$ and $\mathcal{R}(K_c, \alpha')$ still land at a common point, which has to be on the allowable arc in K_c which joins the landing points of $\mathcal{R}(K_c, 2^{k-1}\theta)$ and $\mathcal{R}(K_c, 1/2+2^{k-1}\theta)$; but this arc is in $\mathring{K}_c$, which is a contradiction.

Case where $c \in \mathcal{D}_2$. We can define α and α' as above. If a $2^i\theta_j$ ($1 \le j \le n$, $i \ge 0$) is a proper argument of the tree of c, all other $2^i\theta_{j'}$ ($1 \le j' \le n$) also are, since they are $\sim_c$ equivalent to $2^i\theta_j$. Assume it is the case for some $i_0 > 0$. According to the remark that follows lemma 22.2, all the $2^{i_0-1}\theta_j$ and the $1/2 + 2^{i_0-1}\theta_j$ ($1 \le j \le n$) are $\sim_c$ equivalent. We therefore have the following situation:

Since $\sim_c$ is not crossed, it means that $\mathcal{R}(K_c, \alpha)$ also land at 0, which is impossible since $2^{i_0-1}\theta_j$ is of the form $2^p\alpha$ with $p \ne 0$ and 0 is not periodic for f_c.

Since at least one $2^i\theta)j$ ($i \ge 0$, $1 \le j \le n$) is an argument associated to c by choice of c, the θ_j ($1 \le j \le n$) are proper arguments of the tree of c. ■



2. Finite addresses.

Let c be a point of $\mathcal{D}$. A *finite address* for c is a finite sequence $(c_0, \tau_0, c_1, \tau_1, \ldots, c_{r-1}, \tau_{r-1}, c_r)$ such that

(a) $c_0 = 1/4$; $\tau_0 = 0$ and $c_1 = 0$ if $r \geq 1$.
(b) $c_r = c$.
(c) c_i is the origin of $N(\tau_i)$ for $i = 0, \ldots, r-1$.
(d) $c_{i+1} \in N^*(\tau_i)$ for $i = 0, \ldots, r-1$.

Proposition 22.2. *Let $c \in \mathcal{D}$ be such that $I(c)$ is not reduced to a point, or is of the form $\{p/2^k\}$. Then, c has a unique finite address.*

Proof. Let us set $c'_0 = c$, and define c'_i, τ'_i by induction. For all i, τ'_i is the leader of $I(c'_i)$, and c'_{i+1} is the origin of $N(\tau'_i)$. (Note that the $I(c'_i)$ form an increasing sequence, so that the leader of $I(c'_i)$ is defined for all i.) The number τ'_i is of the form $p'_i/2^{k'_i}$, and the k'_i are strictly decreasing until they vanish. Therefore, there exists r such that $k'_{r-1} = 0$, i.e., $\tau'_{r-1} = 0$, so $c'_r = 1/4$. We then set $c_i = c'_{r-i}$ and $\tau_i = \tau'_{r-1-i}$, and $(c_0, \tau_0, \ldots, \tau_{r-1}, c_r)$ is a finite adress of c. It is clear that a finite address is necessarily obtained in this way, so we have uniqueness. ∎

Remark. Let $c \in \mathcal{D}$ have address $(c_0, \tau_0, c_1, \tau_1, \ldots, c_r)$. We have $c_0 < c_1 < \ldots < c_r$. The set of points before c is $[c_0, c_1]_{N(\tau_0)} \cup \ldots \cup [c_{r-1}, c_r]_{N(\tau_{r-1})}$. If $c' \in]c_{i-1}, c_i]$, the point c' has address $(c_0, \tau_0, \ldots, c_{i-1}, \tau_{i-1}, c')$.

Remark. We can define infinite addresses, by replacing condition (b) by (b'): $\bigcap I(c_r) = I(c)$. We can than show that all point of $\mathcal{D}$ has a finite or infinite address.

3. Seperation points.

Proposition 22.3. *Let c and c' be two points of $\mathcal{D}$ satisying the conditions of proposition 22.2. Then, c and c' have in $\mathcal{D}$ a lower bound $c'' = c \wedge c'$.*

Proof. Let $(c_0, \tau_0, \dots, c_r)$ and $(c'_0, \tau'_0, \dots, c'_{r'})$ be the adresses of respectively c and c', and let k be the largest i such that $\tau_i = \tau'_i$. The points c_{k+1} and c'_{k+1} belong to the strict vein $N^*(\tau_i)$, so they are comparable; let c'' be the least one. It is clear that c'' is the greatest common lower bound of c and c'. ■

Remark. The hypothesis that c and c' satisfy the conditions of proposition 22.2 is not necessary: we can get rid of it by considering infinite addresses if necessary.

Proposition 22.4. *Let c and c' be two points of $\mathcal{D}$ satisying the conditions of proposition 22.2 and set $c'' = c \wedge c'$. We assume that c and c' are not comparable (i.e, $c \not\leq c'$ and $c' \not\leq c$, so that $c'' < c$ and $c'' < c'$). Then, there exist three arguments θ_1, θ_2, θ_3 associated to c'' such that, exchanging c and c' if necessary, we have* $0 < \theta_1 < \arg_-(c) \leq \arg_+(c) < \theta_2 < \arg_-(c') \leq \arg_+(c') < \theta_3 < 1$.

Proof. Let $(c_0, \tau_0, \dots, c_r = c)$ and $(c'_0, \tau'_0, \dots, c'_{r'} = c')$ be the adresses of c and c', and k be the largest i such that $\tau_i = \tau'_i$. We have $c'' \in N(\tau_k)$. Let us set $\tau_* = \tau_{k+1}$ if $c'' = c_{k+1}$ and $\tau_* = \tau_k$ if $c'' < c_{k+1}$, and let us define similarly τ'_*. We have $c'' \in N(\tau_*)$ and $c'' \in N(\tau'_*)$, so there exists four segments θ_1^*, θ_2^*, θ_3^*, θ_4^* associated to c'' such that τ_* is the leader of $[\theta_1^*, \theta_2^*]$ and τ'_* the leader of $[\theta_3^*, \theta_4^*]$. We then have $\theta_1^* < \arg_-(c) \leq \arg_+(c) < \theta_2^*$ and $\theta_3^* < \arg_-(c') \leq \arg_+(c') < \theta_4^*$. On the other hand, $\tau \notin [\theta_3^*, \theta_4^*]$ or $\tau' \notin [\theta_1^*, \theta_2^*]$. It follows that we can extract of $\{\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*\}$ three arguments satisfying the required conditions. ■

For $c \in \mathcal{D}$ and θ an argument associated to c, we define $\hat{\mathcal{R}}(c, \theta)$ in the following way: if $c \in \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{R}(M, \theta)$ lands at c and we set $\hat{\mathcal{R}}(c, \theta) = \overline{\mathcal{R}(M, \theta)} = \mathcal{R}(M, \theta) \cup \{c\}$. If $c \in \mathcal{D}_0$, c is the center of a hyperbolic component W and $\mathcal{R}(M, \theta)$ lands at a point $c' \in \partial W$; we then set $\hat{\mathcal{R}}(c, \theta) = \mathcal{R}(M, \theta) \cup [c, c']_{\overline{W}}$.

Proposition 22.3 has an additional information.

Corollary 22.3. *Let c, c', c'', θ_1, θ_2, θ_3 be as in proposition 22.3. Then c, c' and $\mathcal{R}(M, 0)$ are contained in three distinct connected components of* $\mathbb{C} \setminus \bigcup_{i=1}^{3} \hat{\mathcal{R}}(c'', \theta_i)$.

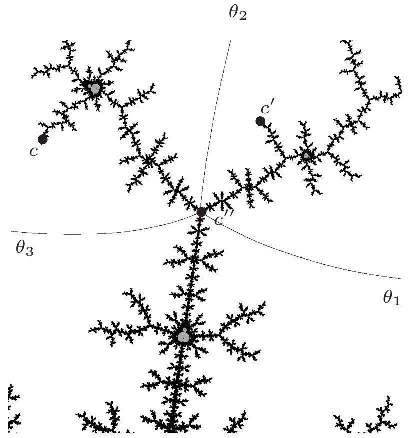
4. The implication (MLC)⟹(HG2).

Theorem 22.4. *If M is locally connected, every connected component of $\overset{\circ}{M}$ is hyperbolic.*

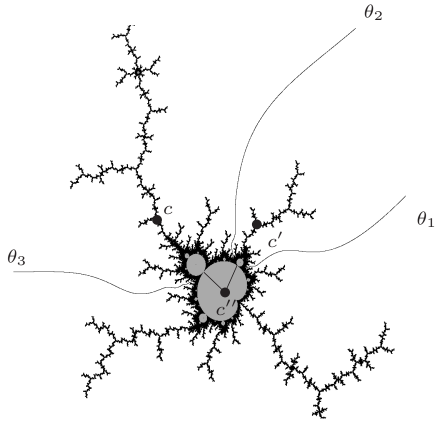
Proof. Assume M is locally connected and let W be a non hyperbolic connected component of $\overset{\circ}{M}$. The set ∂W is uncountable, and since $\mathcal{D}$ is countable, we can find three distinct points x_1, x_2 and x_3 in $\partial W \setminus \mathcal{D}$. The Carathéodory loop $\mathbb{T} \to \partial M$ is surjective; we have $\partial W \subset \partial M$, so we can find t_1, t_2, t_3 such that $\mathcal{R}(M, t_i)$ land at x_i. Permuting the x_i if necessary, we may assume that $0 < t_1 < t_2 < t_3 < 1$. Let $\tau = p/2^k$ and $\tau' = p'/2^{k'}$ be such that:

$$0 < t_1 < \tau < t_2 < \tau' < t_3 < 1; \tag{4}$$

denote by c and c' the landing points of $\mathcal{R}(M, \tau)$ and $\mathcal{R}(M, \tau')$, and set $c'' = c \wedge c'$ (points c and c' satisfy the conditions of proposition 22.2). Since c and c' are maximal in $\mathcal{D}$, they are not comparable. According to proposition 22.3 and



$c'' \in \mathcal{D}_2$



$c'' \in \mathcal{D}_0$

its corollary, we can find three arguments $\theta_1, \theta_2, \theta_3$ asociated to c'' such that c, c' and $\mathcal{R}(M, 0)$ are in three distinct connected components of $\mathbb{C} \setminus Y$, where $Y = \bigcup_{i=1}^{3} \hat{\mathcal{R}}(c'', \theta_i)$. Let us set $X = W \cup \bigcup_{i=1}^{3} \mathcal{R}(M, t_i) \cup \{x_i\}$. The set X is connected. We have $X \cap Y = \emptyset$: indeed, the points of X in $\mathbb{C} \setminus M$ have irrational external arguments whereas those of point in $Y \cap \mathbb{C} \setminus M$ are rational, $X \cap \partial M \subset \partial W \setminus \mathcal{D}$ and $Y \cap \partial M \subset \mathcal{D}$, $X \cap \overset{\circ}{M} \subset W$ non hyperbolic whereas $Y \cap \overset{\circ}{M}$ is empty or contained in a hyperbolic component of center c''. Relation 4 shows that c, c' and $\mathcal{R}(M, 0)$ are in three distinct component of $\mathbb{C} \setminus X$.

Let U be the connected component of $\mathbb{C} \setminus X$ containing Y and V the connected component of $C \setminus Y$ containing X. Every connected component of $\mathbb{C} \setminus X$ other than U is contained in V, so V contains at least two of the three sets $\{c\}$, $\{c'\}$ and $\mathcal{R}(M, 0)$. Contradiction ■

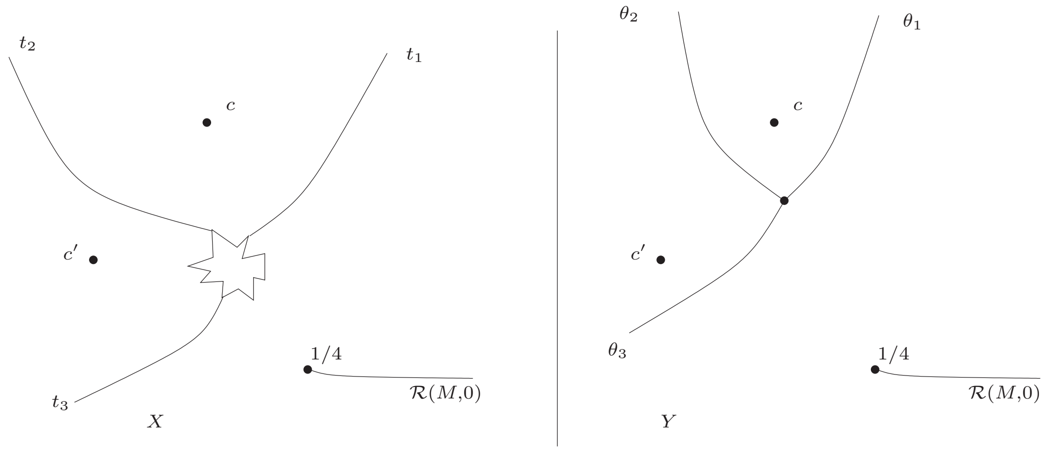


FIGURE 1. Impossible to realize with $X \cap Y = \emptyset$.

CHAPTER 23

Similarity between M and K_c at a Misurewicz point.

By Tan Lei

The french version of the Orsay notes contains a chapter written by Tan Lei. The results have been published in Communication in Mathematical Physics [**Ta**].

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