Corrections and Clarifications to the Second Printing: Appendix

p. 590 In the sentence after Equation A1.7, $\mathbf{g}(a)$ should be $\mathbf{g}(\mathbf{a})$.

p. 592 In the proof of Kantorovitch's theorem, we neglected to justify uniqueness. Please return to the main page and click on the separate link for that proof; we don't give it here to minimize downloading or printing time for those not interested.

p. 593 Proposition A2.1, U is an open ball; the Lipschitz condition holds for all $\mathbf{x}, \mathbf{y} \in U$.

p. 594, second line, g should be $\mathbf{g}$.

p. 595, in the margin note, it would be better to add plus signs before the dots: $1 + |A| + |A|^2 + \dots$, and $I + A + A^2 + \dots$, not $1 + |A| + |A|^2 \dots$, and $I + A + A^2 \dots$.

p. 595 In Equation A2.28, it should be $\mathbf{a}_2 = \mathbf{a}_1 + \vec{\mathbf{h}}_1$, not $\mathbf{a}_2 = \mathbf{a}_1 + \vec{\mathbf{h}}_0$.

p. 596 The two lines at the top of page should be part of the proof, which should start as follows:

Proof. This is a straightforward application of Proposition A2.1, which says that

$$\left|\mathbf{f}(\mathbf{a}_1) - \mathbf{f}(\mathbf{a}_0) - [\mathbf{Df}(\mathbf{a}_0)]\vec{\mathbf{h}}_0\right| \le \frac{M}{2}|\vec{\mathbf{h}}_0|^2, \qquad A2.30$$

but a miracle happens during the computation: the third term in the sum on the left of Equation A2.30 cancels with the second term, since

$$-[\mathbf{Df}(\mathbf{a}_0)]\vec{\mathbf{h}}_0 = [\mathbf{Df}(\mathbf{a}_0)]\overbrace{[\mathbf{Df}(\mathbf{a}_0)]^{-1}\mathbf{f}(\mathbf{a}_0)}^{\vec{\mathbf{h}}_0} = \mathbf{f}(\mathbf{a}_0). \qquad A2.31$$

(Figure A2.1 explains why the cancellation occurs.) So we get ...

p. 597, Lemma A3.1: we are replacing the $\mathbf{a}_i$ by $\mathbf{a}_n$ to be consistent with notation on p. 598. The introduction to the proof of that lemma (last three lines of the page) now read

We will use Equation A2.35 in the form $|\vec{\mathbf{h}}_1| \le k|\vec{\mathbf{h}}_0|$, which by induction gives $|\vec{\mathbf{h}}_i| \le k|\vec{\mathbf{h}}_{i-1}|$, so that ... .

In the third printing we plan to add a new margin note for the proof of Lemma A3.1: Note that while Lemma A2.3 compares the derivative at $\mathbf{a}_1$ to the derivative at $\mathbf{a}_0$, here we compare the derivative at $\mathbf{a}_n$ to the derivative at $\mathbf{a}_0$.

p. 598 Equation A3.10: the last $\le$ should be an equality.

Note: In the proof of the inverse function theorem we did not provide a justification of Equation 2.9.7. Anyone who wants it should please email the authors at jhh8@cornell.edu.

p. 599 In the second margin note, the last line should read: "The last inequality comes from the fact (Equation 2.9.10 and Proposition 1.4.11) that $|\vec{\mathbf{h}}_0(\mathbf{y})| \le |L^{-1}||\mathbf{y}_0 - \mathbf{y}|$" (i.e., $|\vec{\mathbf{h}}_0(\mathbf{y})|$, not $\vec{\mathbf{h}}_0(\mathbf{y})$).

p.600-601 Part (3) of the proof, "proving that $\mathbf{g}$ is an inverse, not just a right inverse," is slightly wrong and is also no longer really necessary, because of the proof (posted on the web page) of uniqueness for Kantorovitch's theorem. The entire discussion of part (3) has been replaced by

(3) Proving that $\mathbf{g}$ is an inverse, not a just right inverse

We have already proved that $\mathbf{f}$ is onto the neighborhood V of $\mathbf{y}_0$; we want to show that $\mathbf{g}(\mathbf{y})$ is the *only* solution $\mathbf{x}$ of $\mathbf{f}_\mathbf{y}(\mathbf{x}) = 0$ with $\mathbf{x} \in W_0$. This is a stronger result than the original statement of Kantorovitch's theorem, but it is exactly the the same result as the modified statement of Kantorovitch's theorem discussed in Remark A2.5; the ball W_0 of the inverse function theorem is the ball U_{-1} discussed there.

p. 602 Statement of Theorem 2.9.10, " Let $B_R(\mathbf{b}) \subset \mathbb{R}^m$ be the ball … " not "Let $B_R(\mathbf{b}) \in \mathbb{R}^m$ be the ball … ."

p.610, Proposition A8.2, part (5): little and big o should be reversed, so to make the proof coherent:

$$5.\ O(|\mathbf{x}|^k)\ o(|\mathbf{x}|^l) = o(|\mathbf{x}|^{k+l})$$

p. 610 In the line immediately before Equation A8.2, there should be a third condition: if $|\mathbf{x}| < \delta_1$.

p. 611 The proof of Proposition A8.2 ends with Equation A.7. There should then be a new subsectionheading, "Composition rules."

p. 615 Theorem A9.2: There exists c between a and $a+h$, not between a and x; $f^{(k+1)}$, not f^{k+1}.

p. 615 Third line of Example A9.4: $|\theta| \le \pi/6$ not $|\theta| < \pi/6$.

p. 615, margin note: $6! = 720$, not 620

p. 615, Equation A9.7 should read (addition of $\frac{1}{I!}$ on the right)

$$f(\mathbf{a}+\vec{\mathbf{h}}) = P^k_{f,\mathbf{a}}(\mathbf{a}+\vec{\mathbf{h}}) + \sum_{I \in \mathcal{I}^{k+1}_n} \frac{1}{I!} D_I f(\mathbf{c})\, \vec{\mathbf{h}}^I.$$

p. 617 In the first line, $\mathbf{h}$ should be $\vec{\mathbf{h}}$. In Theorem 3.5.3, part (a), it would be better to specify that there are $m = k + l$ linearly independent linear functions … .

p. 618 In the two lines after Equation A10.5, it might be clearer to write "in particular, when $\vec{\mathbf{x}} = \vec{\mathbf{e}}_i$ (i.e., when $x_i = 1$ and all the other variables are 0)," rather than "in particular, $\vec{\mathbf{x}} = \vec{\mathbf{e}}_i$, when $x_i = 1$ and all the other variables are 0."

p. 619 We have changed Proposition 3.8.12 to match the new version in Section 3.8:

Proposition 3.8.12 (Frenet frame). *The vectors $\vec{\mathbf{t}}(0), \vec{\mathbf{n}}(0), \vec{\mathbf{b}}(0)$ form the orthonormal basis (Frenet frame) with respect to which our adapted coordinates are computed. Thus the point with coordinates X, Y, Z in the new, adapted coordinates is the point*

$$\mathbf{a} + X\vec{\mathbf{t}}(0) + Y\vec{\mathbf{n}}(0) + Z\vec{\mathbf{b}}(0)$$

in the old x, y, z coordinates.

p. 621, Equation A11.14: the $A_3 s^2$ in the first line should be $A_3^2 s^2$. Equation A11.17: the minus sign in the third entry of the first matrix should be deleted.

p. 622 In the third printing we plan to add a margin note for Equation A12.11:

In Equation A12.11 we use the version of Stirling's formula given in Equation A12.8, using C rather than $\sqrt{2\pi}$, as we have not yet proved that $C = \sqrt{2\pi}$.

p. 624, second line: Exercise 12.1 should be Exercise A12.1 Similarly, Equation 12.11 should be A12.11.

p. 625, Equation A12.15, in the first term of the 3rd line, an open parenthesis is missing; the term should be $\frac{1}{(1-t^2/n)^n}$.

p. 626 Equation A12.21 is correct, but we have rewritten the left-hand side in a form that matches Equation A.12.10, and we have replaced the arrow by the words "converges to":

$$\frac{1}{2^{2n}} \sum_{k=a\sqrt{n}}^{b\sqrt{n}} \binom{2n}{n+k} \quad \text{converges to} \quad \frac{1}{\sqrt{\pi}} \int_a^b e^{-t^2}\, dt.$$

p. 626 Equation A13.1, first line, $y \leq 1$, not $y < 1$ (but this does not affect the result).

p. 627 First paragraph and first margin note: we did not define F, or rather, we defined it in an earlier version of the text. It is defined by $F(x) = \int_0^1 f\begin{pmatrix} x \\ y \end{pmatrix} dy$. In subsequent printings we will deal with this by replacing the next-to-last sentence on page 626 by

However, the inner integral $F(x) = \int_0^1 f\begin{pmatrix} x \\ y \end{pmatrix} dy$ does not make sense when $x = 1$.

p. 628, Last paragraph of margin notes, $f(\mathbf{x}, y)$ should be $f(\mathbf{x}, \mathbf{y})$.

p. 629 First line of Section A.14: "Here we prove Theorem 4.7.5" not "Here we prove Theorem."

p. 630, immediately before and after Equation A14.5: It might be clearer to write "Now define the function $\overline{f}$ that assigns to each $\mathbf{x}$ the maximum of f" (rather than " ... the maximum of the function"), and "define the function $\overline{g}$ that assigns to each $\mathbf{x}$ the maximum of f" (rather than " ... the maximum of the function").

p. 630 and p. 631 Our choice of L to denote the union of cubes was unfortunate, as we also have L denoting the lower integral. In the future we will denote the union by B:

passages affected p. 630: Next, find $N' > N$ such that if B is the union of the cubes $C \in \mathcal{D}_{N'}$ whose closures intersect $\partial\mathcal{D}_N$, then the contribution of $\text{vol}_n B$ to the integral of f is negligible. We do this by finding N' such that

$$\text{vol}_n B \leq \frac{\epsilon}{8 \sup |f|}.$$

Now, find N'' such that every $P \in \mathcal{P}_{N''}$ either is entirely contained in B, or is entirely contained in some $C \in \mathcal{D}_N$, or both.

passage affected p. 631:

$$\left| U_{\mathcal{P}_{N''}}(f) - \int_{\mathbb{R}^n} \overline{g} |d^n \mathbf{x}| \right| \leq 2 \sup |f| \,\text{vol}_n B \leq 2 \sup |f| \frac{\epsilon}{8 \sup |f|} = \frac{\epsilon}{4}.$$

p. 632 The third and fourth margin notes should be on page 633 with the discussion of antisymmetry. The note "We can see from Equation 4.8.9 that exchanging the columns of a 2×2 matrix changes the sign of D" justifies starting the induction at $n = 2$.

p. 634 In the second line of Equation A15.9, the $\tilde{A}$ should be A.

p. 636 We appear to have made a global "find and replace" to change ϕ to φ, in the process changing Φ to φ. Some errant φ's remain. The displayed equation in Theorem 4.10.12 should be

$$\int_Y f(\mathbf{y})\,|d^n\mathbf{y}| = \int_X (f \circ \Phi)(\mathbf{x})\,\big|\det[\mathbf{D}\Phi(\mathbf{x})]\big|\,|d^n\mathbf{x}|.$$

In the last sentence of last margin note, we should have used capital Φ, not φ: "So the maximum distance between two points of $\Phi(C)$ is $\frac{K\sqrt{n}}{2^N}$, i.e., $\Phi(C)$ is contained in the box C' centered at $\Phi(\mathbf{z}_C)$ with side-length $K\sqrt{n}/2^N$." Similarly, in the next to last sentence of the page: " ... to show that $\Phi(\mathcal{D}_N(X))$ is a nested partition."

p.636 The left-hand side of Equation A16.1 should be

$$\int_Y f|d^n\mathbf{y}| \quad \text{not} \quad \int_Y f|d^n\mathbf{x}|.$$

p.637 The first $=$ in Equation A16.5 should be $\leq$. This equation might be clearer with a step added:

$$\begin{aligned} \mathrm{vol}_n\,\Phi(Z) \leq \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n),\\ C\cap Z\neq\emptyset}} \mathrm{vol}_n\,C' \;&=\; \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n),\\ C\cap Z\neq\emptyset}} \left(\frac{K\sqrt{n}}{2^N}\right)^n = \underbrace{\frac{\left(\frac{K\sqrt{n}}{2^N}\right)^n}{\left(\frac{1}{2^N}\right)^n}}_{\substack{\text{ratio } \mathrm{vol}_n\,C'\\ \text{to } \mathrm{vol}_n\,C}} \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n),\\ C\cap Z\neq\emptyset}} \mathrm{vol}_n\,C \\ &= \;(K\sqrt{n})^n\,\mathrm{vol}_n\,A \;\leq\; (K\sqrt{n})^n(\mathrm{vol}_n\,Z + \epsilon). \end{aligned}$$

To go from the third to the fourth term we simultaneously multiply and divide by

$$\mathrm{vol}_n\,C = \left(\frac{1}{2^N}\right)^n.$$

p. 637 More φ's that should be Φ's: in Equation A16.5, the left-hand term of the first line should be $\mathrm{vol}_n\,\Phi(Z)$. In Corollary A16.2, the partition $\Phi(\mathcal{D}_N(X))$. In the third line of the proof, $\Phi(C_1) \subset \Phi(C_2)$.

p. 638 The last line of Lemma A16.3 might be clearer as "around any point $\mathbf{x}$ of C_a ... ," not "around any point of C_a ... "

p. 638 The margin note describing Proposition A16.4 omitted the crucial condition that $\Phi(0) = 0$.

p. 638 In Proposition A16.4, it is correct to say "Let M be a Lipschitz constant for both $[\mathbf{D}\Phi]$ and $[\mathbf{D}\Phi]^{-1}$, but it might be clearer to write $[\mathbf{D}\Phi^{-1}]$ instead of $[\mathbf{D}\Phi]^{-1}$.

p. 638 We have added a comment after Equation A16.9:

Note that although it looks as though $[\mathbf{D}\Phi(0)](\mathbf{x})$ is a matrix times a point, which doesn't make sense, $\mathbf{x}$ is really the vector $(\mathbf{x} - 0)$. Similarly, $\Phi(\mathbf{x})$ is the vector $\Phi(\mathbf{x}) - \Phi(0)$.

p. 639 Margin note: Equation A16.8, not Proposition A16.8

p. 639, right-hand equation in Equation A16.11 right, the M in the denominator should be moved outside the absolute value signs:

$$|\mathbf{x}| \le \frac{2\epsilon}{M\sqrt{n}\big|[\mathbf{D}\Phi(0)]^{-1}\big|}$$

p. 639 Line immediately after Equation A16.13: "Again this follows from A2.1" should be "Again this follows from Proposition A2.1."

p. 639 In Equation A16.15,

$$|\mathbf{x}| \le \frac{2\epsilon}{M(1-\epsilon)\sqrt{n}|[\mathbf{D}\Phi(0)]|^2} \quad \text{not} \quad |\mathbf{x}| \le \frac{2\epsilon|[\mathbf{D}\Phi(0)]|^2}{(1-\epsilon)\sqrt{n}}.$$

p. 640. In Equation A16.16, there should be an M in the denominator:

$$\delta = \frac{2\epsilon}{(1-\epsilon)^2 Mn\big|[\mathbf{D}\Phi(0)]\big|^2}.$$

p. 640, Equation A16.17, all the vol's should be vol_n's.

p. 640, in Equations A16.19, A16.20, and A16.21, there should be a $\mathrm{vol}_n\, C$ in the right-hand side. This part of the text now reads:

If N is the larger of N_1 and N_2, together these give

$$\mathrm{vol}_n\, \Phi(C) < (1+\epsilon)^n \big|\det[\mathbf{D}\Phi(0)]\big|\, \mathrm{vol}_n\, C, \qquad A16.19$$

and since $[\mathbf{D}\Phi(0)] \le M_C|\det[\mathbf{D}\Phi]| < (1+\epsilon)m_C|\det[\mathbf{D}\Phi]|$, we have

$$\mathrm{vol}_n\, \Phi(C) < (1+\epsilon)^{n+1} m_C|\det[\mathbf{D}\Phi|\, \mathrm{vol}_n\, C. \qquad A16.20$$

An exactly similar argument leads to

$$\mathrm{vol}_n\, \Phi(C) > (1-\epsilon)^{n+1} M_C|\det[\mathbf{D}\Phi]|\, \mathrm{vol}_n\, C. \quad \square \qquad A16.21$$

There are also two new margin notes:

In Equation A16.18, $M_C|\det[\mathbf{D}\Phi]|$ is not M_C times $[\mathbf{D}\Phi]|$; it is the last upper bound of $|\det[\mathbf{D}\Phi]|$. Similarly, $m_C|\det[\mathbf{D}\Phi]|$ is the greatest lower bound of $|\det[\mathbf{D}\Phi]|$.

For an appropriate N_1, Proposition A16.4 and Theorem 4.9.1 tell us that

$$\begin{aligned} \mathrm{vol}_n\, C &\le \mathrm{vol}_n\big((1+\epsilon)[\mathbf{D}\Phi(0)]C\big) \\ &= (1+\epsilon)^n|\det[\mathbf{D}\Phi(0)]C|\, \mathrm{vol}_n\, C; \end{aligned}$$

$[\mathbf{D}\Phi(0)]$ is the T of Theorem 4.9.1, which says that

$$\mathrm{vol}_n\, T(A) = |\det[T]|\, \mathrm{vol}_n\, A.$$

p. 641, Equation A16.24: in the fourth, fifth, and sixth lines, $\eta L(K\sqrt{n})^n$ should be $\frac{\eta L(K\sqrt{n})^n}{1-\eta}$. In the fifth and sixth lines, $U_{\Phi(\mathcal{D}_N(\mathbb{R}^n))}$ should be $U_{\Phi(\mathcal{D}_{N_2}(\mathbb{R}^n))}$.

p. 641 The following note has been added to the margin to explain Equation A16.24:

Equation A16.24:

$$M_C\big((f\circ\Phi)|\det[\mathbf{D}\Phi]|\big)$$

is the least upper bound of the product of two functions, $g=(f\circ\Phi),\ h=|\det[\mathbf{D}\Phi]|$. To go from line 4 to line 5, we use

$$M_C(gh)\le (M_C f)(M_C g);$$

we have

$$M_C(f\circ\Phi)=M_{\Phi(C)}(f)$$

and (using Proposition A16.5)

$$M_C|\det[\mathbf{D}\Phi]|\operatorname{vol}_n C\le \frac{1}{1-\eta}\operatorname{vol}_n \Phi(C).$$

To go from the boundary cubes in line 3 to the last term of line 4, note that $M_C(f\circ\Phi)\le L$. The sum over boundary cubes C_∂ is then less than or equal to

$$\begin{aligned}&\sum_{C_\partial} LM_C|\det[\mathbf{D}\Phi]|\operatorname{vol}_n C\le L\sum_{C_\partial}\frac{\operatorname{vol}_n\Phi(C)}{1-\eta}=L\frac{\operatorname{vol}_n\Phi(Z)}{1-\eta}\le\frac{L(K\sqrt{n})^n\operatorname{vol}_n Z}{1-\eta}\\&\le\frac{\eta L(K\sqrt{n})^n}{1-\eta};\end{aligned}$$

Proposition A16.1 says that

$$\operatorname{vol}_n\Phi(Z)\le (K\sqrt{n})^n\operatorname{vol}_n Z,$$

and we know that $\eta=\operatorname{vol}_n Z$.

p. 642, In Equations A16.25 and A16.26, the three occurrences of $\eta L(K\sqrt{n})^n$ should be $\frac{\eta L(K\sqrt{n})^n}{1-\eta}$. There have also been various minor changes in the remainder of the proof, which now reads:

A similar argument about lower sums, using N_3, leads to

$$L_{N_3}\big((f\circ\Phi)|\det[\mathbf{D}\Phi]|\big)\ge\frac{1}{1+\eta}L_{\Phi(\mathcal{D}_{N_3}(\mathbb{R}^n))}(f)-\frac{\eta L(K\sqrt{n})^n}{1-\eta}. \qquad A16.25$$

Denoting by N the larger of N_2 and N_3, we get

$$\begin{aligned}\frac{1}{1+\eta}L_{\Phi(\mathcal{D}_N(\mathbb{R}^n))}(f)-\frac{\eta L(K\sqrt{n})^n}{1-\eta}&\le L_N\big((f\circ\Phi)|\det[\mathbf{D}\Phi]|\big)\\ \le U_N\big((f\circ\Phi)|\det[\mathbf{D}\Phi]|\big)&\le\frac{1}{1-\eta}U_{\Phi(\mathcal{D}_N(\mathbb{R}^n))}(f)+\frac{\eta L(K\sqrt{n})^n}{1-\eta}.\end{aligned} \qquad A16.26$$

We can choose N larger yet so that the difference between upper and lower sums

$$U_{\Phi(\mathcal{D}_N(\mathbb{R}^n))}(f)-L_{\Phi(\mathcal{D}_N(\mathbb{R}^n))}(f), \qquad A16.27$$

is less than η, since f is integrable and $\Phi(\mathcal{D}(\mathbb{R}^n))$ is a nested partition. Denote by a the upper sum in Equation A16.27, by b the lower sum, and by c the term $L(K\sqrt{n})^n$. Then $|a-b| < \eta$, and

$$\begin{aligned}\frac{a+\eta c}{1-\eta} - \frac{b-\eta c}{1+\eta} &= \frac{(1+\eta)(a+\eta c)-(1-\eta)(b-\eta c)}{1-\eta^2} \\ &= \frac{a-b+\eta a+\eta b+2\eta c}{1-\eta^2} < \frac{\eta}{1-\eta^2}(1+a+b+2c),\end{aligned} \qquad A16.28$$

which will be arbitrarily small when η is arbitrarily small, so .... (The remaining N_2 in the last four lines of Section A16 should be N.)

There are two typos in the proof of Proposition A18.1:

p. 645 In Equation 18.4, $\underline{\mathrm{vol}} A_k$ should be $\underline{\mathrm{vol}} A'_k$.

p. 646 In Equation A18.10 it should be the limit of the integral over $\mathbb{R}^n$:

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} [f_k]_R |d^n\mathbf{x}| \geq L/2. \qquad A18.10$$

Three lines after Equation A18.10, $\frac{[f_k]_R}{R}$, not f_k/R.

In addition, we plan to expand on the proof in the third printing. Starting with the second paragraph, it will read as follows, with margin notes in parentheses:

The object is to find a set of positive volume of points $\mathbf{x} \notin B$ such that $\lim_{k\to\infty} f_k(\mathbf{x}) \geq K$, which contradicts the hypothesis. Let $A_k \subset Q$ be the set $A_k = \{\mathbf{x} \in Q \mid f_k(\mathbf{x}) \geq K\}$, so that since the sequence f_k is non-increasing, the sets A_k are nested: $A_1 \supset A_2 \supset \dots$ What we want to show is that the intersection of the A_k is bigger than B.

It is tempting to rephrase this, and say that we need to show that $\mathrm{vol}_n \cap_k (A_k) \neq 0$, and that this is true, since the A_k are nested, and $\mathrm{vol}_n A_k \geq K$ for each k. Indeed, if $\mathrm{vol}_n A_k < K$ then

$$\underbrace{\int_{\mathbb{R}^n} f_k |d^n\mathbf{x}|}_{\geq 2K} = \int_Q f_k |d^n\mathbf{x}| = \int_{A_k} f_k |d^n\mathbf{x}| + \int_{Q-A_k} f_k |d^n\mathbf{x}| < K + K, \qquad A18.1$$

which contradicts the assumption that $\int_Q f_k |d^n\mathbf{x}| \geq 2K$. Thus the intersection should have volume at least K, and since B has volume 0, there should be points in the intersection that are not in B, i.e., points $\mathbf{x} \notin B$ such that $\lim_{k\to\infty} f_k(\mathbf{x}) \geq K$.

(Equation A18.1: since $f_k \leq 1$, and A_k is a subset of the unit cube, which has volume 1, if $\mathrm{vol}_n A_k < K$, then the first term on the right-hand side of Equation A18.1 is less than K. The second term contributes at most K because $f_k \leq 1$, and $Q - A_k$ is a subset of the unit cube.)

The problem with this argument is that A_k might fail to be pavable (see Exercise A18.1), so we cannot blithely speak of its volume. In addition, even if the A_k are pavable, their intersection might not be pavable (see Exercise A18.2). In this particular case this is just an irritant, not a fatal flaw; we need to doctor the A_k's a bit. We can replace $\mathrm{vol}_n(A_k)$ by the *lower volume*, $\underline{\mathrm{vol}}_n(A_k)$, which can be thought of as the lower integral: $\underline{\mathrm{vol}}_n(A_k) = L(\chi_{A_k})$, or as the sum of the volumes of all the disjoint dyadic cubes of all sizes contained in A_k. Even this lower volume is at least K since $f_k(\mathbf{x}) = \inf(f_k(\mathbf{x}), K) + \sup(f_k(\mathbf{x}), K) - K$:

$$\begin{aligned} 2K &\le \int_Q f_k |d^n\mathbf{x}| = \int_Q \inf(f_k(\mathbf{x}), K)|d^n\mathbf{x}| + \int_Q \sup(f_k(\mathbf{x}), K)|d^n\mathbf{x}| - K \\ &\le \int_Q \sup(f_k(\mathbf{x}), K)|d^n\mathbf{x}| = L(\sup(f_k(\mathbf{x}), K)) \le K + \underline{\mathrm{vol}}_n(A_k). \end{aligned} \qquad A18.2$$

(Equation A18.2: the last inequality isn't quite obvious. It is enough to show that

$$L_N(\sup(f_k(\mathbf{x}), K)) \le K + L_N(\chi_{A_k})$$

for any N. Take any cube $C \in \mathcal{D}_N(\mathbb{R}^n)$. Then either $m_C(f_k) \le K$, in which case,

$$m_C(f_k) \,\mathrm{vol}_n\, C \le K \,\mathrm{vol}_n\, C,$$

or $m_C(f_k) > K$. In the latter case, since $f_k \le 1$,

$$m_C(f_k) \,\mathrm{vol}_n\, C \le \mathrm{vol}_n\, C.$$

The first case contributes at most $K \,\mathrm{vol}_n\, Q = K$ to the lower integral. The second contributes at most $L_N(\chi_{A_k}) = \underline{\mathrm{vol}}_n(A_k)$, since any cube for which $m_C(f_k) > K$ is entirely within A_k, and thus contributes to the lower sum. This is why the possible non-pavability of A_k is just an irritant. For typical non-pavable sets, like the rationals or the irrationals, the lower volume is 0. Here there definitely are whole dyadic cubes completely contained in A_k.)

So $\underline{\mathrm{vol}}_n(A_k) \ge K$. Now we want to find points in the intersection of the A_k that are not in B. To do this we adjust our A_k's. First, choose a number N such that the union of all the dyadic cubes in $\mathcal{D}_N(\mathbb{R}^n)$ whose closures intersect B have total volume $< K/3$. Let B' be the union of all these cubes, and let $A'_k = A_k - B'$. Note that the A'_k are still nested, and $\underline{\mathrm{vol}}_n(A'_k) \ge 2K/3$. Next choose ϵ so small that $\epsilon/(1-\epsilon) < 2K/3$, and for each k let $A''_k \subset A'_k$ be a finite union of closed dyadic cubes, such that $\underline{\mathrm{vol}}_n(A'_k - A''_k) < \epsilon^k$. Unfortunately, now the A''_k are no longer nested, so define

$$A'''_k = A''_1 \cap A''_2 \cap \cdots \cap A''_k. \qquad A18.3$$

We need to show that the A'''_k are non-empty; this is true, since

$$\overbrace{\mathrm{vol}\, A'''_k)}^{\mathrm{vol}_n(A''_1 \cap A''_2 \cap \cdots \cap A''_k)} \quad > \quad \overbrace{\underline{\mathrm{vol}} A'_k}^{\underline{\mathrm{vol}}_n(A'_1 \cap A'_2 \cap \cdots \cap A'_k)} \quad -(\epsilon + \epsilon^2 + \cdots + \epsilon^k) \ge \frac{2K}{3} - \frac{\epsilon}{1-\epsilon} > 0. \qquad A18.4$$

(We want to see that if we remove all of B from the A_k, what is left still has positive volume. We start with these:

$$\begin{aligned} &\underline{\mathrm{vol}}_n(A_k) \ge K \\ &\mathrm{vol}_n\, B' < \frac{K}{3} \\ &\underline{\mathrm{vol}}_n A'_k = \underline{\mathrm{vol}}_n(A_k - B') \ge \frac{2K}{3}, \\ &\underline{\mathrm{vol}}_n(A'_k - A''_k) < \epsilon^k. \end{aligned}$$

Since the difference $(A'_k - A''_k)$ is very small, the A''_k almost fill up the A'_k, so their volume is also non-negligible. They are not nested, but the A'''_k are:

$$\begin{aligned} A'''_1 &= A''_1, \\ A'''_2 &= A''_1 \cap A''_2, \\ A'''_3 &= A''_1 \cap A''_2 \cap A''_3, \end{aligned}$$

and so on.

Because the A_k are nested, $\underline{\text{vol}}A_k = \underline{\text{vol}}(\cap_k A_k)$; similarly, $\underline{\text{vol}}A'_k = \underline{\text{vol}}(\cap_k A'_k)$.)

Now the punchline: The A'''_k form a decreasing intersection of compact sets, so their intersection is non-empty (see Theorem A17.1). Let $\mathbf{x} \in \cap_k A'''_k$, then all $f_k(\mathbf{x}) \geq K$, but $\mathbf{x} \notin B$. This is the contradiction we were after. $\square$

p. 648, Lemma A18.4: the displayed equation should be

$$\int_Q (g_p - g_q)^2 |d^n\mathbf{x}| < \epsilon.$$

(i.e., q_p, not q_n.)

p. 648, line between Equations A18.15 and A18.16: $\int_Q (\frac{1}{2}(g_p + g_q))^2 |d^n\mathbf{x}| \geq d_N$, not $\int_Q \frac{1}{2}(g_p + g_q)|d^n\mathbf{x}| \geq d_N$.

p. 649, line -2 of the proof of A19.1 : $\subset$ not $\in$, in the following: "with $C \subset \pi^{-1}(C_1)$ such that $C \cap X \neq \phi$."

p. 650, second line of proof of Theorem 5.2.8: "it is injective because its domain excludes X_1," not "it is injective because its domain excludes X_2."

p. 652, Equation A20.1, the limit should be written

$$\lim_{h\to 0} \frac{f(\mathbf{x} + h\vec{\mathbf{v}}) - f(\mathbf{x})}{h} \quad \text{or} \quad \lim_{h\to 0} \frac{1}{h}\big(f(\mathbf{x} + h\vec{\mathbf{v}}) - f(\mathbf{x})\big).$$

p. 654, last margin note: "can be written $\|\vec{\mathbf{h}}\|_1$," not $\|\vec{\mathbf{h}}\|$.

p. 665 In problem A8.1, the statement of part (a) should read , (a) Show that Proposition 3.4.4 (chain rule for Taylor polynomials) contains the chain rule as a special case, as long as the mappings are continuously differentiable.

p. 655, three lines before Theorem A20.2 we have an ungrammatical sentence. It should be "It says that the exterior derivative with respect to wedge products satisfies an analog of Leibnitz's rule for differentiating products."

p. 655 Three lines before Theorem A20.2: "exterior derivative of a wedge product," not "exterior derivative with respect to wedge products." The title of Theorem A20.2 should be "Exterior derivative of wedge product."

p. 655, Theorem A20.2: there is a d-operator missing on the right before the ψ. The equation should be

$$d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi.$$

p. 655: the margin note has been expanded: Exercise A20.2 asks you to prove this. It is an application of Theorem 6.7.3, and part (e) of Theorem 6.7.3 is a special case of Theorem A20.2, the case where φ is a function and ψ is elementary.

p. 657, Two typos in Equation A21.6: $b_{1,3} = \det \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and the last term in the second line is $b_{3,4}$, not $b_{2,4}$. We have added a margin note near that equation:

Since we are computing the pullback $T^* dy_2 \wedge dy_3$, we take the second and third rows of T, and then select out columns 1 and 2 for $b_{1,2}$, columns 1 and 3 for $b_{1,3}$, and so on.

p. 658 We should have discussed the special case of Definition A21.4, where φ is a 0-form:

In the case where φ is a a 0-form, i.e., a function g, then $\mathbf{f}^* g = g \circ \mathbf{f}$, since

$$\mathbf{f}^* g(P_{\mathbf{x}}) = g(P_{\mathbf{f}(\mathbf{x})} = g(\mathbf{f}(\mathbf{x})) = g \circ \mathbf{f}(P_{\mathbf{x}}).$$

(This justifies the last = in Equation A21.21 on page 660.)

p. 658 In the line immediately before Equation A21.11, "Equation 6.3.7" should be "Equation 6.3.8." The same change should be made in the margin note.

p. 658, In Equation A21.11, the integral in the right-hand side should be over U, not V:

$$\int_{\mathbf{f}(U)} \varphi = \int_U \mathbf{f}^* \varphi.$$

p. 659, last line, Exercise A21.2, not 21.4.

p. 661, Here, to be consistent with the notation in Proposition A22.1 and Lemma A22.2, we are changing Theorem 6.9.2. to concern a k-dimensional manifold, and φ to be a $(k-1)$ form.

p. 661 We have added commentary to Equation A21.22:

$$\begin{aligned} \mathbf{f}^* d(\psi \wedge dx_i) &\overbrace{=}^{\substack{\text{Theorem} \\ A20.2}} \mathbf{f}^* \Big(d\psi \wedge dx_i + \overbrace{(-1)^{k-1} \psi \wedge ddx_i}^{0 \text{ since } ddx_1 = 0} \Big) \\ &\underbrace{=}_{\substack{\text{Prop.} \\ A21.7}} \mathbf{f}^*(d\psi) \wedge \mathbf{f}^* dx_i \underbrace{=}_{\substack{\text{inductive} \\ \text{hypothesis}}} d(\mathbf{f}^*\psi) \wedge \mathbf{f}^* dx_i, \end{aligned}$$

p. 661 In Equation A21.23, $(-1)^{k-1}$ is missing in two places. The equation should be:

$$\begin{aligned} d\mathbf{f}^*(\psi \wedge dx_i) &= d\big(\mathbf{f}^*\psi \wedge \mathbf{f}^* dx_i\big) \\ &= \big(d(\mathbf{f}^*\psi)\big) \wedge \mathbf{f}^* dx_i + (-1)^{k-1} \mathbf{f}^*\psi \wedge d(\mathbf{f}^* dx_i) \qquad A21.23 \\ &= \big(d(\mathbf{f}^*\psi)\big) \wedge \mathbf{f}^* dx_i + (-1)^{k-1} \mathbf{f}^*\psi \wedge dd\mathbf{f}^* x_i = \big(d(\mathbf{f}^*\psi)\big) \wedge \mathbf{f}^* dx_i. \quad \square \end{aligned}$$

p. 661, first margin note of Section A.22: Exercise A22.1, not Exercise 22.1

p. 661, proof of Proposition A22.1: "Recall from Equation A20.15 (in the proof of Theorem 6.7.3 on computing the exterior derivative of a k-form (Equation A20.15) that there exists a constant ... " should be

"Recall from Equation A20.15 (in the proof of Theorem 6.7.3 on computing the exterior derivative of a k-form) that there exists a constant ... "

p. 662, in the first line, " we can guarantee that the difference between the integral of $d\varphi$ over U_+," not " ... over U_1." Similarly, in the first term of Equation A22.8, U_1 should be U_+.

p. 662 It doesn't make any difference, but in keeping with our notation elsewhere in the book we have changed the log's in Equation A22.5 to ln's.

p.662 Equation A22.4: $\mathbb{R}^k_+$ not $\mathbb{R}^k$

p. 663 We have changed the α_i to α_m to avoid confusion with the i of G_i on p. 664.

p. 633, clarification: We have changed the last paragraph of the subsection "Partitions of unity" to read

We will choose our α_n so that in addition to the conditions of Equation A22.10, they have their supports in little subsets in which M has the standard form of Definition 6.6.1 (piece-with-boundary of a manifold): i.e., in little subsets where M is the graph of a function. It will be fairly easy to put these individual pieces into a form where Proposition A22.1 can be applied.

p. 663, first margin note: the statement that partitions of unity are only of theoretical interest is wrong. In fact, the "windows" used in signal processing are precisely a kind of partition of unity.

p. 664 The equation after Equation A22.18 is wrong; $\mathbf{f}^m$ should be replaced by a function $\tilde{\mathbf{f}}^m(\mathbf{u}) = \begin{pmatrix} \mathbf{u} \\ \mathbf{f}^m(\mathbf{u}) \end{pmatrix}$.

p. 663-664 In addition to the above corrections, the section "Choosing good parametrizations" will be rewritten for the third printing, starting after the sentence "Now go back to the Definition 6.6.1 ... ," and a picture has been added. The new version, which we hope will appear in the third printing, reads:

For every $\mathbf{x} \in X$, there exists a ball $U_\mathbf{x}$ of radius $R_\mathbf{x}$ around $\mathbf{x}$ in $\mathbb{R}^n$ such that:

- $U_\mathbf{x} \cap M$ is the graph of a mapping $\mathbf{f} : U_1 \to E_2$, where U_1 is an open subset of the subspace E_1 spanned by the k standard basis vectors that, near $\mathbf{x}$, determine the values of the other variables, and E_2 is the subspace spanned by the other $n-k$. Then if we set $\tilde{\mathbf{f}} = \begin{pmatrix} \mathbf{u} \\ \mathbf{f}(\mathbf{u}) \end{pmatrix}$, we have $\tilde{\mathbf{f}}(U_1) = M \cap U$.

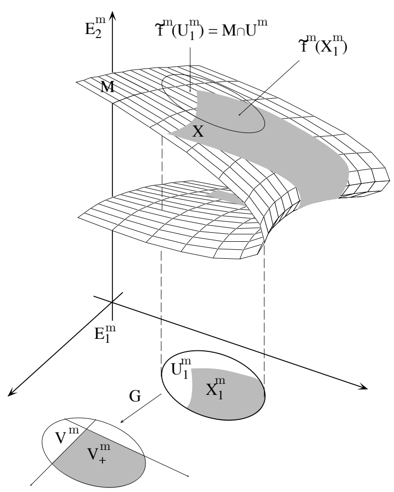
- There is a diffeomorphism $G : U_1 \to V \subset \mathbb{R}^k$ such that

$$G_i(\mathbf{u}) \geq 0,\ i = 1, \dots, k \quad \text{if and only if} \quad \tilde{\mathbf{f}}(\mathbf{u}) \in X \cap U. \qquad A22.15$$

In other words, a point in $X \cap U$ is taken by $G_i \circ \tilde{\mathbf{f}}^{-1}$ to a point in the first quadrant of $\mathbb{R}^k$, where all the coordinates are positive.

Since X is compact, the Heine-Borel theorem (Theorem A17.2l) says that we can cover it by *finitely* many $U_{\mathbf{x}_1}, \dots, U_{\mathbf{x}_N}$ satisfying the properties above.(This is where the assumption that X is compact is used, and it is absolutely essential.) We will label the corresponding sets and functions $U^m = U_{\mathbf{x}_m},\ U_1^m,\ \mathbf{f}^m,\ G^m,\ E_1^m,\ V^m$, and R_m.

To recapitulate: think of X as having a complicated shape, very likely considerably worse than the example we show in Figure A22.2. We cover this awkward shape with N little balls U^m. We choose these U^m so that for each one, $M \cap U^m$ is the graph of a single function $\mathbf{f}^m$. We project each $M \cap U^m$ onto E_1^m to get U_1^m. Thus $\tilde{\mathbf{f}}^m(U_1^m) = M \cap U^m$, and $\tilde{\mathbf{f}}^m(X_1^m) = X \cap U^m$, as shown in Figure A22.2. For each U_1^m there is a function $G^m : U_1^m \to V^m \subset \mathbb{R}^k$, which takes X_1^m to the first quadrant of V^m, denoted V_+^m in the figure. So we have gone from a big, awkward shape to a collection of N little, straight shapes.



Here we have relabeled Figure 6.6.1. Every point in X can be surrounded by a ball that is first projected onto an appropriate E_1^m and then taken by G^m to V^m; points in X are taken to V_+^m, where all coordinates are positive.

We also divide φ up into a sum of forms φ_m with very small support, which we construct with the help of the bump functions: Let $\beta_m : \mathbb{R}^n \to \mathbb{R}$ be the function

$$\beta_m(\mathbf{x}) = \beta_{R_m}(\mathbf{x} - \mathbf{x}_m), \qquad A22.16$$

so that β_m is a C^2 function on $\mathbb{R}^n$ with support in the ball U^m of radius R_m around $\mathbf{x}_m$. Set $\beta(\mathbf{x}) = \sum_{m=1}^N \beta_m$; this corresponds to a finite set of overlapping bump functions, so that we have $\beta(\mathbf{x}) > 0$ on a neighborhood of X. Then the functions

$$\alpha_m(\mathbf{x}) = \frac{\beta_m(\mathbf{x})}{\beta(\mathbf{x})} \qquad A22.17$$

are C^2 on some neighborhood of X. Clearly $\sum_{m=1}^N \alpha_m(\mathbf{x}) = 1$ for all $\mathbf{x} \in X$, so that if we set $\varphi_m = \alpha_m \varphi$, we can write

$$\varphi = \sum_{m=1}^N \alpha_m \varphi = \sum_{m=1}^N \varphi_m. \qquad A22.18$$

Now we can parametrize X using the functions

$$\mathbf{h}_m = \mathbf{f}^m \circ (G^m)^{-1} : V^m \to M,$$

and we can pull the forms $\alpha_m\varphi$ back to $\mathbb{R}^k$ using the pullbacks $\mathbf{h}_m^*(\alpha_m\varphi)$. We have satisfied the conditions of Proposition A22.1: each V_+^m satisfies the conditions for U_+ in that proposition, and using all N of the V_+^m we can account for all of X.

p. 665 in Equation A22.19, all the pullbacks h^* should be $\mathbf{h}_m^*$. In the margin note, $\mathbf{h}^*$ should be $\mathbf{h}_m^*$.

p. 665 In Exercise A8.1, part (a) should read

Show that Proposition 3.4.4 (chain rule for Taylor polynomials) contains the chain rule as a special case, as long as the mappings are continuously differentiable.

p. 666, Exercise A9.3 should read

Prove Equation A9.2 by induction, by first checking that when $k = 0$, it is the fundamental theorem of calculus, and using integration by parts to prove

$$\frac{1}{k!}\int_0^h (h-t)^k g^{(k+1)}(a+t)\,dt = \frac{1}{(k+1)!}g^{(k+1)}(a)h^{k+1} + \\ \frac{1}{(k+1)!}\int_0^h (h-t)^{k+1} g^{(k+2)}(a+t)\,dt.$$

p. 666, Exercise A12.1: in (a) $<$ not $>$; in part (b) specify for $n \geq 2$: “Show that for $n \geq 2$, we have $c_n = \frac{n-1}{n}c_{n-2}$.”

p. 666, Exercise A18.1 first displayed equation

$$\{x \in \mathbb{R} | f(x) \geq 0\} \quad (\text{not} \in R).$$

p. 667 In Exercise A18.1, part (b): the set U_ϵ not $U - \epsilon$.

p. 667, Exercise A18.3: “Show that if f and g are any integrable functions on $\mathbb{R}^n$, then,” not “ ... on $\mathbb{R}^n$, the.” In the displayed equation following the hint, a 2 is needed in front of the second term on the right:

$$2t\int_{\mathbb{R}^n} f(\mathbf{x})g(\mathbf{x})|d^n\mathbf{x}|$$

p. 667, Exercise A21.1 (a): We should have specified that $T : V \to W$ is a linear transformation.