Corrections to the Second Printing: Chapter 6

p. 502, Equation 6.2.5: the equation on the third line has a vertical bar in the right-hand side that doesn't belong there. It should be

$$|d^k\mathbf{x}|(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k)=\sqrt{\det\Big([\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k]^\top[\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k]\Big)}$$

p.503, midpage; $\det(-3)=-3$, not 3. In the line immediately after "Elementary Forms" $dx_{i_1},\wedge\cdots\wedge dx_{i_k}$ should be $dx_{i_1}\wedge\cdots\wedge dx_{i_k}$, without the comma.

p. 505 Replace the sentence after Equation 6.2.16 by: "Each term in this last sum will evaluate φ on k vectors, two of which are equal, and so (by antisymmetry) will return 0." (This change is necessary because since the k vectors are in $\mathbb{R}^n$ with $k>n$ it does not make sense to speak of the determinant.)

p. 506 In Definition 6.2.4, it would be best to reverse left-hand and right-hand sides:

$$(\varphi+\psi)(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k)=\varphi(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k)+\psi(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k);$$

we are defining $(\varphi+\psi)$ in terms of something known, the sum of the two numbers $\varphi(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k)$ and $\psi(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k)$.

p.509 Second margin note: it is the next to last equality, not the last equality.

p. 513 We have added a note: From now on we will use the terms form and form field interchangeably; when we wish to speak of a form that returns the same number regardless of the point at which it is evaluated, we will call it a constant form.

p.516 In Equations 6.3.11 and 6.3.14, the absolute value signs around dt and $ds\,dt$ should be dropped when we are integrating over an oriented domain like $\int_0^a$ and $\int_0^1\int_0^1$.

p.517 Two mistakes in the second margin note: in the first line, dx_1 should be dx; in the next-to-last line, $x\cdot\det v_3$ should be $z\cdot\det v_3$.

p. 519, caption to Figure 6.4.1: in the fifth line, there is an extra vertical line; since $\vec{\mathbf{x}}\cdot\vec{\mathbf{y}}=|\vec{\mathbf{x}}||\,|\vec{\mathbf{y}}|\cos\theta$ should be since $\vec{\mathbf{x}}\cdot\vec{\mathbf{y}}=|\vec{\mathbf{x}}||\vec{\mathbf{y}}|\cos\theta$.

p.519 In Definition 6.4.3, U does not have to be open.

p.520 The left-hand side of Equation 6.4.10 should be $\int_{\gamma(A)}W_{\vec{F}}$.

In Equation 6.4.11, $\vec{F}\begin{pmatrix}x\\y\\z\end{pmatrix}$, not $\vec{F}\begin{pmatrix}z\\y\\z\end{pmatrix}$.

p. 521, Example 6.4.6: we fail to state that $S=\gamma(U)$, and in any case $\gamma(U)$ is not necessarily a surface. The title of this example should be "Integrating a flux form field over a 2-dimensional parametrized domain in $\mathbb{R}^3$)," and "S" and "the surface S" should both be replaced by "$\gamma(U)$."

p.521 in Example 6.4.7, $\vec{F}\begin{pmatrix} x \\ y \\ x \end{pmatrix}$ should be $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. In the last line of Equation 6.4.15, the integral is $\frac{7}{36}$, not $-\frac{1}{4}$.

p. 522, Example 6.4.9: in Equation 6.4.20 there are three errors in the second line. There should be a minus sign, an integral is missing, and $\cos^v$ should be $\cos^2 v$:

$$= -2\pi \int_0^{2\pi} \int_0^r (R^3 u + 3R^2 u^2 \cos v + 3Ru^3 \cos^2 v + u^4 \cos^3 v)\, du\, dv.$$

There are also errors in the paragraph following the equation. The second half of the paragraph should read:

"... and therefore that the change of variables formula might well say that the integrals are equal. This is not true: the absolute value that appears in the change of variables formula isn't present here, which changes the sign of the integral. Really figuring out whether...."

p.523 in Definitions 6.4.10 and 6.4.11, it is not actually necessary that U be open.

p. 528 Perhaps Definition 6.5.6 and the following discussion would be clearer as follows:

Definition 6.5.6 (Orientation of a surface in $\mathbb{R}^3$) To orient a surface in $\mathbb{R}^3$, choose a unit vector field $\vec{N}$ orthogonal to the surface, and such that the vector field $\vec{N}$ depends continuously on the point.

At each point $\mathbf{x}$ there are two vectors $\vec{N}(\mathbf{x})$ orthogonal to the surface; Definition 6.5.6 says to choose one at each point, so that $\vec{N}$ varies continuously. This is possible for an orientable surface ...

p. 529 The first margin note really should be on the preceding page, near the discussion following Definition 6.5.7. We should add that this assumes that the standard basis vectors satisfy the right-hand rule.

p. 529 In the next-to-last paragraph the sentence beginning "Then the orientations coincide if for ... " should specify that the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \in E$ are linearly independent.

p.532 Definition 6.5.15: we should have said explicitly that M is oriented by ω.

p. 532 In the last margin note, the last sentence should be replaced by

Since $U \subset \mathbb{R}^k$ is open, it is necessarily k-dimensional. (We haven't actually defined dimension of a set; what we mean is that $U \subset \mathbb{R}^k$ is a k-dimensional manifold.) Think of a disk and a ball in $\mathbb{R}^3$; the ball can be open but the disk cannot. The "fence" of a disk in $\mathbb{R}^3$ is the entire disk, and it is impossible (Definition 1.5.3) to surround each point of the disk with a ball contained entirely in the disk.

p.534 Equation 6.5.23 might be clearer if we omit the fifth line and add a note. In addition, in the first line, there is a close parenthesis missing (immediately before the equal sign). In the fourth line, the close square bracket is missing. The equation should read:

$$\begin{aligned}
\alpha &= \omega(\mathbf{x})\big(\overrightarrow{D_1}\gamma_1(\mathbf{u}_1), \dots, \overrightarrow{D_k}\gamma_1(\mathbf{u}_1)\big) = \omega(\mathbf{x})([\mathbf{D}\gamma_1(\mathbf{u}_1)]\vec{\mathbf{e}}_1, \dots, [\mathbf{D}\gamma_1(\mathbf{u}_1)]\vec{\mathbf{e}}_k) \\
&= \omega(\mathbf{x})([\mathbf{D}\gamma_2(\mathbf{u}_2)]\left([\mathbf{D}\gamma_2(\mathbf{u}_2)]\right)^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]\vec{\mathbf{e}}_1, \dots, \\
&\qquad [\mathbf{D}\gamma_2(\mathbf{u}_2)]\left([\mathbf{D}\gamma_2(\mathbf{u}_2)]\right)^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]\vec{\mathbf{e}}_k) \\
&= \beta \det\Big[\left([\mathbf{D}\gamma_2(\mathbf{u}_2)]\right)^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]\vec{\mathbf{e}}_1, \dots, \left([\mathbf{D}\gamma_2(\mathbf{u}_2)]\right)^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]\vec{\mathbf{e}}_k)\Big] \\
&= \beta \det\Big(\left([\mathbf{D}\gamma_2(\mathbf{u}_2)]\right)^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]\Big). \qquad 6.5.23
\end{aligned}$$

So $\det\Big(\left([\mathbf{D}\gamma_2(\mathbf{u}_2)]\right)^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]\Big) = \det[\mathbf{D}\Phi] = \alpha/\beta > 0$, which means (Definition 6.5.2) that γ_1 and γ_2 define compatible orientations.

(To go from fourth to the last line of Equation 6.5.23, remember that $T\vec{\mathbf{e}}_i$ is the ith column of T so $[\mathbf{D}\gamma_2(\mathbf{u}_2)]^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]\vec{\mathbf{e}}_i$ is the ith column of $[\mathbf{D}\gamma_2(\mathbf{u}_2)]^{-1}[\mathbf{D}\gamma_1(\mathbf{u}_1)]$; this matrix is the "change of parameters" map $\Phi = \gamma_2^{-1} \circ \gamma_1$.)

p. 535: first line after Equation 6.5.26, Example 6.4.6, not Definition

p.535 In Equation 6.5.27, the limits of integration for x are wrong; instead of from 0 to y it should be 0 to $1-y$. As it turns out, this does not change the final result:

$$\begin{aligned}
\int_P \Phi_{\begin{bmatrix} y \\ -x \\ z \end{bmatrix}} &= \int_T \det\Bigg[\underbrace{\begin{bmatrix} y \\ -x \\ 1-x-y \end{bmatrix}}_{\vec{F}(\gamma(\vec{\mathbf{u}}))}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{\overrightarrow{D_1}\gamma}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\overrightarrow{D_2}\gamma}\Bigg] |dx\, dy| \\
&= \int_T (1-2x)\,|dx\, dy| = \int_0^1 \left(\int_0^{1-y} (1-2x)\, dx\right) dy \qquad 6.5.27 \\
&= \int_0^1 \left[x - x^2\right]_0^{1-y} dy = \int_0^1 (y - y^2)\, dy = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 = \frac{1}{6}. \quad \triangle
\end{aligned}$$

p.536 In Equation 6.5.30, the integral is 6π, not 7π.

p.537 Three lines above Definition 6.6.1: "if you drew the (x, y)-axes on a piece of rubber," not "if you drew the (x, y)-plane on a piece of rubber."

p.537 Next-to-last line of Definition 6.6.1: $\mathbf{f}(X_1)$, not $f(X_1)$

p.540 Replace the last sentence before Definition 6.6.8 by the sentence

At $\mathbf{x}$, the tangent space $T_{\mathbf{x}}(\partial X) = \ker[\mathbf{D}g(\mathbf{x})] \cap T_{\mathbf{x}}M$ has dimension one less than the dimension of $T_{\mathbf{x}}M$; it subdivides the tangent space $T_{\mathbf{x}}M$ into the outward-pointing vectors and the inward-pointing vectors.

(Change $\ker[\mathbf{D}G(\mathbf{x})]$ to $\ker[\mathbf{D}g(\mathbf{x})]$ and "one less than the dimension of $T_{\mathbf{x}}X$" to "one less than the dimension of $T_{\mathbf{x}}M$." Replace "it subdivides the tangent space into" by "it subdivides the tangent space $T_{\mathbf{x}}M$ into.")

p. 545, 6th line: Example 6.5.22, not Definition 6.5.22

p.546 Theorem 6.7.3, part (a) U must be open. Part (c): recall that a constant form returns the same number regardless of the point at which it is evaluated. Elementary forms are a special case of constant forms. In part (e), note that the form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is an elementary form. The general case, taking the exterior derivative of the wedge product of a k-form and an l-form, is given by Theorem A20.2:

$$d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^k \varphi \wedge d\psi,$$

since in (e), ψ is a constant form, with exterior derivative 0 by part (c).

p. 547 line before Equation 6.7.10: "if f is a function," not "if f and g are functions".

p. 548 First line: the 2-form should be $x_1 x_3^2 \, dx_1 \wedge dx_2$, not $x_1 x_3^2 \, dx_1 \wedge d_2$

p. 548 The caption to Figure 6.7.2 now reads

The vector field $\vec{F}_2$. Note that because the vectors are scaled by $\frac{1}{|\mathbf{x}|}$, they get smaller the further they are from the origin. If $\vec{F}_2$ represents the flow of an incompressible fluid, then just as much flows through the unit circle as flows through a larger circle.

p. 549 The top of the page now reads

The scaling $\frac{\mathbf{x}}{|\mathbf{x}|^n}$, which means that the vectors get smaller the further they are from the origin (as shown for $n = 2$ in Figure 6.7.2) is chosen so that the flux outwards through a series of ever-larger spheres all centered at the origin remains constant. In fact, the flux is equal to $\text{vol}_{(n-1)} \, S^{n-1}$.

First line of Equation 6.7.18: the left-hand side should be $d\omega_n$, not ω_n. In Theorem 6.7.7, U is open.

p. 550 Definition 6.8.1, second line: $U \subset \mathbb{R}^3$ not $U \subset \mathbb{R}^n$.

p. 555: the bottom margin note ("The flow is out if the box has the standard orientation. If not, the flow is in.") belongs on the next page, with the discussion of divergence. In the caption of Figure 6.8.2, "it rotates" should be "the paddle rotates."

p.556 Remark: more precisely, div grad f is the Laplacian, and grad div $\vec{F}$ minus curl curl $\vec{F}$ gives the Laplacian acting on each coordinate of a vector field.

Last margin note: Brouwer, not Brower.

p. 558: in the first paragraph of the first margin note, $D_2 f = -2x^2 dx_2$ should be $D_2 f = -2x_2 dx_2$. Further on in the same note, it would be more in keeping with our notation to write $-2x_2 \, dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n$ rather than $-2x_2 \wedge dx_2 \wedge dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n$, and $2x_2 \, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. rather than $2x_2 \wedge dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.; i.e., omitting the wedge between the coefficient and the form. However, it is not incorrect to use the wedge; in both cases, the coefficient is a function, which is a 0-form field.

p. 560 On the far right of Equation 6.9.20, there should be parentheses to avoid ambiguity: $\sum_{i=0}^{m-1} \big(f(x_{i+1}) - f(x_i)\big)$. Similarly, on the far left of Equation 6.9.21, $\sum_{i=0}^{m-1} \big(f(x_{i+1}) - f(x_i)\big)$.

p.561, immediately after second subheading. "We will now describe a situation where the proof in Section 6.9 really does work" should be "We now describe a situation where this informal proof really does work."

p. 561, Equation 6.9.23, the $P^o_{x_i}$ should be $P^o_{\mathbf{x}_i}$.

p. 561, second line of subsection "A situation where the easy proof works": form on $\mathbb{R}^k$, not in $\mathbb{R}^k$.

p. 561, last line: where $x_1 \le 0$, not $x_1 \ge 0$.

p. 561 We plan to change the caption to Figure 6.9.4 for the third printing; it will read

The integral of $d\varphi$ over U_- equals the integral of φ over the boundary of U_-. The boundary of U_- consists of its eastern border (the intersection of the y-axis and U_-) and its curved border $(\partial U \cap U_-)$, but only the eastern border contributes to the integral of φ over ∂U_-, since $\varphi = 0$ on the curved border. Here the support of φ is shown smaller than U, but the support could be all of U; since U is open, and φ vanishes outside U, we have $\varphi = 0$ on the boundary of U.

p. 561 In the paragraph beginning "As in the case of Riemann sums," approximately $\epsilon^{-(k+1)}$ such parallelograms" should be "on the order of $\epsilon^{-(k+1)}$ such parallelograms (precisely that many if X has volume 1)."

p. 562 in Equations 6.9.31 and 6.9.32, U_1 should be U_-.

p. 563 In the line before Equation 6.9.35, U_1 should be U_-.

p. 564 in Equation 6.10.5, the integral is over ∂U, not ∂D.

p. 565, margin note: the second line of the three-line displayed equation is missing an open parenthesis. Parentheses could also be added to the first line. Next-to-last line of the long margin note: "the flux of curl $\vec{F}$," not "the flux of a vector field $\vec{F}$." The margin note has been modified; it now reads

The $\vec{N}|d^2\mathbf{x}|$ in the left-hand side of Equation 6.10.9 takes the parallelogram $P_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}})$ and returns the vector

$$\vec{N}(\mathbf{x})|\vec{\mathbf{v}} \times \vec{\mathbf{w}}|,$$

since the integrand $|d^2\mathbf{x}|$ is the element of area; given a parallelogram, it returns its area, i.e., the length of the cross-product of its sides. When integrating over S, the only parallelograms $P_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}})$ we will evaluate the integrand on are tangent to S at $\mathbf{x}$, and with compatible orientation. Since $\vec{\mathbf{v}}, \vec{\mathbf{w}}$ are tangent to S, $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$ is perpendicular to S. So $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$ is a multiple of $\vec{N}(\mathbf{x})$, which is also perpendicular to the surface. (It is a positive multiple because $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$ and $\vec{N}(\mathbf{x})$ have the same orientation.) In fact

$$\vec{\mathbf{v}} \times \vec{\mathbf{w}} = |\vec{\mathbf{v}} \times \vec{\mathbf{w}}|\vec{N}(\mathbf{x}),$$

since $\vec{N}(\mathbf{x})$ has length 1. So

$$\begin{aligned}&\big(\mathrm{curl}\vec{F}(\mathbf{x})\big) \cdot \big(\vec{N}(\mathbf{x})\big)\ |d^2\mathbf{x}|(\vec{\mathbf{v}}, \vec{\mathbf{w}})\\ &= \big(\mathrm{curl}\,\vec{F}(\mathbf{x})\big) \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})\\ &= \det[\mathrm{curl}\,\vec{F}(\mathbf{x}), \vec{\mathbf{v}}, \vec{\mathbf{w}}],\end{aligned}$$

i.e., the flux of curl $\vec{F}$ acting on $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$.

p. 565 The beginning of the third paragraph will be changed in the third printing, to read

In Equation 6.10.9, $\vec{T}(\mathbf{x})\,|d^1\mathbf{x}|$ is a complicated way of expressing the identity. Since the element of arc length $|d^1\mathbf{x}|$ takes a vector and returns its length, $\vec{T}(\mathbf{x})\,|d^1\mathbf{x}|$ takes a vector and returns $\vec{T}(\mathbf{x})$ times its length.

p. 566 Equation 6.10.13 should be

$$t \mapsto \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix};$$

the first line of Equation 6.10.14 should be

$$\int_{C_1} W_{\vec{F}} = \int_0^{2\pi} \begin{bmatrix} (\sin t)^3 \\ \cos t \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cos t \\ \sin t \\ 0 \end{bmatrix} dt.$$

(Note that in addition to the vectors having three entries, the integral is from 0 to 2π.) In the second line of the equation, the integral should also be from 0 to 2π. The end result is correct.

p. 567 Equation 6.10.21: p is pressure

p. 570 A margin note has been added near Equation 6.11.6:

Actually, a function can have gradient 0 without being globally constant; for example, the function defined on the separate intervals $(0, 1$ and $(2, 3)$ and equal to 1 on the first and 2 on the other, has grad $= 0$.

p. 572 A margin note has been added near Equation 6.11.16:

A function f that is a potential can be thought of as an altitude function. Equation 6.11.16 says that we can determine the altitude of $\mathbf{x}$ (i.e., $f(\mathbf{x})$) by measuring how hard we have to work to climb to $\mathbf{x}$ (the work of $\vec{F}$ over $\gamma(\mathbf{x})$.

p. 572 At the bottom of the page, "Theorem 6.11.5 asserts that $\vec{F} = \vec{\nabla} f$,where ... " should be

Theorem 6.11.5 asserts that $\vec{F} = \vec{\nabla} f$, and Equation 6.11.16 suggests how to find f; set ... .

p. 573, in the second line of Equation 6.11.23, $3ab$ should be $3abc$.

p. 573, Exercise 6.1.2, "the integrand takes a rectangle $a < x < b, /c < y < d$ ... " should be "the integrand takes a rectangle $a < x < b,\ c < y < d$ ... ." The exercise should include the comment, "again, the limit should not depend on how the domain is decomposed into rectangles."

p. 584 Exercise 6.10.1 should have said that $\vec{F}$ is C^1.

p. 587, Exercise 6.11.2, part (b), "and the other through $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$."