THE HENON MAPPING IN THE COMPLEX DOMAIN

John H. Hubbard

Department of Mathematics
Cornell University
Ithaca, New York

This paper reports on results of A. Douady, J. Hubbard and R. Oberste-Vorth.

1. INTRODUCTION

In this paper we will try to describe the behavior of the mapping $F: \mathbb{C}^2 \to \mathbb{C}^2$;

$$F: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x^2 + c - ay \\ x \end{bmatrix}, \quad a \neq 0,$$

under iteration; these maps will be called the <u>Henon family</u>.

The mapping F has constant Jacobian a, and is invertible; in fact the inverse is given by

$$F^{-1}: \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} y \\ \frac{1}{a}(y^2 + c - x) \end{bmatrix}$$

and is also polynomial of degree 2.

The mapping F is not quite as arbitrary as it might appear. Any polynomial map $F: \mathbb{C}^2 \to \mathbb{C}^2$ of degree 2 can be written $F = F_0 + F_1 + F_2$ with F_i homogeneous of degree i. In order for F to have constant Jacobian, it is necessary that F_2 have a one-dimensional kernel and a 1-dimensional

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image. Any automorphism $F: \mathbb{C}^2 \to \mathbb{C}^2$ for which the kernel and the image of F_2 are linearly independent can be conjugated to an element of the Henon family.

2. HISTORY AND MOTIVATION

a) Henon first considered the Henon family while trying to understand the Lorentz equations. He constructed a Poincaré section of the Lorentz equations, and then tried to find a simple mapping whose qualitative properties would be similar.

After some experimentation, he came up with an element of the Henon family which appeared to have a strange attractor similar to a section of the attractor found by Lorentz.

After much work, the theory is still fragmentary.

b) In 1925 (work completed by Bieberback in 1932) Fatou gave examples of injective analytic mappings

$$g: \mathbb{C}^2 \to \mathbb{C}^2,$$

whose images omit an open subset of $\mathbb{C}^2$.

The existence of such domains $U = g(\mathbb{C}^2)$ shows that many key results of complex analysis in one dimension fail in higher dimensions, for instance:

<u>Montel</u>: Any family $F_n: U \to \bar{\mathbb{C}}$ of meromorphic functions which omit three values is equicontinuous.

The following is one way (as far as I know the only way) of constructing such mappings g.

1. Find a mapping F of the Henon family with an attractive fixed point $\vec{x}_0$.
2. If the eigenvalues λ_i of $d_{\vec{x}_0}F$ satisfy $|\lambda_1| < |\lambda_2|$ and

$\lambda_1 \neq \lambda_2^n$ for all n = 2,3,..., then there exists

$g: (\mathbb{C}^2, 0) \to (\mathbb{C}^2, \vec{x}_0)$ such that

$g(\vec{u}) = \vec{x}_0 + \vec{u} + o(|\vec{u}|)$, and

$F \circ g = g \circ d_{\vec{x}_0} F.$

This is a result of S. Sternberg (~1959); if $|\lambda_1| > |\lambda_2|^2$, you can show:

Proposition (Oberste-Vorth). The limit

$$g(\vec{u}) = \lim_{n \to \infty} F^{-n}((d_{x_0} F)^{\circ n}(\vec{u}) + \vec{x}_0)$$

exists for all $\vec{u} \in \mathbb{C}^2$, and defines a Fatou-Bieberbach mapping.

If $|\lambda_1|$ and $|\lambda_2|$ are farther apart, it is harder to construct g.

Such mappings have largely been viewed as pathological. Calabi asked me what the image looked like. The results of this paper grew out of his question.

3. THE RELATION WITH THE THEORY OF POLYNOMIALS

In the study of iteration of polynomials of one variable, extending to complex values of the variable has been very useful, even if the original polynomials was real. We hope the same thing will happen in this case, essentially for the same reason.

There is essentially nothing you can say about real polynomials which is independent of the coefficients, largely because virtually all features independent of conjugation, such as periodic cycles, are liable to disappear as the parameters are varied. In the complex domain, the behavior is far more uniform.

Let $P(z) = z^d + a_{d-1}z^{d-1} + \ldots + a_0$ be a polynomial. The most useful construction is the function $\phi_P(z)\colon (\mathbb{C},\infty) \to (\mathbb{C},\infty)$ such that

$$\phi_P(P(z)) = (\phi_P(z))^d \text{ and}$$

$$\phi_P(z) = z + o(1) \text{ near } \infty .$$

The function $\phi_P(z)$ is constructed as follows

$$\phi_P(z) = \lim_{n\to\infty} (P^{\circ n}(z))^{1/d^n}$$

(This is a standard "scattering theory construction": go to ∞ by P, and return by the unperturbed map $P_0\colon z \to z^d$.) In order to give a meaning to the $(1/d^n)$-power, write the limit above as an infinite product

$$\phi_P(z) = z \cdot \frac{(P(z))^{1/d}}{z} \cdot \frac{P(^{\circ 2}(z))^{1/d^2}}{(P(z))^{1/d}} \cdot \ldots ,$$

and note that

$$\frac{(P^{\circ(k+1)}(z))^{1/d^{k+1}}}{P^{\circ k}(z))^{1/d^k}} = \left(1 + \frac{a_{d-1}(P^{\circ k}(z))^{d-1} + \ldots + a_0}{(P^{\circ k}(z))^d}\right)^{1/d^{k+1}} .$$

Since the denominator is larger than the numerator for large z, we can define the root to be the principal branch. It is easy to show that the infinite product converges.

If we don't want to worry about branches of roots, we can define

$$h_P(z) = \lim_{n\to\infty} \frac{1}{d^n} \log_+ |P^{\circ n}(z)|,$$

where $\log_+(x) = \sup(\log(x),0)$.

It is quite easy to show that this limit exists and is continuous on all of $\mathbb{C}$, harmonic on $\mathbb{C} - K_p$, where

$$K_p = \{z | P^{\circ n}(z) \not\to \infty\}.$$

In fact, h_p is the <u>Green's function</u> of K_p.

We will define functions ϕ and h analogous to these on $\mathbb{C}^2$.

4. RATES OF ESCAPE FOR THE HENON FAMILY

Look at the formula for the Henon mappings. If x is reasonably large, and large with respect to y, then the predominant behavior is that <u>the</u> x-<u>coordinate gets squared</u>. That motivates the following proposition.

<u>Proposition 1</u>. The limit

$$h_+\begin{bmatrix}x\\y\end{bmatrix} = \lim_{n\to\infty} \frac{1}{2^n} \log_+ \left| F^{\circ n}\begin{bmatrix}x\\y\end{bmatrix}_1 \right|,$$

where $\begin{bmatrix}x\\y\end{bmatrix}_1 = x$, exists and defines a continuous function on $\mathbb{C}^2$, harmonic on

$$U_+ = \{\begin{bmatrix}x\\y\end{bmatrix} | h_+\begin{bmatrix}x\\y\end{bmatrix} > 0\}.$$

We have

$$h_+(F\begin{bmatrix}x\\y\end{bmatrix}) = 2h_+\begin{bmatrix}x\\y\end{bmatrix} .$$

The behavior of h_+ is partially described below.

<u>Proposition 2</u>. The mapping

$$h_+ : U_+ \to \mathbb{R}_+$$

is a trivial fibration whose fibers are homeomorphic to the complement of a solenoid in S^3.

The following sketch of a proof will explain what is meant by a solenoid.

<u>Idea of Proof</u>. We will first choose a region $V_+ \subset \mathbb{C}^2$ for which the points surely escape. One such choice is

$$V_+ = \{[\begin{smallmatrix}x\\y\end{smallmatrix}] \,\big|\, |y| < \alpha|x|^2,\ |x| > \beta\},$$

with $\alpha = \frac{1}{6|a|}$, and $\beta = \sup(\sqrt{2|c|}, 3\sqrt[3]{2}|a|, 3)$. Then $V_+ \subset U_+$, and $U_+ = \bigcup_{n\geq 0} f^{-n}(V_+)$. It is fairly clear that

$$h_+([\begin{smallmatrix}x\\y\end{smallmatrix}]) = \log|x| + o(\log(|x|^2 + |y|^2)),$$

uniformly in V_+.

Let $U_+(s) = \{x \in U_+ | h_+(x) = s\}$, and $V_+(s) = U_+(s) \cap V_+$. Then $V_+(s)$ is for large s very nearly the set

$$\{[\begin{smallmatrix}x\\y\end{smallmatrix}] \,\big|\, |x| = e^s,\ |y| < \alpha|x|^2\},$$

and as such is a solid torus. The key point is: <u>How does $F(V_+(x)) \subset V_+(2s)$ look?</u>

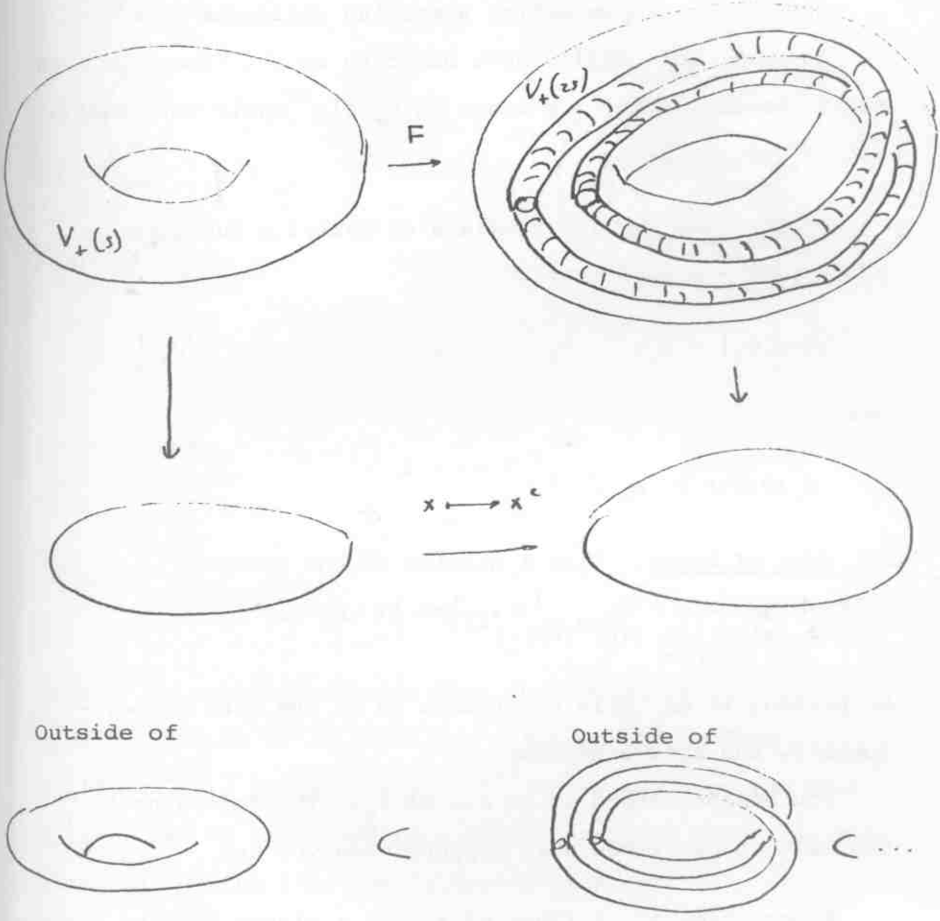
<u>Answer</u>: The map of f wraps $V_+(s)$ into $V_+(2s)$ by winding it around twice, as in the first picture on the following page.

Now we can think of

$$U_+ = V_+(s) \cup F^{-1}(V_+(2s) \cup F^{-2}(V_+(4s)) \cup \ldots,$$

where each of the terms in the increasing union is a solid torus winding twice around inside the next.

This is a little hard to imagine, but can be done if you recall that the outside, in S^3, of an unknotted solid torus is also a solid torus. So consider the second diagram:



This realizes an increasing union as above, and the union is the complement of a solenoid. Q.E.D.

5. ANGLES OF ESCAPE

In the case of polynomials, there existed an analytic function ϕ_p defined near ∞ such that

$$\log|\phi_p| = h_p.$$

Question: Can we define something analogous in $\mathbb{C}^2$?

Answer: No! Still, what ought to be the fibers of ϕ d exist; however, they are dense in $U_+(s)$. Their intersectio with $V_+(s)$ are dense.

Proposition 3. There exists an analytic function $\phi_+: V_+ \to \mathbb{C} - \bar{D}$ such that

$$\log|\phi_+| = h_+,$$

and

$$\phi_+(F(\vec{x})) = (\phi_+(\vec{x}))^2.$$

Idea of Proof. Give a meaning to the roots in

$$\phi_+(\vec{x}) = \lim_{n\to\infty} ((F^{\circ n}(\vec{x}))_1)^{1/2^n}$$

by passing to an infinite product as in the case of polynomials, and show convergence.

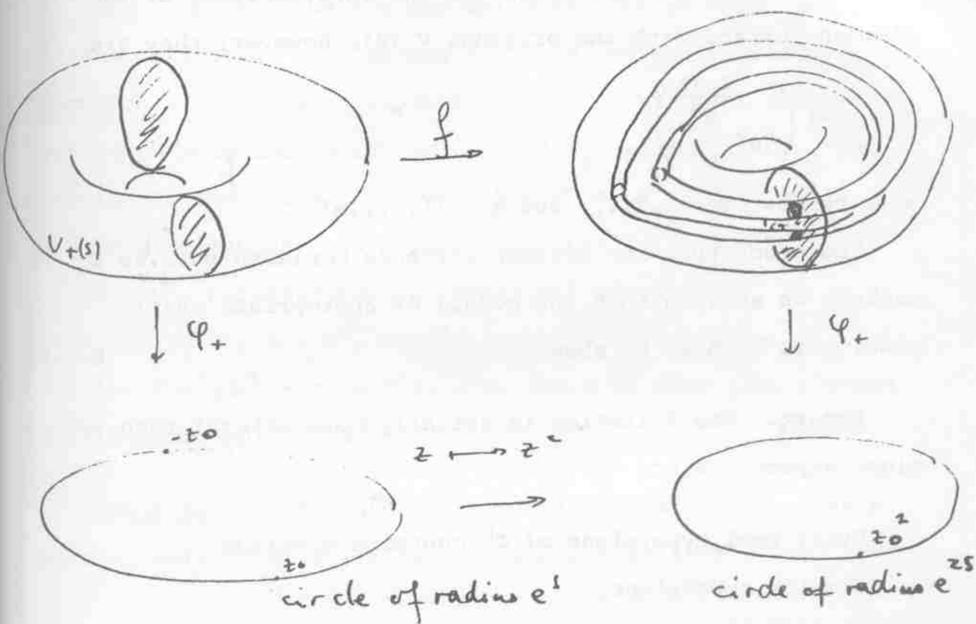
You cannot extend ϕ_+ to all of U_+. The following proposition describes what happens when you try.

Proposition 4. A fiber of ϕ_+, as a closed Riemann su in $V_+(s)$, can be continued to a Riemann surface isomorphi $\mathbb{C}$ and dense in $U_+(s)$. These Riemann surfaces foliate $U_+($ and the mapping

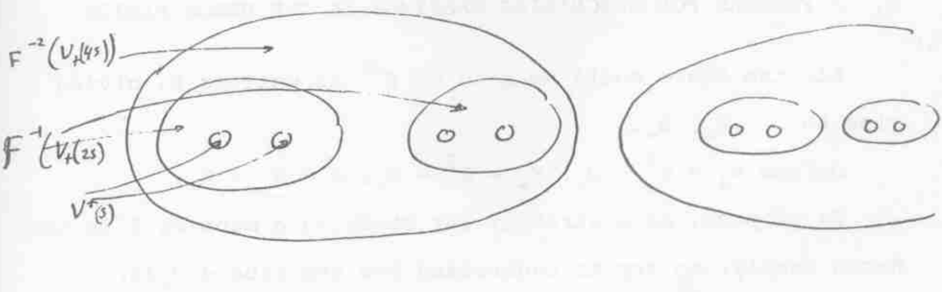
$$\vec{x} \to \phi_+(\vec{x})/|\phi_+(\vec{x})|$$

induces a bijection of the set of leaves onto the (non-Hausdorff) group $\mathbb{R}/\mathbb{Z}[\frac{1}{2}]$.

Idea of the Proof. Consider again the picture



Since F is analytic, we see that the two discs $\phi_+^{-1}(z_0) \cup \phi_+^{-1}(-z_0)$ are in $F^{-1}(V_+(2s))$ two subdiscs of the disc $F^{-1}(\phi_+^{-1}(z_0^2))$. Continuing into $F^{-2}(V_+(4s)),\ldots$ we see that $\phi_+^{-1}(z_0)$ can be continued to a Riemann surface with a stratification looking like the following drawing.



The shaded internal discs are the intersections of the Riemann surface with the original $V_+(s)$; however, they are

$$\phi_+^{-1}\left(z_0 e^{\frac{2\pi i k}{2^m}}\right)$$

as m ranges over 1,2,... and $k \in \{1,\ldots,m\}$.

The proof that the Riemann surfaces are isomorphic to $\mathbb{C}$ depends on showing that the moduli of appropriate annuli grow; this is easy to show. Q.E.D

Remark. The foliation is actually more natural than you might expect. Since

> Every real hyperplane of $\mathbb{C}^n$ contains a unique complex hyperplane,

the tangent space to $U_+(s)$ at a point contains a unique complex direction. The leaves are simply the integral curves of this field of directions. Of course, this could be said of the level surfaces of any real valued function, but in general such fields of directions have no integral curves.

6. A PROGRAM FOR DESCRIBING MAPPINGS IN THE HENON FAMILY

All the above could be said of F^{-1} as well as F, giving rise to U_-, V_-, h_-.

Define $K_+ = \mathbb{C}^2 - U_+$, $K_- = \mathbb{C}^2 - U_-$, $K = K_+ \cap K_-$.

We propose, as a strategy for studying a mapping F in the Henon family, to try to understand how the fibers $U_+(s)$ collapse onto ∂K_+ as $s \to 0$.

As an example of such a description, we propose the following conjecture.

Let $\sigma: S^3 \to S^3$ be the map which is more or less implicit in the definition of a solenoid. More precisely define a sequence $T_i \subset S^3$, $i = \ldots,-1,0,1,\ldots$ of tori, each decomposing S^3 into two solid tori T_i^+ and T_i^-, with $T_i^+ \subset T_{i+1}^+$ and winding around twice. Suppose that $X^+ = \cap T_i^+$ and $X^- = \cap T_i^-$ are both solenoids, and define σ so that $\sigma(T_{i+1}^+) = T_i^+$.

On $\mathbb{R}^4$, parametrized in "spherical coordinates" by (r,s) with $r \in [0,\infty)$, $s \in S^3$, consider the map $g: (r,s) \to (r^2,\sigma(s))$.

Let $Y = (\mathbb{R}^4 - \mathrm{cone}(X^-)) \cup B$, and note that $g(Y) \subset Y$ and Y is homeomorphic to $\mathbb{R}^4$.

Conjecture. If F has an attractive fixed point, then F is conjugate to $g: Y \to Y$ by a homeomorphism $\phi: \mathbb{C}^2 \to Y$.

This would have, among others, the consequence that there exists a Fatou-Bieberbach domain whose boundary is a topological manifold, and that there are infinitely many analytic embeddings of $\mathbb{C}$ into the boundary whose images are dense.

Computer pictures support the above conjecture.