LOCAL CONNECTIVITY OF JULIA SETS AND BIFURCATION LOCI: THREE THEOREMS OF J.-C. YOCCOZ

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Introduction

This paper consists of three parts, each devoted to a result due to J.-C. Yoccoz. The proofs are all somewhat modified, but they are also deeply influenced by Yoccoz's work.

The three results all bear on the local connectivity of Julia sets and bifurcation sets for complex analytic polynomials. However, each is of great interest in its own right and has many applications to other topics.

Notation. If P is a polynomial in one variable, then

$$K_P = \{ z \in \mathbb{C} \mid \text{the sequence } P^{\circ k}(z) \text{ is bounded} \}.$$

We will write $P_c(z) = z^2 + c$, and K_c, etc., when discussing quadratic polynomials specifically. If P is monic of degree d with K_P connected, there is then a unique conformal mapping $\phi_P : \mathbb{C} - K_P \to \mathbb{C} - \overline{D}$ which satisfies $\phi_P(P(z)) = (\phi_P(z))^d$ and tangent to the identity at ∞. We call $R_P(\theta) = \phi_P^{-1}(\{re^{2\pi i\theta}, r > 1\})$ the external ray of K_P at angle θ.

In the quadratic parameter space, let

$$M = \{ c \in \mathbb{C} \mid c \in K_c \} = \{ c \in \mathbb{C} \mid K_c \text{ is connected} \}.$$

For background about these notions, see [DH1] or [M]. The main outstanding problem about quadratic polynomials is the following question:

CONJECTURE MLC. The set M is locally connected.

All the results in this paper touch on this conjecture.

In the first part, we will prove an inequality relating the combinatorial rotation number to the analytic derivative at a repelling fixed point of a polynomial. If P is a polynomial of degree d with K_P connected, and z_0 is a repelling fixed point of P, then Theorem I.A asserts that $K_P - \{z_0\}$ has finitely many components $X_0, \dots, X_{q'-1}$, written in their natural circular order. The mapping P permutes them and preserves their order; it therefore maps locally each X_i to $X_{i+p'}$. We call p'/q' the *combinatorial rotation number* of P at z_0, and $m = \gcd(p', q')$ the *cycle number*; let $p = p'/m$, $q = q'/m$. Yoccoz's inequality is then the following statement:

THEOREM I.B. There is a branch τ of $\log P'(z)$ satisfying

$$\frac{\operatorname{Re}\tau}{|\tau - 2\pi i p/q|^2} \geq \frac{mq}{2\log d}\,.$$

Pommerenke [P] and Levin [L] have independently proved somewhat weaker versions of this inequality.

Theorem I.B has consequences about local connectivity in the parameter space.

THEOREM I.C. Let $P_c(z) = z^2 + c$ be a quadratic polynomial with an indifferent cycle. Then c has a basis of connected neighborhoods in M.

Note that this is not a result which says that the parameter space looks like the Julia set: it is quite possible for K_c not to be locally connected for polynomials with indifferent cycles. Theorem I.C will be proved in Part I only when the multiplier of the indifferent cycle is not 1; and even then the proof is somewhat different depending on whether the multiplier is rational or irrational. The case where the multiplier is 1 will be proved at the end of Part III, and will require results about *Mandelbrot-like families* [DH1].

The second part contains the most important result of the paper. One of the oldest ideas in dynamical systems is *Markov Partitions*. Given a dynamical system $f: X \to X$, cut up the dynamical space into pieces $X = \bigsqcup A_i$, $i \in I$. Let $B_I = I^{\mathbb{N}}$ be the set of sequences $(i_0, i_1, \dots)$ of elements of I, equipped with the shift mapping $s: B_I \to B_I$. Then we can assign to a point

$x \in X$ the sequence of pieces which its orbit visits. This provides a mapping $\sigma : X \to B_I$, which is the central construction of symbolic dynamics.

This idea is only interesting if the image $\sigma(X)$ can be understood, and more significantly if the identifications induced by σ can be understood. With very few exceptions, this has been impossible unless the dynamical system was assumed to be hyperbolic (the only exception the author is aware of is Guckenheimer's proof of no wandering domains for interval mappings with negative Schwarzian derivative). Theorem 9.5 gives a criterion under which the "Markov philosophy" works, even when there is a critical point in the dynamical space.

The method leading to this result was pioneered in [BH2], in the language of *Puzzles and Tableaux*. Part of the objective of this paper is to show that Yoccoz's argument can be rewritten in that language. There have already been several other uses of Theorem 9.5 (I am aware of work by Faught, Lyubich, Shishikura and Światec); in the case of quadratic polynomials it yields the following result.

THEOREM II. If $c \in M$ is not infinitely renormalizable, and does not have an indifferent periodic point, then K_c is locally connected.

The third part is devoted to transplanting the results of Part II to the parameter space. A similar result occurs in [BH2], where it is much easier. For quadratic polynomials, Yoccoz proves the following result.

THEOREM III. If $c \in M$ is not infinitely renormalizable, then c has a basis of closed connected neighborhoods in M.

This paper is largely based on lectures and manuscript notes by Yoccoz. The formulations in this paper were obtained in collaboration with B. Branner, A. Douady, D. Faught and M. Shishikura. More specifically, Douady provided one central idea in the proof of Theorem III, and Faught provided another, as well as the proof described here for the non-recurrent case.

Conversations with L. Goldberg, J. Luo, C. McMullen and J. Milnor were also very helpful. Carsten Peterson [Pe], Habib Jellouli and Milnor [M2] have also written up parts of Yoccoz's results, and I have had access to their notes. No research can take place without adequate funding, and I thank the NSF for grant DMS-8901729.

Part I. The Yoccoz Inequality

Let P be a monic polynomial of degree d with K_P connected, and $z \in K_P$ a repelling fixed point with $P'(z) = \omega$. In this part we will give two results which complement each other; the proof of the second is a refinement of the proof of the first. The first result is actually due to A. Douady.

THEOREM I.A. There are finitely many external rays of K_P which land at z; they are all periodic with the same period.

This theorem is proved in subsection 2.3.

The second result gives an inequality about the derivative at such a fixed point; in order to state it, we need the notion of the *combinatorial rotation number*. The q' external rays of K_P which land at z are transitively permuted by P, and this permutation must preserve their circular order since P is a local homeomorphism at z; thus the permutation must be a circular permutation, which shifts each ray to the one which is p' further counterclockwise for some $p' < q'$. Write $p'/q' = p/q$ in lowest terms, and call $m = \gcd(p', q')$ the *cycle number of* P *at* z; m is the number of cycles of rays landing at z, and we call p/q the *combinatorial rotation number of* P *at* z.

THEOREM I.B. There exists a branch τ of $\log P'(z)$ which satisfies

$$\frac{\operatorname{Re} \tau}{|\tau - 2\pi i p/q|^2} \geq \frac{mq}{2 \log d}.$$

This theorem is proved in subsection 3.6.

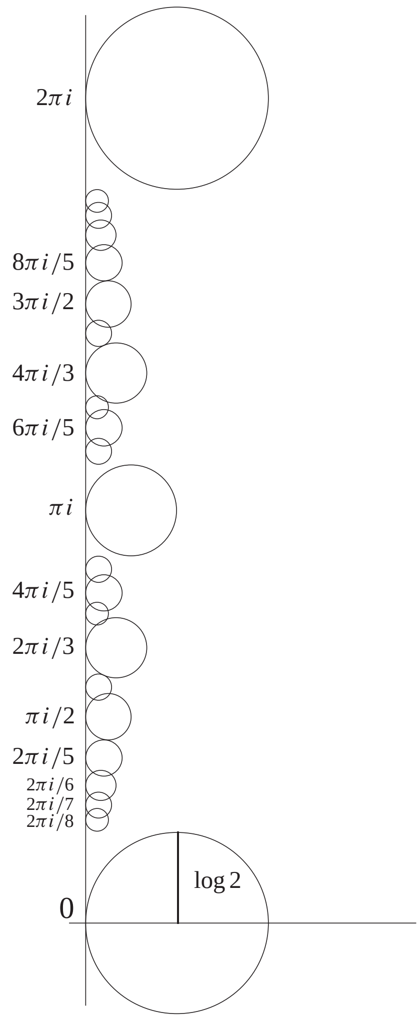
A geometric formulation of this inequality is that τ belongs to the closed disc of radius $(\log d)/(mq)$ tangent to the imaginary axis at $2\pi i p/q$. Indeed, for $a \in \mathbb{R}$, the equation

$$\frac{\operatorname{Re} z}{|z - ia|^2} = \frac{1}{2r}$$

is the equation of the circle of radius r tangent in the right half-plane to the imaginary axis at ia. Translating by ia, it is enough to show that

$$\frac{\operatorname{Re} z}{|z|^2} = \frac{1}{2r}$$

is the equation of the circle centered at r with radius r; we leave this to the reader. In particular, Theorem I.B says that the derivative and the "combinatorial derivative" $e^{2\pi i p/q}$ are not too different. The following drawing illustrates the discs in which τ must be when $d = 2$ and $m = 1$.



1. The linearizing coordinate

Let $\lambda\colon \mathbb{C} \to \mathbb{C}$ be a linearizing map at z, i.e., λ satisfies $\lambda(0) = z$ and $\lambda(\omega\zeta) = P(\lambda(\zeta))$.

REMARK 1.1. Such a mapping exists, and in fact one is given by the formula

$$\lambda(\zeta) = \lim_{k\to\infty} P^{\circ k}\left(\frac{\zeta}{\omega^k} + z\right).$$

It is uniquely determined by $\lambda'(0)$.

Set $\tilde{K} = \lambda^{-1}(K_P)$.

PROPOSITION 1.2. For each component U of $\mathbb{C} - \tilde{K}$, the mapping

$$\lambda|_U : U \to \mathbb{C} - K_P$$

is a universal covering map.

PROOF. First notice that the components of $\mathbb{C} - \tilde{K}$ are simply connected. Otherwise, there would be a compact component of $\tilde{K}$, and by multiplying by ω^{-n} for n sufficiently large, this compact set could be brought into the set on which λ is an isomorphism; this would show that K_P is not connected.

Suppose that $\lambda(\zeta) = w_0 \notin K_P$, and that $W \subset \mathbb{C} - K_P$ is a simply connected neighborhood of w_0. To finish the proof, we need to show that there is a branch μ of λ^{-1} defined on W and with $\mu(w_0) = \zeta$. Consider $w_n = \lambda(\zeta/\omega^n)$; each of these satisfies $P^n(w_n) = w_0$. There is a branch g_n of P^{-n} defined on W such that $g_n(w_0) = w_n$, since

$$P^n : P^{-n}(W) \to W$$

is a covering map (proper and a local homeomorphism).

The limit $\mu(w) = \lim_{n\to\infty} \omega^n(g_n(w) - z)$ is easily seen to exist and to solve the problem.

2. The external rays landing at a repelling fixed point

There exists a unique analytic mapping $\phi_P : \mathbb{C} - K_P \to \mathbb{C} - \overline{D}$ tangent to the identity at infinity and such that $\phi_P(P(z)) = (\phi(z))^d$; this mapping is often called the *Böttcher coordinate* [DH1, M]. By proposition 1.2, there exists on each component U of $\mathbb{C} - \tilde{K}$ an analytic branch

$$\tilde{\phi}_U = \log(\phi_P \circ \lambda) : U \to H,$$

where H is the right half-plane. The mapping $\tilde{\phi}_U$ is unique up to addition of a multiple of $2\pi i$, and is an isomorphism; let ψ_U be the inverse map.

Let $G_P : \mathbb{C} \to \mathbb{R}$ be the Green's function of K_P, and $\tilde{G} = G_P \circ \lambda$. Then $\tilde{G}(\omega\zeta) = d\tilde{G}(\zeta)$. Moreover, $\tilde{G} = \operatorname{Re}\tilde{\phi}_U$ in each component U of $\mathbb{C} - \tilde{K}$, so $\tilde{G}(\psi_U(w)) = \operatorname{Re} w$.

PROPOSITION 2.1. Each component U of $\mathbb{C} - \tilde{K}$ is periodic under multiplication by ω.

PROOF. Let $\mathbf{T}$ be the torus obtained as the quotient of $\mathbb{C} - \{0\}$ by the multiplicative group generated by ω, and $\pi : \mathbb{C} - \{0\} \to \mathbf{T}$ be the canonical projection. If the components $\omega^n(U)$ are all distinct, then π is injective on U. We will show that this is impossible by a length-area inequality.

Let

$$U' = \bigcup_{n\in\mathbb{Z}} \omega^n(U).$$

There is a component U_0 of U' for which

$$m = \sup_{\substack{\zeta\in U' \\ |\zeta|=1}} \tilde{G}(\zeta)$$

is realized at $\zeta_0 \in U_0$; we may assume that $U = U_0$.

LEMMA 2.2. If $\operatorname{Re}(w) \geq md^n$, then $|\psi_U(w)| \geq |\omega|^n$.

PROOF. We have

$$\tilde{G}\left(\frac{\psi_U(w)}{\omega^n}\right) = \frac{\tilde{G}(\psi_U(w))}{d^n} \geq \frac{md^n}{d^n}.$$

Since $\psi_U(w)/\omega^n \in U'$, it follows that $|\psi_U(w)|/|\omega|^n \geq 1$, as required.

Suppose $\tilde{\phi}_U(w_0) = \zeta_0$, where $w_0 = m + iv_0$. We will apply the length-area argument to a long rectangle

$$R = \{\, \tilde{w} \mid \log m \leq \operatorname{Re}\tilde{w} \leq \log m + L \text{ and } |\operatorname{Im}\tilde{w} - \arg w_0| \leq \delta \,\}$$

and its image in $\mathbf{T}$ under

$$\eta_U : \tilde{w} \mapsto \pi(\psi_U(e^{\tilde{w}})).$$

For δ sufficiently small, the image of every horizontal line of R under η_U will cross the line on $\mathbf{T}$ corresponding to the unit circle at least $(L/\log d) - 1$ times, making at least that many full turns around the torus. We will endow $\mathbf{T}$ with the euclidean metric $|d\zeta|/|\zeta|$; the image of a horizontal line then has length at least $((L/\log d) - 1)\log|\omega|$.

We get

$$\begin{aligned}
\operatorname{Area}(\mathbf{T}) \geq \operatorname{Area}(\eta_U(R)) &= \int_{\arg w_0-\delta}^{\arg w_0+\delta} \int_m^{m+L} |\eta_U'(u+iv)|^2\, du\, dv \\
&\geq \frac{1}{L}\int_{\arg w_0-\delta}^{\arg w_0+\delta} \left(\int_m^{m+L} |\eta_U'(u+iv)|\, du\right)^2 dv \\
&\geq \frac{1}{L}\int_{\arg w_0-\delta}^{\arg w_0+\delta} \left(\log|\omega|\left(\frac{L}{\log d} - 1\right)\right)^2 dv.
\end{aligned}$$

The last term tends to infinity with L, but the first term is bounded (in fact, $\operatorname{Area}(\mathbf{T}) = 2\pi \log|\omega|$). This is a contradiction, and proves proposition 2.1.

2.3 Proof of Theorem I.A. We now know that for each component U of $\mathbb{C} - \tilde{K}$ there exists q such that $\omega^q U = U$. Clearly the quotient of U by the action of ω^q is an annulus; let γ be its unique closed geodesic, $\tilde{\gamma}$ its inverse image in U and ρ the image of $\tilde{\gamma}$ under λ. At one end $\tilde{\gamma}$ tends to 0 and at the other to ∞, since it is obtained by iterating ω^q on a compact set (the closure of one lift of γ). Since $\tilde{G}(\omega\zeta) = d\tilde{G}(\zeta)$ it follows that $\tilde{G}$ tends to 0 at one end and to ∞ at the other; since it is a Poincaré geodesic in $\mathbb{C} - K_P$, ρ is an external ray landing at z, periodic of period q.

This shows that there is at least one periodic external ray landing at z corresponding to each component. There is clearly also at most one. The fact that there are only finitely many cycles will follow from Proposition 3.3 and 3.5.

3. The Yoccoz Inequality

3.1 A length-area inequality for annuli on a torus. The torus $\mathbf{T} = \mathbb{C} - \{0\}/(\omega)$ is also given by

$$\mathbf{T} = \mathbb{C}/(2\pi i\mathbb{Z} + \tau\mathbb{Z}),$$

where τ is a branch of $\log\omega$ with $\operatorname{Re}(\tau) > 0$. If p and q are coprime (i.e., $m = 1$), then the segment $[0, 2\pi i p + q\tau]$ projects to a simple closed curve on $\mathbf{T}$, which we call the (p, q)-curve.

PROPOSITION 3.2. Let $A \subset \mathbf{T}$ be an embedded annulus homotopic to a (p, q)-curve. Then

$$\operatorname{mod} A \leq \frac{2\pi \operatorname{Re} \tau}{|2\pi i p + \tau q|^2}.$$

PROOF. Let B be the annulus

$$\{ z \mid 0 < \operatorname{Im}(z) < h \}/\mathbb{Z},$$

where $\mathbb{Z}$ acts by translation, with $h = \operatorname{mod} B = \operatorname{mod} A$, and $f\colon B \to A$ a conformal mapping. Then using Schwarz's lemma and the fact that for any fixed y, the simple closed curve

$$x \mapsto f(x, y), \qquad 0 \leq x \leq 1$$

has length at least $|2\pi i p + \tau q|$, we find

$$\begin{aligned} 2\pi \operatorname{Re}(\tau) &= \operatorname{Area}(\mathbf{T}) \geq \operatorname{Area}(A) = \int_B |f'(x, y)|^2 \, dx \, dy \\ &= \int_0^h \left(\int_0^1 |f'(x, y)|^2 \, dx \right) dy \geq \int_0^h \left(\int_0^1 |f'(x, y)| \, dx \right)^2 dy \\ &\geq \int_0^h |2\pi i p + \tau q|^2 \, dy = h|2\pi i p + \tau q|^2. \end{aligned}$$

This is the required inequality.

When the cycle number $m > 1$, we need the following generalization of 3.2.

PROPOSITION 3.3. Let $A_i \subset \mathbf{T}$ be disjoint embedded annuli homotopic to a (p, q)-curve. Then

$$\sum_i \operatorname{mod} A_i \leq \frac{2\pi \operatorname{Re} \tau}{|2\pi i p + \tau q|^2}.$$

The modifications to the proof above are left to the reader.

3.4 The structure of U/ω^q. Let U be a component of $\mathbb{C}-\tilde{K}$ of period q, with combinatorial rotation number p/q.

PROPOSITION 3.5. The image of U in $\mathbf{T}$ is an annulus of modulus $\pi/q \log d$, and of homotopy class $(-p, q)$ for an appropriate choice of $\tau = \log \omega$.

PROOF. The mapping ψ_U transforms $\omega^q : U \to U$ into the map $H \to H$ which is simply multiplication by d^q. Thus we simply need to know what the modulus of the quotient of H by this (simple) Moebius transformation is; the principal branch of the logarithm maps H to the strip $\mathrm{Re}(z) < \pi$, and transforms the multiplication into translation by $q \log d$. This computes the modulus.

Next we compute the homotopy class of the image of U in $\mathbf{T}$. Let U_0, $U_1, \dots, U_{q-1}$ be the orbit of U under ω, *in counter-clockwise order*, so that $\omega U_i = U_{i+p}$, taking the indices modulo q. The inverse image of U_i under the exponential map is a union

$$\exp^{-1} \bigcup_{i=1,\dots,q-1} U_i = \bigcup_{i\in\mathbb{Z}} U_i',$$

and there is a unique such labeling up to translation by a multiple of q if we require that $\exp(U_i') = U_{i \bmod q}$ and that U_j' separates U_i' from U_k' if and only if $i < j < k$.

The condition on the combinatorial rotation number says that for any choice τ of $\log \omega$, there exists an integer l such that for any $i \in \mathbb{Z}$ and any $\zeta \in U_i'$, $\tau + \zeta \in U_{i+lq}$. Since we can change l by 1 by adding $2\pi i$ to τ, we may assume that $l = 0$; this is the appropriate choice of τ in the statement.

Indeed, with this choice we have that if $\zeta \in U_0'$, then $\zeta + q\tau \in U_{pq}'$, so that $\zeta + q\tau - 2\pi i p \in U_0'$. But the image in $\mathbf{T}$ of a curve joining ζ to $\zeta + q\tau - 2\pi i p$ in U_0' is a simple closed curve generating the fundamental group of the annulus.

3.6 Proof of Yoccoz's inequality. This is just a matter of putting Propositions 3.3 and 3.5 together. This gives

$$\frac{m\pi}{q \log d} \le \frac{2\pi \,\mathrm{Re}\,\tau}{|\tau q - 2\pi i p|^2}$$

which can easily be rewritten in the form required in the theorem.

4. Local connectivity of M for polynomials with indifferent fixed points

In order to prove anything about the Mandelbrot set M we will need to delve into its combinatorics. This is a difficult subject to write about, because much of what is known is folklore, acquired originally from intimate knowledge of computer experiments rather than from rigorous proofs. Many of the facts we require, or minor variants, can be found in [DH1].

I will try to collect the facts that are needed, with sketches of proofs not readily available elsewhere.

4.1 Facts about M. If P_{c_0} has a rationally indifferent cycle $z_0, \dots, z_{k-1}$ with derivative $\omega = e^{2\pi i t}$ and $t \in \mathbb{Q}/\mathbb{Z}$, then exactly two external rays of M land at c_0, except when $k = 1$ and $t = 0$, which corresponds to $c_0 = 1/4$, where only the ray at angle 0 (i.e., the positive real axis) lands.

This can happen in two ways: If $t = p/q \neq 0$, two hyperbolic components U_1 and U_2 of $\operatorname{int} M$ have closures which touch at c_0; for $c \in U_1$ the polynomial P_c has an attractive cycle of length k, and for $c \in U_2$ the polynomial has an attractive cycle of length qk. The cycle of length k varies analytically with c in a neighborhood of c_0, and the cycle of length qk forms a ramified covering space of the c-plane near c_0. Moreover, the multiplier of the cycle is an analytic function of c in a neighborhood of c_0.

If $t = 0$ and $k > 1$, then c_0 is the cusp of a hyperbolic component of $\operatorname{int} M$. At such a point, two cycles of length k coalesce, and a path surrounding c_0 lifts to paths exchanging the two cycles.

If P_{c_0} has a rationally indifferent fixed point, at the end of external rays R_{θ_1} and R_{θ_2}, we call the region in the c-plane cut out by these rays together with c_0, and not including 0, the *wake* W_{c_0} of c_0. If U is a hyperbolic component of $\operatorname{int} M$, and $\phi_U : \overline{D} \to \overline{U}$ is the interior parameterization of U, we set

$$c_U = \phi_U(0), \quad c_{U,t} = \phi_U\left(e^{2\pi i t}\right)$$

and call the intersection

$$L_{U,p/q} = M \cap W_{c_{U,p/q}}$$

the p/q-*limb* of U.

PROPOSITION 4.2. (a) Every point of M in the wake of $c_{U,0}$ is either in $\overline{U}$ or in one of the limbs of U.

(b) There exists a function $\eta_U : \mathbb{N} \to \mathbb{R}$ with $\eta_U(q) \to 0$ as $q \to \infty$, such that

$$\operatorname{diam} L_{p/q}(U) \leq \eta_U(q).$$

PROOF. Let $z_0 = 0, \dots, z_{k-1}$ be the superattractive cycle for the polynomial at the center c_U of U. There are unique analytic functions $\zeta_0(c), \dots, \zeta_{k-1}(c)$, defined in $W = W_{c_0}$ such that for all $c \in W$ the points $\zeta_0(c), \dots, \zeta_{k-1}(c)$ form a periodic cycle for P_c, and such that $\zeta_j(c_U) = z_j$.

For $c \in (W - \overline{U}) \cap M$, this cycle is repelling, and by Theorem 3.6, finitely many external rays $R_c(\theta_0), \dots, R_c(\theta_{q-1})$ of K_c land at $\zeta_1(c)$, with some combinatorial rotation number p/q. The situation where the fixed point $\zeta_1(c)$ of $P_c^{\circ k}$ is repelling, at the end of finitely many rays of given angles $\theta_0, \dots, \theta_{q-1}$, is structurally stable. It holds on an open subset of the wake W, bounded by

- external rays of M where, for the corresponding polynomial, one of the rays of angle θ_i lands on the critical point, and
- points c where the cycle becomes indifferent.

Thus every point $c \in (W - \overline{U}) \cap M$ is in the wake of the point of $c_{U,p/q} \in \partial U$ where the cycle $\zeta_0(c_{U,p/q}), \dots, \zeta_{k-1}(c_{U,p/q})$ is rationally indifferent with derivative $e^{2\pi i p/q}$, and p/q is the combinatorial rotation number of the fixed point $z_0(c)$ of P_c^k.

For part (b) we will apply the Yoccoz inequality to the fixed point $\zeta_j(c)$ of P_c^k. This point is an analytic function of $c \in W$, and we can define a branch $\tau(c)$ of $\log(P_c^k)'(\zeta_j(c))$ in $W - \gamma$, where $\gamma = \phi_U[0, 1]$. The image $\tau(L_{U,p/q})$ must be contained in a disc of radius $k \log 2$ tangent to the imaginary axis. Since the derivative of τ does not vanish on $\overline{U} - c_0$, there exists a unique section $\sigma\colon (0, 2\pi i) \to \overline{U}$, which extends to a neighborhood of $(0, 2\pi i)$ in the complex plane.

To get uniformity, we would like this extension to be defined on a neighborhood of 0 and $2\pi i$. When c_0 is not primitive, this can be done, because the multiplier of the cycle is analytic near c_0. When c_0 is primitive, we can extend σ if we replace the c-plane by a double cover ramified above c_0, since the multiplier of the cycle is locally an analytic function on such a the double cover.

Choose a compact neighborhood of $[0, 2\pi i]$ in the domain of σ; on this compact neighborhood the derivative of σ is bounded by some constant C. As soon as q is large enough, the disc of radius $(k \log 2)/q$ will be in this compact neighborhood, so its image will have diameter smaller than $C(k \log 2)/(2q)$.

REMARK 4.3. We actually get a bound on the size of the limbs: there exists a constant C_U, depending on U, such that if c_0 is not primitive, then

$$\operatorname{diam} L_{U,p/q} \le C_U \frac{k \log 2}{q},$$

and if c_0 is primitive, then

$$\operatorname{diam} L_{U,p/q} \le C_U \left(\frac{k \log 2}{q}\right)^{1/2}.$$

Proposition 4.2 has the following consequences for the local connectivity of M. Let P_{c_0} be a quadratic polynomial with an indifferent cycle $z_0, \dots, z_{k-1}$ with multiplier $e^{2\pi i t}$.

COROLLARY 4.4. If t is irrational, then c_0 has a basis of connected neighborhoods in M.

PROOF. The point c_0 is in the boundary ∂U of a unique hyperbolic component U of the interior of M; let W be the wake of the root of U.

A basis of connected neighborhoods of c_0 in M can now be defined as follows. Find rational numbers p_1/q_1 and p_2/q_2 with $p_1/q_1 \le t \le p_2/q_2$ and $p_2/q_2 - p_1/q_1 < \epsilon$. As soon as ϵ is small enough, the denominators of all rational numbers $r \in [p_1/q_1, p_2/q_2]$ are large. Let V be the region in the closed unit disc cut off by the line joining $e^{2\pi i p_1/q_1}$ to $e^{2\pi i p_2/q_2}$, and consider the union of $\phi_U^{-1}(V)$ and the union of all the r-limbs of U with $r \in [p_1/q_1, p_2/q_2]$ and rational. This is a closed neighborhood of c_0, and as ϵ tends to 0, the diameter tends to 0.

COROLLARY 4.5. If P_{c_0} has a rationally indifferent cycle with multiplier $e^{2\pi i p/q} \ne 1$, then c_0 has a basis of closed connected neighborhoods in M.

PROOF. It is shown in [DH1] (and was already known to Fatou) that there exist two hyperbolic components U_1 and U_2 of the interior of M such that $c_0 = \overline{U_1} \cap \overline{U_2}$, and that the point is at internal angle p/q on U_1 and at internal angle 0 on U_2. Consider neighborhoods V_1 of $e^{2\pi i t}$ and V_2 of 1 in $\overline{D}$ cut off by a chord. The union of $\tilde{V}_1 = \phi_{U_1}(V_1)$, $\tilde{V}_2 = \phi_{U_2}(V_2)$ and the limbs of U_1 attached to $\tilde{V}_1$ (except the limb attached at $e^{2\pi i t}$), and the limbs of U_2 attached to $\tilde{V}_2$, together form a closed neighborhood of c_0. As the diameters of V_1 and V_2 tend to 0, the diameters of the attached limbs also tend to 0 as above, and the diameters of the neighborhoods tend to 0.

Part II. Tableaux and Sums of Moduli

Although the main result of this part applies to many objects besides quadratic polynomials, we will introduce the main notions in that context.

A quadratic polynomial P with K_P connected will be called renormalizable if there is an open subset $U \subset \mathbb{C}$ containing 0 and an integer $k > 1$ such that one component U' of $P^{-k}(U)$ is relatively compact in U, and the restriction

$$P_U \equiv P^k|_{U'}: U' \to U$$

is polynomial-like of degree 2 [DH2], with its own filled-in Julia set

$$K_{P_U} = \{ z \in U' \mid P_U^n(z) \in U' \text{ for all } n \geq 0 \}$$

also connected.

It is easy to check that this last condition is equivalent to requiring that $P^{nk}(0) \in U'$ for all $n \geq 0$, or that P_U be hybrid-equivalent [DH2] to a quadratic polynomial Q with K_Q connected.

We will add one condition to the definition of renormalizability; McMullen has recently discovered non-standard and unexpected examples of renormalizable polynomials. We will require that either the "internal" fixed point α does not belong to K_{P_U}, or that if it does, it has combinatorial rotation number 0 for P_U, and thus corresponds to the "external" fixed point of P_U.

The polynomial-like mapping P_U might itself be renormalizable, and so forth. Thus we can speak of a polynomial being m times renormalizable, or even infinitely many times renormalizable. The Feigenbaum polynomial is the simplest example of an infinitely-many times renormalizable polynomial. The local connectivity of the Julia set for the Feigenbaum polynomial is not known (at least to the author); but there definitely are infinitely renormalizable polynomials the Julia set of which is not locally connected.

Now Yoccoz's theorem can be stated.

THEOREM II. If P is not infinitely renormalizable, and does not have an indifferent fixed point, then K_P is locally connected.

The proof will be given in section 11.

5. Quadratic puzzles

Let P be a quadratic polynomial with K_P connected. The unique fixed external ray lands at a fixed point β with combinatorial rotation number 0. Suppose the other fixed point α is repelling with the combinatorial rotation number $p/q \in (0, 1)$.

REMARK 5.1. Write $P_c(z) = z^2 + c$. Then the number p/q is related to the position of c in M as follows. Let M_0 be the set of polynomials with an attractive fixed point (the inside of the cardioid). Then $M - \overline{M_0}$ consists of the limbs $M_{p/q}$, i.e., components which touch M_0 at the point $e^{2\pi i p/q}/2 - e^{4\pi i p/q}/4$ (the corresponding polynomial has a fixed point with derivative $e^{2\pi i p/q}$). The polynomials P_c with $c \in M_{p/q}$ are precisely those for which the K_c is connected, the fixed point $\alpha(c)$ is repelling and has combinatorial rotation number p/q.

Let h_P be the Green's function of K_P. Choose $R > 0$, which will remain fixed for the rest of the paper, and let U_0 be the region

$$U_0 = \{ z \mid h_P(z) < R \}.$$

Let $\theta_1, \theta_2, \ldots, \theta_q$ be the external angles of the fixed point α, and $R_{\theta_1}, \ldots, R_{\theta_q}$ the corresponding rays. Let $\Gamma_0 \subset U_0$ be the graph formed by the parts of the rays $R_{\theta_1}, \ldots, R_{\theta_q}$ in U_0.

Define $U_n = P^{-1}(U_{n-1})$ and $\Gamma_n = P^{-1}(\Gamma_{n-1})$. The sequence $U_0 \supset U_1 \supset \cdots$ together with the graphs $\Gamma_n \subset U_n$ is called the *puzzle* $\mathcal{P}$ of P. The set $\mathcal{P}(N)$ of pieces at depth N of the puzzle $\mathcal{P}$ is the set of closures of components of $U_N - \Gamma_N$ in U_N.

Since α is the only point of $\Gamma_0 \cap K_P$, we see that the only points of K_c which are in the boundary of a piece of the puzzle are the pre-images of α. We will make the assumption that *the forward image of the critical point does not contain* α; then each preimage of α is in exactly q pieces of $\mathcal{P}(n)$ for all sufficiently large n.

REMARK 5.2. This assumption is largely for convenience; those polynomials which fail to satisfy it are in fact especially simple, in particular the Julia sets of such polynomials are known to be locally connected [DH1].

An *end* x of the puzzle is a nested sequence

$$x = (X_0 \supset X_1 \supset X_2 \supset \cdots)$$

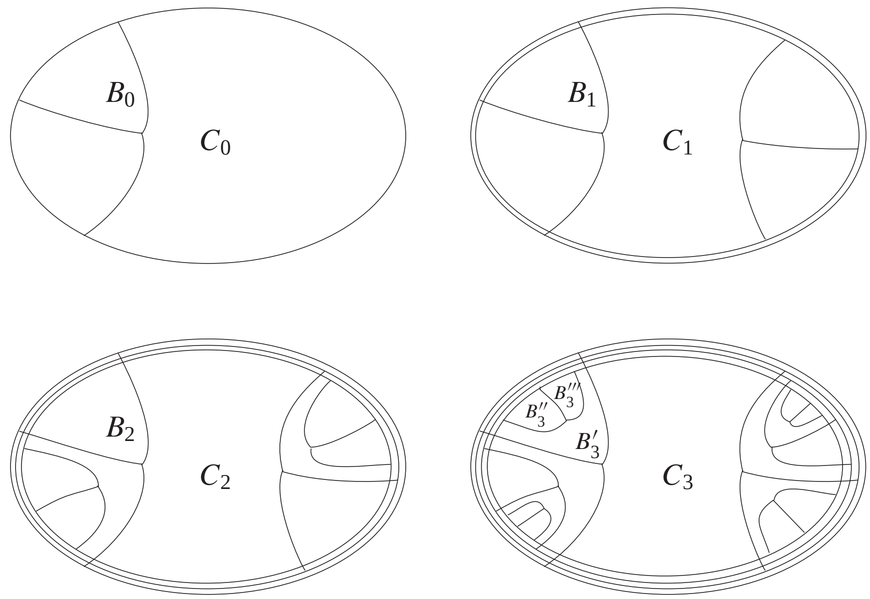


Figure 5.4.. The first three levels of a puzzle in the 1/3-limb.

of pieces $X_n \in \mathcal{P}(n)$. We will denote by $\mathcal{E}_P$ the set of ends of the puzzle of P. The set $\mathcal{E}_P$ has a natural topology as a Cantor set; we will have no use for it. Clearly P induces a mapping $\hat{P}\colon \mathcal{E}_P \to \mathcal{E}_P$.

For any end x, we will denote by $X_n(x)$ its piece at depth n; if $z \in K_P$ is a point which is not an inverse image of α, so that it is in the interior of every piece which contains it, we will similarly denote by $X_n(z)$ the piece at depth n containing z.

There are two ends of particular interest: the *critical end* $C_0 \supset C_1 \supset \cdots$ consisting of the pieces containing the critical point, and the critical value end $B_0 \supset B_1 \supset \cdots$ consisting of the pieces containing the critical value.

To each end x we associate its *impression* $\mathcal{I}(x)$: the compact connected subset

$$\mathcal{I}(x) = \bigcap_n X_n(x) \subset \mathbb{C}.$$

EXAMPLE 5.3. It is quite easy to construct the topology of a puzzle to a given depth "by hand". The only information needed to construct the puzzle at depth $n+1$ is the puzzle at depth n, and the knowledge of which piece contains the critical value. Figure 5.4 shows levels 0–3 of the puzzle for a polynomial where the combinatorial rotation number at α is 1/3.

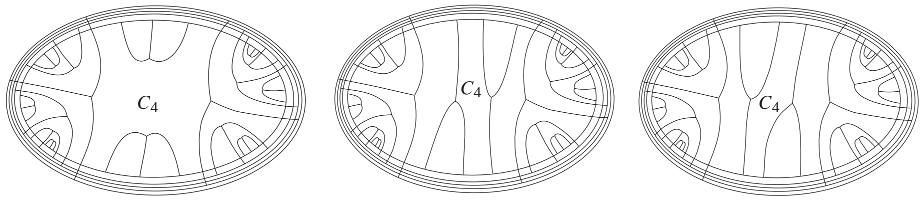


Figure 5.5.. The three possible choices for level 4.

Notice that the pairs (U_i, Γ_i) are all homeomorphic for $i \leq 3$ when the polynomial is in the 1/3-limb, because the graphs Γ_1 and Γ_2 do not cut B_0 into pieces. But Γ_3 cuts B_2 into 3 pieces, and the topology of the pair (U_4, Γ_4) depends on the choice of which of the three components will be B_3. The three choices are illustrated in Figure 5.5, corresponding from left to right to the choices of B_3', B_3'' and B_3''' shown in Figure 5.4.

The two Figures on page 375 show puzzles representing the first of these choices, and the Figures on page 376 show puzzles representing the second and third choices.

The main reason for constructing puzzles is the following result.

PROPOSITION 5.6. Let P be a quadratic polynomial such that for each end $x \in \mathcal{E}_P$, the impression $\mathcal{I}(x)$ is a point. Then K_P is locally connected and $\operatorname{int} K_P = \emptyset$.

In fact, the main result will be:

THEOREM 5.7. (a) If P is not renormalizable, the impression of each end of its puzzle is a point.

(b) If P is renormalizable, then the ends of its puzzle which are preimages of the critical end have impressions which are homeomorphic to K_{P_1} for some quadratic polynomial P_1 with K_{P_1} connected; the impressions of the other ends are points.

Theorem 5.7 will be proved in several steps. The case where the critical end is non-recurrent will be dealt with in section 7. Proposition 7.2 shows that if the critical end is recurrent, then only the critical nest needs study. The proof of (a) is given in section 11, whereas Corollary 8.2 shows (b).

Proposition 5.6 requires the following lemma.

LEMMA 5.8. For any piece X of the puzzle, the intersection $K_P \cap X$ is connected.

PROOF. We will proceed by induction on the level. At level 0, there are exactly q pieces $Q_0, \dots, Q_{q-1}$, and the intersections $Q_i \cap K_P$ are the closures of the components of $K_P - \alpha$. Let W_1 and W_2 be disjoint open sets with $Q_i \cap K_P = (W_1 \cap K_P) \cup (W_2 \cap K_P)$. Then α must belong to either W_1 or W_2, say W_1, and set $W_2' = W_2 \cap \operatorname{int} Q_i$, so that $W_2 \cap K_P = W_2' \cap K_P$. Then K_P is contained in the union of the disjoint open sets W_2' and $W_1 \cup \bigcup_{j \neq i} \operatorname{int} Q_j$. But K_P is connected, so $W_2' \cap K_P$ is empty.

Let Q be a piece at level $n > 0$, and $Q' = P(Q)$. If the critical point ω_P is not in Q, then P restricts to a homeomorphism $K_P \cap Q$ onto $K_P \cap Q'$ so $K_P \cap Q$ is connected by induction. If $\omega_P \in K_P \cap Q$ and $K_P \cap Q$ is not connected, take a component L which does not contain ω_P. Then $P(L)$ does not contain $P(\omega_P) \in K_P \cap Q'$, hence $P(L)$ is not a component $K_P \cap Q'$, and there is a point $z' \in P(L)$ such that every neighborhood of z' contains points of $K_P - P(L)$. If $y \in L$ is an inverse image of z, clearly P is not a local homeomorphism $K_P \to K_P$ at y. But $P\colon K_P \to K_P$ is a local homeomorphism at every point except ω_P.

PROOF OF PROPOSITION 5.6. Any point $z \in K_P$ is in some (possibly several) end $X_0 \supset X_1 \supset X_2 \supset \cdots$ of the puzzle of P. If z is in the interior of all of the X_n, the intersections $X_i \cap K_c$ form a basis of connected neighborhoods of z since the diameters of the pieces tend to 0. If z is on the boundary of some X_n, then z is a preimage of α. As such, it is at every sufficiently large depth in the interior of the union of exactly q pieces of the puzzle, and together these pieces form a form a connected neighborhood of z. These neighborhoods as the depth increases again form a basis of connected neighborhoods of z.

6. Tableaux of ends

The *nest* of an end $x = X_0 \supset X_1 \supset X_2 \supset \cdots$ is the sequence of possibly degenerate annuli $A_0(x), A_1(x), A_2(x), \dots$ where $A_i(x) = X_i - \operatorname{int}(X_{i+1})$.

Degenerate annuli occur when $\partial X_k \cap \partial X_{k+1} \neq \emptyset$; in that case we set the modulus of the degenerate annulus to be 0.

As in [BH2], the reason to introduce nests is the following result from complex analysis.

PROPOSITION 6.1. If for some nest $A_0(x), A_1(x), A_2(x), \ldots$ the series

$$\sum_{n=0}^{\infty} \operatorname{mod}(A_n)$$

is divergent, then diam $X_n \to 0$ as $n \to \infty$, and $\mathcal{J}(x)$ is a point.

PROOF. See [BH2], Props. 5.4 and 5.5.

The annuli above are complicated subsets of $\mathbb{C}$, with wiggly boundaries trying to approximate fractals, and they are difficult to understand in that way. On the other hand, they map to each other under P, and the modulus behaves in a simple way under analytic mappings. The tableau of an end is designed to encode how the various annuli map to each other.

The *tableau* $T(x)$ of an end x is the two-dimensional array of pieces

$$T_{n,k}(x) = X_n(\hat{P}^k(x)).$$

The mapping P maps up one and to the right.

In the tableau we will be particularly interested in the *critical positions*: the positions for which the corresponding piece contains the critical point. A critical position (n, m) will be called *strictly critical* if the critical point is in $T_{n+1,m}$ and *semi-critical* if the critical point is in $T_{n,m} - T_{n+1,m}$. The marked grid is the array $\mathbf{N}^2$ with the critical positions marked. In particular, the *critical marked grid*, the marked grid of the critical tableau, has its 0th column entirely marked.

Proposition 6.2 describes the internal consistencies which the marked grids of ends for a single polynomial must satisfy. The last two are best stated as a comparison between the tableau of an arbitrary end and the critical end, but the case where the arbitrary end is also the critical end is particularly interesting.

PROPOSITION 6.2. The ends of a quadratic puzzle have marked grids which obey the following three tableau rules:

(Ta) Each column is either entirely strictly critical, or entirely non-critical, or has a unique semi-critical position, with everything above strictly critical and everything below non-critical;

(Tb) If the (n, k) position of a marked grid is critical, then the $(i, k + j)$ position of that marked grid is of the same nature as the (i, j) position of the critical marked grid for $i + j \leq n$.

(Tc) If, for the critical grid, the position (n, k) is strictly critical and the positions $(n + i, i)$ are not critical for $0 < i < k$, and if in the tableau T of some end the position $(n + k, l)$ is semi-critical, then the position $(n, l + k)$ of that grid is semi-critical.

PROOF. The condition (Ta) is simply that the pieces of an end are nested.

The condition (Tb) is also immediate: If the position (n, k) of a tableau is critical, then the piece corresponding to the position (i, k) is C_i for $i \leq n$. Hence the pieces above and to the right of these coincide, since they are images by P of the same (critical) pieces.

The first part of the hypothesis of (Tc) says that

$$P^k(C_{n+k}) = C_n$$
$$P^k(C_{n+k+1}) = C_{n+1},$$

and that the restrictions of P^k to C_{n+k} and to C_{n+k+1} are proper of degree 2. In particular, this shows that

$$C_{n+k+1} = P^{-k}(C_{n+1}) \cap C_{n+k}.$$

If the position $(n, l + k)$ of the tableau T is strictly critical, then the position $(n + 1, l + k)$ is C_{n+1}, so that the position $(n + k + 1, l)$ is a component of $P^{-k}(C_{n+1})$, contained in C_{n+k} but not C_{n+k+1}. We just saw that there is no such piece, so position $(n, l + k)$ is semi-critical.

REMARK 6.3. Rules (a) and (b) were trivial. But rule (b) still contains a lot of information: many of the proofs, especially in real 1-dimensional dynamics, are largely concerned with the question: if a point is close to the critical point, then for some number of moves its orbit stays close to that orbit. In our case, because of our “combinatorial distance” defined by the pieces of the puzzle, this is easy, but if we tried to use epsilons, the statement would become enormously more difficult. Actually, rule (c) is also of this nature, saying exactly that if a point is close but not too close to the critical point, then its orbit remains close but not too close.

PROPOSITION 6.4. The tableaux of polynomials in $M_{p/q}$ satisfy two further conditions:

(Td) There can be at most $q-1$ consecutive non-critical positions in the 0th row. Moreover, the position $(0,k),\ldots,(0,k+q-1)$ are non critical if and only if $(0,k-1)$ is strictly critical.

(Te) There are no semi-critical positions in the rows $1,\ldots,q-1$.

PROOF. (Td) Suppose the kth column is an end x, and that the position $(0,k)$ is not critical. Then x is in one of the pieces $Q_1,\ldots,Q_{q-1}$, say Q_j. Then $P^{q-j}(x)$ is in Q_0, and hence the position $(0,k+q-j)$ is critical, and will be the first critical position in the 0th row after $(0,k)$.

Moreover, ends in C_1 are the only ones which are mapped to Q_1; thus the only way to have $j=1$ in the previous argument is for the end corresponding to the $(k-1)$th column to start with a strictly critical position. This proves (Td).

To see (Te), we observe that for each $i=2,\ldots,q$, the piece C_i is the only piece of $\mathcal{P}(i)$ contained in C_{i-1}.

REMARK 6.5. It isn't quite true that every marked grid satisfying (Ta)–(Te) is the marked grid of a quadratic puzzle. Faught [F] has completely cleared up the problem. Note that to depth q the puzzle is completely determined. A critical marked grid satisfying rules (Ta) through (Te) is the marked grid of a polynomial if and only if one can assign a piece to each position (i,j) with $i<q$.

7. Reduction to the critical nest, off-critical ends and non-recurrent polynomials

7.1 The role of the critical nest. Just as in [BH2], the only end which really requires study is the critical end.

PROPOSITION 7.2. (a) If the critical nest of a quadratic puzzle is divergent, all ends are points.

(b) If the critical nest is convergent, then the convergent nests are precisely the inverse images of the critical nest.

PROOF. There are two cases to consider. Either the tableau $T(x)$ of an end x has critical positions arbitrarily deep, or it does not. If it does, the modulus of a nest is at most that of the critical nest; this is exactly Thm. 4.3 (b) of [BH2]. The other case (called the off-critical case), is proved in Proposition 7.5 below. This shows (a).

Part (b) is the same as Thm. 4.3 (c) of [BH2].

7.3 Ends which avoid the critical end are points. Let P be a quadratic polynomial. An end x of the puzzle will be called *off-critical* if there is a critical piece C_m such that the orbit of x never enters C_m. It should be clear that this coincides with:

(1) the tableau $T(x)$ does not have critical positions arbitrarily deep, and
(2) the orbit of x does not accumulate on the critical end.

Off-critical ends need a somewhat different (and easier) treatment from other ends. Yoccoz uses an argument using hyperbolic geometry, which requires some rather delicate doctoring of the pieces; we will present an argument using tableaux, due to D. Faught.

At every level n, there is only one piece Y_n which is contained in C_0 but not in the interior of C_0: the piece containing the fixed point α. These pieces form a nest $y = Y_0 \supset Y_1 \supset \cdots$. There are two possibilities:

(1) The nest y is the critical nest, so that the critical nest is periodic of period q, or
(2) the nest y is not the critical nest, and its impression is the single point $\{\alpha\}$. Indeed, the branch of P^{-q} fixing α will then map Y_n isomorphically to Y_{n+q}, and since α is repelling, this branch of P^{-q} is strongly contracting.

Suppose that the second alternative occurs, so there is a first n such that $Y_n \neq C_n$. Then P^q is injective on Y_n, so that if an end x is in $Y_n - Y_{n+q}$, then $P^q(x)$ is in C_0 but not in Y_n. Given any end x which is not a preimage of $\{\alpha\}$, and any integer N, there exists $N' > N$ such that $y = P^{N'}(x) \in Y_m - Y_{m+1} \subset C_0$ for some m. Then $z = P^{n-m}(y) \in Y_n - Y_{n+q}$ so that $P^q(z) \in C_0 - Y_n$. We have proved the following result.

PROPOSITION 7.4. For any end x which is not an inverse image of y, there exist infinitely many columns of the tableau $T(x)$ in which the piece at level 0 is C_0, and the piece at level n is not Y_n.

Let us apply this result to off-critical ends.

PROPOSITION 7.5. The impression of an off-critical end is a point.

PROOF. If the first alternative above occurs, then there is a fixed m such that no forward image of x is in $Y_m = C_m$, so there are infinitely many columns of $T(x)$ which have C_0 at level 0, and a piece compact in C_0 at level m, since x is off-critical. If the second alternative occurs, the same is true by Proposition 7.4.

Since there are only finitely many pieces at depth m, the annuli one can get by removing a relatively compact piece at level m from the interior of C_0 have moduli bounded below. The annuli in the 0th column which map to them also have moduli bounded below, since the end is off-critical, and the degree of the mapping is bounded above.

Proposition 7.5 is enough to prove Theorem 5.7 for some polynomials. We will call a polynomial P *combinatorially non-recurrent* if the orbit of the critical value end $b = B_0 \supset B_1 \supset \cdots$ never enters C_N for some N.

PROPOSITION 7.6. If a polynomial P is non-recurrent, then all its ends are points. In particular, K_P is locally connected.

PROOF. For the off-critical ends, in particular the critical value end, this is proved in Proposition 7.5.

The impression of the critical end is also a point, since for any annulus in the critical value end, its inverse image in the critical end has half its modulus.

For all other ends, the argument goes as in 7.2.

REMARK 7.7. (a) In the case where the critical point is strictly preperiodic, Proposition 7.6 was already known [DH1], using the existence of the *orbifold metric* on J_c.

(b) Yoccoz claims that such an expanding metric exists in this case as well.

8. Periodic tableaux

The critical tableau of quadratic polynomial P may have the kth column entirely critical for some $k > 0$. In that case, the tableau of the polynomial is periodic by rule (Tb), and the critical nest is certainly convergent. This case can be analyzed just like the periodic case of [BH2].

PROPOSITION 8.1. Let a polynomial P have a tableau periodic of least period k, and let N be so large that $X_{n,l} \neq C_n$ when $n > N$ and $0 < l < k$. Then the mapping

$$f = P_c^k|_{C_{N+k}} : C_{N+k} \to C_N$$

is polynomial-like of degree 2, and the critical point of f does not escape.

Conversely, if a quadratic polynomial is k-renormalizable and k is the smallest integer with this property, then there exists N such that $P_c^k(C_n) = C_{n-k}$ for all $n > N$.

PROOF. The first part is obvious. For the second part, observe that if P is renormalizable, then the domain of the renormalization can be chosen to contain the critical point of P. Let K' is the filled-in Julia set of the renormalized mapping. Clearly no preimage of $\alpha(c)$ belongs to K', unless $\alpha \in K'$; our extra condition on renormalizability implies that if this occurs, then α and its inverse images do not disconnect K', so that $K' \subset \cap C_n$, which is the desired conclusion.

COROLLARY 8.2. (a) If a polynomial is renormalizable, then the critical end of its puzzle is homeomorphic to $K_{P'}$ for some quadratic polynomial P' with connected Julia set.

(b) All ends of the puzzle are then points except for the preimages of the critical end.

PROOF. This follows immediately from Propositions 8.1 and 7.2.

9. Divergence criteria for tableaux

There are two essential differences between quadratic tableaux and the ones for cubics: some annuli are degenerate, and others contain the critical point in their interiors. These present the main difficulty, because we cannot compute their moduli from those of their images.

9.2 Weighting a marked grid. In this section we will describe a combinatorial way of assigning weights to the entries of the critical nest. These will be lower bounds for the moduli of the corresponding annuli. Throughout this section we will consider the critical tableau of some quadratic polynomial.

Consider the function $t : \mathbf{N}^* \to \mathbf{N}$, defined as follows:

$$t(n) = \begin{cases} i & \text{where } i \text{ is maximal with } (i, j) \text{ critical, } i + j = n, \text{ and } j > 0 \\ 0 & \text{if there is no such integer.} \end{cases}$$

REMARK 9.2. The function t, together with the first two tableau rules, completely determines the marked grid. Yoccoz writes his combinatorics entirely in terms of the function t.

We find it difficult to deal with bare combinatorics, so we have devised the following suggestive language to help describe tableaux. We will say that $t(n)$ is the parent of n so that the set

$$\{ m \in \mathbb{N} \mid t(m) = n \}$$

is the set of children of n. Children can be good or bad: we will say that m is a *good child* of $t(m)$ if the position $(t(m), m - t(m))$ is strictly critical, and a bad child if $(t(m), m - t(m))$ is semi-critical. The reason for this distinction is the following lemma.

LEMMA 9.3. If m is a good child of $t(m)$, then the annulus $C_m - C_{m+1}$ is a double cover of $C_{t(m)} - C_{t(m)+1}$.

PROOF. The mapping $P^{m-t(m)}$ makes the piece maps C_m and C_{m+1} as ramified double covers to the pieces at the positions $(t(m), m - t(m))$ and $(t(m)+1, m-t(m)+1)$ respectively. The piece at position $(t(m), m-t(m))$ is $C_{t(m)}$ in any case, but the piece at position $(t(m) + 1, m - t(m) + 1)$ is $C_{t(m)+1}$ only if $(t(m), m - t(m))$ is strictly critical, i.e., if m is a good child. It follows that if m is a good child, then $P^{m-t(m)}$ maps the annulus A_m to the annulus $A_{t(m)}$ as an unramified double cover.

Choose $n_0 \geq 0$, and define weights $w_{n_0}(n)$ as follows:

$$w_{n_0}(n) = \begin{cases} 0 & \text{if } n \text{ is not a descendant of } n_0 \\ 0 & \text{if } n \text{ is a bad child of } t(n) \\ 1 & \text{if } n = n_0 \\ \frac{1}{2} w_{n_0}(t(n)) & \text{if } n \text{ is a good child of } t(n). \end{cases}$$

In the genealogical language above, the level n_0 is the founder of the dynasty who amassed the original wealth, and among his descendants good children inherit half the wealth of their parents, whereas bad children inherit nothing.

The justification for this rule is the following lemma.

LEMMA 9.4. If mod $A_{n_0} = M$, then mod $A_n \leq M w_{n_0}(n)$ for all $n \geq 0$.

PROOF. This is immediate from Lemma 9.3.

In Theorem 9.5 we will show that in the recurrent non-periodic case, the total fortune of all levels is infinite. Define the *generation* of a level n to be that i such that $t^i(n) = n_0$. The obvious idea is that each level should have at least two good children. If that were the case, then the fortune would increase, or at least be preserved from generation to generation, and summing over generations would yield an infinite total fortune. Unfortunately, although all levels must indeed have two children, sometimes there will be bad ones among them, and accordingly the fortune can decrease from one generation to the next. The notion of nobility (genetic goodness) is defined to address this problem.

THEOREM 9.5. A critical tableau satisfying rules (Ta), (Tb), (Tc), and which is non-recurrent, with weighting w_{n_0} for some n_0, is convergent if and only if it is periodic.

LEMMA 9.6. Every level has at least two children.

PROOF. See [BH], Lemma 4.6.

Define a level to be noble if it contains no semi-critical positions.

LEMMA 9.7. (a) Every level has at least one good child.
(b) An only good child is noble.
(c) Children of a noble parent are noble and good.

PROOF. (a) For any level n, let (n, k) be the first strictly critical position on the nth row with $k > 0$; by recurrence there is such a position. By rule (Ta) there are no critical positions of the form $(n+i, k-i)$ with $0 < i < k$, so $n+k$ is a good child of n.

(b) Suppose that m is a good child of n and that (m, k) is semi-critical, with k minimal for this property. Then there are no strictly critical positions of the form $(m+i, k-i)$ with $0 < i < k$, because by rule (Ta) they would give rise to a semi-critical position (m, k') with $0 < k' < k$.

By rule (Tc), we find that the position $(n, k+m-n)$ is also semi-critical, showing that n is not noble; thus noble parents have only noble children. Moreover, there are no critical positions of the form $(l, k+m+1-l)$ with $l > n$. Now find the first strictly critical position (n, p) with $p > k+m-n+1$; the level $n+p$ is another good child of n.

Since children of noble parents are good by the definition of nobility, all parts of Lemma 9.7 are proved.

REMARK 9.8. The proof above actually shows that any youngest good child is noble; the child need not be an only good child.

PROOF OF THEOREM 9.5. Let us define the modified fortune $w'_{n_0}(n)$ of the level n to be $w_{n_0}(n)$ if n is not noble, and $2w_{n_0}(n)$ if n is noble. Then the modified fortune of the Nth generation

$$W(N) = \sum_{G(n)=N} w'_{n_0}(n)$$

is non-decreasing: each level of generation N either is:

- not noble and has at least two good children, in which case the modified fortune of the children is at least that of the parent, or
- not noble and has a single good child, in which case the modified fortune of the child is equal to that of the parent;
- noble, in which case it has at least two children by Lemma 9.6 both of which are noble and good by Lemma 9.7 (c). Thus their modified fortune equals that of their parent.

Since the generations are disjoint, and generation n_0 has a positive fortune, the total modified fortune is infinite. But the total fortune is at least half the total modified fortune, hence it is also infinite.

10. The proof of Theorem II.A for finitely-renormalizable polynomials

To relate the combinatorics of tableaux to properties of polynomials, we need to weight a tableau, choosing n_0 so that the corresponding annulus in the nest is non-degenerate. In this section we will explain how this is done.

PROPOSITION 10.1. Let P be a non-renormalizable quadratic polynomial and suppose α has combinatorial rotation number p/q. Then there exists a smallest integer ν such that the entry $(0, q\nu)$ in the critical tableau of P is the first semi-critical entry in the 0th row, and the annulus at depth $q\nu$ in the critical nest is non-degenerate.

PROOF. Since P is not renormalizable, there exists k such that the position (k, q) is semi-critical. It follows from rules (Tb) and (Td) that the critical positions on the line $\{(i, j) \mid i + j = q + k\}$ are precisely the ones for which j is a positive multiple of q, and all these positions are semi-critical by rule (Tc). The positions (i, jq) were then all strictly critical for $jq < q + k$, and rule (Td) says that $k+q$ is a multiple of q. If we set $\nu = (k+q)/q$, then the νq-th position is the first semi-critical position in the 0th row.

Any semi-critical position at level 0 corresponds to a non-degenerate annulus. If any entry of a diagonal corresponds to a non-trivial annulus, then all the entries of that diagonal do. The result follows.

We will call the number ν the *rank* of P. Although it is not essential to our purposes, we will describe the meaning of the rank in the parameter space in section 15.

We can give an estimate on the annulus above. For every $c \in M - M_0$, denote by $Z[c]$ the union of the non-critical pieces at level 1 contained in the critical piece $C_0[c]$ at level 0.

PROPOSITION 10.2. (a) For every $p/q \in \mathbb{Q}/\mathbb{Z}$ with $p/q \neq 0$, and every neighborhood V of the root of the limb $M_{p/q}$, there exists $K > 0$ such that

$$\operatorname{mod}(C_0[c] - Z[c]) \geq K \qquad \text{for all } c \in M_{p/q} - V.$$

(b) If $c \in M_{p/q} - V$ has rank ν, then the annulus at depth $q\nu$ in the critical nest has modulus at least $K/2^\nu$.

PROOF. Part (a) simply follows from the fact that on the complement of a neighborhood of the root, both $C_0[c]$ and $Z[c]$ vary continuously with c.

Part (b) is an immediate consequence of the following lemma:

LEMMA 10.3. If U, U' are simply connected Riemann surfaces, $f : U' \to U$ is a proper analytic mapping of degree d (i.e., a ramified cover), $Z \subset U$ is a connected, simply connected compact subset and $Z' \subset U'$ is a component of $f^{-1}(A)$, then

$$\operatorname{mod}(U' - Z') \geq \frac{1}{d} \operatorname{mod}(U - Z).$$

PROOF. This is easy from the characterization 13.1 of the modulus by extremal length. We can also assume that $U = D_1$ the unit disc and $Z = \overline{D}_r$. Let r_i be the absolute values of the critical values of f, in ascending order. Then the region

$$A_i = \{ z \in D_1 \mid r_i < |z| < r_{i+1} \}$$

lifts to a collection of annuli in $U' - Z'$, each of which covers its image with degree at most d, and at least one of which surrounds Z'; call such an annulus A_i'. Then

$$\operatorname{mod}(U - Z) = \sum \operatorname{mod} A_i \leq d \sum \operatorname{mod} A_i \leq d \operatorname{mod}(U' - Z').$$

REMARK 10.4. We don't know that excluding V in part (a) is necessary, but the external rays which land at $\alpha(c)$ do not vary continuously in a neighborhood of the root, so that the easy proof above does not go through without modification. When we come to use Proposition 10.2 in 13.2, we will have another reason to exclude such a neighborhood, which really is necessary, so we don't lose much generality by doing it now.

Let $c \in M_{p/q}$ be a polynomial of rank ν, and weight its tableau by $w_{q\nu}$. Motivated by Lemma 9.4, and Theorems 9.5 and 10.2, we make the following definitions.

DEFINITION 10.5. (a) The nth annulus in the critical nest is *contributing* if its weight is non-zero.

(b) Its *combinatorial modulus* is then $2^{-\nu} w_{\nu q}(n)$.

11. The proof of Theorem II.A for finitely-renormalizable polynomials

We are now in a position to prove Theorem II. Consider first the non-renormalizable case. By Theorem 7.6 we may assume that the polynomial is recurrent. By Theorem 6.1 is enough to show that all nests are divergent, and by Theorem 7.2 it is enough to show that the critical nest is divergent. By Lemma 9.4 if we can show that there is one non-trivial annulus in the critical nest, then the divergence of the series of combinatorial weights implies the divergence of the critical nest. This last divergence is proved in 9.5, and the existence of a non-degenerate annulus is proved in Proposition 10.1.

This finishes the proof of Theorem II in the non-renormalizable case. In order to treat the finitely renormalizable case, we first want to abstract a little the setting in which tableaux arise.

Let U be a Riemann surface, $U' \subset U$ a relatively compact open subset, $f: U' \to U$ a proper mapping with a single ordinary critical point ω.

PROPOSITION 11.1. Let $\Gamma \subset U$ be a closed subset such that

- $f(\Gamma) \subset \Gamma$;
- the components of $U - \Gamma$ are simply connected; and
- the orbit of ω does not intersect Γ.

If puzzles, nests, ends and tableaux are defined exactly as in sections 5 and 6, then these tableaux satisfy rules (Ta), (Tb), (Tc).

In particular, if there is one non-degenerate annulus in the critical nest, and if the critical tableau is recurrent non-periodic, then the impressions of all ends are points.

PROOF. We have simply abstracted the properties actually used in the proofs above.

Let P be a finitely renormalizable polynomial. We will define a new puzzle adapted to the situation, satisfying the conditions of Proposition 11.1. Let $Q = P^k|_U: U \to U'$ be the "last" renormalization of P, and so that the polynomial Q' hybrid equivalent to Q is not renormalizable. For definiteness sake let U be chosen so that the critical point of P is in U.

Let α_P be the fixed point of P with non-zero combinatorial rotation number (as before), and let α_Q be the fixed point of Q corresponding to the fixed point of Q' with non-zero combinatorial rotation number. We define the graph $\Gamma = \Gamma' \cup \Gamma''$ as follows:

(1) $\Gamma' = \Gamma'_0 \cup \cdots \cup \Gamma'_{k-1}$ is the union of the external rays of K landing at the points $\alpha_Q, P(\alpha_Q), \ldots, P^{k-1}(\alpha_Q)$.

(2) $\Gamma'' = \Gamma''_0 \cup \cdots \cup \Gamma''_l$ where Γ''_0 is the graph formed by the union of the rays landing at α_P, $\Gamma''_n = P^{-1}(\Gamma''_{n-1})$, and l is the smallest number

such that one component of Γ''_l separates Γ'_1 (the component of Γ' associated to the critical value) from all other components of Γ' and of Γ''_m with $m < l$. Because of the density of inverse images of α_P, there is such a number l.

Restrict the mapping to a region U bounded by some equipotential. The pair U, Γ satisfies the conditions of Proposition 11.1.

Moreover, the argument of Proposition 10.1 carry over to this situation. Let q be the denominator of the combinatorial rotation number of Q at α_Q. In the critical tableau, every kqth position in the 0th row will be strictly critical until one reaches a first semi-critical position $(0, \nu kq)$. The graph was carefully chosen so that the non-critical pieces at level 1 contained in the critical piece at level 0 and which intersect K_Q are contained in the interior of the critical piece at level 0.

Therefore the critical piece at depth νkq contains the next in its interior, giving a non-degenerate annulus in the critical nest.

We leave to the reader to check that if the tableau is non-recurrent an argument similar to Proposition 7.6 goes through. The tableau cannot be periodic, since in that case Q would be renormalizable.

This finally proves Theorem II.

Part III. Annuli in Parameter Space

The object of this part is to prove the following theorem, also due to Yoccoz (with a substantially different proof):

THEOREM III. Let $c \in M$ be a point such that P_c is not infinitely renormalizable. Then c has a basis of closed connected neighborhoods in M.

If P_c has an attractive cycle, it is in the interior of M so of course has a basis of connected neighborhoods in M. If P_c has an indifferent cycle with multiplier different from 1, the result was proved in Theorem I.C. We will first prove the result for polynomials to which the tableau argument applies, and in fact we will give the proof only for non-renormalizable polynomials, and leave the finitely renormalizable case to the reader. We will then prove the case where P_c has an indifferent cycle with multiplier 1, using ideas from tableaux and the theory of Mandelbrot-like families.

This leaves the case of non-recurrent polynomials. There are three rather different proofs available in that case. Yoccoz has a proof using an expanding metric on the Julia set.

Theorem III will be proved by showing that the parameter plane carries an analog of the puzzle, which we will call the parapuzzle. Moreover, for any $c \in M_{p/q}$, the critical value nest, i.e., the nest of c in the puzzle of P_c, and the nest of c in the parapuzzle are "homeomorphic". In particular, to each annulus A surrounding c in the dynamical plane there corresponds an annulus MA in the parameter space also surrounding c; and these annuli have disjoint interiors. This does not simply make Theorem III trivial, however: A and MA do not have the same modulus. Yoccoz manages to estimate the modulus $\operatorname{mod} MA$ in terms of $\operatorname{mod} A$; we will also do this, by a different argument using extremal length.

12. The $M_{p/q}$ parapuzzle

Let M_0 be the cardioid of M, i.e., the set of c such that P_c has an attractive fixed point. Every limb of M_0 has a root on the cardioid, at which two external rays of angles $\theta^+_{p/q}$, $\theta^-_{p/q}$ land. Let $W_{p/q}$ be the wake of $c_{M_0,p/q}$, i.e., the closed region in the c-plane cut out by these rays. Recall that for all $c \in \operatorname{int} W_{p/q}$, the repelling fixed point $\alpha(c)$ is the endpoints of exactly q external rays, with angles $2^i\theta^+_{p/q}$, $i = 0, \dots, q-1$.

Emphasizing the dependence on parameters, for each $c \in \operatorname{int} U_{p/q}$ we set

$$U_0[c] = \{\, z \in \mathbb{C} \mid |h_c(z)| < R \,\}$$

as in section 5, and

$$\Gamma_0[c] = \partial U_0[c] \cup \text{ the external rays landing on } \alpha[c].$$

Further, as in 5 define

$$U_n[c] = P_c^{-1}(U_{n-1}) \quad \text{and} \quad \Gamma_n[c] = P_c^{-1}(\Gamma_{n-1}).$$

REMARK 12.1. The use of square brackets is designed to avoid the following ambiguity: in Part II, we denoted by $X_i(z)$ the piece of the puzzle of P at depth i containing z. The piece will now be denoted $X_i[c](z)$, with the square bracket indicating the polynomial the puzzle of which is being used.

To transfer this data to the parameter space, we will use a very general way of defining a locus in the parameter space when we already have a locus in each dynamical plane.

DEFINITION 12.2. The *p/q-parapuzzle* is the sequence of regions $MU_0 \supset MU_1 \supset \cdots$ in the parameter plane, with the graphs $M\Gamma_0, M\Gamma_1, \ldots$, where

$$MU_i = \{ c \mid c \in U_i[c] \}$$

and

$$M\Gamma_i = \{ c \mid c \in \Gamma_i[c] \}.$$

The Figures on pages 377 and 378 show various parts of a $1/3$-parapuzzle.

This definition is perfectly precise, but perhaps not as explicit as one might wish. The following description may help to understand it better.

PROPOSITION 12.3. We have

$$MU_n = \left\{ c \in U_{p/q} \;\middle|\; h_M(c) \le \frac{R}{2^n} \right\}$$

and $M\Gamma_n$ is a graph, composed of those values of c such that $P_c^n(c) = \alpha[c]$ and for each of these the q external rays of M which land there.

PROOF. The first part follows immediately from the formula $h_c(c) = h_M(c)$. The points where $P_c^n(c) = \alpha[c]$ are clearly in $M\Gamma_n$ since $\alpha[c] \in \Gamma_n[c]$, and these are the only points of $M\Gamma_n \cap M$.

We will call $\mathcal{M}_n$ the set of closures of components of $MU_n - M\Gamma_n$; in keeping with the notation of Part II, $MX_n(c)$ will denote the piece of the parapuzzle at depth n containing c.

Of course, there is no dynamics in the parameter plane, and the parapuzzle is not "self-similar". On the contrary, all quadratic puzzles are reflected in the parapuzzle. Everyone has observed that near $c \in \partial M$, the set M "looks like" K_c near c. There are many ways of making this precise; the way of saying it for puzzles is the following:

PROPOSITION 12.4. For every $c \in M_{p/q}$, with nest $MX_0(c) \supset MX_1(c) \supset \cdots$, we have:

(a) $\Phi_M(MX_i(c) - M) = \phi_c(B_i[c] - K_c)$;

(b) the analytic isomorphism $\Phi_M^{-1} \circ \phi_c \colon B_i[c] - K_c \to MX_i(c) - M$ extends to a homeomorphism $\partial B_i[c] \to \partial MX_i(c)$;
(c) if $\partial B_i[c] \cap \partial B_{i'}[c] = \emptyset$, then $\partial MX_i \cap \partial MX_{i'} = \emptyset$.

PROOF. For all c in the interior of a piece MX of the parapuzzle at depth i, the point c is not in the graph $\Gamma_i[c]$. In particular, this graph is "structurally stable" for c in that region, made up of rays at the same angles, and with rays at the same angles coming together. The boundary of this piece is formed by values of c where c is on the closures of the rays forming the boundary of the critical value piece. Thus the boundary of X and of the critical value piece for all $c \in \operatorname{int} X$ are made up of the same pieces of the rays at the same angles. This proves (a) and (b), and (c) follows without difficulty.

REMARK 12.5. We have carefully abstained from claiming that the isomorphism $\Phi_M^{-1} \circ \phi_c$ extends to a homeomorphism $B_i[c] \to MX_i(c)$. This is false; the topology, never mind the complex structure, of these sets do not coincide. Part (b) only guarantees an extension to finitely many points.

13. Adding disks to the parapuzzle

Choose $c \in M$. On the complements of M and K_c, the nest of c in the parapuzzle and in the puzzle $\mathcal{P}[c]$ coincide. However, the corresponding pieces are glued together in different ways, to it is unclear that there is any relation between the annuli in the dynamical plane and the parameter plane. Of course, the pieces are also all glued together differently in the dynamical plane for different c's.

In this section we will demonstrate a way to get uniform estimates on the moduli of annuli in the parameter space in terms of those in the dynamical plane.

First we need to recall the definition of the modulus of an annulus using extremal length. Given a conformal measurable metric $\rho = \rho(z)|dz|$, we define Area_ρ to be the area of A for the associated element of area $\rho^2 dx\, dy$, and if $\gamma \colon I \to \mathbb{C}$ is a rectifiable curve, we set $\text{Length}_\rho(\gamma)$ the length with respect to the metric.

Let A be an annulus, and let S be the family of rectifiable curves joining one boundary component to the other (one end to the other if there is no obvious boundary).

PROPOSITION 13.1. The modulus of A is given by the formula

$$\operatorname{mod}(A) = \sup_{\substack{\text{conformal} \\ \text{metrics } \rho}} \frac{\left(\inf_{\gamma \in S} \operatorname{Length}_{\rho}(\gamma)\right)^2}{\operatorname{Area}_{\rho}(A)}.$$

PROOF. This is a standard "Grötzsch"-type proof. We may as well assume that $A = B_h/\mathbb{Z}$, here $B_h = \{ z \in \mathbb{C} \mid 0 < \operatorname{Im} z < h \}$, so that $\operatorname{mod} A = h$. In the obvious coordinate $z = x + iy$, we can write

$$\begin{aligned}
\operatorname{Area}_{\rho}(A) &= \int_0^1 \left(\int_0^h \rho^2(x, y)\, dy \right) dx \\
&= \frac{1}{h} \int_0^1 \left(\int_0^h \rho^2(x, y)\, dy \int_0^h 1^2\, dy \right) dx \\
&\geq \frac{1}{h} \int_0^1 \left(\int_0^h \rho(x, y)\, dy \right)^2 dx \\
&\geq \frac{1}{h} \left(\inf_{\gamma \in S} \operatorname{Length}_{\rho}(\gamma) \right)^2.
\end{aligned}$$

This proves the result, since it shows that the modulus h is at least the required ratio, but the euclidean metric clearly realizes equality.

So to get a lower bound on the modulus of an annulus (usually what one wants), we need to find a conformal metric, together with an upper bound for the area, and a lower bound for the length of curves joining the boundary components.

Theorem III follows from the somewhat stronger Proposition 13.2. Choose a neighborhood V of the root of $M_{p/q}$, and recall the Definition 10.5 of a contributing annulus and its combinatorial modulus.

PROPOSITION 13.2. There exists a constant C, depending only on p/q and V, such that for every $c \in M_{p/q} - V$, and each contributing annulus $B_i[c] - B_{i+1}[c]$ of combinatorial modulus $1/2^m$, the corresponding annulus $MX_i(c) - MX_{i+1}(c)$ in the nest of c for the parapuzzle has modulus at least $C/2^m$.

The ideas of the proof below are largely due to Douady and Faught.

PROOF. We will use a metric $\mu = \mu_1 + \mu_2$, where

$$\mu_1 = 2^{m+\nu} |d \log \phi_M|,$$

as one might expect. From Proposition 12.4, we see that the area of $MX_i(c) - MX_{i+1}(c)$ for μ_1 is then bounded above by $2^m C$, where C is a constant depending only on p/q and R. Also, the μ_1-length of any curve joining $MX_{i+1}(c)$ to $\partial MX_i(c)$ without entering M is bounded below by $\inf\{(\log R)/2, \ell_{p/q}\}$, where $\ell_{p/q}$ is the smallest difference of the angle of a ray landing on $\alpha[c]$ and the angle of a ray landing on $-\alpha[c]$. These angles depend only on the limb $M_{p/q}$ containing c.

The metric μ_2 will be $\mu_2 = |d \log(P_c^i(c)) - \alpha[c]|$, but we must be careful to restrict it to an appropriate region.

First choose $r > 0$ such that for all $c' \in W_{p/q}$, the disc $D_r(\alpha[c'])$ is contained in $U_0[c']$ and does not intersect the $q-1$ non-critical pieces of depth 1 in the critical piece $C_0[c']$.

Next choose $r_1 < r$ and set

$$A = \inf_{c \in M_{p/q} - V} \; \inf_{i=1,2} \; \inf_{\{z \in R_c(\theta_i) \cap (\mathbb{C} - D_{r_1})\}} h_c(z).$$

We have $A > 0$, because for $c' \in M_{p/q} - V$, the external rays landing at $\alpha[c']$ depend continuously on c', as parameterized curves. However, this is *not true* at the root of $M_{p/q}$, and if V had not been removed, the number A would vanish. This is the only place in the proof where V plays a role.

Define the subsets $\mathcal{D}_\rho$ and $\mathcal{C}_0$ of $\mathbb{C}^2$ by the formulas

$$\mathcal{D}_\rho = \{\, (c', z) \mid c' \in W_{p/q} \text{ and } z \in D_\rho(\alpha[c'])\,\}$$

and

$$\mathcal{C}_0 = \{\, (c', z) \mid c' \in W_{p/q} \text{ and } z \in C_0[c']\,\}.$$

LEMMA 13.3. There exists a constant L and an analytic branch

$$\Phi \colon \mathcal{D}_r \cap \mathcal{C}_0 \to \mathbb{C}$$

of $\log(z - \alpha[c'])$ such that

$$|\operatorname{Im} \Phi(c', z)| \le L$$

for $(c', z) \in \mathcal{D}_r - \mathcal{D}_{r_1}$.

PROOF. The existence of Φ simply reflects the fact that for each $c' \in W_{p/q}$, the "sector" of $D_r(\alpha[c'])$ cut out by $C_0(\alpha[c'])$ does not separate $\alpha[c']$ from infinity. The bound L follows immediately from compactness.

REMARK 13.4. The sector mentioned above changes with c', and we cannot guarantee $L < 2\pi$, or even that the intersection of the image of Φ with vertical lines has length at most 2π. The number $L - 2\pi$ measures the difference of the rates at which external rays spiral around $\alpha[c']$

Suppose $MX_i(c)$ is a piece of the parapuzzle such that $X_i(c) - X_{i+1}(c)$ is a contributing annulus, of combinatorial modulus $1/2^m$. Then for all $c' \in MX_i(c)$, the piece $B_i[c']$ maps by $P_{c'}^i$ to $C_0[c']$, with degree m, and with $B_{i+1}[c']$ mapping to a non-critical piece.

Thus $\partial B_i[c']$ contains precisely 2^m inverse images of $\alpha[c']$, and the puzzle at level $i+1$ inside $B_i[c']$ cuts out a number of relatively compact subpieces, which may map either with or without ramification to one of the non-critical pieces of level 1 contained in $C_0[c']$. Moreover, the images of these pieces do not intersect $D_r(\alpha[c'])$, so the inverse image

$$P_{c'}^{-i}(D_r(\alpha[c']))$$

consists of various components, which may contain one or more inverse images of $\alpha[c']$. A sketch of what such a piece might look like is represented in Figure 13.5.

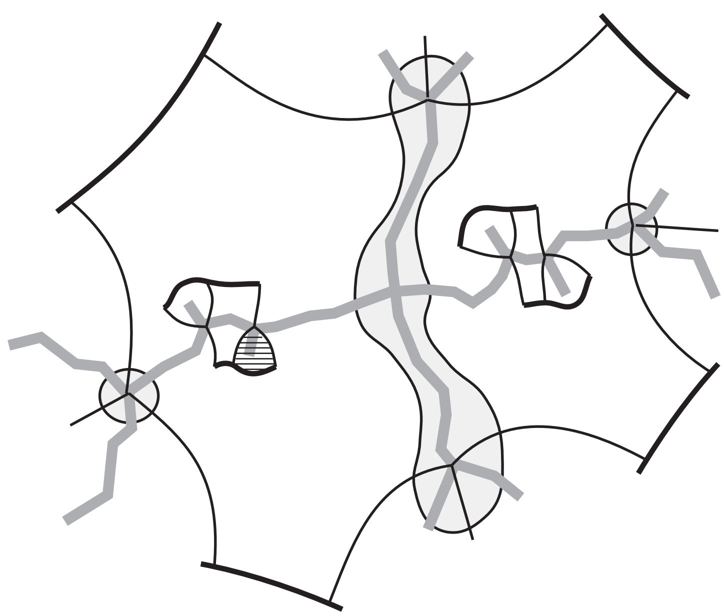


Figure 13.5.. A contributing annulus.

This picture will serve just as well for the parapuzzle piece $MX_i(c)$. This time the 2^m points of $MX_i(c) \cap M$ are points c' such that $P_{c'}$ is a polynomial for which the critical point is strictly preperiodic, and the piece $MX_{i+1}(c)$ *does not intersect any inverse image of* D_r *under the mapping* $p_i : c \mapsto P_c^i(c) - \alpha(c)$.

For each component U of

$$p_i^{-1}(D_r) \cap MX_i$$

containing a point of $\partial MX_i \cap M$, let us choose a measurable section s_U of $\phi_i : c \mapsto \Phi(c, p_i(c))$, as follows.

For each point of $M \cap \partial MX_i(c_0)$, and almost every $s \in [r_1, r]$, choose a component γ_s of $(\log|p_i(c)|)^{-1}(s)$ which is an arc separating that point from $MX_{i+1}(c_0)$. This is possible, because except for finitely many values of s (p_i is an algebraic fucntion, and has only finitely many critical values) the locus $(\log|p_i(c)|)^{-1}(s)$ is a union of arcs separating all the points of $M \cap \partial MX_i(c_0)$ from $MX_{i+1}(c_0)$. Actually, the choice of γ_s can be made measurable (in fact, continuous except on finitely many curves), so as to define a section of ϕ_i.

We are finally able to define μ_2: on each U as above, it is $d \log p_i$ on the image of s_U, extended by 0.

REMARK 13.6. We constructed a section of ϕ_i, because we do not know whether the more natural idea of a section of p_i exists. As far as we know, p_i might fail to be injective on a curve γ_s, and the consideration of the logarithm may be essential.

We do not know whether p_i is injective on U; it might have critical points. Thus if we had failed to restrict μ_2 to the image of s_u, we would not know how to bound the area. It turns out that this doesn't matter; the μ_2 as defined does just as well.

LEMMA 13.7. The area of U for μ_2 is at most $2L(\log r/r_1)$.

PROOF. The support of $\mu_2 \cap U$ maps injectively by ϕ_i to th region

$$\{ z \in \mathbb{C} \mid \log r_1 \le \operatorname{Re} z \le \log r \text{ and } |\operatorname{Im} z| \le L \}.$$

Thus the area of $MX_i - MX_{i+1}$ for $\mu = \mu_1 + \mu_2$ is bounded above by a constant times 2^m, where the constant depends only on c (or more precisely on p/q, ν, r, r_1, and L.

Thus the following lemma ends the proof of the theorem.

LEMMA 13.8. Any curve joining MX_{i+1} to ∂MX_i has length at least $\inf\{A, \ell_{p/q}, \log r/r_1\}$.

PROOF. This is clear from our definition of the section. Any curve joining MX_{i+1} to ∂MX_i must hit the boundary at a point c' with some potential. Either this potential is lower than A, and then c' belongs to a component of $p_i(D_r - D_{r_1})$ containing a point of $\partial MX_i \cap M$, and the curve must have cut all the curves γ_s separating that point from MX_{i+1}, and hence must have length at least $\log r/r_1$. Otherwise, either this path intersects M, in which case its length is at least A for μ_1 since it must climb from potential 0 to potential A, or it doesn't intersect M, in which case its length is at least $\ell_{p/q}$ for μ_1.

14. Three further results

To complete the results promised in the introduction, we need three further theorems. Each is important in its own right, but we have shown similar results earlier in the paper, and will only sketch the arguments; therefore we will collect these rather dissimilar statements in a single section.

14.1 Non-recurrent polynomials in the parameter space. Let $c \in M_{p/q}$ be a point such that P_c is a combinatorially non-recurrent polynomial.

THEOREM 14.2. The parapuzzle pieces $MX_i(c)$ form a basis of neighborhoods of c.

PROOF. In Proposition 7.6, we showed that there is an infinite sequence of disjoint annuli in the critical nest with moduli bounded below. Precisely the same method used in the proof of Proposition 13.2 allows us to show that the corresponding sequence of annuli in the parapuzzle nest of c also have moduli bounded below.

14.3 Finitely renormalizable polynomials in the parameter space. In this subsection we will transfer the arguments of section 11 to the parameter space, only sketching the arguments.

THEOREM 14.4. Let $c \in M$ be a point such that P_c is finitely renormalizable. Then c has a basis of connected neighborhoods in M.

PROOF. Recall from section 11 the definition of $\Gamma = \Gamma' \cup \Gamma''$. Two of the rays of Γ'_1 bound the region containing the critical value c, and the corresponding rays in the parameter space land at the same point, and cut out a region W of the parameter plane such that for all $c' \in W$, the rays with the same angles as those of Γ form a puzzle. We can form in W an enriched parapuzzle by the same construction as above. The arguments as were used to prove Proposition 13.2 can be slightly modified to the new parapuzzle, to prove Theorem 14.4.

14.5 Roots of primitive components of M. There is one more result announced in the introduction which remains to be proved.

THEOREM 14.6. If $c \in M$ is a point such that P_c has an indifferent cycle with multiplier 1, then c has a basis of connected neighborhoods in M.

PROOF. The point c is the root of a primitive component U of the interior of M; let us first suppose that the center c_0 of U is exactly once renormalizable, and call M' the renormalized copy of M centered at c_0.

Choose a critical piece $C_m[c_0]$ of the puzzle for c_0 such that $P_{c_0}: C_m[c_0] \to C_{m-k}[c_0]$ is polynomial like of degree 2 and represents the renormalization. Then the piece $MX_m(c_0)$ parameterizes a Mandelbrot-like family, and M' is precisely the connectedness locus for this family. In other words,

$$M' = \bigcap_{n \geq m} MX_n(c_0).$$

Now consider the points on ∂U with internal angles $\pm 1/n$, and choose for each a ray landing there (we know there are exactly two). Call these rays $R_{\pm n}$, and join their landing points by the geodesic γ_n in U. Then the union of the rays $R_{\pm n}$ and of γ_n forms a curve cutting $\mathbb{C}$ into two parts; let V_n be the part containing c.

We now claim that the sets $MX_n(c_0) \cap V_n \cap M$ form a basis of connected neighborhoods of c in M. They are clearly connected; it is enough to show that the intersection of all these neighborhoods is $\{c\}$. The intersection must be in M', since this is the intersection of the pieces; but by Proposition 4.2 the diameters of $V_n \cap M'$ tend to 0, since these consist of a small piece of U, and limbs which are attached at points with interior angles with large denominators.

This settles the case where c_0 is once renormalizable. If it is several times renormalizable, we must as above restrict to an appropriate part of the parameter space, in which we can use a richer puzzle and parapuzzle. We leave the details to the reader.

15. Sublimbs of $M_{p/q}$ and ranks of polynomials

In this section we will develop some further combinatorial properties of M; in that sense it is a continuation of section 4. There we studied the components of the complement of a hyperbolic component; here we will study the components of the complement of a renormalization of M.

Let U be a hyperbolic component of int M, and c_U the center of U. Let M_U be the renormalized copy of M centered at c_U, and $c_{U,t}$ the renormalization by c_U of the point at external angle t of ∂M; this makes sense for $t \in \mathbb{Q}/\mathbb{Z}$ by [DH1] (and for many irrational values of t also, by the results of this paper).

The polynomial $P_{c_{U,0}}$ has an indifferent cycle whose multiplier has a rational argument. This argument is equal to 1 if and only if c_0 is the cusp of a primitive component of int M.

We will assume that U is not the main cardioid M_0 of M.

PROPOSITION 15.1. If $t \neq 0$ is a dyadic angle, then $c_{U,t}$ is the extremity of precisely q external rays of M if $q \geq 2$, and of 2 external rays if $q = 1$.

PROOF. For c in the wake of $c_{U,0}$, there is a unique repelling cycle $\zeta_0(c)$, $\dots$, $\zeta_{k-1}(c)$ depending analytically on c, which at $c_{U,0}$ merges with the attracting cycle in U.

The combinatorial rotation number of P^k at a point of this cycle is constant throughout the wake of $c_{U,0}$, and since the same rays land at the indifferent cycle of P_{c_0}, we see that this rotation number is p/q. Moreover, precisely q rays land at each point of the cycle when $q > 1$, and precisely 2 if $q = 1$.

For $c \in M_U$, the points of the repelling cycle $\zeta_0(c), \dots, \zeta_{k-1}(c)$ are the "external" fixed points of the renormalizations. Note that there are qk renormalizations, and that q renormalizations share the same "external" fixed point.

By definition, under the polynomial $P_{U,t}$ the critical value lands after mqk iterations on a point of the cycle, where m is the power of 2 in the denominator of t. Therefore the same number q of external rays land at the critical value (in the dynamical plane) as at a point of the cycle. According to [DH1], this is also the number of rays which land at $c_{U,t}$ in the parameter space, in fact they are the same rays.

PROPOSITION 15.2. The set $(W_{c_{U,0}} \cap M) - M_U$ consists of components $S_{U,t,j}$, where $t \in \mathbb{Q}/\mathbb{Z}$ runs through dyadic angles different from 0, and

$$j = \begin{cases} 1 & \text{if } U \text{ is primitive} \\ 1, \ldots, q & \text{otherwise.} \end{cases}$$

The subset $S_{U,t,j}$ touches M_U at $c_{U,t}$, and for fixed t, the $S_{U,t,j}$ are separated by the rays landing at $c_{U,t}$. To make things definite, we may number these counterclockwise from the sector containing M_U.

PROOF. For any $c \in W_{c_{U,0}}$, consider in the dynamical plane the two rays with the same external angles as $c_{U,0}$ in the parameter plane. These two rays touch at the point $\zeta_1(c)$, and cut the plane into two pieces, one of which contains the critical value. Call W_1 that component, form W_2 by adding to W_1 a small disc around $\zeta_1(c)$, which is mapped injectively to a larger disc by $P_c^{\circ qk}$, and finally form W by cutting off W_2 at an equipotential so low that it intersects the rays ∂W_1 inside the disc.

Then $P^{\circ qk}$ maps the boundary of W strictly outside itself, and defines a polynomial-like mapping which we will call $P_{c,U}$. The set M_U in the parameter space is exactly the set of $c \in W_{c_{U,0}}$ for which the corresponding filled-in Julia set $K_{c,U}$ is connected.

If $c \notin M_U$, the critical value c escapes under iteration. Let m be the first iteration such that $P_{c,U}^m(c) \notin W_1$. Then on the previous iteration $P_{c,U}^{m-1}(c)$ must have been in the part of $W \cap K_c$ which escapes in one iteration. This is the region in the dynamical plane cut off by the rays landing at the (unique) inverse image of $\zeta_1(c)$ under $P_{c,U}$; these angles are precisely those which land in the parameter space at $c_{U,1/2}$.

Continuing this way, we see that in order for the critical value to escape, it must be in a part of W cut off by rays landing at the inverse images of $\zeta_1(c)$ under $P_{c,U}^{-m}$, which will have the same angles as those landing on a point $c_{U,t}$ for some t dyadic, with denominator 2^m. This proves Proposition 15.2, since rays cutting off c in the dynamical plane will also cut off c in the parameter space.

We will call the set $S_{U,t,j}$ a *satellite* of M_U. We will be especially interested in the satellites of the components $U_{p/q}$ of int M which touch the cardioid M_0 at the point of internal angle p/q.

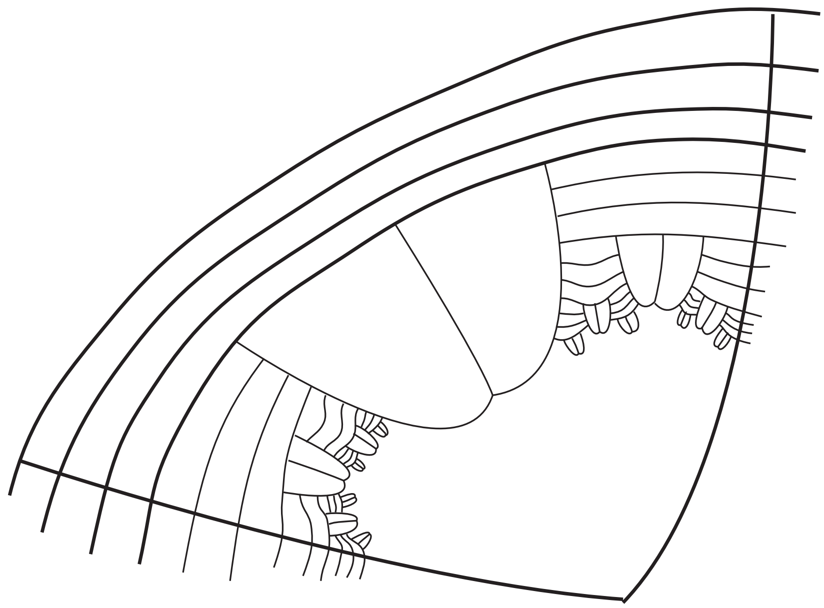


Figure 15.3.. The critical value end to depth 12 for a polynomial in $M_{1/3}$ of rank ≥ 4.

For $U_{p/q}$, the rank gives an alternative way of understanding satellites. The polynomials P_c for $c \in M_{U_{p/q}}$ are precisely those the tableau of which has the q-th column entirely critical. All other $c \in M_{p/q}$ have a rank $\nu[c]$ defined in section 10 as the smallest integer such that position $(0, q\nu)$ of the tableau is semi-critical.

If the polynomial P_c with $c \in M_{p/q}$ has rank ν, then the critical value nest of the puzzle of P_c has the following structure to depth $q\nu$: there is one component of Γ_q in the piece B_0, and if $\nu > 1$ then $B_q[c]$ is the component of the complement with $\alpha[c]$ in its closure. At level $2q$ there are two components of $\Gamma_{2q}[c]$ contained in $B_q[c]$, separated by the one component $\Gamma_q[c]$, and again if $\nu > 2$ then $B_{2q}[c]$ is the component of the complement with $\alpha[c]$ in its closure. More generally, at level nq there are 2^{n-1} components of $\Gamma_{nq}[c]$, which alternate with the components $\Gamma_{q(n-1)}[c]$. The picture 15.3 shows what the critical value end looks like down to depth 12, for a polynomial in $M_{1/3}$ with rank $\nu \geq 4$.

To say that the polynomial has rank ν is exactly to say that the piece $B_{q\nu}$ is in one of the $2^{\nu-1}(q-1)$ components of $B_{q\nu-1} - \Gamma_{q\nu}$ not containing $\alpha[c]$. This is equivalent (by the proof of Proposition 15.2) to saying that c is in a satellite $S_{U_{p/q},t,j}$ where t is a dyadic number with 2^ν in the denominator, and where the numerator tells which of the $2^{\nu-1}$ components of $\Gamma_{nq}[c] \cap B_{(n-1)q}[c]$ is separating the critical value from $\alpha[c]$.

EXAMPLE 15.4. If $p/q = 1/3$, then the point $c_{1/3,1/2}$ is the rather noticeable triple point near the top of M. It is at the end of the 3 rays at angles $9/56$, $11/56$ and $15/56$. In [D], Douady provides the following algorithm to compute two of these angles:

- Find the two rays landing at $c_{1/3,0}$, in this case $1/7 = .\overline{001}$ and $2/7 = .\overline{010}$. (All "decimals" are in base 2.) The *low repeating block* is 001 and the *high repeating block* is 010.
- Write $t = .\epsilon_0\epsilon_1\ldots$, and note that since t is dyadic there are two ways of doing this. Then replace ϵ_i by the low or the high repeating block depending on whether ϵ_i is 0 or 1.

For instance, $1/2 = .0\,\overline{1} = .1\overline{0}$, so two of the angles of $c_{1/3,1/2}$ are $.001\,\overline{010} = 9/56$ and $.010\,\overline{001} = 15/56$.

These are also the angles of the rays in the dynamical plane bounding the two extreme rays of $\Gamma_3 \cap B_0$, as can easily be checked from the fact that these must be appropriate $1/8$'s of $2/7$ and $1/7$ respectively. It is easy to figure out which eighths in this case, and a precise way of doing this in general is described in [DH3].

BIBLIOGRAPHY

[**DH1**] A. Douady and J. Hubbard, *Etude dynamique des polynômes complexes* I & II, Publ. Math. d'Orsay (1984-85).

[**DH2**] A. Douady and J. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. ENS Paris **18** (1985), 287–343.

[**DH3**] A. Douady and J. Hubbard, *Itération des polynômes quadratiques complexes*, CRAS Paris (1982), 123–126.

[**BH1**] B. Branner and J. Hubbard, *The iteration of cubic polynomials, Part I: The global topology of parameter space*, Acta Math **160** (1988), 143–206.

[**BH2**] B. Branner and J. Hubbard, *The iteration of cubic polynomials, Part II: Patterns and parapatterns*, Acta Math (to appear).

[**D**] A. Douady, *Algorithms for computing angles in the Mandelbrot set*, in "Chaotic Dynamics and Fractals, Atlanta 1984" (Barnsley and Demko, eds.), Academic Press, 1986.

[F] D. Faught, *Local connectivity in a family of cubic polynomials*, Thesis, Cornell University (1992).

[L] G. M. Levin, *On the Pommerenke's Inequality for the eigenvalue of a fixed point*, Manuscript, to appear in Colloquium Mathematicum, Wroclaw.

[M] J. Milnor, *Dynamics in one complex variable, Introductory Lectures*, preprint, Institute for Mathematical sciences, SUNY at Stony Brook, 1990, #5.

[M2] J. Milnor, *Local connectivity of Julia sets: Expository Lectures*, preprint, Institute for Mathematical sciences, SUNY at Stony Brook, 1992, #11.

[P] C. Pommerenke, *On conformal mapping and the iteration of rational functions*, Complex Variables, Th. and Appl. **5** no. 224 (1986).

[Pe] C. Petersen, *The Pommerenke-Levin-Yoccoz Inequality*, preprint 1991.

[Y] J.-C. Yoccoz, *Sur la connectivité locale de M*, unpublished, 1989.