0.1 Reading Mathematics

The most efficient logical order for a subject is usually different from the best psychological order in which to learn it. Much mathematical writing is based too closely on the logical order of deduction in a subject, with too many definitions without, or before, the examples which motivate them, and too many answers before, or without, the questions they address.—William Thurston

Many students do well in high school mathematics courses without reading their texts. At the college level you are expected to read the book. Better yet, read ahead. If you read a section before listening to a lecture on it, the lecture will be more comprehensible, and if there is something in the text you don't understand, you will be able to listen more actively and ask questions.

Reading mathematics is different from other reading. We think the following guidelines can make it easier. First, keep in mind that there are two parts to understanding a theorem: understanding the statement, and understanding the proof. *The first is more important than the second.*

What if you don't understand the statement? If there's a symbol in the formula you don't understand, perhaps a δ, look to see whether the next line continues, "where δ is such and such." In other words, read the whole sentence before you decide you can't understand it. In this book we have tried to define all terms before giving formulas, but we may not have succeeded everywhere.

If you're still having trouble, *skip ahead to examples.* This may contradict what you have been told—that mathematics is sequential, and that you must understand each sentence before going on to the next. In reality, although mathematical writing is necessarily sequential, mathematical understanding is not: you (and the experts) never understand perfectly up to some point and not at all beyond. The "beyond," where understanding is only partial, is an essential part of the motivation and the conceptual background of the "here and now." You may often (perhaps usually) find that when you return to something you left half-understood, it will have become clear in the light of the further things you have studied, even though the further things are themselves obscure.

Many students are very uncomfortable in this state of partial understanding, like a beginning rock climber who wants to be in stable equilibrium at all times. To learn effectively one must be willing to leave the cocoon of equilibrium. So *if you don't understand something perfectly, go on ahead and then circle back.*

In particular, an example will often be easier to follow than a general statement; you can then go back and reconstitute the meaning of the statement in light of the example. Even if you still have trouble with the general statement, you will be ahead of the game if you understand the examples. We feel so strongly about this that we have sometimes flouted mathematical tradition and given examples before the proper definition.

The Greek Alphabet

Greek letters that look like Roman letters are not used as mathematical symbols; for example, A is capital a, not capital α. The letter χ is pronounced "kye," to rhyme with "sky"; φ, ψ and ξ may rhyme with either "sky" or "tea."

α	A	alpha
β	B	beta
γ	Γ	gamma
δ	Δ	delta
ϵ	E	epsilon
ζ	Z	zeta
η	H	eta
θ	Θ	theta
ι	I	iota
κ	K	kappa
λ	Λ	lambda
μ	M	mu
ν	N	nu
ξ	Ξ	xi
o	O	omicron
π	Π	pi
ρ	P	rho
σ	Σ	sigma
τ	T	tau
υ	Υ	upsilon
φ, ϕ	Φ	phi
χ	X	chi
ψ	Ψ	psi
ω	Ω	omega

Read with pencil and paper in hand, making up little examples for yourself as you go on.

Some of the difficulty in reading mathematics is notational. A pianist who has to stop and think whether a given note on the staff is A or F will not be able to sight-read a Bach prelude or Schubert sonata. The temptation, when faced with a long, involved equation, may be to give up. You need to take the time to identify the "notes."

Learn the names of Greek letters—not just the obvious ones like alpha, beta, and pi, but the more obscure psi, xi, tau, omega. The authors know a mathematician who calls all Greek letters "xi" (ξ), except for omega (ω), which he calls "w." This leads to confusion. Learn not just to recognize these letters, but how to pronounce them. Even if you are not reading mathematics out loud, it is hard to think about formulas if $\xi, \psi, \tau, \omega, \varphi$ are all "squiggles" to you.

Sum and product notation

Sum notation can be confusing at first; we are accustomed to reading in one dimension, from left to right, but something like

$$\sum_{k=1}^{n} a_{i,k} b_{k,j} \qquad 0.1.1$$

In Equation 0.1.3, the symbol $\sum_{k=1}^{n}$ says that the sum will have n terms. Since the expression being summed is $a_{i,k}b_{k,j}$, each of those n terms will have the form ab.

requires what we might call two-dimensional (or even three-dimensional) thinking. It may help at first to translate a sum into a linear expression:

$$\sum_{i=0}^{\infty} 2^i = 2^0 + 2^1 + 2^2 \ldots \qquad 0.1.2$$

or

$$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j}. \qquad 0.1.3$$

Two $\sum$ placed side by side do not denote the product of two sums; one sum is used to talk about one index, the other about another. The same thing could be written with one $\sum$, with information about both indices underneath. For example,

$$\sum_{i=1}^{3}\sum_{j=2}^{4}(i+j) = \sum_{\substack{i \text{ from } 1 \text{ to } 3,\\ j \text{ from } 2 \text{ to } 4}} (i+j)$$
$$= \left(\sum_{j=2}^{4} 1+j\right) + \left(\sum_{j=2}^{4} 2+j\right) + \left(\sum_{j=2}^{4} 3+j\right) \qquad 0.1.4$$
$$= \big((1+2)+(1+3)+(1+4)\big)$$
$$+ \big((2+2)+(2+3)+(2+4)\big)$$
$$+ \big((3+2)+(3+3)+(3+4)\big);$$

this double sum is illustrated in Figure 0.1.1.

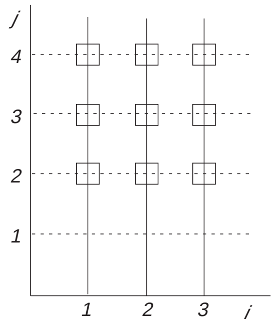

FIGURE 0.1.1.

In the double sum of Equation 0.1.4, each sum has three terms, so the double sum has nine terms.

Rules for product notation $\prod$ are analogous to those for sum notation:

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n; \quad \text{for example,} \quad \prod_{i=1}^{n} i = n!. \qquad 0.1.5$$

Proofs

When Jacobi complained that Gauss's proofs appeared unmotivated, Gauss is said to have answered, *You build the building and remove the scaffolding.* Our sympathy is with Jacobi's reply: he likened Gauss to *the fox who erases his tracks in the sand with his tail.*

We said earlier that it is more important to understand a mathematical statement than to understand its proof. We have put some of the harder proofs in Appendix A; these can safely be skipped by a student studying multivariate calculus for the first time. We urge you, however, to read the proofs in the main text. By reading many proofs you will learn what a proof is, so that (for one thing) you will know when you have proved something and when you have not.

In addition, a good proof doesn't just convince you that something is true; it tells you why it is true. You presumably don't lie awake at night worrying about the truth of the statements in this or any other math textbook. (This is known as "proof by eminent authority"; you assume the authors know what they are talking about.) But reading the proofs will help you understand the material.

If you get discouraged, keep in mind that the content of this book represents a cleaned-up version of many false starts. For example, John Hubbard started by trying to prove Fubini's theorem in the form presented in Equation 4.5.1. When he failed, he realized (something he had known and forgotten) that the statement was in fact false. He then went through a stack of scrap paper before coming up with a correct proof. Other statements in the book represent the efforts of some of the world's best mathematicians over many years.