Matrix Factorizations for Complete Intersections

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Linear Algebra:	Vector Spaces finite dimensional	over a Field e.g. $\mathbb{C}$	tool: basis

Algebra:	Modules finitely generated	over a Ring e.g. $\mathbb{C}[x_1, \ldots, x_n]$	tool: ?

Can we use bases to study the structure of modules? They rarely exist: only free modules have bases.

Instead of a basis, we have to consider *generators*.

Example.

$N = (xy, xz)$ is an ideal in $\mathbb{C}[x, y, z]$.

It is generated by $f := xy$ and $g := xz$.

$\{f, g\}$ is not a basis since we have the relation

$$zf - yg = z(xy) - y(xz) = 0\,.$$

N does not have a basis.

Generators give very little information
about the structure of a module.
Usually there are relations on the generators,
and relations on these relations, etc. .

Basic Question.

How do we describe the structure of a module?

Hilbert's Approach (1890, 1893):

use Free Resolutions.

Definition. Let R be a commutative noetherian ring (e.g. $\mathbb{C}[x_1, \dots, x_n]$, $\mathbb{C}[[x_1, \dots, x_n]]$, or their quotients). A sequence

$$\mathbb{F}: \quad \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0$$

of f.g. free R-modules is a ***free resolution*** of a f.g. R-module N if:

(1) $\mathbb{F}$ is an exact complex, that is, $\mathrm{Ker}(d_i) = \mathrm{Im}(d_{i+1}) \ \forall i$.

(2) $N \cong \mathrm{Coker}(d_1) = F_0/\mathrm{Im}(d_1)$, that is,

$$\cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \to \cdots \to F_1 \xrightarrow{d_1} F_0 \to N \to 0 \text{ is exact.}$$

$d = \{d_i\}$ is the ***differential*** of the resolution.

Example. $S = \mathbb{C}[x, y, z]$

$N = (xy, xz)$ has a free resolution

$$0 \longrightarrow S \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} N \longrightarrow 0\,.$$

$$\cdots \longrightarrow F_2 \xrightarrow{d_2 = \begin{pmatrix} \text{relations} \\ \text{on the} \\ \text{relations} \\ \text{in } d_1 \end{pmatrix}} F_1 \xrightarrow{d_1 = \begin{pmatrix} \text{relations} \\ \text{on the} \\ \text{generators} \\ \text{of } N \end{pmatrix}} F_0 \xrightarrow{\begin{pmatrix} \text{generators} \\ \text{of } N \end{pmatrix}} N \longrightarrow 0$$

A free resolution of a module N is a description of the structure of N.

We would like to construct a free resolution as efficiently as possible, that is, at each step we would like to pick a ***minimal*** system of relations.

Example. $S = \mathbb{C}[x, y, z]$
$N = (xy, xz)$ has free resolutions

$$0 \longrightarrow S \xrightarrow{\begin{pmatrix} z \\ -y \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} N \longrightarrow 0$$

$$0 \longrightarrow S \xrightarrow{\begin{pmatrix} -y \\ 1 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} z & yz \\ -y & -y^2 \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} xy & xz \end{pmatrix}} N \longrightarrow 0\,.$$

The concept of *minimality* makes sense in two main cases

local	graded
e.g. $\mathbb{C}[[x_1, \ldots, x_n]]$	e.g. $\mathbb{C}[x_1, \ldots, x_n]$ $\deg(x_i) = 1 \ \forall i$

because of Nakayama's Lemma.

In these cases, a minimal free resolution of a module N exists and is unique up to an isomorphism.

The ranks of the free modules in the minimal free resolution are the *Betti numbers* $b_i(N)$ of N.

Notation: $S = \mathbb{C}[[x_1, \ldots, x_n]]$, $R = S/J$.

Hilbert's Syzygy Theorem. Every module over S has a finite minimal free resolution. Its length is $\leq n$.

Serre's Regularity Criterion.

Every f.g. R- module has a finite free resolution

$\iff$ $\mathbb{C}$ has a finite free resolution

$\iff$ R is a regular ring (that is, J is generated by linear forms).

Over a quotient ring
most minimal free resolutions are infinite.

Notation: $S = \mathbb{C}[[x_1, \ldots, x_n]]$

Question.

What happens over a hypersurface $R = S/(f)$?

The answer involves matrix factorizations. A *matrix factorization* of an element $f \in S$ is a pair of square matrices d, h with entries in S, such that

$$dh = hd = f\,\mathrm{Id}\,.$$

Let $f \in S = \mathbb{C}[[x_1, \ldots, x_n]]$, and set $R = S/(f)$.
Eisenbud (1980) introduced the concept of ***matrix factorization*** d, h of f. Its ***MF-module*** is

$$M := \mathrm{Coker}(d) = R^b/\mathrm{Im}(d)\,.$$

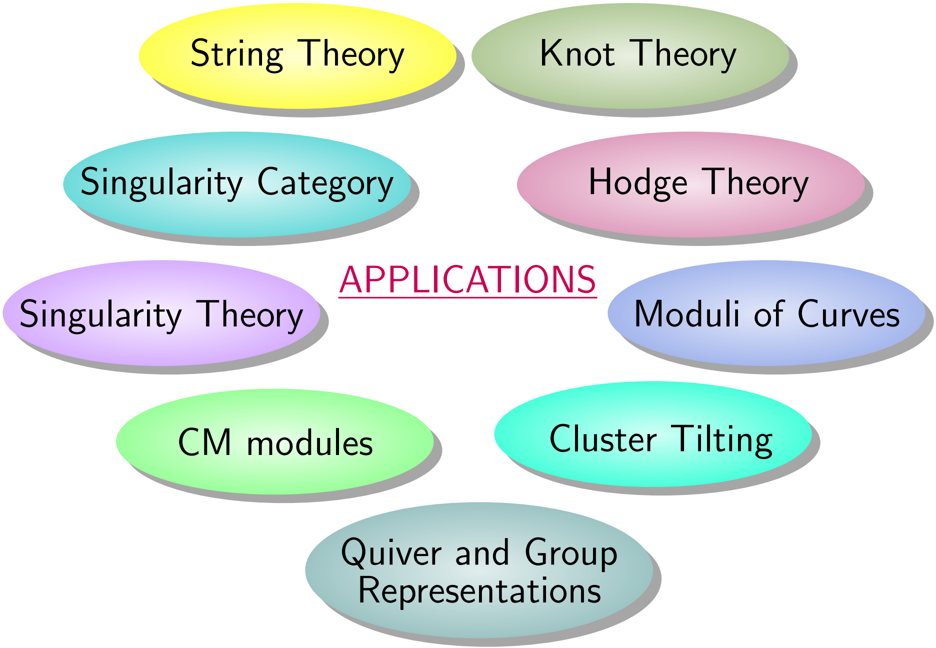
(1) $\cdots \xrightarrow{d} R^b \xrightarrow{h} R^b \xrightarrow{d} R^b \xrightarrow{h} R^b \xrightarrow{d} R^b$
is the minimal free resolution of M over R.

(2) Asymptotically, every minimal free resolution over R is of type (1): if $\mathbb{F}$ is a minimal free resolution over R, then $\forall\, s \gg 0$ the truncation

$$\mathbb{F}_{\geq s}: \quad \cdots \longrightarrow F_{s+1} \xrightarrow{d_{s+1}} F_s$$

is described by a matrix factorization.

(3) $0 \to S^b \xrightarrow{d} S^b$ is the minimal free resolution of M over S.


APPLICATIONS
String Theory
Knot Theory
Singularity Category
Hodge Theory
Singularity Theory
Moduli of Curves
CM modules
Cluster Tilting
Quiver and Group Representations

Notation: $S = \mathbb{C}[[x_1, \ldots, x_n]]$

We considered the question:

What is the structure of minimal free resolutions over a hypersurface $R = S/(f)$?

Next Question. *What happens over a quotient ring $R = S/(f_1, \ldots, f_c)$?*

Hope: even though a minimal free resolution is infinite, it might be the case that its structure is encoded in finite data. The Serre-Kaplansky Problem embodies this view.

The Serre-Kaplansky Problem. *Is the Poincaré series $\sum_{i\geq 0} b_i^R(\mathbb{C})t^i$ rational?*

Here:

R is a local ring with residue field $\mathbb{C}$.

$b_i^R(\mathbb{C})$ are the Betti numbers ($=$ the ranks of the free modules) in the minimal free resolution of $\mathbb{C}$ considered as an R-module.

The Serre-Kaplansky Problem. *Is the Poincaré series* $\sum_{i\geq 0} b_i^R(\mathbb{C})t^i$ *rational?*

This was one of the central questions in Commutative Algebra for many years.

Anick (1980) found a counterexample:

$$R = \mathbb{C}[x_1, \ldots, x_5]/\big(\,(x_1, \ldots, x_5)^3, x_1^2,\ x_2^2,\ x_4^2,\ x_5^2,\ x_1x_2,\ x_4x_5,\ x_1x_3 + x_3x_4 + x_2x_5\big)\,.$$

The quest for rings with rational Poincaré series keeps going ... Recently, Herzog and Huneke proved rationality over $\mathbb{C}[x_1, \ldots, x_n]/J^m$ for every $m \geq 2$ and every graded ideal J.

Notation: $S = \mathbb{C}[[x_1, \dots, x_n]]$

We are discussing:

Question. *What is the structure of minimal free resolutions over* $R = S/(f_1, \dots, f_c)$*?*

Anick's example shows that we should impose some conditions on $f_1, \dots, f_c$.

A main class of interest are the complete intersection rings, and we will focus on:

Question.

What is the structure of minimal free resolutions over a complete intersection $R = S/(f_1, \dots, f_c)$?

Notation: $S = \mathbb{C}[[x_1, \ldots, x_n]]$

Definition. Let Q be a quotient ring of S. An element $g \in Q$ is a *non-zerodivisor* if

$$gw = 0,\ w \in Q \implies w = 0\,.$$

We say that $f_1, \ldots, f_c$ is a *regular sequence* if

f_i is a non-zerodivisor on $S/(f_1, \ldots, f_{i-1}) \quad \forall i$,

and also $S/(f_1, \ldots, f_c) \neq 0$.

The quotient $R = S/(f_1, \ldots, f_c)$ is called a *complete intersection*.

Numerical Results

Let $R = \mathbb{C}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_c)$ be a complete intersection.

Theorem. (Tate, 1957) (Gulliksen, 1980)
The Poincaré series of every f.g. module over R is rational.

Theorem. (Avramov-Gasharov-Peeva, 1997)
The Betti numbers of every f.g. module over R are eventually non-decreasing.

Numerical results indicate that minimal free resolutions over R are highly structured.

Let $f_1, \ldots, f_c \in S = \mathbb{C}[[x_1, \ldots, x_n]]$ be a regular sequence, and set $R = S/(f_1, \ldots, f_c)$.
Eisenbud and Peeva introduced the concept of ***matrix factorization*** d, h. Its ***MF-module*** is

$$M := \mathrm{Coker}(d) .$$

(1) We constructed the infinite minimal free resolution of M over R.

Hypersurface Case: The minimal free resolution is

$$\cdots \xrightarrow{d} R^b \xrightarrow{h} R^b \xrightarrow{d} R^b \xrightarrow{h} R^b \xrightarrow{d} R^b \ .$$

It has constant Betti numbers and is periodic of period 2.
General Case: The Betti numbers grow polynomially. There is a pattern for the odd differentials, and another for the even differentials.

Let $f_1, \ldots, f_c \in S = \mathbb{C}[[x_1, \ldots, x_n]]$ be a regular sequence, and set $R = S/(f_1, \ldots, f_c)$.
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(1) We constructed the infinite minimal free resolution of M over R.

(2) Asymptotically, every minimal free resolution over R is of type (1): if $\mathbb{F}$ is a minimal free resolution over R, then $\forall\, s \gg 0$ the truncation

$$\mathbb{F}_{\geq s}: \ \cdots \to F_{s+1} \to F_s$$

is described by a matrix factorization.

(3) We constructed the minimal free resolution of M over S. It has length c.

The results hold over a graded or local complete intersection with infinite residue field.

An expository paper (joint with J. McCullough)

with Open Problems on Infinite Free Resolutions

is available at

http://math.cornell.edu/~irena