COUNTEREXAMPLES TO THE EISENBUD-GOTO REGULARITY CONJECTURE

JASON MCCULLOUGH AND IRENA PEEVA

Abstract. Our main theorem shows that the regularity of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the degree; this holds over any field k. In particular, we provide counterexamples to the longstanding Regularity Conjecture, also known as the Eisenbud-Goto Conjecture (1984). We introduce a method which, starting from a homogeneous ideal I, produces a prime ideal whose projective dimension, regularity, degree, dimension, depth, and codimension are expressed in terms of numerical invariants of I. The method is also related to producing bounds in the spirit of Stillman's Conjecture, recently solved by Ananyan-Hochster.

1. Introduction

Hilbert's Syzygy Theorem provides a nice upper bound on the projective dimension of homogeneous ideals in a standard graded polynomial ring: projective dimension is smaller than the number of variables. In contrast, there is a doubly exponential upper bound on the Castelnuovo-Mumford regularity in terms of the number of variables and the degrees of the minimal generators. It is the most general bound on regularity in the sense that it requires no extra conditions. The bound is nearly sharp since the Mayr-Meyer construction leads to examples of families of ideals attaining doubly exponential regularity. On the other hand, for reduced, irreducible, smooth (or nearly smooth) projective varieties over an algebraically closed field, regularity is well controlled by several upper bounds in terms of the degree, codimension, dimension or degrees of defining equations. As discussed in the influential paper [BM] by Bayer and Mumford, "the biggest missing link" between the general case and the smooth case is to obtain a "decent bound on the regularity of all reduced equidimensional ideals". The longstanding Regularity Conjecture 1.2, by Eisenbud-Goto [EG] (1984), predicts a linear bound in terms of the degree for non-degenerate prime ideals over an algebraically closed field. In subsection 1.7 we give counterexamples to the Regularity Conjecture 1.2.

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Our main Theorem 1.9 is much stronger and shows that the regularity of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the degree; this holds over any field k (the case $k = \mathbb{C}$ is particularly important). We provide a family of prime ideals P_r, depending on a parameter $r \in \mathbf{N}$, whose degree is singly exponential in r and whose regularity is doubly exponential in r. For this purpose, we introduce an approach, outlined in subsection 1.5, which starting from a homogeneous ideal I, produces a prime ideal P whose projective dimension, regularity, degree, dimension, depth, and codimension are expressed in terms of numerical invariants of I.

1.1 Motivation and Conjectures. This subsection provides an overview of regularity conjectures and related results. We consider a standard graded polynomial ring $U = k[z_1, \ldots, z_p]$ over a field k, where all variables have degree one. Projective dimension and regularity are well-studied numerical invariants that measure the size of a Betti table. Let L be a homogeneous ideal in the ring U, and let $\beta_{ij}(L) = \dim_k \operatorname{Tor}_i^U(L,k)_j$ be its graded Betti numbers. The *projective dimension*

$$\operatorname{pd}(L) = \max\left\{ i \;\middle|\; \beta_{ij}(L) \neq 0 \right\}$$

is the index of the last non-zero column of the Betti table $\beta(L) := \big(\beta_{i,i+j}(L)\big)$, and thus it measures its width. The height of the table is measured by the index of the last non-zero row and is called the *(Castelnuovo-Mumford) regularity* of L; it is defined as

$$\operatorname{reg}(L) = \max\left\{ j \;\middle|\; \beta_{i,\,i+j}(L) \neq 0 \right\} .$$

By [EG, Theorem 1.2], (see also [Pe, Theorem 19.7]) for any $q \geq \operatorname{reg}(L)$, the truncated ideal $L_{\geq q}$ is generated in degree q and has a linear minimal free resolution. A closely related invariant $\operatorname{maxdeg}(L)$ is the maximal degree of an element in a minimal system of homogeneous generators of L. Note that $\operatorname{maxdeg}(L) \leq \operatorname{reg}(L)$.

Alternatively, regularity can be defined using local cohomology, see for example, the expository papers [Ch, Ei] and the books [Ei2, La2].

Hilbert's Syzygy Theorem (see for example, [Ei3, Corollary 19.7] or [Pe, Theorem 15.2]) provides a nice upper bound on the projective dimension of L:

$$\operatorname{pd}(L) < p.$$

However, the general (not requiring any extra conditions) regularity bound is doubly exponential:

$$\operatorname{reg}(L) \leq \left(2 \operatorname{maxdeg}(L)\right)^{2^{p-2}} .$$

It is proved by Bayer-Mumford [BM] (using results in Giusti [Gi] and Galligo [Ga]) if $\operatorname{char}(k) = 0$, and by Caviglia-Sbarra [CS] in any characteristic. This bound is nearly the best possible, due to examples based on the Mayr-Meyer construction [MM]; for example, there exists an ideal L in $10r+1$ variables for which $\mathrm{maxdeg}(L) = 4$ and

$$\mathrm{reg}(L) \geq 2^{2^r}$$

by [BM, Proposition 3.11]. Other versions of the Mayr-Meyer ideals were constructed by Bayer-Stillman [BS] and Koh [Ko].

Still more examples of ideals with high regularity have been constructed by Caviglia, Chardin-Fall, and Ullery [Ca, CF, Ul]. For more details about regularity, we refer the reader to the expository papers [BM, Ch, Ei] and the books [Ei2, La2].

In sharp contrast, a much better bound is expected if $L = I(X)$ is the vanishing ideal of a geometrically nice projective scheme $X \subset \mathbb{P}_k^{p-1}$. The following elegant bound was conjectured by Eisenbud, Goto, and others, and has been very challenging.

The Regularity Conjecture 1.2. (Eisenbud-Goto [EG], 1984) *Suppose that the field k is algebraically closed. If $L \subset (z_1, \ldots, z_p)^2$ is a homogeneous prime ideal in U, then*

$$\mathrm{reg}(L) \ \leq \ \deg(U/L) - \mathrm{codim}(L) + 1\,, \tag{1.3}$$

where $\deg(U/L)$ *is the multiplicity of U/L (also called the degree of U/L, or the degree of X), and* $\mathrm{codim}(L)$ *is the codimension (also called height) of L.*

The condition that $L \subset (z_1, \ldots, z_p)^2$ is equivalent to requiring that X is not contained in a hyperplane in $\mathbb{P}_k^{p-1}$. Prime ideals that satisfy this condition are called *non-degenerate*.

The Regularity Conjecture holds if U/L is Cohen-Macaulay by [EG]. It is proved for curves by Gruson-Lazarsfeld-Peskine [GLP], completing classical work of Castelnuovo. It also holds for smooth surfaces by Lazarsfeld [La] and Pinkham [Pi], and for most smooth 3-folds by Ran [Ra]. In the smooth case, Kwak [Kw] gave bounds for regularity in dimensions 3 and 4 that are only slightly worse than the optimal ones in the conjecture; his method yields new bounds up to dimension 14, but they get progressively worse as the dimension goes up. Other special cases of the conjecture and also similar bounds in special cases are proved by Brodmann [Br], Brodmann-Vogel [BV], Eisenbud-Ulrich [EU], Herzog-Hibi [HH], Hoa-Miyazaki [HM], Kwak [Kw2], and Niu [Ni].

The following variations of the Regularity Conjecture have been of interest:

Eisenbud and Goto further conjectured that the hypotheses in 1.2 can be weakened to say that X is reduced and connected in codimension 1. This was proved for curves by Giaimo [Gia]. Examples show that the hypotheses cannot be weakened much further: The regularity of a reduced equidimensional X cannot be bounded by its degree, as [EU, Example 3.1] gives a reduced equidimensional union of two irreducible complete intersections whose regularity is much larger than its degree.

Example 3.11 in [Ei] shows that there is no bound on the regularity of non-reduced homogeneous ideals in terms of multiplicity, even for a fixed codimension. See [Ei2, Section 5C, Exercise 4] for an example showing that the hypothesis that the field k is algebraically closed is necessary.

In 1988 Bayer and Stillman [BS, p. 136], made the related conjecture that the regularity of a reduced scheme over an algebraically closed field is bounded by its degree (which is the sum of the degrees of its components). This holds if L is the vanishing ideal of a finite union of linear subspaces of $\mathbf{P}_k^{p-1}$ by a result of Derksen-Sidman [DS].

It is a very basic problem to get an upper bound on the degrees of the defining equations of an irreducible projective variety. The following weaker form of the Regularity Conjecture provides an elegant bound.

Conjecture 1.4. (Folklore Conjecture) *Suppose that the field k is algebraically closed. If L is a homogeneous non-degenerate prime ideal in U, then*

$$\operatorname{maxdeg}(L) \leq \deg(U/L)\,.$$

1.5. Our Approach. Fix a polynomial ring $S = k[x_1, \dots, x_n]$ over a field k with a standard grading defined by $\deg(x_i) = 1$ for every i. As discussed above, there exist examples of homogeneous ideals with high regularity (for example, based on the Mayr-Meyer construction), but they are not prime. Motivated by this, we introduce a method which, starting from a homogeneous ideal I, produces a prime ideal P whose projective dimension, regularity, maxdeg, multiplicity, dimension, depth, and codimension are expressed in terms of numerical invariants of I. The method has two ingredients: Rees-like algebra and Step-by-step Homogenization.

In Section 3, we consider the prime ideal Q of defining equations of the Rees-like algebra $S[It, t^2]$. This was inspired by Hochster's example in [Be] which, starting with a family of three-generated ideals in a regular local ring, produces prime ideals with embedding dimension 7, Hilbert-Samuel multiplicity 2, and arbitrarily many minimal generators. In contrast to the usual Rees algebra, whose defining equations are difficult to find in general (see for example [Hu], [KPU]), those of the Rees-like algebra are given explicitly in Proposition 3.2. Furthermore, one can obtain the graded Betti numbers of Q using a mapping cone resolution described in Theorem 3.10.

We introduce Step-by-step Homogenization in Section 4. The ideal Q is homogeneous but in a polynomial ring that is not standard graded. We change the degrees of the variables to 1 and homogenize the ideal; we do this one variable at a time, in order to not drop the degrees of the defining equations. One usually needs to homogenize a Gröbner basis in order to obtain a generating set of a homogenized ideal, but we show that in our case it suffices to homogenize a minimal set of generators. Our Step-by-step Homogenization method is expressed in Theorem 4.5, which can be applied to any non-degenerate prime ideal that is homogeneous in a positively graded polynomial ring in order to obtain a homogeneous prime ideal in a standard graded polynomial ring. Its key property is the preservation of the graded Betti numbers, which usually change after homogenization. Applying this to the ideal Q we produce a prime ideal P, by Proposition 4.8.

A set of generators of P is defined in Construction 2.4, and we prove in Proposition 2.9 that it is minimal. The key and striking property of the construction of the ideal P is that it has a nicely structured minimal free resolution (coming from the minimal free resolution of Q), which makes it possible to express its regularity, multiplicity, and other invariants in terms of invariants of I. We prove the following properties of P:

Theorem 1.6. *Let k be any field. Let I be an ideal generated minimally by homogeneous elements $f_1, \dots, f_m$ (with $m \geq 2$) in the standard graded polynomial ring $S = k[x_1, \dots, x_n]$.*

The ideal P, defined in Construction 2.4, is homogeneous in the standard graded polynomial ring

$$R = S[y_1, \dots, y_m, u_1, \dots, u_m, z, v]$$

with $n+2m+2$ variables. It is minimally generated by the elements listed in (2.5) and (2.6), (by Proposition 2.9). It is prime and non-degenerate, (by Proposition 4.8). Furthermore:

(1) *The maximal degree of a minimal generator of P is*

$$\operatorname{maxdeg}(P) = \max \left\{ 1 + \operatorname{maxdeg}\big(\operatorname{Syz}_1^S(I)\big),\ 2\big(\operatorname{maxdeg}(I) + 1\big) \right\}.$$

(2) *The multiplicity of R/P is*

$$\deg(R/P) = 2 \prod_{i=1}^{m} \big(\deg(f_i) + 1\big).$$

(3) *The Castelnuovo-Mumford regularity, the projective dimension, the depth, the codimension, and the dimension of R/P are:*

$$\begin{aligned} \operatorname{reg}(R/P) &= \operatorname{reg}(S/I) + 2 + \sum_{i=1}^{m} \deg(f_i) \\ \operatorname{pd}(R/P) &= \operatorname{pd}(S/I) + m - 1 \\ \operatorname{depth}(R/P) &= \operatorname{depth}(S/I) + m + 3 \\ \operatorname{codim}(P) &= m \\ \dim(R/P) &= m + n + 2\,. \end{aligned}$$

Property (1) holds by Corollary 2.10. Property (2) holds by Theorem 5.2. The properties listed in (3) are proved in Section 5. Above, we used $\operatorname{reg}(R/P) = \operatorname{reg}(P) - 1$ and $\operatorname{pd}(R/P) = \operatorname{pd}(P) + 1$. Since $\operatorname{depth}(R/P) \geq m + 3$, we may use Bertini's theorem, see [Fl], to reduce the number of variables by at least $m + 2$ and thus obtain a prime ideal P' in a polynomial ring R' with at most $n + m$ variables, instead of $n + 2m + 2$ variables, and with $\dim(R'/P') \leq n$. Note that factoring out linear homogeneous non-zerodivisors preserves projective dimension, regularity, and degree.

1.7. Counterexamples and the Main Theorem. We provide the following counterexamples to the Regularity Conjecture 1.2. They are also counterexamples to the weaker Conjecture 1.4 and the Bayer-Stillman Conjecture. For this, we use properties (1) and (2) in Theorem 1.6.

Counterexamples 1.8. The counterexamples in (1) and (2) hold over any field k.

(1) For $r \geq 1$, Koh constructed in [Ko] an ideal I_r generated by $22r - 3$ quadrics and one linear form in a polynomial ring with $22r - 1$ variables, and such that $\operatorname{maxdeg}(\operatorname{Syz}_1(I_r)) \geq 2^{2^{r-1}}$. His ideals are based on the Mayr-Meyer construction in [MM]. By Theorem 1.6, I_r leads to a homogeneous prime ideal P_r (in a standard graded polynomial ring R_r) whose multiplicity and maxdeg are:

$$\deg(R_r/P_r) \leq 4 \cdot 3^{22r-3} < 4^{22r-2} < 2^{50r}$$
$$\operatorname{maxdeg}(P_r) \geq 2^{2^{r-1}} + 1 > 2^{2^{r-1}}.$$

Therefore, Conjecture 1.4 predicts

$$2^{2^{r-1}} + 1 \leq 4 \cdot 3^{22r-3},$$

which fails for $r \geq 9$. Moreover, the difference

$$\operatorname{reg}(P_r) - \deg(R_r/P_r) \geq \operatorname{maxdeg}(P_r) - \deg(R_r/P_r) > 2^{2^{r-1}} - 2^{50r}$$

can be made arbitrarily large by choosing a large r.

(2) Alternatively, we can use the Bayer-Stillman example in [BS, Theorem 2.6] instead of Koh's example. For $r \geq 1$, they constructed a homogeneous ideal I_r (using $d = 3$ in their notation) generated by $7r+5$ forms of degree at most 5 in a polynomial ring with $10r+11$ variables and such that $\operatorname{maxdeg}(\operatorname{Syz}_1(I_r)) \geq 3^{2^{r-1}}$. The example is based on the Mayr-Meyer construction in [MM]. By Theorem 1.6, I_r leads to a homogeneous prime ideal P_r whose multiplicity is $\deg(R_r/P_r) \leq 2 \cdot 6^{7r+5}$ and with $\operatorname{maxdeg}(P_r) \geq 3^{2^{r-1}} + 1$. Therefore, Conjecture 1.4 predicts $3^{2^{r-1}} + 1 \leq 2 \cdot 6^{7r+5}$, which fails for $r \geq 8$.

(3) In Section 4, we give two examples of 3-dimensional projective varieties in $\mathbb{P}^5$ for which the Regularity Conjecture 1.2 fails. These examples cannot prove Theorem 1.9 but are small enough to be computable with Macaulay2 [M2].

We remark that from Counterexamples 1.8(1),(2) it follows that we can obtain counterexamples using the Rees algebras $S[I_r t]$ (instead of the Rees-like algebras $S[I_r t, t^2]$); this is proved in [CMPV]. In that paper we also construct counterexamples which do not rely on the Mayr-Meyer construction.

What next? The bound in the conjecture is very elegant, so it is certainly of interest to study if it holds when we impose extra conditions on the prime ideal.

Suppose $\mathrm{char}(k) = 0$ and $X \subset \mathbb{P}_k^{p-1}$ is a smooth variety. In this case the Regularity Conjecture is open and Kwak-Park [KP] and Noma [No] reduced it to the Castelnuovo's Normality Conjecture that X is r-normal for all $r \geq \deg(X) - \mathrm{codim}(X)$. However, other bounds are known. Bertram-Ein-Lazarsfeld [BEL] obtained an important bound that implies

$$\mathrm{reg}\, X \leq 1 + (s-1)\mathrm{codim}\, X$$

if X is cut out scheme-theoretically by equations of degree $\leq s$. Later this bound was proved by Chardin-Ulrich [CU] for X satisfying weaker conditions. See [Ch2] for an overview. These results were generalized in [DE] to a large class of projective schemes. On the other hand, Mumford proved in the appendix of [BM, Theorem 3.12] that if X is reduced, smooth, and pure dimensional then

$$\mathrm{reg}\, X \leq (\dim X + 1)(\deg X - 2) + 2\,.$$

Note that the above bounds are different in flavor than the Regularity Conjecture: they are not linear in the degree (or the degree of the defining equations) since there is a coefficient involving the dimension or codimension.

In [BM] Bayer and Mumford pointed out that the main missing piece of information between the general case and the geometrically nice smooth case is that we do not have yet a reasonable bound on the regularity of all reduced equidimensional ideals. Thus, instead of imposing extra conditions on the ideals we may weaken the bound, which is linear in the Regularity Conjecture. If the residue field k is algebraically closed and L is a non-degenerate prime ideal, then $\deg(U/L) \geq 1 + \mathrm{codim}(U/L)$ (see for example, [EG, p. 112]). So instead of a bound on regularity involving multiplicity and codimension, we could look for a bound in terms of the multiplicity alone. Counterexamples 1.8(1)(or 2) prove the main result in our paper:

Main Theorem 1.9. *Over any field k (the case $k = \mathbb{C}$ is particularly important), the regularity of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the multiplicity, i.e., for any polynomial $\Theta(x)$ there exists a non-degenerate homogeneous prime ideal L in a standard graded polynomial ring V over the field k such that* $\mathrm{reg}(L) > \Theta\big(\deg(V/L)\big)$.

Proof. In the notation and under the assumptions of Counterexmaples 1.8(1), we have

$$\operatorname{reg}(P_r) > 2^{2^{r-1}} > 2^{1/2\cdot \deg(R_r/P_r)^{1/50}} = \left(\sqrt{2}\right)^{\deg(R_r/P_r)^{1/50}}.$$

The function $f(x) = \left(\sqrt{2}\right)^{x^{1/50}}$ is not bounded above by any polynomial in x. $\square$

It is natural to wonder if there exists any bound in terms of the multiplicity. In [CMPV] we prove the existence of such a bound using the recent result of Ananyan-Hochster [AH2] that Stillman's Conjecture holds. However, the bound obtained in this way is very large.

Question 1.10. *Suppose the field k is algebraically closed. What is an optimal function $\Phi(x)$ such that* $\operatorname{reg}(L) \leq \Phi(\deg(L))$ *for any non-degenerate homogeneous prime ideal L in a standard graded polynomial ring over k?*

In the spirit of [BS] it would be nice if $\Phi(x)$ is singly exponential.

Next we will explain how Question 1.10 is related to Stillman's Conjecture, which asks whether there exists an upper bound on the regularity of homogeneous ideals generated by m forms of degrees $a_1, \ldots, a_m$ (independent of the number of variables). Let I be an ideal in a standard graded polynomial ring S over a field K minimally generated by homogeneous forms of degrees $a_1, \ldots, a_m$. We may enlarge the base field K to an algebraically closed field k without changing the regularity. Let $\Phi(x)$ be a function such that $\operatorname{reg}(L) \leq \Phi(\deg(L))$ for any non-degenerate homogeneous prime ideal L in a standard graded polynomial ring over k. Let P be the prime ideal associated to I according to our method, and apply Theorem 1.6. Then

$$\operatorname{reg}(I) \leq \operatorname{reg}(P) \leq \Phi(\deg(R/P)) = \Phi\left(2\prod_{i=1}^{m}(a_i+1)\right).$$

Thus, $\Phi\left(2\prod_{i=1}^{m}(a_i+1)\right)$ provides a bound on the regularity in terms of the degrees $a_1, \ldots, a_m$ of the generators.

Bounds for Stillman's Conjecture which are better than those obtained in [AH2] were obtained for all ideals generated by quadrics by Ananyan-Hochster in [AH]. They have also announced bounds in the cases of generators of degree at most three, or generators of degree at most four and $\operatorname{char}(k) \neq 2$. See the expository papers [FMP, MS] for a discussion of other results in this direction.

There is an equivalent form of Stillman's Conjecture that replaces projective dimension by regularity; the equivalence of the two conjectures was proved by Caviglia. Motivated by this, we discuss projective dimension of prime ideals in Section 6. Theorem 6.2 provides an analogue to Theorem 1.9.

2. Definition of the ideal P, starting from a given ideal I

In this section, we introduce notation which will be used in the rest of the paper. Starting form a homogeneous ideal I, we write generators for a new ideal, which we denote by P. We will study the properties of P in the next sections.

Notation 2.1. If N is a graded module and $p \in \mathbf{Z}$, denote by $N(-p)$ the shifted module for which $N(-p)_i = N_{i-p}$ for all i.

If $(\mathbf{V}, d)$ is a complex, we write $\mathbf{V}[-p]$ for the shifted complex with $\mathbf{V}[-p]_i = \mathbf{V}_{i+p}$ and differential $(-1)^p d$.

For a finitely generated graded U-module N, we denote by $\operatorname{Syz}_i^U(N)$ the i^{th} syzygy module.

Assumptions and Notation 2.2. Consider the polynomial ring

$$S = k[x_1, \ldots, x_n]$$

over a field k with a standard grading defined by $\deg(x_i) = 1$ for every i. Let I be a homogeneous ideal minimally generated by forms $f_1, \ldots, f_m$ of degrees $a_1, \ldots, a_m$, where $m \geq 2$. We denote by $(\mathbf{F}, d^F)$ the minimal graded S-free resolution of $\operatorname{Syz}_2^S(S/I) = \operatorname{Syz}_1^S(I)$. Thus the minimal graded S-free resolution of S/I has the form

$$\mathbf{F}' : \quad \mathbf{F}[2] \xrightarrow{d_0^F = (c_{ij})} F_{-1} := S(-a_1) \oplus \cdots \oplus S(-a_m) \xrightarrow{d_{-1}^F = (f_1 \ \dots \ f_m)} F_{-2} := S\,,$$

and in particular $\mathbf{F}$ is a truncation of $\mathbf{F}'$.

Denote by $\xi_1, \ldots, \xi_m$ a homogeneous basis of $F_{-1} = S(-a_1) \oplus \cdots \oplus S(-a_m)$ such that $d_{-1}^F(\xi_i) = f_i$ for every i. Fix a homogeneous basis $\mu_1, \ldots, \mu_{\operatorname{rank} F_0}$ of F_0 that is mapped by the differential d_0^F to a homogeneous minimal system of generators of $\operatorname{Ker}(d_{-1}^F)$. Let $C = (c_{ij})$ be the matrix of the differential d_0^F in these fixed homogeneous bases. Thus $\operatorname{Syz}_1^S(I)$ is generated by the elements

$$\left\{ \sum_{i=1}^{m} c_{ij} \xi_i \;\middle|\; 1 \leq j \leq \operatorname{rank}(F_0) \right\}. \tag{2.3}$$

In matrix form, these elements correspond to the entries in the matrix product $(\xi_1, \xi_2, \ldots, \xi_m)\, C$.

Construction 2.4. In the notation and under the assumptions of 2.2, we will define an ideal P. The motivation for this construction is outlined in subsection 1.5 of the Introduction. We consider the standard graded polynomial ring

$$R = S[y_1, \ldots, y_m, u_1, \ldots, u_m, z, v] = k[x_1, \ldots, x_n, y_1, \ldots, y_m, u_1, \ldots, u_m, z, v]\,.$$

Let P be the ideal generated by

$$\left\{ y_i y_j u_i^{a_i} u_j^{a_j} - zv f_i f_j \;\middle|\; 1 \leq i, j \leq m \right\} \tag{2.5}$$

and

$$\left\{ \sum_{i=1}^{m} c_{ij} y_i u_i^{a_i} \,\middle|\, 1 \le j \le \operatorname{rank}(F_0) \right\}. \tag{2.6}$$

The degrees of these generators are

$$\begin{aligned} \deg(y_i y_j u_i^{a_i} u_j^{a_j} - zv f_i f_j) &= \deg(f_i) + \deg(f_j) + 2 \\ \deg\left(\sum_{i=1}^{m} c_{ij} y_i u_i^{a_i}\right) &= 1 + \deg\left(\sum_{i=1}^{m} c_{ij}\xi_i\right), \end{aligned} \tag{2.7}$$

where $\sum_{i=1}^{m} c_{ij}\xi_i$ belongs to the minimal system of homogeneous generators (2.3) of $\operatorname{Syz}_1^S(I)$.

The ideal P is homogeneous. It is non-degenerate since there are no linear forms among the generators listed above.

Example 2.8. Let $S = k[x_1, x_2, x_3]$ and $I = (x_1x_2,\ x_1x_3,\ x_2x_3)$. Computation with Macaulay2 shows that the minimal free resolution of I is

$$0 \longrightarrow S^2(-3) \xrightarrow{\begin{pmatrix} -x_3 & 0 \\ x_2 & -x_2 \\ 0 & x_1 \end{pmatrix}} S^3(-2)\,.$$

The ideal P is generated by

$$y_1^2u_1^4 - zvx_1^2x_2^2, \quad y_2^2u_2^4 - zvx_1^2x_3^2, \quad y_3^2u_3^4 - zvx_2^2x_3^2,$$
$$y_1y_2u_1^2u_2^2 - zvx_1^2x_2x_3, \quad y_1y_3u_1^2u_3^2 - zvx_1x_2^2x_3, \quad y_2y_3u_2^2u_3^2 - zvx_1x_2x_3^2$$

and

$$-x_3y_1u_1^2 + x_2y_2u_2^2, \quad -x_2y_2u_2^2 + x_1y_3u_3^2$$

in the ring $R = k[x_1, x_2, x_3, y_1, y_2, y_3, u_1, u_2, u_3, z, v]$. Here are some numerical invariants of I and P computed by Macaulay2 [M2] and illustrating Theorem 1.6:

$$\begin{aligned} \operatorname{pd}(R/P) &= 4 & \operatorname{pd}(S/I) &= 2 \\ \operatorname{depth}(R/P) &= 7 & \operatorname{depth}(S/I) &= 1 \\ \operatorname{reg}(R/P) &= 9 & \operatorname{reg}(S/I) &= 1 \\ \operatorname{codim}(P) &= 3 & & \\ \deg(R/P) &= 54 = 2 \times 3^3\,. & & \end{aligned}$$

Proposition 2.9. *In the notation and under the assumptions of 2.4, the set of generators (2.5) and (2.6) of P is minimal.*

Proof. Suppose that one of the considered generators is an R-linear combination of the others. This remains the case after we set $z = v = 0$ and $u_1 = \cdots = u_m = 1$.

Thus, an element g in one of the sets

$$\mathcal{A} := \{y_i y_j \,|\, 1 \le i, j \le m\}$$
$$\mathcal{B} := \left\{ \sum_{i=1}^{m} c_{ij} y_i \,\middle|\, 1 \le j \le \operatorname{rank}(F_0) \right\}$$

is an $S[y_1, \ldots, y_m]$-linear combination of the other elements in these sets. We will work over the ring $S[y_1, \ldots, y_m]$.

By Construction 2.2, we have $c_{ij} \in (x_1, \ldots, x_n)$. Hence, $\mathcal{B} \subset (x_1, \ldots, x_n)$, and it follows that $g \notin \mathcal{A}$.

Let $g \in \mathcal{B}$. Since $\mathcal{A}$ generates $(y_1, \ldots, y_m)^2$ it follows that g is an S-linear combination of the elements in $\mathcal{B}$. This contradicts the fact that the columns of the matrix C (in the notation of Construction 2.2) form a minimal system of generators of $\operatorname{Syz}_1^S(I)$. □

Recall from the Introduction that for a finitely generated graded module N (over a positively graded polynomial ring), we denote by $\operatorname{maxdeg}(N)$ the maximal degree of an element in a minimal system of homogeneous generators of N.

Corollary 2.10. *In the notation and under the assumptions of 2.4,*

$$\operatorname{maxdeg}(P) = \max\left\{ 1 + \operatorname{maxdeg}\big(\operatorname{Syz}_1^S(I)\big),\ 2\big(\operatorname{maxdeg}(I) + 1\big) \right\}$$

Proof. Apply (2.7), and note that the maximal degree of an element in (2.6) is $\operatorname{maxdeg}\big(\operatorname{Syz}_1^S(I)\big) + 1$. □

3. Rees-like algebras

Given a homogeneous ideal I (in the notation of 2.2), we will define a prime ideal Q using a Rees-like construction. We will give an explicit set of generators of Q and then study its minimal free resolution.

Construction 3.1. In the notation and under the assumptions of 2.2, we will construct a prime ideal Q. We introduce a new polynomial ring

$$T = S[y_1, \ldots, y_m, z]$$

graded by $\deg(z) = 2$ and $\deg(y_i) = \deg(f_i) + 1$ for every i.

Consider the graded homomorphism (of degree 0)

$$\begin{aligned} \varphi:\ T &\longrightarrow S[It, t^2] \subset S[t] \\ y_i &\longmapsto f_i t \\ z &\longmapsto t^2\,, \end{aligned}$$

where t is a new variable and $\deg(t) = 1$. The homogeneous ideal $Q = \operatorname{Ker}(\varphi)$ is prime. Note that $Q \cap S[z] = 0$ since $S[z]$ maps isomorphically to $S[t^2]$.

Proposition 3.2. *In the notation above and 2.2, the ideal Q is generated by the elements*

$$\left\{ y_i y_j - z f_i f_j \,\middle|\, 1 \le i, j \le m \right\} \tag{3.3}$$

and

$$\left\{ \sum_{i=1}^{m} c_{ij} y_i \,\middle|\, 1 \le j \le \operatorname{rank}(F_0) \right\}. \tag{3.4}$$

Proof. First note that the elements in (3.3) and (3.4) are in $Q = \operatorname{Ker}(\varphi)$ since

$$\varphi\left(y_i y_j - z f_i f_j\right) = f_i t f_j t - t^2 f_i f_j = 0$$
$$\varphi\left(\sum_{i=1}^{m} c_{ij} y_i\right) = t \sum_{i=1}^{m} c_{ij} f_i = 0$$

by (2.3).

Let $e \in Q$. We may write $e = f + g$, where $f \in (y_1, \ldots, y_m)^2$ and $g \in S[z]\operatorname{Span}_k\{1, y_1, \ldots, y_m\}$. Using elements in (3.3) we reduce to the case when $f = 0$, so $e = h(z) + \sum_{i=1}^{m} h_i(z) y_i$ with $h(z), h_1(z), \ldots, h_m(z) \in S[z]$. Then

$$0 = \varphi(e) = h(t^2) + \sum_{i=1}^{m} h_i(t^2) t f_i \in S[t]$$

implies that $h(z) = 0$ since $h(t^2)$ contains only even powers of t while $\sum_{i=1}^{m} h_i(t^2) t f_i$ contains only odd powers of t. Thus $e \in (y_1, \ldots, y_m)$ and we may write

$$e = z^p \sum_{i=1}^{m} g_i y_i + \text{ (terms in which } z \text{ has degree} < p)$$

for some $p \ge 0$ and $g_1, \ldots, g_m \in S$. We will argue by induction on p that e is in the ideal generated by the elements in (3.4). Suppose $e \ne 0$. We consider

$$0 = \varphi(e) = t^{2p} t \sum_{i=1}^{m} g_i f_i + \text{ (terms in which } t \text{ has degree} \le 2p - 1),$$

and conclude that $\sum_{i=1}^{m} g_i f_i = 0$. As $\operatorname{Syz}_1^S(I)$ is generated by the elements in (2.3), it follows that $\sum_{i=1}^{m} g_i y_i$ is in the ideal generated by the elements in (3.4). The element

$$e - z^p \sum_{i=1}^{m} g_i y_i \in \operatorname{Ker}(\varphi)$$

has smaller degree with respect to the variable z. The base of the induction is $e = 0$. □

Remark 3.5. We remark that Proposition 3.2 and its proof hold much more generally in the sense that S does not need to be a standard graded polynomial ring. In this paper, we will only use Proposition 3.2 as it is stated above.

Corollary 3.6. *The set of generators in Proposition 3.2 is minimal.*

Proof. Suppose that one of the considered generators is a T-linear combination of the others. This remains the case after we set $z = 0$, and then we can apply the proof of Proposition 2.9. $\square$

In the rest of this section, we focus on the minimal graded free resolution of T/Q over T.

Observation 3.7. We work in the notation and under the assumptions of 3.1. Since Q is a non-degenerate prime, z is a non-zerodivisor on T/Q. Let

$$\overline{T} := T/(z) = k[x_1, \dots, x_n, y_1, \dots, y_m]$$

and denote by $\overline{Q} \subset \overline{T}$ the homogeneous ideal (which is the image of Q) generated by

$$\left\{ y_i y_j \,\middle|\, 1 \leq i, j \leq m \right\}$$

and

$$\left\{ r_j := \sum_{i=1}^{m} c_{ij} y_i \,\middle|\, 1 \leq j \leq \operatorname{rank}(F_0) \right\}.$$

It follows that the graded Betti numbers of T/Q over T are equal to those of $\overline{T}/\overline{Q}$ over $\overline{T}$.

We are grateful to Maria Evelina Rossi, who pointed out that $\overline{T}/\overline{Q}$ is the Nagata idealization of S with respect to the ideal I (see [Na] for the definition of Nagata idealization).

Construction 3.8. We remark that the minimal graded free resolution of T/Q over T is not a mapping cone. However, we will construct the minimal graded free resolution of $\overline{T}/\overline{Q}$ over $\overline{T}$ using a mapping cone. Minimality is proved in Theorem 3.10. We work in the notation and under the assumptions of 3.7. Consider the ideals

$$M := \left(r_j \,\middle|\, 1 \leq j \leq \operatorname{rank}(F_0) \right)$$
$$N := (y_1, \dots, y_m)^2,$$

so $\overline{Q} = M + N$.

There is a short exact sequence

$$0 \longrightarrow M/(M \cap N) \xrightarrow{\gamma} \overline{T}/N \longrightarrow \overline{T}/(M+N) = \overline{T}/\overline{Q} \longrightarrow 0, \tag{3.9}$$

where γ is the homogeneous map (of degree 0) induced by $M \subset \overline{T}$. Let $(\mathbf{B}, d^B)$ and $(\mathbf{G}, d^G)$ be the graded minimal free resolutions of $M/(M \cap N)$ and $\overline{T}/N$, respectively.

Let $\zeta : \mathbf{B} \longrightarrow \mathbf{G}$ be a homogeneous lifting of γ. Its mapping cone $\mathbf{D}$ is a graded free resolution of $\overline{T}/\overline{Q}$ over $\overline{T}$. It is the complex with modules

$$D_q = G_q \oplus B_{q-1}$$

and differential

$$\begin{array}{c} \\ G_{q-1} \\ B_{q-2} \end{array} \begin{array}{c} \begin{array}{cc} G_q & \qquad B_{q-1} \end{array} \\ \begin{pmatrix} d_q^G & \zeta_{q-1} \\ 0 & (-1)^{q-1} d_{q-1}^B \end{pmatrix} \end{array}.$$

Thus, as a bigraded (graded by homological degree and by internal degree) module

$$\mathbf{D} = \mathbf{G} \oplus \mathbf{B}[1]\,.$$

We will describe the resolutions $\mathbf{G}$ and $\mathbf{B}$.

The resolution $\mathbf{G}$ may be expressed as $\overline{T} \otimes \mathbf{G}'$, where $\mathbf{G}'$ is the Eliahou-Kervaire resolution (or the Eagon-Northcott resolution) that resolves minimally $k[y_1, \ldots, y_m]/(y_1, \ldots, y_m)^2$ over the polynomial ring $k[y_1, \ldots, y_m]$.

Next, we consider the resolution $\mathbf{B}$. Set $b = \operatorname{rank}(B_0) = \operatorname{rank}(F_0)$. Choose a basis $\rho_1, \ldots, \rho_b$ of B_0 such that ρ_j maps to r_j for every j. Note that $(y_1, \ldots, y_m)$ annihilates each $r_j \in M/(M \cap N)$, and so

$$(y_1, \ldots, y_m)\rho_j \in \operatorname{Syz}_1^{\overline{T}}\big(M/(M \cap N)\big)$$

for every j. We want to find $(h_1, \ldots, h_b) \in B_0$ that minimally generate the syzygy module. We can reduce to the case where every $h_j \in S$ since $(y_1, \ldots, y_m)B_0 \subseteq \operatorname{Syz}_1^{\overline{T}}(M/(M \cap N))$. Let $(h_1, \ldots, h_b) \in S^b$. As

$$\begin{aligned} B_0 &\longrightarrow \big(M/(M \cap N)\big) \\ \rho_j &\longmapsto r_j = \sum_{i=1}^{m} c_{ij} y_i \quad \text{for } 1 \le j \le b\,, \end{aligned}$$

we have

$$d^B\Big(\sum_{j=1}^{b} h_j \rho_j\Big) = \sum_{j=1}^{b} h_j d^B(\rho_j) = \sum_{j=1}^{b} h_j r_j = \sum_{j=1}^{b} h_j \sum_{i=1}^{m} c_{ij} y_i = \sum_{i=1}^{m} \Big(\sum_{j=1}^{b} h_j c_{ij}\Big) y_i.$$

On the other hand, in the free resolution $\mathbf{F}'$ (see 2.2 for notation) we have

$$\begin{aligned} F_0 &\longrightarrow F_{-1} \\ \mu_j &\longmapsto \sum_{i=1}^{m} c_{ij} \xi_i \quad \text{for } 1 \le j \le b\,, \end{aligned}$$

where $\xi_1, \dots, \xi_m$ is a basis of F_{-1} such that $d^F(\xi_i) = f_i$ and $\mu_1, \dots, \mu_b$ is a basis of F_0; therefore,

$$d^F\Big(\sum_{j=1}^{b} h_j\mu_j\Big) = \sum_{j=1}^{b} h_j d^F(\mu_j) = \sum_{j=1}^{b} h_j \sum_{i=1}^{m} c_{ij}\xi_i = \sum_{i=1}^{m}\Big(\sum_{j=1}^{b} h_j c_{ij}\Big)\xi_i\,.$$

It follows that $\sum_{j=1}^{b} h_j\rho_j$ is in $\mathrm{Syz}_1^{\overline{T}}(M/(M\cap N))$ if and only if $\sum_{j=1}^{b} h_j c_{ij} = 0$ for every i, if and only if $\sum_{j=1}^{b} h_j\mu_j$ is in $\mathrm{Syz}_2^S(I)$. We have proved:

$$\mathrm{Syz}_1^{\overline{T}}\big(M/(M\cap N)\big) = (y_1, \dots, y_m)B_0 + \mathrm{Syz}_2^S(I)\otimes_S \overline{T}\,.$$

Therefore,

$$\mathbf{B} = \mathbf{K}_{\overline{T}}(y_1, \dots, y_m)\otimes_{\overline{T}}\big(\mathbf{F}(-1)\otimes_S \overline{T}\,\big)\,,$$

where

- $\mathbf{K}_{\overline{T}}(y_1, \dots, y_m)$ is the Koszul complex on $y_1, \dots, y_m$ over $\overline{T}$.
- $\mathbf{F}$ is the minimal S-free resolution of $\mathrm{Syz}_1^S(I)$ by 2.2.

We remark that the acyclicity of the tensor product of complexes above follows from

$$H_q\big(\mathbf{K}_{\overline{T}}(y_1, \dots, y_m)\otimes_{\overline{T}}(\mathbf{F}\otimes_S \overline{T})\big) \cong H_q\big(\overline{T}/(y_1, \dots, y_m)\otimes_{\overline{T}}(\mathbf{F}\otimes_S \overline{T})\big) \cong H_q\big(\mathbf{F}\big) = 0$$

for $q > 0$. Also, note that the shift $\mathbf{F}(-1)$ is explained by

$$\begin{aligned}\deg(\rho_j) &= \deg(r_j) = \deg(c_{ij}) + \deg(y_i) = \deg(c_{ij}) + \deg(f_i) + 1\\ &= \deg(c_{ij}) + \deg(\xi_i) + 1 = \deg(\mu_j) + 1\,.\end{aligned}$$

Theorem 3.10. *In the notation and under the assumptions above, the graded minimal $\overline{T}$-free resolution of $\overline{T}/\overline{Q}$ can be described as a bigraded (graded by homological degree and by internal degree) module by*

$$\mathbf{D} = \mathbf{G}\ \oplus\ \big(\mathbf{K}_{\overline{T}}(y_1, \dots, y_m)\otimes \mathbf{F}(-1)\big)[1]\,,$$

where

- $\mathbf{G} = \overline{T}\otimes \mathbf{G}'$*, where* $\mathbf{G}'$ *is the Eliahou-Kervaire resolution (or the Eagon-Northcott resolution) that minimally resolves* $k[y_1, \dots, y_m]/(y_1, \dots, y_m)^2$ *over* $k[y_1, \dots, y_m]$.
- $\mathbf{K}_{\overline{T}}(y_1, \dots, y_m)$ *is the Koszul complex on* $y_1, \dots, y_m$ *over* $\overline{T}$.
- $\mathbf{F}$ *is the minimal* S*-free resolution of* $\mathrm{Syz}_1^S(I)$ *by 2.2.*
- $\mathbf{F}[1](-1)$ *is obtained from* $\mathbf{F}$ *by shifting according to Notation 2.1, (we shift the resolution* $\mathbf{F}$ *one step higher in homological degree and increase the internal degree by* 1*).*

Proof. We will prove that the free resolution $\mathbf{D}$ obtained in Construction 3.8 is minimal by showing that the map γ can be lifted to a minimal homogeneous map

$\zeta:\ \mathbf{B}\longrightarrow\mathbf{G}$. We will show by induction on the homological degree q that ζ_q can be chosen so that

$$\operatorname{Im}(\zeta_q)\subseteq(x_1,\dots,x_n)G_q$$

for all $q\geq 0$. This property holds in the base case $q=0$ since we may choose $\zeta_0(\rho_j)=r_j=\sum_{i=1}^{m}c_{ij}y_i$ (where $\rho_1,\dots,\rho_b$ is the basis of B_0 chosen in Construction 3.8) and $c_{ij}\in(x_1,\dots,x_n)$ for all i and j by Construction 2.2.

Consider $q\geq 1$. Let $\tau_1,\dots,\tau_p$ be a homogeneous basis of B_q. For each $1\leq r\leq p$, we will define $\zeta_q(\tau_r)$. As $d^G_{q-1}\big(\zeta_{q-1}(d^B_q(\tau_r))\big)=\zeta_{q-2}\big(d^B_{q-1}(d^B_q(\tau_r)\big)=0$ and $\mathbf{G}$ is a resolution, there exists a homogeneous $g_r\in G_q$ with $d^G_q(g_r)=\zeta_{q-1}\left(d^B_q(\tau_r)\right)$. By induction hypothesis, we conclude that $d^G_q(g_r)\in(x_1,\dots,x_n)G_{q-1}$. We may write $g_r=h_r+e_r$, where $h_r\in(x_1,\dots,x_n)G_q$ and $e_r\in G'_q$ in view of the decomposition $\mathbf{G}=\overline{T}\otimes\mathbf{G}'=S\otimes\mathbf{G}'=(x_1,\dots,x_n)\mathbf{G}\oplus\mathbf{G}'$ as $k[y_1,\dots,y_m]$-modules induced by the decomposition $S=(x_1,\dots,x_n)\oplus k$. Hence,

$$d^G_q(e_r)=d^G_q(g_r)-d^G_q(h_r)\in(x_1,\dots,x_n)G_{q-1}\,.$$

Note that the differential in the Eliahou-Kervaire resolution $\mathbf{G}'$ preserves both summands in the considered decomposition. Therefore,

$$d^G_q(e_r)\in\big((x_1,\dots,x_n)G_{q-1}\big)\cap G'_{q-1}=0\,.$$

Thus, we can define $\zeta_q(\tau_r)=h_r\in(x_1,\dots,x_n)G_q$. □

4. **Step-by-step Homogenization**

Recall that a polynomial ring over k is called *standard graded* if all the variables have degree 1. The method of Step-by-step Homogenization, given by Theorem 4.5, can be applied to any non-degenerate prime ideal M in a positively graded polynomial ring W in order to obtain a non-degenerate prime ideal M' in a standard graded polynomial ring W' (with more variables). Its key property is that the graded Betti numbers are preserved; note that the graded Betti numbers usually change after homogenizing an ideal.

Motivation 4.1. The ideal Q (defined in the previous section) is a prime ideal in the polynomial ring T, which is not standard graded. Our goal is to construct a prime ideal in a standard graded ring. We may change the degrees of the variables $y_1,\dots,y_m,z$ to 1, but then Q is no longer homogeneous and we have to homogenize it. We change the degrees of $y_1,\dots,y_m,z$ one variable at a time and homogenize at each step using new variables $u_1,\dots,u_m,v$; this step-by-step homogenization assures that the degrees of the generators in Proposition 3.2 do not get smaller after homogenization. Usually in order to obtain a generating set of a homogenized ideal one needs to homogenize a Gröbner basis, but in our case it suffices to homogenize a minimal set of generators by Lemma 4.2. We will see in Proposition 4.8 that the ideal P, as defined in Construction 2.4, is obtained from Q in this way.

Consider a polynomial ring $\widetilde{W} = k[w_1, \ldots, w_q]$ positively graded by $\deg(w_i) \in \mathbb{N}$. Let $g \in \widetilde{W}$. We write g as a sum $g = g_1 + \cdots + g_p$ of homogeneous components. Consider $\widetilde{W}[s]$, where s is a new variable of degree 1. Recall that the $\widetilde{W}$*-homogenization* of g is the polynomial

$$g_{hom} = \sum_{1 \le j \le p} s^{\deg(g) - \deg(g_j)} g_j \in \widetilde{W}[s]\,.$$

One-step Homogenization Lemma 4.2. *Consider a polynomial ring $W = k[w_1, \ldots, w_q]$ positively graded with $\deg(w_i) \in \mathbb{N}$ for every i. We say that a polynomial is W-homogeneous if it is homogeneous with respect to the grading of W. Let M be a W-homogeneous prime ideal, and let $\mathcal{K}$ be a minimal set of W-homogeneous generators of M. Suppose $\deg(w_1) > 1$ and $w_1 \notin M$.*

Consider M as an ideal in $\widetilde{W} = k[w_1, \ldots, w_q]$, where $\deg(w_1) = 1$ and all other variables have the same degree as in W (thus, $\widetilde{W}$ and W are the same polynomial ring but with different gradings). Consider $\widetilde{W}[s]$, where s is a new variable of degree 1, and let M_{hom} be the ideal generated by the $\widetilde{W}$-homogenizations of the elements in $\mathcal{K}$. Then:

1. *$\widetilde{W}$-homogenization of the elements in $\mathcal{K}$ is obtained by replacing the variable w_1 by $w_1 s^{\deg_W(w_1)-1}$, (which we call relabeling). In particular, $\widetilde{W}$-homogenization preserves the degrees (with respect to the W-grading) of these elements.*
2. *The ideal M_{hom} is prime.*
3. *The graded Betti numbers of M over W are the same as those of M_{hom} over $\widetilde{W}[s]$.*

Proof. Since M is prime and $w_1 \notin M$ by assumption, none of the elements in $\mathcal{K}$ is divisible by w_1, and so $\widetilde{W}$-homogenization preserves their degrees with respect to the W-grading. We have proved (1).

To simplify the notation, set $U = W/M$ and $a = s^{\deg_W(w_1)-1}$, where $\deg_W(w_1)$ is taken with respect to the W-grading. Observe that we have graded isomorphisms (of degree 0):

$$\begin{aligned} \widetilde{W}[s]/M_{hom} &= k[w_1, \ldots, w_q, s]/M_{hom} \\ &\cong k[w_2, \ldots, w_q, s, u]/\big(\alpha(\mathcal{K})\big) \\ &\cong k[w_1, w_2, \ldots, w_q, s, u]/(M,\, w_1 - au) = W[s,u]/(M,\, w_1 - au) \\ &\cong U[s,u]/(w_1 - au)\,, \end{aligned} \tag{4.3}$$

where:

- u is a new variable of degree 1.
- The first isomorphism is

$$k[w_1, w_2, \ldots, w_q, s] \xrightarrow{w_1 \mapsto u} k[w_2, \ldots, w_q, s, u]\,.$$

  Its purpose is just to rename the variable w_1 in order to make the rest of the notation clearer.
- The second isomorphism is induced by the isomorphism

$$\begin{aligned} \alpha:\ k[w_1, w_2, \ldots, w_q, s, u]/(w_1 - au) &\longrightarrow k[w_2, \ldots, w_q, s, u] \\ w_1 &\longmapsto au\,. \end{aligned}$$

Statement (2) can be proved using Buchberger's algorithm to show that the ideal M_{hom} is the homogenization $(\, g_{hom} \,|\, g \in M\,)$ of M in $\widetilde{W}[s]$, and thus $\widetilde{W}[s]/M_{hom}$ is a domain. We are grateful to David Eisenbud, who suggested the following alternative: by [Ei3, Exercise 10.4], $U[s,u]/(w_1 - au)$ is a domain. For completeness, we present a proof of that exercise. Localizing at a, we get the homomorphism

$$\begin{aligned} \psi:\ U[s,u] &\longrightarrow U[s]_a \\ u &\longmapsto w_1 a^{-1}\,. \end{aligned}$$

Clearly, $w_1 - au \in \mathrm{Ker}(\psi)$. Let $g \in \mathrm{Ker}(\psi)$. Write $g = \sum_{i=0}^{p} h_i u^i$ with $h_i \in U[s]$. Then

$$0 = \psi(g) = \sum_{i=0}^{p} h_i w_1^i a^{-i} = a^{-p} \sum_{i=0}^{p} h_i w_1^i a^{p-i}$$

in $U[s]_a$. Therefore, $a^r \sum_{i=0}^{p} h_i w_1^i a^{p-i} = 0$ in $U[s]$ for some power r. Since a is a non-zerodivisor in $U[s]$, we conclude that $\sum_{i=0}^{p} h_i w_1^i a^{p-i} = 0$ in $U[s]$. Hence, $h_p w_1^p = af$ for some $f \in U[s]$. As M is prime and $w_1 \notin M$, we have that w_1^p is a non-zerodivisor on U. Since a, w_1^p is a homogeneous regular sequence on $U[s]$, it follows that $h_p = ae$ for some $e \in U[s]$. Therefore, $g - eu^{p-1}(au - w_1) \in \mathrm{Ker}(\psi)$ and has smaller degree (in the variable u) than g. Proceeding in this way, we conclude $g \in (w_1 - au)$. Thus, $\mathrm{Ker}(\psi) = (w_1 - au)$.

(3) The graded Betti numbers of W/M over W are equal to those of $W[s,u]/M$ over $W[s,u]$, and hence equal to those of $W[s,u]/(M,\, w_1 - au)$ over $W[s,u]/(w_1 - au)$ since $w_1 - au$ is a homogeneous non-zerodivisor. Hence, they are equal to the graded Betti numbers of $\widetilde{W}[s]/M_{hom}$ over $\widetilde{W}[s] \cong W[s,u]/(w_1 - au)$ by (4.3). □

Example 4.4. We will illustrate how Lemma 4.2 works and compare it to the traditional homogenization in the simple example of the twisted cubic curve. We will use notation that is different than in the rest of the paper.

We consider the defining ideal of the affine monomial curve parametrized by (t, t^2, t^3). It is the prime ideal E that is the kernel of the map

$$\begin{aligned} W := k[x,y,z] &\longrightarrow k[t] \\ x &\longmapsto t \\ y &\longmapsto t^2 \\ z &\longmapsto t^3 \,. \end{aligned}$$

This ideal is

$$E = (x^2 - y,\ xy - z)\,.$$

It is graded with respect to the grading defined by $\deg(x) = 1,\ \deg(y) = 2,\ \deg(z) = 3$. The graded Betti numbers of E over W are $\beta_{0,2} = 1, \beta_{0,3} = 1, \beta_{1,5} = 1$ and thus $\operatorname{reg}(E) = 4$.

Applying Lemma 4.2 two times, in the defining equations of E we replace the variable y by yu and we replace the variable z by zv^2. Thus, we obtain the homogeneous prime ideal

$$E' = (x^2 - yu,\ xyu - zv^2)$$

in the ring $W' = k[x, y, z, u, v]$ which is standard graded (all variables have degree one). The graded Betti numbers of E' over W' (and thus also the regularity) are the same as the graded Betti numbers of E over W.

On the other hand, the traditional homogenization (that is, taking projective closure) of E is obtained by homogenizing a Gröbner basis. The generators $x^2 - y,\ xy - z$ and the element $xz - y^2$ form a minimal Gröbner basis with respect to the degree-lex order. Homogenizing them with a new variable w we obtain the homogeneous prime ideal

$$E'' = (x^2 - yw,\ xy - zw,\ xz - y^2)$$

in the ring $W'' = k[x, y, z, w]$ which is standard graded. We have fewer variables in the ring W'' than in W'. However:

(1) One needs a Gröbner basis computation in order to obtain the generators of E'', while the generators of E' are obtained from those of E.
(2) The Betti numbers of E'' over W'' are $\beta'_{0,2} = 3,\ \beta'_{1,3} = 2$, and so they are different than those of E over W; moreover $\operatorname{reg}(E'') = 2$, which is smaller than $\operatorname{reg}(E)$.

Step-by-step Homogenization Theorem 4.5. *Consider a polynomial ring* $W = k[w_1, \dots, w_p]$ *positively graded with* $\deg(w_i) \in \mathbb{N}$ *for every* i. *Suppose* $\deg(w_i) > 1$ *for* $i \le q$ *and* $\deg(w_i) = 1$ *for* $i > q$ *(for some* $q \le p$*). Let* M *be a homogeneous non-degenerate prime ideal, and let* $\mathcal{K}$ *be a minimal set of homogeneous generators* *of M. Consider the homogenous map (of degree 0)*

$$\nu : W = k[w_1, \ldots, w_p] \longrightarrow W' := k[w_1, \ldots, w_p, v_1, \ldots, v_q]$$
$$w_i \longmapsto w_i v_i^{\deg_W(w_i)-1} \quad \textit{for } 1 \le i \le q\,,$$

where $v_1, \ldots, v_q$ are new variables and W' is standard graded. The ideal $M' \subset W'$ generated by the elements of $\nu(\mathcal{K})$ is a homogeneous non-degenerate prime ideal in W'. Furthermore, the graded Betti numbers of W'/M' over W' are the same as those of W/M over W.

We say that M' is obtained from M by *Step-by-step Homogenization* or by *relabeling* (the latter is motivated by a similar construction, called relabeling of monomial ideals, in [GPW]).

Proof. We will homogenize repeatedly, applying Lemma 4.2 at each step. The proof is by induction on an invariant j defined below. For the base case $j = 0$, set $N(0) := M$ and $Z(0) := W$.

Suppose that by induction hypothesis, we have constructed a non-degenerate prime ideal $N(j)$ that is homogeneous in the polynomial ring $Z(j) = k[w_1, \ldots, w_p, v_1, \ldots, v_j]$ graded so that $\deg(w_1) = \cdots = \deg(w_j) = 1$ and $\deg(w_{j+1}) > 1$. Now, change the grading of the ring so that $\deg(w_{j+1}) = 1$, but all other variables retain their degrees. Let $N(j+1)$ be the ideal $N(j)_{hom}$, defined in Lemma 4.2 using a new variable v_{j+1} of degree 1, in the ring $Z(j+1) = Z(j)[v_{j+1}]$. It is generated by the homogenizations of the generators of $N(j)$ obtained in the previous step. By Lemma 4.2, the ideal $N(j+1)$ is non-degenerate, prime, and homogenous in $Z(j+1)$. Furthermore, the graded Betti numbers are preserved.

The process terminates at $W' := Z(q)$ and $M' := N(q)$. □

Example 4.6. Using our Step-by-step Homogenization method, we can produce the following counterexample to the Regularity Conjecture 1.2. It does not prove Theorem 1.9 but has the advantage of being small enough to be computed by Macaulay2 [M2]. Consider the ideal $I := I_{2,(2,1,7)}$ constructed in [BMN] in the standard graded polynomial ring

$$S = k[u, v, w, x, y, z]\,;$$

the ideal is

$$I = (u^{11}, v^{11}, u^2w^9 + v^2x^9 + uvwy^8 + uvxz^8)\,.$$

We computed with Macaulay2 over the fields $k = \mathbb{Z}_2$, $k = \mathbb{Z}_{32003}$, and $k = \mathbb{Q}$. Consider the prime ideal M that defines the Rees algebra $S[It]$. Let $M \subset W := S[w_1, w_2, w_3]$ be the defining prime ideal of $S[It]$, where $\deg(w_i) = 12$ for $i = 1, 2, 3$. Computation shows that $\operatorname{maxdeg}(M) = 418$. Apply the Step-by-step Homogenization, described in Theorem 4.5, and obtain a homogeneous non-degenerate prime ideal M' in a standard graded polynomial ring W' with 12 variables. It has multiplicity $\deg(W'/M') = 375$, which is smaller than $\operatorname{maxdeg}(M') = 418$. It has small codimension $\operatorname{codim}(M') = 2$. The computation shows that $\operatorname{pd}(W'/M') = 5$, and by the Auslander-Buchsbaum Formula we conclude $\operatorname{depth}(W'/M') = 7$.

We enlarge the field and make it algebraically closed; note that primeness is preserved because our prime ideal comes from a Rees algebra. By Bertini's Theorem, see [Fl], there exists a regular sequence of 6 generic linear forms so that primeness is preserved after factoring them out. The dimension of the obtained projective variety $X \subset \mathbb{P}^5$ is 3. Its degree is 375 and its regularity is ≥ 418.

Note that Kwak [Kw2] proved the inequality $\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim} X + 1$ if $X \subset \mathbb{P}^5$ is 3-dimensional, non-degenerate, irreducible, and *smooth*.

Example 4.7. We are grateful to Craig Huneke who suggested the following way of producing counterexamples with smaller multiplicity. In the case when all the generators of the ideal I have the same degree, the Rees algebra $S[It]$ has a graded presentation using a standard graded polynomial ring. The following is our smallest counterexample in terms of both dimension and degree.

Consider the ideal $I := I_{2,(2,1,2)}$ constructed in [BMN] in the standard graded polynomial ring

$$S = k[u, v, w, x, y, z]\,;$$

the ideal is

$$I = (u^6, v^6, u^2w^4 + v^2x^4 + uvwy^3 + uvxz^3)\,.$$

As in Example 4.6, we consider the defining prime ideal $M \subset W = S[w_1, w_2, w_3]$ of the Rees algebra $S[It]$, except now $\deg(w_i) = 1$ for $i = 1, 2, 3$. Computation with Macaulay2 [M2] shows that $\operatorname{maxdeg}(M) = 38$, $\deg(W/M) = 31$, and $\operatorname{pd}(W/M) = 5$. As $\dim(W) = 9$, we may apply Bertini's Theorem to obtain a projective threefold in $\mathbb{P}^5$ whose degree is 31 but its regularity is 38.

In light of the previous two examples, it would be interesting to find out if the Regularity Conjecture 1.2 or some other small bound holds for all projective surfaces. Recall that the conjecture holds for all *smooth* surfaces by Lazarsfeld [La] and Pinkham [Pi].

We now apply the Step-by-step Homogenization to the Rees-like algebras introduced in Section 3.

Proposition 4.8. *The ideal P, defined in Construction 2.4, is the Step-by-step Homogenization of the ideal Q defined in 3.1. The ideal P is prime. The graded Betti numbers of R/P over R are equal to those of T/Q over T, where T is the ring defined in 3.1.*

Proof. Recall that $\deg(z) = 2$ and $\deg(y_i) = a_i + 1$ for every i by Construction 3.1. Applying the Step-by-step Homogenization to the ideal Q replaces all instances of

y_i in the considered generators (3.3) and (3.4) of Q with $y_i u_i^{a_i}$, and similarly z is replaced by zv, where $u_1, \dots, u_m, v$ are new variables of degree 1. Thus, in the notation of 2.4, we obtain the ideal P in the standard graded polynomial ring R. Apply Theorem 4.5. □

5. Multiplicity and other numerical invariants

In this section, we compute the multiplicity, regularity, projective dimension, depth, and codimension of P using the free resolution in Theorem 3.10. For this purpose, we briefly review the concept of Euler polynomial.

Notation 5.1. Consider a polynomial ring $W = k[w_1, \dots, w_q]$ positively graded with $\deg(w_i) \in \mathbb{N}$. Fix a finite graded complex $\mathbf{V}$ of finitely generated W-free modules. We may write $V_i = \oplus_{j \in \mathbf{Z}} W(-j)^{b_{ij}}$. Suppose $b_{ij} = 0$ if $j < 0$ or $i < 0$. The *Euler polynomial* of $\mathbf{V}$ is

$$\mathrm{E}_{\mathbf{V}} = \sum_{i \geq 0} \sum_{j \geq 0} (-1)^i b_{ij} t^j \, .$$

Let N be a graded finitely generated W-module, and let $\mathbf{V}$ be a finite graded free resolution of N. Since every graded free resolution of N is isomorphic to the direct sum of the minimal graded free resolution and a trivial complex (see for example, [Ei3, Theorem 20.2]), it follows that the Euler polynomial does not depend on the choice of the resolution, so we call it the *Euler polynomial* of N. We factor out a maximal possible power of $1 - t$ and write

$$\mathrm{E}_{\mathbf{V}} = (1-t)^c \, h_{\mathbf{V}}(t) \, ,$$

where $h_{\mathbf{V}}(1) \neq 0$. If $\deg(w_1) = \dots = \deg(w_q) = 1$ and the module $N \neq 0$ is cyclic, then $h_{\mathbf{V}}(1) = \deg(N)$ and $c = \mathrm{codim}(N)$, (see for example, [Pe, Theorem 16.7]).

Theorem 5.2. *In the notation of 2.4, the multiplicity of R/P is*

$$\deg(R/P) = 2 \prod_{i=1}^{m} \big(\deg(f_i) + 1\big) \, .$$

Proof. By Proposition 4.8, the graded Betti numbers of R/P over R are equal to those of T/Q over T, and thus are equal to the graded Betti numbers of $\overline{T}/\overline{Q}$ over $\overline{T}$ by Observation 3.7. Therefore, we can compute the multiplicity of R/P using the free resolution $\mathbf{D}$ in Theorem 3.10. We will use the notation in Theorem 3.10 and Notation 5.1.

Recall that the resolution $\mathbf{G}'$ resolves $Y := k[y_1, \dots, y_m]/(y_1, \dots, y_m)^2$ and that $\deg(y_i) = a_i + 1$. Since

$$\frac{E_{\mathbf{G}'}}{\prod_{i=1}^{m} (1 - t^{a_i + 1})}$$

is the Hilbert series $1+\sum_{i=1}^m t^{a_i+1}$ of Y, it follows that the Euler polynomial of $\mathbf{G}$ is

$$E_{\mathbf{G}} = E_{\mathbf{G}'} = \left[\prod_{i=1}^m (1-t^{a_i+1})\right]\left[1+\sum_{i=1}^m t^{a_i+1}\right]. \tag{5.3}$$

The Euler polynomial of the Koszul complex is

$$E_{\mathbf{K}(y_1,\dots,y_m)} = \prod_{i=1}^m (1-t^{a_i+1})$$

since $\deg(y_i) = a_i+1$. Note that according to 2.2 we have

$$E_{\mathbf{F}} = E_{\mathbf{F}'} + \sum_{i=1}^m t^{a_i} - 1\,,$$

where $\mathbf{F}'$ is the minimal S-free resolution of S/I. We conclude

$$\begin{aligned} E_{\mathbf{B}[1]} &= E_{\mathbf{K}(y_1,\dots,y_m)\otimes\mathbf{F}(-1)[1]} \\ &= (-1)t\left[\prod_{i=1}^m (1-t^{a_i+1})\right]\left[E_{\mathbf{F}'} + \sum_{i=1}^m t^{a_i} - 1\right], \end{aligned} \tag{5.4}$$

where the factor $(-1)t$ reflects the shifts of the homological and internal degrees.

By (5.3), (5.4), and Theorem 3.10 it follows that the Euler polynomial of the graded free resolution $\mathbf{D}$ is

$$\begin{aligned} E_{\mathbf{D}} = E_{\mathbf{G}} + E_{\mathbf{B}[1]} &= \left[\prod_{i=1}^m (1-t^{a_i+1})\right]\left[1+\sum_{i=1}^m t^{a_i+1} - tE_{\mathbf{F}'} - t\sum_{i=1}^m t^{a_i} + t\right] \\ &= \left[\prod_{i=1}^m (1-t^{a_i+1})\right]\left[1+t-tE_{\mathbf{F}'}\right]. \end{aligned}$$

Since $\operatorname{codim}(I) \geq 1$, we have $E_{\mathbf{F}'}(1) = 0$. Therefore, evaluating the second factor $1+t-tE_{\mathbf{F}'}$ above at $t=1$, we get 2. Hence,

$$E_{\mathbf{D}} = (1-t)^m h_{\mathbf{D}}(t)$$

and

$$h_{\mathbf{D}}(t) = \left(\prod_{i=1}^m (1+t+\cdots+t^{a_i})\right)\left[1+t-tE_{\mathbf{F}'}\right].$$

By 5.1, the multiplicity is

$$h_{\mathbf{D}}(1) = 2\prod_{i=1}^m (a_i+1)\,. \qquad \square$$

Proof of Theorem 1.6 (3). By Proposition 4.8, the graded Betti numbers of R/P over R are equal to those of T/Q over T, and thus are equal to the graded Betti numbers of $\overline{T}/\overline{Q}$ over $\overline{T}$ by Observation 3.7. Thus, we can compute the considered numerical invariants of R/P using the minimal graded free resolution $\mathbf{D}$ in Theorem 3.10.

Since $\operatorname{reg}(\mathbf{F}) = \operatorname{reg}(S/I) + 2$, from Theorem 3.10 we obtain:

$$\begin{aligned}\operatorname{reg}(R/P) = \operatorname{reg}(\overline{T}/\overline{Q}) &= \max\big\{\operatorname{reg}(\mathbf{F}) + \operatorname{reg}(\mathbf{K}_{\overline{T}}),\ \operatorname{reg}(\mathbf{G})\big\} \\ &= \operatorname{reg}(\mathbf{F}) + \operatorname{reg}(\mathbf{K}_{\overline{T}}) = \operatorname{reg}(S/I) + 2 + \sum_{i=1}^{m} \deg(f_i).\end{aligned}$$

The projective dimension of R/P is equal to that of $\overline{T}/\overline{Q}$, so from Theorem 3.10 we obtain

$$\operatorname{pd}(R/P) = m + \operatorname{pd}(\operatorname{Syz}_1^S(I)) + 1 = m + \operatorname{pd}(S/I) - 1\,,$$

(where the summand +1 comes from the shifting in the mapping cone resolution).

By the Auslander-Buchsbaum Formula, the depth of R/P is

$$\begin{aligned}\operatorname{depth}(R/P) &= -\operatorname{pd}(R/P) + \operatorname{depth}(R) \\ &= -m - \operatorname{pd}(S/I) + 1 + \operatorname{depth}(S) + 2m + 2 \\ &= m + 3 + \operatorname{depth}(S/I)\,.\end{aligned}$$

The codimension of P is equal to that of $\overline{Q}$, so by the proof of Theorem 5.2 it follows that $\operatorname{codim}(P) = m$. Therefore, the dimension of R/P is

$$\dim(R/P) = \dim(R) - \operatorname{codim}(P) = m + n + 2\,.$$

□

6. **Projective Dimension**

In the notation of the Introduction, the analogue to Question 1.10 for projective dimension is:

Question 6.1. *Suppose the field k is algebraically closed. What is an optimal function $\Psi(x)$ such that $\operatorname{pd}(L) \le \Psi(\deg(L))$ for any non-degenerate homogeneous prime ideal L in a standard graded polynomial ring over k?*

Any such bound must be rather large by the following theorem, which is the projective dimension analogue of our Main Theorem 1.9.

Theorem 6.2. *Over any field k (in particular, over $k = \mathbb{C}$), the projective dimension of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the multiplicity, i.e., for any polynomial $\Theta(x)$ there exists a non-degenerate homogeneous prime ideal L in a standard graded polynomial ring V over the field k such that $\operatorname{pd}(L) > \Theta\big(\deg(V/L)\big)$.*

Proof. In [BMN, Corollary 3.6] there is a family of ideals I_r (for $r \geq 1$), each in a polynomial ring S_r, with three generators in degree r^2 and such that

$$\mathrm{pd}(S_r/I_r) \geq r^{r-1}\,.$$

Applying our method to these ideals yields prime ideals P_r in polynomial rings R_r with $\mathrm{codim}(P_r) = 3$, and

$$\deg(R_r/P_r) = 2(r^2+1)^3$$
$$\mathrm{pd}(R_r/P_r) \geq r^{r-1} + 2\,.$$

In this case, a polynomial function in the multiplicity yields a polynomial function in r, and so it cannot bound the projective dimension which is exponential in r. □

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Mathematics Department, Iowa State University, Ames, IA 50011, USA

Mathematics Department, Cornell University, Ithaca, NY 14853, USA