October 2018

RESEARCH STATEMENT

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My primary work is in Commutative Algebra, and my primary research is focused on Free Resolutions and Hilbert Functions. I have also done work on the many connections of Commutative Algebra with Algebraic Geometry, Combinatorics, Computational Algebra, Noncommutative Algebra, and Subspace Arrangements, and I remain very interested in these fields as well.

INTRODUCTION TO FREE RESOLUTIONS

Research on free resolutions is a core and beautiful area in Commutative Algebra. It contains a number of challenging conjectures and open problems; many of them are discussed in my book [Pe1]. The idea to describe the structure of a module by a free resolution was introduced by Hilbert in his famous paper [Hi]; this approach was present in the work of Cayley [Ca] as well. A *free resolution* of a finitely generated module T over a commutative noetherian ring R is an exact sequence

$$\mathbf{F}: \quad \ldots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

of homomorphisms of free finitely generated R-modules such that $F_0/\mathrm{Im}(d_1) \cong T$. The maps d_i are called *differentials*. Hilbert's Syzygy Theorem shows that if R is a polynomial ring, then every finitely generated graded R-module has a finite free resolution (that is $F_j = 0$ for $j \gg 0$); similarly every finitely generated module over a regular local ring has a finite free resolution.

If R is local, or if both R and T are graded, there exists a minimal free resolution $\mathbf{F}_T$ which is unique up to an isomorphism and is contained in any free resolution of T. The rank of the free module F_i in $\mathbf{F}_T$ is called the i'th *Betti number* and is denoted by $b_i^R(T)$. It may be expressed as

$$b_i^R(T) = \dim \mathrm{Tor}_i^R(T,k) = \dim \mathrm{Ext}_R^i(T,k)\,,$$

where k is the residue field of R. The Betti numbers are among the most studied numerical invariants of T. In the graded case we have graded Betti numbers $b_{i,j}^R(T)$ and Hilbert showed how to use them in order to compute the Hilbert function that measures the size of the module T. The submodule $\mathrm{Im}(d_i) = \mathrm{Ker}(d_{i-1})$ of F_{i-1} is called the i'th *syzygy module* of T; its minimal resolution is a truncation of $\mathbf{F}_T$.

Hilbert's insight was that the properties of the minimal free resolution $\mathbf{F}_T$ are closely related to the invariants of the module T. The key point is that the map $d_0: F_0 \to T$ sends a basis of F_0 to a minimal system $\mathcal{G}$ of generators of T, the first differential d_1 describes the minimal relations $\mathcal{R}$ among the generators $\mathcal{G}$, the second differential d_2 describes the minimal relations on the relations $\mathcal{R}$, etc. Hence, the resolution can be interpreted as

$$\ldots \to F_2 \xrightarrow{d_2 = \begin{pmatrix}\text{a minimal system}\\ \text{of relations}\\ \text{on the}\\ \text{relations } \mathcal{R}\end{pmatrix}} F_1 \xrightarrow{d_1 = \begin{pmatrix}\text{a minimal system}\\ \mathcal{R} \text{ of relations}\\ \text{on the}\\ \text{generators } \mathcal{G}\end{pmatrix}} F_0 \xrightarrow{d_0 = \begin{pmatrix}\text{a minimal}\\ \text{system } \mathcal{G} \text{ of}\\ \text{generators}\\ \text{of } T\end{pmatrix}} T\,.$$

Thus, the resolution is a way of describing the structure of T.

The condition of minimality is important. The mere existence of free resolutions suffices for foundational issues such as the definition of Ext and Tor, and there are various methods of producing resolutions uniformly (for example, the Bar resolution). But without minimality, resolutions are not unique, and the very uniformity of constructions like the Bar resolution implies that they give little insight into the structure of the modules resolved. In contrast, the minimal free resolution $\mathbf{F}_T$ encodes a lot of properties of T; for example, over a regular local (or graded) ring the Auslander-Buchsbaum Formula expresses the depth of T in terms of the length (called projective dimension) of $\mathbf{F}_T$, while non-minimal resolutions do not measure depth.

Constructing a free resolution may be interpreted as repeatedly solving systems of R-linear equations. This process is implemented in the computer algebra system Macaulay2.

Since the days of Cayley and Hilbert, minimal free resolutions have played many important roles in mathematics. They now appear in fields as diverse as Algebraic Geometry, Combinatorics, Computational Algebra, Invariant Theory, Mathematical Physics, Noncommutative Algebra, Number Theory, and Subspace Arrangements. For many years, they have been both central objects and fruitful tools in Commutative Algebra.

ABSTRACTS OF SOME OF MY PAPERS GROUPED BY TOPIC

1. Regularity. Regularity is a numerical invariant which measures the complexity of the structure of homogeneous ideals in a polynomial ring. We will work over a standard graded polynomial ring U over a field k. Let L be a homogeneous ideal in U, and let $b_{i,j}(L) = \dim_k \operatorname{Tor}_i^U(L,k)_j$ be its graded Betti numbers. The *Castelnuovo-Mumford regularity* of L is

$$\operatorname{reg}(L) = \max\left\{ j \,\middle|\, b_{i,\,i+j}(L) \neq 0 \text{ for some } i \right\}.$$

Alternatively, regularity can be defined using local cohomology, (see the books [Ei3] and [La2]).

The projective dimension $\operatorname{pd}(L) = \max\left\{ i \,\middle|\, b_{i,j}(L) \neq 0 \text{ for some } j \right\}$ and regularity are the main numerical invariants that measure the complexity of the minimal free resolution of L. Hilbert's Syzygy Theorem implies that both regularity and projective dimension are finite. It also provides a nice upper bound on the projective dimension: $\operatorname{pd}(L)$ is smaller than the number of variables. In contrast, Bayer-Mumford [BM] (using results by Giusti and Galligo) and Caviglia-Sbarra [CS] proved a doubly exponential upper bound on the regularity of homogeneous ideals. The bound is in terms of the number of variables and the degrees of the minimal generators of L. It is nearly sharp since the Mayr-Meyer construction [MM] leads to examples of families of ideals attaining doubly exponential regularity; such examples were constructed by Bayer-Mumford [BM], Bayer-Stillman [BS], and Koh [Ko]. It was expected that much better upper bounds hold for the defining ideals of geometrically nice projective varieties over an algebraically closed field. In the smooth case, important bounds were obtained by Bayer-Mumford [BM], Bertram-Ein-Lazarsfeld [BEL], Chardin-Ulrich [CU], and others.

As discussed in the influential paper [BM] by Bayer and Mumford, "the biggest missing link" between the general case and the smooth case is to obtain a "decent bound on the regularity of all reduced equidimensional ideals". In particular, there has been a lot of interest in producing a bound on the regularity of all prime ideals (the ideals that define irreducible projective varieties). The longstanding Regularity Conjecture predicts the following elegant linear bound in terms of the degree:

Regularity Conjecture 1.1. (Eisenbud-Goto, 1984) [EG] *Suppose that the field k is algebraically closed. If $L \subset (z_1,\ldots,z_p)^2$ is a homogeneous prime ideal in $U = k[z_1,\ldots,z_p]$, then*

$$\operatorname{reg}(L) \ \leq\ \deg(U/L) - \operatorname{codim}(L) + 1\,,$$

where $\deg(U/L)$ *is the degree of* U/L *(also called the multiplicity of* U/L*), and* $\operatorname{codim}(L)$ *is the codimension of* L.

Note that the conjectured upper bound does not depend on the number of variables p. The condition that $L \subset (z_1, \ldots, z_p)^2$ is equivalent to requiring that the projective variety $V(L)$ is not contained in a hyperplane in $\mathbf{P}_k^{p-1}$. Prime ideals that satisfy this condition are called *non-degenerate*.

The Regularity Conjecture is proved for curves by Gruson-Lazarsfeld-Peskine [GLP], completing classical work of Castelnuovo. It is also proved for smooth surfaces by Lazarsfeld [La1] and Pinkham [Pi], and for most smooth 3-folds by Ran [Ra]. It holds if U/L is Cohen-Macaulay by a result of Eisenbud-Goto [EG]. In the smooth case, Kwak ([Kw] gave bounds for regularity in dimensions 3 and 4 that are only slightly weaker. Many other special cases and related bounds have been proved as well.

In [MP] Jason McCullough and I construct counterexamples to the Regularity Conjecture. We provide a family of prime ideals P_r, depending on a parameter r, whose regularity $\operatorname{reg}(P_r)$ is doubly exponential in r and whose degree is singly exponential in r. Our main theorem is much stronger:

Theorem 1.2. (McCullough-Peeva, [MP]) *Let k be a field. The regularity of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the degree, i.e., for any polynomial $f(x) \in \mathbf{R}[x]$ there exists a non-degenerate homogeneous prime ideal Y in a standard graded polynomial ring V over the field k such that* $\operatorname{reg}(Y) > f\big(\deg(V/Y)\big)$.

Note that the theorem holds over any field k (the case $k = \mathbf{C}$ is particularly important). For this purpose, we introduce an approach, which starting from a homogeneous ideal I, produces a prime ideal P whose projective dimension, regularity, degree, dimension, depth, and codimension are expressed in terms of numerical invariants of I. The construction involves two new concepts:

(1) Rees-like algebras which, unlike the standard Rees algebras, have well-structured defining equations and minimal free resolutions;

(2) a new homogenization technique for prime ideals which, unlike classical homogenization, preserves graded Betti numbers.

Theorem 1.2 gives rise to the question whether there exists a bound on regularity of prime ideals (in a polynomial ring over an algebraically closed field) in terms of the degree alone. In a joint work [CCMPV] with Caviglia, Chardin, McCullough, and Varbaro we prove the existence of such a bound using a recent breakthrough result of Ananyan and Hochster [AH].

2. Free Resolutions over Complete Intersections. Minimal free resolutions over a local complete intersection R have attracted attention ever since the elegant construction of the minimal free resolution of the residue field k by Tate in 1957 [Ta]. The next impressive result was Gulliksen's proof [Gu] in 1974 that for every finitely generated R-module N, the Poincaré series $\sum_i b_i^R(N)t^i$ (where $b_i^R(N)$ are the Betti numbers) is rational and its denominator divides $(1-t^2)^c$ (where c is the codimension of R). For this purpose, he showed that $\operatorname{Ext}_R(N,k)$ can be regarded as a finitely generated graded module over a polynomial ring $k[\chi_1, \ldots, \chi_c]$, graded by $\deg(\chi_i) = 2$. This also implies that the even Betti numbers $b_{2i}^R(N)$ are eventually given by a polynomial in i, and the odd Betti numbers are given by another polynomial. In 1989 Avramov [Av] proved that the two polynomials have the same leading coefficient and the same degree. In 1997 Avramov, Gasharov and Peeva [AGP] showed that the truncated Betti sequence $\{b_i^R(N)\}_{i \geq q}$ is either strictly increasing or constant for $q \gg 0$ and proved further properties of the Betti numbers. By [Ei1, AGP] we have examples of minimal resolutions over R that have intricate structure but exhibit stable patterns when sufficiently truncated. Such examples and the numerical results above motivate the study of resolutions of high syzygies. Note that the minimal resolution of a syzygy module of N is a truncation of the minimal resolution of N.

The theory of matrix factorizations was introduced by Eisenbud [Ei1] to describe the structure of minimal free resolutions of high syzygies over a hypersurface. A *matrix factorization* of an element $f \neq 0$ in a regular local ring S is a pair (d, h) of maps of finitely generated S-free modules $A_0 \xrightarrow{h} A_1 \xrightarrow{d} A_0$ such that $hd = f \cdot \mathrm{Id}_{A_1}$ and $dh = f \cdot \mathrm{Id}_{A_0}$. This concept has many other applications: A major advance was made by Orlov [Or1, Or2] who showed that matrix factorizations could be used to study Kontsevich's homological mirror symmetry by giving a new description of singularity categories. Matrix factorizations have also been used in the study of Cohen-Macaulay modules and Singularity Theory [BGS, BHU, CH, Kn], cluster tilting [DH], Khovanov-Rozansky homology [KR1, KR2], moduli of curves [PVa], Hodge Theory [BFK], quiver and group representations, and other topics. Starting with Kapustin and Li [KL], who followed an idea of Kontsevich, physicists discovered amazing connections with String Theory – see [As] for a survey. Despite all this work on applications, progress on the structure of minimal free resolutions over complete intersections was scant.

As mentioned above, in 1980 Eisenbud [Ei1] described the minimal free resolutions of high syzygies over a hypersurface. In 2000, Avramov and Buchweitz analyzed the codimension 2 case. But the general case (of higher codimensions) was elusive. In contrast, non-minimal resolutions have been known for over 45 years from the work of Shamash [Sh]. Eisenbud and I have wondered, for many years, how to describe the eventual patterns in the minimal resolutions of modules over complete intersections of higher codimension. With the theory developed in our research monograph [EP1] we have found an answer. For this purpose, we introduce a new concept of higher matrix factorization (d, h) with respect to a regular sequence; this extends the theory of matrix factorizations of a non-zerodivisor. For a finitely generated module N over a local complete intersection, we show that any high syzygy of N is a higher matrix factorization module and we construct its minimal resolution.

Recently, we obtained in [EP2] a description of the structure of Cohen-Macaulay modules (of any codimension) over a regular local ring in terms of higher matrix factorizations; in the codimension one case a description in terms of matrix factorizations was given in [Ei1].

Consider a finitely generated module M annihilated by a regular sequence $f_1, \ldots, f_c$ in a regular local ring S with a residue field k, and set $R = S/(f_1, \ldots, f_c)$. Let E be the exterior algebra over k generated by c elements. The homotopies for the f_i on an S-free resolution of M induce a structure of graded E-module on $\mathrm{Tor}^S(M, k)$. In [ESP], Eisenbud, Schreyer, and I show that, when M is a high R-syzygy, the structure of $\mathrm{Tor}^S(M, k)$ carries a lot of interesting information. The Castelnuovo-Mumford regularity of the E-module $\mathrm{Tor}^S(M, k)$ is 1. The Betti numbers of the 0-linear strand of the minimal E-free graded resolution of $\mathrm{Tor}^S(M, k)$ are given by the even Betti numbers of M over the complete intersection R, and the Betti numbers of the 1-linear strand are given by the odd Betti numbers of M over R. In [EPS2] we provide a counterexample to Eisenbud's conjecture (since 1980) that the CI operators commute on a sufficiently high truncation of the minimal free resolution of any module over a complete intersection. In [EPS3] we study resolutions over quadratic complete intersections.

In [AGP] we introduce a new homological dimension: CI-dimension (complete intersection dimension). CI-dimension localizes and stands between the projective dimension and the Gorenstein dimension (introduced by Auslander and Bridger). A fundamental homological result for modules of finite projective dimension is the Auslander-Buchsbaum Formula (see [Ei2, Theorem 19.9]); we establish an analogue to it for modules of finite CI-dimension. We prove that a ring Q is a complete intersection if and only if its residue field has finite CI-dimension; this result is an analogue to the Auslander-Buchsbaum-Serre Criterion that Q is regular if and only if its residue field has finite projective dimension (see [Ei2, Theorem 19.12]). The class of modules of finite CI-dimension contains all modules of finite projective dimension and all modules over a complete intersection. For the study of such modules we develop some new cohomological tools and constructions over quantum symmetric algebras; in particular, we construct an extension of Manin's quantum Koszul complex [Man]. We find a place in the minimal resolution of a module of finite CI-dimension after which stable behavior develops; beyond this place the sequence of Betti numbers is either constant or strictly increasing, and gaps between consecutive numbers grow polynomially.

3. Finite regularity and Koszul Algebras. Consider a positively graded commutative algebra R that is finitely generated over a field k. Let T be a graded finitely generated R-module, and denote by $b^R_{i,j}(T)$ its graded Betti numbers. The complexity of the minimal free resolution of T may be measured by two important numerical invariants: its *projective dimension* $\max\{i \,|\, b^R_{i,j}(T) \neq 0 \text{ for some } j\}$ and its *Castelnuovo-Mumford regularity* $\max\{j \,|\, b^R_{i,i+j}(T) \neq 0 \text{ for some } i\}$. Projective dimension measures the number of systems of R-linear equations that have to be solved in order to build a minimal free resolution of T. Regularity measures the range of degrees involved in solving these systems of equations. Hilbert's Syzygy Theorem and the Auslander-Buchsbaum-Serre Theorem (see [Ei2, Theorem 19.7] and [Ei2, Theorem 19.12]) establish (3)⇒(1) and (2)⇒(3) in the next result.

Theorem 3.1. *The following conditions are equivalent:*
(1) *Every finitely generated graded R-module has finite projective dimension.*
(2) *The residue field has finite projective dimension.*
(3) *R is a polynomial ring over a field.*

Theorem 3.1 provides a characterization of the algebras R over which all modules have finite projective dimension. In [AP] we characterize the algebras R over which all modules have finite regularity. Our result establishes a generalized version of a conjecture of Avramov and Eisenbud [AE].

Theorem 3.2. (Avramov-Peeva, [AP]) *The following conditions are equivalent:*
(1) *Every finitely generated graded R-module has finite regularity.*
(2) *The residue field has finite regularity.*
(3) *R is a polynomial ring over a Koszul algebra.*

Koszul algebras, appearing in Theorem 3.2(3) above, are defined by the vanishing of the regularity of the residue field. They have been of high interest due to their extraordinary homological properties and to their appearance in many cases of interest in Algebra, Combinatorics, and Topology (for example, see [BGS, Ke, Man, Pr]).

4. Hilbert Schemes over Quotient Rings, and Deformations. Throughout, k is an algebraically closed field of characteristic zero and $S = k[x_1, \ldots, x_n]$ is a polynomial ring graded by $\deg(x_i) = 1$ for all i. We start with a brief introduction to Hilbert functions. If I is a homogeneous ideal in S, then the quotient $R := S/I$ inherits the grading by $R_i = S_i/I_i$ for all i. The size of a homogeneous ideal J in R is measured by the *Hilbert function* $\mathrm{Hilb}_{R/J}(i) = \dim(R_i/J_i)$ for $i \in \mathbf{Z}$. Hilbert's insight was that $\mathrm{Hilb}_{R/J}$ is determined by finitely many of its values. He proved that there exists a polynomial (called the *Hilbert polynomial*) $g(t) \in \mathbf{Q}[t]$ such that $\mathrm{Hilb}_{R/J}(i) = g(i)$ for $i \gg 0$. If S/J (here $R = S$) is the coordinate ring of a projective algebraic variety X, then the degree of the Hilbert polynomial equals the dimension of X, and the leading coefficient determines yet another important invariant – the degree (multiplicity) of X. Hilbert functions for monomial ideals in the ring $k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$ have been extensively studied in Combinatorics since each such Hilbert function counts the number of faces in a simplicial complex.

Lex ideals are fruitful tools in the study of Hilbert functions. They are monomial ideals defined in a simple way: Denote by $>_{lex}$ the lexicographic order on the monomials in S extending $x_1 > \ldots > x_n$. A monomial ideal L in S is *lex* if the following property holds: if $m \in L$ is a monomial and $q >_{lex} m$ is a monomial of the same degree, then $q \in L$ (that is, for each $i \geq 0$ the vector space L_i is either zero or is spanned by lex-consecutive monomials of degree i starting with x_1^i). A formula for the Hilbert function of a lex ideal can be derived easily.

A core result in Commutative Algebra is Macaulay's Theorem 4.1, which characterizes numerically the Hilbert functions of homogeneous ideals in the polynomial ring S:

Theorem 4.1. (Macaulay, [Ma]) *For every homogeneous ideal in S there exists a lex ideal with the same Hilbert function.*

Lex ideals also play an important role in the study of Hilbert schemes. Grothendieck [Gr] introduced the Hilbert scheme $\mathcal{H}_{r,g}$ that parametrizes subschemes of $\mathbf{P}^r$ with a fixed Hilbert polynomial g. The structure of the Hilbert scheme is known to be very complicated. In [HM] Harris and Morrison state Murphy's Law for Hilbert Schemes: "There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme." The main structural result on $\mathcal{H}_{r,g}$ is Hartshorne's Theorem:

Theorem 4.2. (Hartshorne, [Ha]) *The Hilbert scheme $\mathcal{H}_{r,g}$ is connected.*

Jointly with coauthors, we began the study of Hilbert schemes over some quotient rings in [GMP, MPe, PS1, PS2]. Our results are over exterior algebras, Clements-Lindström rings, and Veronese rings. It is natural to consider such rings because it is known that Macaulay's Theorem 4.1 holds over them [Ka, Kr, CL, GMP]. We study the Hilbert scheme that parametrizes all homogeneous ideals with a fixed Hilbert function (instead of a fixed Hilbert polynomial; note that the Hilbert polynomial vanishes if the ring is artinian). A minor modification of the proof of Theorem 4.2 shows that the Hilbert scheme $\mathcal{H}_S^h$, that parametrizes all homogeneous ideals in S with a fixed Hilbert function h, is connected.

Theorem 4.3.

(1) (Peeva-Stillman, [PS1]) *Let E be an exterior algebra on variables $e_1, \ldots, e_n$ over k. The Hilbert scheme $\mathcal{H}_E^h$, that parametrizes all homogeneous ideals in E with a fixed Hilbert function h, is connected.*

(2) (Murai-Peeva, [MPe]) *Let C be a Clements-Lindström ring $k[x_1, \ldots, x_n]/P$, where P is an ideal generated by powers of the variables. The Hilbert scheme $\mathcal{H}_C^h$, that parametrizes all homogeneous ideals in C with a fixed Hilbert function h, is connected.*

(3) (Gasharov-Murai-Peeva, [GMP]) *Let $V = S/J$ be a Veronese ring (here, J is the defining ideal of a Veronese toric variety). The Hilbert scheme $\mathcal{H}_V^h$, that parametrizes all homogeneous ideals in V with a fixed Hilbert function h, is connected.*

In 4.2 and all three cases in 4.3, the authors prove that every homogeneous ideal with a fixed Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h. A deformation connects two ideals $J_{t=0}$ and $J_{t=1}$ in the sense that we have a family of homogeneous ideals J_t varying with the parameter $t \in [0,1]$ so that the Hilbert function is preserved; in this case, the ideals J_t form a path on the Hilbert scheme. Hartshorne's proof [Ha] of Theorem 4.2 relies on deformations called "distractions" which use generic change of coordinates and polarization. Unfortunately, polarization does not work over the exterior algebra E and over a Veronese ring V. In addition, generic change of coordinates does not work over a Clements-Lindström ring C. We prove the results in Theorem 4.3 by building new types of deformations.

In [PS2] we introduce the notion of flip and show that the basic flips form a basis of the tangent space at a monomial point in the Hilbert scheme $\mathcal{H}_E^h$ over an exterior algebra E. This implies that the tangent space has a basis consisting of directions tangent to Gröbner deformations. Such a structure of the tangent space is surprising since it does not hold in the polynomial case (over S).

5. Maximal Betti Numbers for a fixed Hilbert Function. We maintain the notation in Subsection 4. Analyzing the paths on a Hilbert scheme may shed light on whether there exists an object with maximal Betti numbers. The following result is a corollary of the proof of Theorem 4.2.

Theorem 5.1. (Bigatti, Hulett, Pardue; see [Pa]) *A lex ideal in $S = k[x_1, \ldots, x_n]$ attains maximal Betti numbers among all homogeneous ideals with the same Hilbert function.*

This result was quite surprising when it was discovered since examples were known in which no ideal attains minimal Betti numbers. It is natural to ask if Theorem 5.1 holds over the rings considered in Theorem 4.3. This is a difficult question since minimal resolutions over exterior algebras, Clements-Lindström rings, or Veronese rings are infinite (in contrast, Theorem 5.1 is about finite resolutions) and so they are considerably more intricate. Furthermore, the paths on the Hilbert scheme constructed in the proofs of Theorem 4.3, do not give information on how the Betti numbers change along a path. We use different techniques to obtain maximal Betti numbers: over a Clements-Lindström ring C we construct special changes of coordinates and use them to provide a construction that starting with a monomial ideal yields a lex-closer ideal with bigger Betti numbers, (the construction may not yield a path between the two ideals on the Hilbert scheme), and over a Veronese ring V we analyze the minimal free resolution of a Borel ideal using mapping cones. The following result holds:

Theorem 5.2. *We use the notation in Theorem 4.3.*

(1) (Aramova-Herzog-Hibi, [AHH]) *Every lex ideal in an exterior algebra E attains maximal Betti numbers among all homogeneous ideals with the same Hilbert function.*
(2) (Murai-Peeva, [MPe]) *Every lex ideal in a Clements-Lindström ring C attains maximal Betti numbers among all homogeneous ideals with the same Hilbert function.*
(3) (Gasharov-Murai-Peeva, [GMP]) *Every lex ideal in a Veronese ring V attains maximal Betti numbers among all homogeneous ideals with the same Hilbert function.*

6. Subspace Arrangements. Consider the topology of the complement of a subspace arrangement. Much fewer results are known for real subspace arrangements than for complex ones because Algebraic Geometry methods can be applied in the complex case, but not in the real case. In [PRW], Reiner, Welker, and I introduce an approach to express the ranks of the cohomology of the complement of a real diagonal subspace arrangement by the Betti numbers of a minimal free resolution. Our approach is different from the well-known method for computing the cohomology of such a complement using a formula of Goresky and MacPherson [GM]. In [PW] we relate the (co)homological properties of real coordinate subspace arrangements and square-free monomial ideals. Using Goresky-MacPherson's Formula for the cohomology of the complement of a subspace arrangement leads to the result that dim $\widetilde{\mathrm{H}}^i(\mathcal{M}; k)$, where k is a field and $\mathcal{M}$ is the complement of a real coordinate subspace arrangement, is the sum of the Betti numbers of the i-strand in the minimal resolution of a certain square-free monomial ideal.

In the introduction in [Hir], Hirzebruch wrote: "The topology of the complement of an arrangement of lines in the projective plane is very interesting, the investigation of the fundamental group of the complement is very difficult." The cohomology algebra of the complement of a central complex hyperplane arrangement is the well-known Orlik-Solomon algebra [OS] (here, "central" means that all hyperplanes contain the origin). The fundamental group of the complement is interesting, complicated, and few results are known about it. In [Pe2] I focus on the approach to describe the ranks for the lower central series of such a fundamental group via the Betti numbers of the linear strand of the minimal free resolution of the field of complex numbers over the Orlik-Solomon algebra. This is related to results of Kohno [Koh], Falk-Randell [FR], Shelton-Yuzvinsky [SY].

7. Toric Rings. The study of toric rings is an area of active research involving rich interplay between Algebraic Geometry, Commutative Algebra, and Combinatorics.

Complete intersection ideals are ideals whose generators have sufficiently general coefficients. They might be regarded as generic among all ideals with fixed small number of generators. In [PSt1] we introduce an entirely different notion of genericity: toric ideals whose generators are generic with respect to the exponents in their terms– not their coefficients. For toric rings we define a notion of genericity which ensures nicely structured homological behavior. We construct the minimal free resolution (over a polynomial ring) of a generic toric ring.

In [GPW1], Welker and I consider the infinite minimal free resolution of the residue field k over a generic toric ring. For many years, one of the central questions in Commutative Algebra was the Serre-Kaplansky Problem whether the Poincaré series $\sum_{i\geq 0} \dim_k(\mathrm{Tor}_i^Q(k,k))t^i$ of a finitely generated commutative local noetherian ring Q (with residue field k) is rational. In 1980 Anick (see [An]) constructed a ring with transcendental Poincaré series. Our main result in [GPW1] provides a positive answer to this problem (in the graded case) for generic toric rings. We also prove an analogue to a result (for monomial ideals) of Eisenbud, Reeves, Totaro [ERT] on vanishing of certain Betti numbers: we show that the rate of a generic toric ring is the maximum degree of a minimal generator of the toric ideal minus one.

In [PRS] we introduce an approach which relates the Betti numbers of the residue field over a toric ring to Combinatorics and Noncommutative Algebra: We consider the toric ring as given in non-commutative variables and then identify monomials with facets of simplicial complexes. This makes it possible to use a uniform non-pure shelling of the toric ring.

In [PSt2] we introduce an approach to study the free resolutions of toric rings via integer-points-free bodies. This leads to an upper bound $2^{\text{codimension}} - 1$ on the projective dimension of an arbitrary toric ring. The existence of a bound in terms of the codimension is surprising and has no analogues for other classes of rings. In the codimension 2 case the integer-points-free bodies have simple structure; using this we construct the minimal free resolution of the toric ring.

The study of ideals with the same multigraded Hilbert function as a given toric ideal J was initiated by Arnold, who showed that the structure of such ideals is encoded in continued fractions in the case when J defines a monomial curve in $\mathbf{A}^3$. In [PS3] Stillman and I study the toric Hilbert scheme $\mathcal{H}_J$ which parametrizes all ideals with the same multigraded Hilbert function as a given toric ideal J. Reeves and Stillman [RS2] proved that the lex ideal is a smooth point on Grothendieck's Hilbert scheme (over $k[x_1,\ldots,x_n]$). Usually no lex ideal exists on $\mathcal{H}_J$, but we show that the toric ideal J is a smooth point on $\mathcal{H}_J$. We also show that if J has codimension two, then its toric Hilbert scheme $\mathcal{H}_J$ is two dimensional and smooth.

In [GP] we solve a conjecture raised by Sturmfels. We prove that if a toric ideal J has codimension two, then the toric Hilbert scheme $\mathcal{H}_J$ has one component and this component is the closure of the orbit of J under the torus action. For monomial curves in $\mathbf{A}^3$ this result was proved by Arnold, Korkina, Post, and Roelofs [Ko, KPR].

8. Monomial Resolutions. In the early 1960's, Kaplansky posed the problem of finding the minimal free resolution of a monomial ideal in a polynomial ring $k[x_1,\ldots,x_n]$, where k is a field. Despite the existence of helpful combinatorial structure in monomial ideals, this problem turned out to be very difficult. For many years, the well-known Stanley-Reisner correspondence introduced by Hochster and Reisner [Ho, Re] was the main progress.

The Stanley-Reisner theory is based on computing the Betti numbers of a monomial ideal by simplicial complexes. In [GPW2], we introduce a new idea inspired by topological combinatorics of subspace arrangements. We introduce the lcm-lattice of a monomial ideal. We show that it plays the same role in describing the homology of the ideal as the role of the intersection lattice in describing the cohomology of the complement of a complex subspace arrangement. Namely:

- the lcm-lattice determines the Betti numbers (analogue of the Goresky-MacPherson Formula for subspace arrangements [GM]);
- the lcm-lattice determines the maps in the minimal free resolution up to relabeling;
- the lcm-lattice together with the additional data, which pairs of minimal monomial generators are relatively prime, determine the algebra structure of the Tor-algebra (analogue of results by De Concini-Procesi [DP] for subspace arrangements).

In [BPS], we forward an idea in an entirely different direction. We introduce an elegant approach for resolving a monomial ideal by encoding the whole resolution (including the differential maps) into a single simplicial complex. We prove that generically such a resolution is minimal and comes from the boundary of a polytope. For a non-generic ideal we introduce the technique of resolving by "deforming to the generic case". It provides a non-minimal resolution whose length is less or equal to the number of variables (our resolution is usually much shorter and smaller than Taylor's resolution). The idea of simplicial resolutions was generalized to cellular resolutions [BSt], and to resolutions supported on a CW-complex [BW]. Such resolutions were built in several cases of interest.

It is well-known how the Hilbert function changes when we add the squares of the variables to a monomial ideal. In [MPS] we describe how the minimal free resolution changes.

In [PV] Velasco and I introduce nearly Scarf monomial ideals and construct their minimal free resolutions. The main application of such resolutions is that Velasco [Ve] uses them to give the first examples of minimal monomial resolutions that cannot be supported by any CW-complex. We also introduce the concept of frame of a monomial free resolution. We prove that the problem of constructing a minimal monomial free resolution is equivalent to the problem of building its frame. Thus, the concept of frame can encode the minimal free resolution of any monomial ideal, while CW-cellular complexes cannot.

One might expect that the homological properties of quadratic monomial ideals are simple. However, it turns out that their minimal free resolutions are so complicated that it is beyond reach to obtain a structural description; we do not even know how to express the important numerical invariant regularity. The technique of polarization reduces the study of minimal resolutions of quadratic monomial ideals to the study of monomial ideals generated by square-free monomials (of the form $x_i x_j$ with $i \neq j$). The minimal generators of such an ideal can be encoded in a graph as follows: if G is a graph with no loops on vertices $\{1, \dots, n\}$, then its edge ideal is

$$I_G = \left(\{x_i x_j \,\middle|\, \{i,j\} \text{ is an edge in } G\} \right)$$

in the polynomial ring $k[x_1, \dots, x_n]$ over a field k. The study of edge ideals has attracted a lot of research. It is an area with fascinating interactions between Commutative Algebra and Combinatorics. The goal is to understand the relations between the algebraic properties of an edge ideal and the combinatorial properties of its graph (or the complement graph). For example, Fröberg's Theorem [Fr] characterizes combinatorially the edge ideals whose regularity is as minimal as possible: $\mathrm{reg}(I_G) = 2$ if and only if the complement graph G^c is chordal. If an ideal J generated in degree p attains minimal regularity $\mathrm{reg}(J) = p$, we say that its *minimal free resolution is linear*. Following Francisco, Hà, and Van Tuyl, we are interested to characterize the edge ideals whose powers have linear resolutions. They asked whether the powers I_G^s have a linear resolution for all $s \geq 2$ if and only if G^c has no induced 4-cycles. In [NP], Nevo and I construct a counterexample, and introduce a conjecture in this direction. The conjecture is proved for claw-free graphs by Nevo [Ne] and in some other special cases, but the general case is open.

References

[AH] T. Ananyan, M. Hochster: Small Subalgebras of Polynomial Rings and Stillman's Conjecture, arXiv:1610.09268.

[An] D. Anick: Counterexample to a conjecture of Serre, *Ann. Math.* **115** (1982), 1–33.

[AHH] A. Aramova, J. Herzog, and T. Hibi: Gotzmann theorems for exterior algebras and combinatorics, *J. Alg.* **191** (1997), 174–211.

[As] P. Aspinwall: Some Applications of Commutative Algebra to String Theory, in *Commutative Algebra* (Editor: I. Peeva), Springer, 2013, 25–56.

[Av] L. L. Avramov: Modules of finite virtual projective dimension, *Invent. math.* **96** (1989), 71–101.

[AB] L. Avramov and R.-O. Buchweitz: Homological algebra modulo a regular sequence with special attention to codimension two, *J. Alg.* **230** (2000), 24–67.

[AE] L. L. Avramov, D. Eisenbud: Regularity of modules over a Koszul algebra, *J. Alg.* **153** (1992), 85–90.

[AGP] L. L. Avramov, V. Gasharov, I. Peeva: Complete intersection dimension, *Publ. Math. l'IHÉS* **86** (1997), 67-114.

[AP] L. L. Avramov, I. Peeva: Finite regularity and Koszul algebras, *Amer. J. Math.* **123** (2001), 275–281.

[BHU] J. Backelin, J. Herzog, B. Ulrich: Linear maximal Cohen-Macaulay modules over strict complete intersections, *J. Pure Appl. Alg.* **71** (1991), 187–202.

[BFK] M. Ballard, D. Favero, L. Katzarkov: A category of kernels for equivariant factorizations and its implications for Hodge theory, *Publ. Math. Inst. Hautes Études Sci.* **120** (2014), 1–111.

[BW] E. Batzies and V. Welker: Discrete Morse theory for cellular resolutions, *J. Reine Angew. Math.* **543** (2002), 147–168.

[BM] D. Bayer and D. Mumford: What can be computed in Algebraic Geometry?, *Computational Algebraic Geometry and Commutative Algebra*, Symposia Mathematica, Volume XXXIV, Cambridge University Press, Cambridge, 1993, 1–48.

[BPS] D. Bayer, I. Peeva, B. Sturmfels: Monomial resolutions, *Math. Res. Letters* **5** (1998), 31–46.

[BS] D. Bayer and M. Stillman: On the complexity of computing syzygies. Computational aspects of commutative algebra, *J. Symbolic Comp.* **6** (1988), 135–147.

[BSt] D. Bayer, B. Sturmfels: Cellular resolutions of monomial modules, *J. Reine Angew. Math.* **502** (1998), 123–140.

[Be] J. Becker: On the boundedness and the unboundedness of the number of generators of ideals and multiplicity, *J. Alg.* **48** (1977), 447–453.

[BGS] A. Beilinson, V. Ginzburg, W. Soergel: Koszul duality patterns in representation theory, *J. Amer. Math. Soc.* **9** (1996), 473–527.

[BEL] A. Bertram, L. Ein, R. Lazarsfeld: Vanishing theorems, a theorem of Severi, and the equations defining projective varieties, *J. Amer. Math. Soc.* **4** (1991), 587–602.

[BGS] R.-O. Buchweitz, G.-M. Greuel, F.-O. Schreyer: Cohen-Macaulay modules on hypersurface singularities II, *Invent. Math.* **88** (1987), 165–182.

[CCMPV] G. Caviglia, M. Chardin, J. McCullough, I. Peeva, and M. Varbaro: Regularity of prime ideals, *M. Zeit.*, to appear.

[CS] G. Caviglia, E. Sbarra: Characteristic-free bounds for the Castelnuovo-Mumford regularity, *Compositio Math.* **141** (2005), 1365–1373.

[Ca] A. Cayley: On the theory of elimination, *Cambridge and Dublin Math. J.* **3** (1848) 116–120; Collected papers Vol I, Cambridge U. Press, 1889, 370–374.

[CU] M. Chardin and B. Ulrich: Liaison and Castelnuovo-Mumford regularity, *Amer. J. Math.* **124** (2002), 1103–1124.

[CL] G. Clements, B. Lindström: A generalization of a combinatorial theorem of Macaulay, *J. Comb. Theory* **7** (1969), 230–238.

[DH] H. Dao, C. Huneke: Vanishing of Ext, cluster-tilting modules and finite global dimension of endomorphism rings, *Amer. J. Math.* **135** (2013), 561–578.

[DP] C. De Concini, C. Procesi: Wonderful models of subspace arrangements, *Selecta Math. New Series* **1** (1995), 459–494.

[Ei1] D. Eisenbud: Homological algebra on a complete intersection with an application to group representations, *Trans. Amer. Math. Soc.* **260** (1980), 35–64.

[Ei2] D. Eisenbud: *Introduction to commutative algebra with a view towards algebraic geometry*, Springer, 1995.

[Ei3] D. Eisenbud: *The geometry of syzygies. A second course in commutative algebra and algebraic geometry*, *Graduate Texts in Math.* **229**, Springer-Verlag, New York, 2005.

[EG] D. Eisenbud and S. Goto: Linear free resolutions and minimal multiplicity, *J. Alg.* **88** (1984), 89–133.

[EP1] D. Eisenbud, I. Peeva: Minimal free resolutions over complete intersections, research monograph, *Lecture Notes in Math.* **2152**, Springer, 2016.

[EP2] D. Eisenbud, I. Peeva: Layered resolutions of Cohen-Macaulay modules, submitted.

[ERT] D. Eisenbud, A. Reeves, B. Totaro: Initial ideals, Veronese subrings, and rates of algebras, *Adv. Math.* **109** (1994), 168–187.

[EPS] D. Eisenbud, F.-O. Schreyer, I. Peeva: Tor as a module over an exterior algebra, *J. Eur. Math. Soc.*, to appear.

[EPS2] D. Eisenbud, F.-O. Schreyer, I. Peeva: Non-commutative CI operators, submitted.

[EPS3] D. Eisenbud, F.-O. Schreyer, I. Peeva: Quadratic complete intersections, submitted.
[FR] M. Falk and R. Randell: The lower central series of a fiber-type arrangement, *Invent. Math.* **82** (1985), 77–88.
[GMP] V. Gasharov, S. Murai, I. Peeva: Hilbert schemes and maximal Betti numbers over Veronese rings, *Math. Zeitschrift* **267** (2011), 155–172.
[GP] V. Gasharov, I. Peeva: Deformations of codimension 2 toric varieties, *Compositio Math.* **123** (2000), 225-241.
[GPW1] V. Gasharov, I. Peeva, V. Welker: Rationality for generic toric rings, *Math. Zeit.* **233** (2000), 93–102.
[GPW2] V. Gasharov, I. Peeva, V. Welker: The lcm lattice in monomial resolutions, *Math. Res. Letters* **6** (1999), 521–532.
[GM] M. Goresky and R. MacPherson: *Stratified Morse theory*, Springer, 1988.
[GLP] L. Gruson, R. Lazarsfeld, and C. Peskine: On a theorem of Castelnuovo and the equations defining projective varieties, *Invent. Math.* **72** (1983), 491–506.
[Gu] T. Gulliksen: A change of ring theorem with applications to Poincaré series and intersection multiplicity, *Math. Scand.* **34** (1974), 167–183.
[HM] J. Harris, I. Morrison: Moduli of curves, *Graduate Texts in Mathematics* **187**, Springer, New York, 1998.
[Ha] R. Hartshorne: Connectedness of the Hilbert scheme, *Publ. Math. l'IHÉS* **29** (1966), 5-48.
[Hi] D. Hilbert: Über the Theorie von algebraischen Formen, *Math. Ann.* **36** (1890), 473–534. An English translation is available in Hilbert's Invariant Theory Papers, Math. Sci. Press, 1978.
[Hir] F. Hirzebruch: Arrangements of lines and algebraic surfaces, In: *Arithmetic and Geometry*, vol. II, *Progress in Math.* **36**, Birkhäuser, Boston, 1983, 113–140.
[Ho] M. Hochster: Cohen-Macaulay rings, combinatorics and simplicial complexes, in *Ring Theory II* (B.R. McDonald and R. Morris, eds.), Lecture Notes Pure and Appl. Math. **26**, Marcel Dekker, 1977, 171–223.
[Hu] C. Huneke: On the symmetric and Rees algebra of an ideal generated by a d-sequence, *J. Alg.* **62** (1980), 268–275.
[KL] A. Kapustin, Y. Li: D-branes in Landau-Ginzburg models and algebraic geometry, *J. High Energy Phys.* **12** (2003).
[Ka] G. Katona: A theorem for finite sets, in *Theory of Graphs* Academic Press, New York (1968), 187–207.
[Ke] G. Kempf: Some wonderful rings in algebraic geometry, *J. Alg.* **134** (1990), 222–224.
[KR1] M. Khovanov and L. Rozansky: Matrix factorizations and link homology, *Fund. Math.* **199** (2008), 1–91.
[KR2] M. Khovanov and L. Rozansky: Matrix factorizations and link homology II, *Geometry&Topology* **12** (2008), 1387–1425.
[Kn] H. Knörrer: Cohen-Macaulay modules on hypersurface singularities, *Invent. Math.* **88** (1987), 153–164.
[Koh] T. Kohno: Série de Poincaré-Koszul associée aux groupes de tresses pures, *Invent. Math.* **82** (1985), 57–75.
[Ko] E. Korkina: Classification of A-graded algebras with 3 generators, *Indag. Math.* **3** (1992), 27–40.
[KPR] E. Korkina, G. Post, and M. Roelofs: Classification of generalized A-graded algebras with 3 generators, *Bul. Sci. Math.* **119** (1995), 267–287.
[Kr] J. Kruskal: The number of simplices in a complex, *Mathematical Optimization Techniques* (R. Bellman, ed.), University of California Press, Berkeley/Los Angeles (1963), 251–278.
[KPU] A. Kustin, C. Polini, B. Ulrich: Bernd Rational normal scrolls and the defining equations of Rees algebras, *J. Reine Angew. Math.* **650** (2011), 23–65.
[Kw] S. Kwak: Castelnuovo regularity for smooth subvarieties of dimensions 3 and 4, *J. Alg. Geom.* **7** (1998), 195–206.
[La1] R. Lazarsfeld: A sharp Castelnuovo bound for smooth surfaces, *Duke Math. J.* **55** (1987), 423–438.
[La2] R. Lazarsfeld: *Positivity in Algebraic Geometry I and II, A Series of Modern Surveys in Mathematics*, **48** and **49**, Springer-Verlag, Berlin, 2004.
[Ma] F. Macaulay: Some properties of enumeration in the theory of modular systems, *Proc. London Math. Soc.* **26** (1927), 531-555.
[Man] Y. Manin: Some remarks on Koszul algebras and quantum groups, *Ann. Inst. Fourier (Grenoble)* **37** (1987), 191–205.
[MM] E. Mayr and A. Meyer: The complexity of the word problem for commutative semigroups and polynomial ideals, *Adv. Math.* **46** (1982), 305–329.
[MP] J. McCullough, I. Peeva: Counterexamples to the Eisenbud-Goto Regularity Conjecture, *J. Amer. Math. Soc.* **31** (2018), 473–496..
[MPS] J. Mermin, I. Peeva, M. Stillman: Ideals containing the squares of the variables, *Adv. Math.* **217** (2008), 2206–2230.
[MPe] S. Murai, I. Peeva: Hilbert schemes and Betti numbers over a Clements-Lindström rings, *Compositio Math.* **148** (2012), 1337–1364.
[OS] P. Orlik, L. Solomon: Combinatorics and topology of complements of hyperplanes, *Invent. Math.* **56** (1980), 167–189.
[Or1] D. Orlov: Triangulated categories of singularities and D-branes in Landau- Ginzburg models, *Trans. Mat. Inst. Steklova* **246**, (2004), 240–262; translation in *Proc. Steklov Inst. Math.* **246** (2004), 227–248.
[Or2] D. Orlov: Derived categories of coherent sheaves and triangulated categories of singularities, *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin*, Vol. II, *Progr. Math.* **270** (2009), Birkhäuser, 503–531.
[OP] G. Ottaviani and R. Paoletti: Syzygies of Veronese embeddings, *Compositio Math.* **125** (2001), 31–37.
[Pa] K. Pardue: Deformation classes of graded modules and maximal Betti numbers, *Ill. J. Math.* **40** (1996), 564–585.
[Pe1] I. Peeva: *Graded syzygies*, Springer-Verlag London Ltd., London, 2011.

[Pe2] I. Peeva: Hyperplane arrangements and linear strands in resolutions, *Trans. Amer. Math. Soc.* **355** (2003), 609–618.

[PRS] I. Peeva, V. Reiner, B. Sturmfels: How to shell a monoid, *Math. Annalen* **310** (1998), 379–393.

[PRW] I. Peeva, V. Reiner, V. Welker: Cohomology of real diagonal subspace arrangements via resolutions, *Compositio Math.* **117** (1999), 107–123.

[PS1] I. Peeva, M. Stillman: Connectedness of Hilbert schemes, *J. Alg. Geom.* **14** (2005), 193–211.

[PS2] I. Peeva, M. Stillman: Flips and Hilbert schemes over exterior algebras, *Math. Ann.* **339** (2007), 545–557.

[PS3] I. Peeva, M. Stillman: Toric Hilbert schemes, *Duke Math. J.* **111** (2002), 419–449.

[PSt1] I. Peeva, B. Sturmfels: Generic lattice ideals, *J. Amer. Math. Soc.* **11** (1998), 363–373.

[PSt2] I. Peeva, B. Sturmfels: Syzygies of codimension 2 lattice ideals, *Math. Zeit.* **229** (1998), 163–194.

[PV] I. Peeva, M. Velasco: Frames and degenerations for monomial resolutions, *Trans. Amer. Math. Soc.* **363** (2011) 2029–2046.

[PW] I. Peeva, V. Welker: Coordinate subspace arrangements and monomial ideals, in *Proc. of the Osaka meeting on Computational Commutative Algebra and Combinatorics*, *Adv. Studies in Pure Math.* **33** (2002), 65–74.

[Pi] H. Pinkham: A Castelnuovo bound for smooth surfaces, *Invent. Math.* **83** (1986), 321–332.

[PVa] A. Polishchuk, A. Vaintrob: Matrix factorizations and Cohomological Field Theories, *J. Reine Angew. Math.* **714** (2016), 1–122.

[Pr] S. Priddy: Koszul resolutions, *Trans. Amer. Math. Soc.* **152** (1970), 39–60.

[Ra] Z. Ran: Local differential geometry and generic projections of threefolds, *J. Differential Geom.* **32** (1990), 13–137.

[RS] A. Reeves, M. Stillman: Smoothness of the lexicographic point, *J. Alg. Geom.* **6** (1997), 235–246.

[Re] G. Reisner: Cohen-Macaulay quotients of polynomial rings, *Adv. Math.* **21** (1976), 30–49.

[Sh] J. Shamash: The Poincaré series of a local ring, *J. Alg.* **12** (1969), 453–470.

[SY] B. Shelton, S. Yuzvinsky: Koszul algebras from graphs and hyperplane arrangements, *J. London Math. Soc.* **56** (1997), 477–490.

[Ta] J. Tate: Homology of Noetherian rings and of local rings, *Ill. J. Math.* **1** (1957), 14–27.

[Ve] M. Velasco: Minimal free resolutions that are not supported by a CW-complex, *J. Alg.* **319** (2008), 102–114.