A local Ramsey theory for block sequences

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Outline

1. Ramsey theory for block sequences
2. (Local) Ramsey theory on $\mathbb{N}$
3. Local Ramsey theory for block sequences
4. Pathological pure states on $\mathcal{B}(\ell_2)$

Notation

Throughout, B is a Banach space with normalized Schauder basis (e_n), and $E = \mathrm{span}_F(e_n)$, for F a countable subfield of $\mathbb{R}$ (or $\mathbb{C}$) so that the norm on E takes values in F.

- For $x, y \in E$ (or B), we write $x < y$ if $\max \mathrm{supp}(x) < \min \mathrm{supp}(y)$, where $\mathrm{supp}(x) = \{n : x = \sum_n a_n e_n \Rightarrow a_n \neq 0\}$.
- A sequence $x_0 < x_1 < x_2 < \cdots$ is a block sequence (of (e_n)).
- $\mathrm{bb}^\infty(E)$ is the set of normalized block sequences in E, and $\mathrm{bb}^\infty(B)$ those in B. These are Polish spaces.
- For $n \in \mathbb{N}$, and X a block sequence, X/n is the tail of X with supports above n.
- For $X, Y \in \mathrm{bb}^\infty(E)$ (or $\mathrm{bb}^\infty(B)$) write $X \preceq Y$ if X is a block sequence of Y, and $X \preceq^* Y$ if there is some n for which $X/n \preceq Y$.

Games with block vectors

Definition

For $Y \in \mathrm{bb}^\infty(E)$,

- $G[Y]$ denotes the Gowers game below Y: Players I and II alternate with I going first.
  - I plays $Y_k \preceq Y$,
  - II responds with a vector $y_k \in \mathrm{span}_F(Y_k)$ such that $y_k < y_{k+1}$.
- $F[Y]$ denotes the infinite asymptotic game below Y: Players I and II alternate with I going first
  - I plays $n_k \in \mathbb{N}$,
  - II responds with a vector $y_k \in \mathrm{span}_F(Y)$ such that $n_k < y_k < y_{k+1}$.

In both games, the outcome is the block sequence (y_k).

For $Y \in \mathrm{bb}^\infty(B)$, the games are defined similarly, with II playing block vectors. We denote these games $G^*[Y]$ and $F^*[Y]$.

Gowers' dichotomy

Theorem (Gowers, 1996)

Whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(B)$ *is analytic,* $X \in \mathrm{bb}^\infty(B)$, *and* $\Delta = (\delta_n) > 0$, *then there is a* $Y \preceq X$ *such that either*

- *every* $Z \preceq Y$ *is in* $\mathbb{A}^c$, *or*
- *II has a strategy in* $G^*[Y]$ *for playing into* $\mathbb{A}_\Delta$.

- $\mathbb{A}_\Delta = \{(z_n) \in \mathrm{bb}^\infty(B) : \exists (z'_n) \in \mathbb{A} \forall n (\|z_n - z'_n\| < \delta_n)\}$ is the Δ-expansion of $\mathbb{A}$.

Rosendal's dichotomy

The proof of Gowers' theorem was greatly simplified by Rosendal. He showed that one can work in E and obtain an exact result, avoiding Δ-expansions, at the cost of introducing $F[Y]$.

Theorem (Rosendal, 2010)

Whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic and $X \in \mathrm{bb}^\infty(E)$, there is a $Y \preceq X$ such that either

- *I has a strategy in $F[Y]$ for playing into $\mathbb{A}^c$, or*
- *II has a strategy in $G[Y]$ for playing into $\mathbb{A}$.*

- From this, Gowers' result can be derived with minimal use of Δ-expansions.

Infinite Ramsey theory on $\mathbb{N}$

Theorem (Silver, 1970)

If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ is analytic and $X \in [\mathbb{N}]^\infty$, then there is a $Y \in [X]^\infty$ such that either $[Y]^\infty \cap \mathbb{A} = \emptyset$ or $[Y]^\infty \subseteq \mathbb{A}$.

- Here, $[X]^\infty$ is the set of all infinite subsets of $[X]$.
- This result was the culmination of work of Ramsey, Nash-Williams, Galvin, and Prikry.
- (Shelah & Woodin, 1990) The assumption of “analytic” can be upgraded to “in $\mathbf{L}(\mathbb{R})$” (think, “every reasonably definable set”), under large cardinal hypotheses.

Local Ramsey theory

Theorem (Silver, 1970)

If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ is analytic, then for any $X \in [\mathbb{N}]^\infty$, there is a $Y \in [X]^\infty$ such that either $[Y]^\infty \cap \mathbb{A} = \emptyset$ or $[Y]^\infty \subseteq \mathbb{A}$.

Local Ramsey theory concerns "localizing" the witness Y above. That is, finding families $\mathcal{H} \subseteq [\omega]^\omega$ such that, provided the given X is in $\mathcal{H}$, Y can also be found in $\mathcal{H}$.

Local Ramsey theory (cont'd)

Definition

- $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ is a coideal if it is the complement of a (non-trivial) ideal. Equivalently, it is a non-empty family such that
  - $X \in \mathcal{H}$ and $X \subseteq^* Y \implies Y \in \mathcal{H}$,
  - $X \cup Y \in \mathcal{H} \implies X \in \mathcal{H}$ or $Y \in \mathcal{H}$.
- A coideal $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ is selective (or a happy family) if whenever $X_0 \supseteq X_1 \supseteq \cdots$ are in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X/n \subseteq X_n$ for all $n \in X$.

Examples (of selective coideals)

- $[\mathbb{N}]^\infty$
- $\mathcal{U}$ a selective (or Ramsey) ultrafilter (exist under CH, MA)
- $[\mathbb{N}]^\infty \setminus \mathcal{I}$ where $\mathcal{I}$ is the ideal generated by an infinite a.d. family

Local Ramsey theory (cont'd)

Theorem (Mathias, 1977)

Let $\mathcal{H} \subseteq [\mathbb{N}]^\infty$ be a selective coideal. If $\mathbb{A} \subseteq [\mathbb{N}]^\infty$ is analytic, then for any $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either $[Y]^\infty \cap \mathbb{A} = \emptyset$ or $[Y]^\infty \subseteq \mathbb{A}$.

- Used by Mathias to establish the non-existence of infinite analytic maximal a.d. (mad) families in $[\mathbb{N}]^\infty$.
- Used by Todorcevic to give a proof ("Topics in Topology", 1997) of a result (Bourgain, Fremlin, & Talagrand, 1978) concerning separable Rosenthal compacta.
- The corresponding result for sets in $\mathbf{L}(\mathbb{R})$, under large cardinals, is due to Todorcevic (see Farah, "Semiselective coideals", 1997).
- Local Ramsey theory for topological Ramsey spaces has been developed by Di Prisco, Mijares, Nieto (2015).

A local Ramsey theory for block sequences?

Motivating question: Are there local forms of Gowers' and Rosendal's dichotomies?

Possible obstacles:

- What is a "coideal" of block sequences?
- Coideals on $\mathbb{N}$ witness the pigeonhole principle. There is no pigeonhole principle here...
- This is due to the existence of disjoint asymptotic sets in the unit sphere. Even "up to ϵ", there is no such principle in general, due to the existence of separated asymptotic sets.

Families of block sequences

We begin with the discrete case:

Definition

- By a family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$, we mean a non-empty set which is upwards closed with respect to $\preceq^*$.
- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ has the (p)-property if whenever $X_0 \succeq X_1 \succeq \cdots$ in $\mathcal{H}$, there is an $X \in \mathcal{H}$ such that $X \preceq^* X_n$ for all n.
- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ is full if whenever $D \subseteq E$ and $X \in \mathcal{H}$ is such that for all $Y \in \mathcal{H} \upharpoonright X$, there is $Z \preceq Y$ with $\mathrm{span}_F(Z) \subseteq D$, then there is $Z \in \mathcal{H} \upharpoonright X$ with $\mathrm{span}_F(Z) \subseteq D$.

A full family with the (p)-property is a (p^+)-family.

- Fullness says that $\mathcal{H}$ witnesses a pigeonhole principle wherever it holds "$\mathcal{H}$-frequently" below an element of $\mathcal{H}$.
- Obviously, $\mathrm{bb}^\infty(E)$ itself is a (p^+)-family.

A local Rosendal dichotomy

Theorem (S.)

Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(E)$ be a (p^+)-family. Then, whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- *I has a strategy for playing $F[Y]$ into $\mathbb{A}^c$, or*
- *II has a strategy for playing $G[Y]$ into $\mathbb{A}$.*

- The proof closely follows Rosendal's, using "combinatorial forcing" to obtain the result for open sets.
- Fullness is necessary; it is implied by the theorem for clopen sets.
- Under large cardinal hypotheses, we can extend this result to sets $\mathbb{A}$ in $\mathbf{L}(\mathbb{R})$, though we need an additional assumption on $\mathcal{H}$. (For details, see my talk at Toposym next week.)

Examples of (p^+)-families

What are non-trivial examples of (p^+)-families?
A filter in $\mathrm{bb}^\infty(E)$ is a family $\mathcal{F}$ so that whenever $X, Y \in \mathcal{F}$, there is a $Z \in \mathcal{F}$ with $Z \preceq X$ and $Z \preceq Y$.

Theorem (S.)

- (CH or MA) *There exists (p^+)-filters.*
- *However, it is also consistent that they do not exist.*

- (p^+)-filters are constructed under CH or MA by a transfinite induction of length $2^{\aleph_0}$. One can also force with $(\mathrm{bb}^\infty(E), \preceq^*)$ to add a (p^+)-filter generically.
- The existence of full filters implies the existence of selective ultrafilters on $\mathbb{N}$, and models without selective ultrafilters are well-known to exist (e.g., Kunen, 1976).

Families of block sequences in Banach spaces

In $\mathrm{bb}^\infty(B)$, (p)-families are defined as before.

Definition

- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(B)$ is almost full if whenever $D \subseteq B_1$ is closed, $X \in \mathcal{H}$ is such that for all $Y \in \mathcal{H} \upharpoonright X$, there is $Z \preceq Y$ with $\overline{\mathrm{span}}(Z) \subseteq D$, and $\epsilon > 0$, there is $Z \in \mathcal{H} \upharpoonright X$ with $\overline{\mathrm{span}}(Z) \subseteq D_\epsilon$.
- A family $\mathcal{H} \subseteq \mathrm{bb}^\infty(B)$ is spread if whenever $X \in \mathcal{H}$ and $I_0 < I_1 < I_2 < \cdots$ is a sequence of finite sets in $\mathbb{N}$, then there is a $Y = (y_n) \in \mathcal{H} \upharpoonright X$ such that $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$.

An almost full family with the (p)-property is a (p^*)-family.

Lemma

A spread (p^*)-family which is invariant under small perturbations, will be a (p^+)-family when restricted to block sequences over any countable subfield $\mathbb{R}$ (or $\mathbb{C}$).

A local Gowers' dichotomy

Theorem (S.)

Let $\mathcal{H} \subseteq \mathrm{bb}^\infty(B)$ be a spread (p^*)-family which is invariant under small perturbations. Then, whenever $\mathbb{A} \subseteq \mathrm{bb}^\infty(E)$ is analytic, $X \in \mathcal{H}$ and $\Delta > 0$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- *every block sequence of Y is in $\mathbb{A}^c$, or*
- *II has a strategy in $G^*[Y]$ for playing into $\mathbb{A}_\Delta$.*

- Again, under large cardinal hypotheses, we can extend this result to sets $\mathbb{A}$ in $\mathbf{L}(\mathbb{R})$, with an additional hypothesis on $\mathcal{H}$. (This was done for $\mathcal{H} = \mathrm{bb}^\infty(B)$ by Lopez-Abad (2005).)

Pure states on $\mathcal{B}(\ell_2)$

Definition

- A state on $\mathcal{B}(\ell_2)$ is a positive linear functional τ with $\tau(I) = 1$.
- A pure state is an extreme point in the (weak*-compact convex) set of states.

Example

- If v is a unit vector, then $\tau(T) = \langle Tv, v\rangle$ defines a vector state.
- If (e_n) is an orthonormal basis, and $\mathcal{U} \in \beta\mathbb{N}$, then $\tau_{\mathcal{U}}(T) = \lim_{n\to\mathcal{U}}\langle Te_n, e_n\rangle$ defines a diagonalizable pure state.

- Anderson (1980) conjectured that every pure state on $\mathcal{B}(\ell_2)$ is diagonalizable.
- (Akemann & Weaver, 2008): (CH) There is a counterexample.

A non-diagonalizable pure state

Using our local Gowers dichotomy, and the theory of quantum filters of Farah, Weaver, and Bice, we show:

Theorem (S.)

If $\mathcal{F}$ is a quantum filter of projections in the Calkin algebra whose preimage in $\mathcal{B}(\ell_2)$ is generated by a spread (p^*)-family of block projections, then $\mathcal{F}$ corresponds to a non-diagonalizable pure state.

- Such families $\mathcal{F}$ are easily constructed under CH or MA, or by forcing with projections in the Calkin algebra, though non-diagonalizable pure states were already known to exist in these settings (Farah & Weaver).
- One can show that any $\mathcal{F}$ satisfying the hypotheses of the theorem is a (genuine!) filter, but the existence of such families is independent of ZFC (Bice, 2011).

More examples?

I have the following examples of (p^+)-families:

- (ZFC) $\mathrm{bb}^\infty(E)$.
- (CH, MA, ...) (p^+)-filters.
- (CH) There is a “mad” family $\mathcal{A}$ of block sequences of E so that the set $\mathcal{H}$ of all X whose span has ∞-dimensional intersection with ∞-many elements of $\mathcal{A}$ is a (p^+)-family.
- (ZFC) If E contains a block sequence equivalent to the standard basis of c_0, or ℓ_p for $1 \leq p < \infty$, respectively, then the set $\mathcal{H}$ of all block sequences which have such a block subsequence is a (p^+)-family.
- (ZFC) ???

I would be grateful for more examples of these families, particularly within Banach space theory.