Turbulence and Essential Equivalence of Subspaces

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Smooth Classification

An analytic equivalence relation E is smooth if it is Borel reducible to equality on some Polish space X (w.l.o.g. $X = \mathbb{R}$).
i.e., E-classes can be definably classified by *real number invariants*.

Example

E_0 is the Borel equivalence relation on 2^ω given by

$$x \, E_0 \, y \iff \exists n \in \omega \forall m \geq n (x(m) = y(m)).$$

It is well-known (in this room) that E_0 is not smooth.

Corollary

If E is a Borel equivalence relation and $E_0 \leq_B E$, then E is not smooth.

Orbit Equivalence Relations

For G a Polish group acting continuously on a Polish space X, let E_G (sometimes X/G) be the orbit equivalence relation

$$x \; E_G \; y \iff \exists g \in G(y = g \cdot x).$$

This is an analytic equivalence relation (it may fail to be Borel).

Classification by Countable Structures

An analytic equivalence relation is classifiable by countable structures if it is Borel reducible to the isomorphism relation on the space of countable models of some first-order theory, e.g., groups, graphs, etc.

i.e., E-classes can be definably classified by invariants which are countable groups, graphs, etc.

Fact

Essentially countable Borel equivalence relations (in particular, E_0 and all smooth ones) are classifiable by countable structures.

Turbulence

Hjorth isolated a *dynamical* condition for Polish group actions which precludes classification by countable structures.

For G a Polish group acting continuously on a Polish space X, we say that the action of G is turbulent if

- every orbit is dense;
- every orbit is meager;
- every local orbit is somewhere dense.

Turbulence (cont'd)

Theorem (Hjorth, 1996)

Let G be a Polish group acting turbulently on a Polish space X. Then, E_G is not classifiable by countable structures.

Corollary

Let G be a Polish group acting turbulently on a Polish space X, and E an analytic equivalence relation. If $E_G \leq_B E$, then E is not classifiable by countable structures.

Turbulence: Examples

Example

Let G be a proper Polishable subgroup of $(\mathbb{R}^\omega, +)$ such that for every $\vec{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$, there is a $g \in G$ which agrees with $\vec{x}$ on its first n coordinates, e.g., c_0 and ℓ^p $(1 \leq p < \infty)$. Then the action of G by translation on $\mathbb{R}^\omega$ is turbulent.

Example

The same holds for G a proper Polishable subgroup of $(\mathbb{T}^\omega, \cdot)$, where $\mathbb{T}$ is the unit circle group.

Example

If X is a separable infinite dimensional Banach space, and Y a proper linear subspace of X which is dense and Polishable, then the action of Y on X by translation is turbulent.

Turbulence: Examples (cont'd)

Let $[0,1]^\omega/c_0$ be the restriction of $\mathbb{R}^\omega/c_0$ to $[0,1]^\omega$.
Note that this is no longer (a priori) an orbit equivalence relation.

Proposition

$[0,1]^\omega/c_0$ *is Borel bireducible with* $\mathbb{T}^\omega/G_0$*, for* $G_0 = \{(z_n)_n : \lim_n z_n = 1\}$*.*

Proof.

To reduce $[0,1]^\omega/c_0$ to $\mathbb{T}^\omega/G_0$, map $(\alpha_n)_n$ to $(e^{(i\pi/2)\alpha_n})_n$.
For the reverse reduction, reduce $\mathbb{T}^\omega/G_0$ to $([-1,1]^2)^\omega/(c_0 \times c_0)$ by embedding $\mathbb{T}$ into $[-1,1]^2$.
Then, contract $[-1,1]$ to $[0,1]$ and alternate coordinates to reduce $([-1,1]^2)^\omega/(c_0 \times c_0)$ to $[0,1]^\omega/c_0$. □

Bounded Operators on a Hilbert Space

Fix a separable infinite dimensional complex Hilbert space H, and let $\mathcal{B}(H)$ denote the space of all bounded linear operators on H.

We will investigate Borel equivalence relations occurring on $\mathcal{B}(H)$.

Caution: $\mathcal{B}(H)$ is not separable in the operator norm. Instead, we consider it with the strong operator topology, in which its Borel structure is standard, though it fails to be Polish. In fact, there is no Polish topology on $\mathcal{B}(H)$ which makes addition continuous and preserves this Borel structure.

Essential Equivalence

An operator K on H is compact if it maps bounded sets to sets with compact closure; equivalently K is a norm limit of finite rank operators. Denote by $\mathcal{K}(H)$ the set of compact operators.

Fact

$\mathcal{K}(H)$ is a proper norm closed ideal in $\mathcal{B}(H)$. In fact, it is the only one.

Let $\equiv_{\text{ess}}$ on $\mathcal{B}(H)$ denote equivalence modulo compact operators, or essential equivalence. One can show that $\equiv_{\text{ess}}$ is Borel.

Essential Equivalence (cont'd)

Proposition

$[0,1]^\omega/c_0 \leq_B \equiv_{ess}$. *Thus,* $\equiv_{ess}$ *is not classifiable by countable structures.*

Proof.

Fix an orthonormal basis $(e_n)_{n\in\omega}$ for H. Consider the (continuous) map $[0,1]^\omega \to \mathcal{B}(H)$ given by $\alpha = (\alpha_n)_n \mapsto T_\alpha$, where for $v \in H$

$$T_\alpha v = \sum_{n=0}^{\infty} \alpha_n \langle v, e_n \rangle e_n.$$

Such operators are diagonal with eigenvalues α_n, and are compact if and only if the sequence of eigenvalues converges to 0. Applying this to $T_\alpha - T_\beta$ for $\alpha, \beta \in [0,1]^\omega$ shows that this map is a reduction. □

Motivating Theorems

Theorem (Weyl–von Neumann, 1930's)

If S and T are bounded self-adjoint operators on H, then S and T are unitarily equivalent modulo compact if and only if S and T have the same essential spectrum.

Theorem (Ando–Matsuzawa, 2014)

The Weyl–von Neumann correspondence is a Borel reduction from unitary equivalence modulo compact of self-adjoint operators to equality of closed subsets of $\mathbb{R}$.

Theorem (Kechris–Sofranidis, 2001)

The conjugation action of the unitary group $U(H)$ on itself, and on self-adjoint operators of norm 1, is (generically) turbulent.

Subspaces and Projections

A projection $P \in \mathcal{B}(H)$ is an operator satisfying $P = P^2 = P^*$. Denote by $\mathcal{P}(H)$ the set of projections.

Via $P \leftrightarrow \mathrm{ran}(P)$, projections are in bijective correspondence with closed subspaces of H.

We will consider the restriction of $\equiv_{\mathrm{ess}}$ to $\mathcal{P}(H)$, or essential equivalence of subspaces.

Fact

$\mathcal{P}(H)$ is a Polish space in the strong operator topology.

Essential Equivalence of Projections

Proposition

E_0 is Borel reducible to $\equiv_{ess}$ on $\mathcal{P}(H)$.

Proof (sketch).

Fix an orthonormal basis $(e_n)_{n\in\omega}$. The map $2^\omega \to \mathcal{P}(H) : x \mapsto P_x$ where P_x is the projection onto $\overline{\text{span}}\{e_n : n \in x\}$ is a reduction. $\square$

A Twist for Non-Classifiability

But in fact, we can show much more:

Theorem (S.)

$[0,1]^\omega/c_0$ is Borel reducible to $\equiv_{ess}$ on $\mathcal{P}(H)$. Consequently, the latter is not classifiable by countable structures.

A Twist for Non-Classifiability (cont'd)

The reduction of $[0,1]^\omega/c_0$ to $\equiv_{\text{ess}}$ on $\mathcal{P}(H)$ is given by the map $[0,1]^\omega \to \mathcal{P}(H) : \alpha = (\alpha_n)_n \to P_\alpha$, where P_α is the projection onto $\overline{\text{span}}\{e_{2n} + \alpha_n e_{2n+1} : n \in \omega\}$.

The flavor of the proof: we establish a decomposition for $P_\alpha - P_\beta$:

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3,$$

where, for $v \in H$

$$T_0 v = \sum_{n=0}^{\infty} \left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right] \langle v, e_{2n} \rangle e_{2n}, \qquad T_2 v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] \langle v, e_{2n} \rangle e_{2n},$$

$$T_1 v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] \langle v, e_{2n+1} \rangle e_{2n+1}, \quad T_3 v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2} \right] \langle v, e_{2n+1} \rangle e_{2n+1},$$

$$S_0 v = \sum_{n=0}^{\infty} \langle v, e_{2n+1} \rangle e_{2n}, \qquad S_1 v = \sum_{n=0}^{\infty} \langle v, e_{2n} \rangle e_{2n+1}.$$

A Twist for Non-Classifiability (cont'd)

The upshot of the decomposition

$$P_\alpha - P_\beta = T_0 + S_0 T_1 + S_1 T_2 + T_3$$

is that T_0, T_1, T_2, and T_3 are diagonal operators whose eigenvalues go to 0 when $\alpha_n - \beta_n \to 0$, and thus $P_\alpha - P_\beta$ is compact.

Conversely, if $P_\alpha - P_\beta$ is compact, then $(P_\alpha - P_\beta)e_n \to 0$ in norm. Using

$$(P_\alpha - P_\beta)e_{2n} = \left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2}\right] e_{2n} + \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2}\right] e_{2n+1},$$
$$(P_\alpha - P_\beta)e_{2n+1} = \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2}\right] e_{2n} + \left[\frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2}\right] e_{2n+1},$$

orthogonality of e_{2n} and e_{2n+1}, and a series of inequalities, one obtains that $\alpha_n - \beta_n$ must converge to 0 if the displayed sequences do.

Related Results

Proposition

For $1 \leq p < \infty$,

1. $[0,1]^\omega/\ell^p$ *is Borel bireducible to the orbit equivalence relation of the turbulent action of* $G_p = \{(z_n)_n \in \mathbb{T}^\omega : \sum_{n=0}^{\infty} |z_n - 1|^p < \infty\}$ *by translation on* $\mathbb{T}^\omega$.
2. $[0,1]^\omega/\ell^p$ *is Borel reducible to equivalence modulo Schatten* p*-class in* $\mathcal{B}(H)$ *(or* $\mathcal{K}(H)$*).*

Theorem (S.)

E_1 *is Borel reducible to equivalence modulo finite dimensions in* $\mathcal{P}(H)$*. Consequently, the latter is not Borel reducible to the orbit equivalence relation of any Polish group action.*

Problems

Problem

Is $\equiv_{ess}$ on $\mathcal{B}(H)$ (or $\mathcal{P}(H)$) Borel bireducible with $[0,1]^\omega/c_0$? Likewise for equivalence modulo Schatten p-class and $[0,1]^\omega/\ell^p$.

Problem

Is unitary equivalence modulo Schatten p-class (of self-adjoint operators) smooth? Classifiable by countable structures?

Thanks!