Backwards and Forwards:
A Taste of Model Theory

Iian B. Smythe
ibs24@cornell.edu

Department of Mathematics
University of Manitoba

Department of Mathematics
Cornell University

Canadian Undergraduate Mathematics Conference 2011

Outline

Motivation

- What do groups, rings, vector spaces over a field, boolean algebras, lattices, partial orders, etc... all have in common?
- Each is a *mathematical structure*, with certain distinguished relations, functions and constants, acting in a prescribed way upon an underlying set.
- We want to formalize this in such a way that captures all of the above examples.
- In order to do this, we must first formalize the mathematical languages in which we 'talk' about these structures.

First-order languages

Definition

A *first-order language* $\mathcal{L}$ is a set of symbols consisting of:

- a set $\mathrm{Rs}_{\mathcal{L}}$ of *relation symbols*,
- a set $\mathrm{Fs}_{\mathcal{L}}$ of *function symbols*,
- a set $\mathrm{Cs}_{\mathcal{L}}$ of *constant symbols*,
- a (countable) set $\mathrm{Vb} = \{x_0, x_1, x_2, \dots\}$ of *variables*,
- the equality symbol $=$,
- the logical symbols $\neg$, $\vee$, and $\forall$ ($\wedge$, $\rightarrow$, $\leftrightarrow$ and $\exists$ are optional).

To each function symbol and each relation symbol, we associate a natural number, called the *arity* of that symbol.

- Warning: A language consists of relation, function and constant *symbols*, not actual relations, functions and constants.

First-order languages: Examples

We will look at the following examples throughout this presentation:

- The *language of groups*, $\mathcal{L}_G$, consists of no relation symbols, a binary function symbol $\times$, a unary function symbol $^{-1}$, and a constant symbol e.
- The *language of rings* (with unity), $\mathcal{L}_R$, consists of no relation symbols, binary function symbols $+$ and $\times$, a unary function symbol $-$, and constant symbols 0 and 1. (A language with no relation symbols is called an *algebraic language*.)
- The *language of set theory*, $\mathcal{L}_\in$, consists of a single binary relation symbol $\in$, and no function or constant symbols.
- The *language of partial orders*, $\mathcal{L}_<$, consists of a single binary relation symbol $<$, and no function or constant symbols.

Syntax: Terms

Definition

Given a (first-order) language $\mathcal{L}$, the set of $\mathcal{L}$-*terms* is the smallest set of strings of symbols from $\mathcal{L}$ such that:

- any constant symbol c is a term,
- any variable x_i is a term, and
- if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then so is $f(t_1, \ldots, t_n)$.

- Terms play a similar role in formal languages as nouns play in (western) natural languages.

Syntax: Formulas and sentences

Definition

Given a language $\mathcal{L}$, the set of $\mathcal{L}$-*formulas* is the smallest set of strings of symbols from $\mathcal{L}$ such that:

- if t_0 and t_1 are $\mathcal{L}$-terms, then $t_0 = t_1$ is an $\mathcal{L}$-formula,
- if R is an n-ary relation symbol of $\mathcal{L}$ and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then $R(t_1, \ldots, t_n)$ is an $\mathcal{L}$-formula,
- if φ and ψ are $\mathcal{L}$-formulas, and x a variable, then the following are $\mathcal{L}$-formulas: $\neg\varphi$, $\varphi \vee \psi$, and $\forall x \varphi$.

- Given an $\mathcal{L}$-forumla φ, the variables which are not quantified over are called the *free variables* of φ.
- A formula with no free variables is called a *sentence*, and any set of $\mathcal{L}$-sentences is called an $\mathcal{L}$-*theory*.

Syntax: Examples

- In the language of groups, the following is formula (in fact, it is a sentence):
$$\forall x((x \times x^{-1} = e) \wedge (x^{-1} \times x = e)).$$
- In the language of partial orders, the following is a formula, but not a sentence (it has a free variable, z):
$$\forall x \exists y((x < y) \wedge ((y < z) \vee \neg(z = x))).$$

Structures

Definition

Given a language $\mathcal{L}$, an *$\mathcal{L}$-structure* $\mathfrak{A}$ consists of:

- a set A, called the *universe* of $\mathfrak{A}$,
- for each n-ary relation symbol $R \in \mathrm{Rs}_{\mathcal{L}}$, an n-ary relation $R^{\mathfrak{A}}$ on A (i.e. a subset of A^n),
- for each n-ary function symbol $f \in \mathrm{Fs}_{\mathcal{L}}$, an n-ary function $f^{\mathfrak{A}} : A^n \to A$, and
- for each constant symbol $c \in \mathrm{Cs}_{\mathcal{L}}$, a constant $c^{\mathfrak{A}} \in A$.

If $\mathrm{Rs}_{\mathcal{L}} = \{R_i\}_{i \in I}$, $\mathrm{Fs}_{\mathcal{L}} = \{f_j\}_{j \in J}$, and $\mathrm{Cs}_{\mathcal{L}} = \{c_k\}_{k \in K}$, we will often write: $\mathfrak{A} = \langle A; \{R_i^{\mathfrak{A}}\}_{i \in I}, \{f_j^{\mathfrak{A}}\}_{j \in J}, \{c_k^{\mathfrak{A}}\}_{k \in K} \rangle$

- The $R^{\mathfrak{A}}$, $f^{\mathfrak{A}}$ and $c^{\mathfrak{A}}$ are the *interpretations* of those symbols in $\mathfrak{A}$. (We will omit the superscripts when they are understood.)

Structures: Examples

- If G is a group, then $\langle G; \times, ^{-1}, e\rangle$, where $\times$, $^{-1}$ and e are given the obvious interpretations, is an $\mathcal{L}_G$-structure.
- If $<$ is interpreted as the usual order on $\mathbb{R}$, then $\langle \mathbb{R}; <\rangle$ is an $\mathcal{L}_<$-structure.
- Of course, apart from agreeing with the arity of the symbols, we have no constraints on their interpretations. Thus, an $\mathcal{L}$-structure in general need not look like the intended objects of study.

Semantics

Definition

Given an $\mathcal{L}$-structure $\mathfrak{A}$, a map $\alpha : \mathrm{Vb} \rightarrow A$ is called an $\mathfrak{A}$-*assignment*.

Definition

If t is an $\mathcal{L}$-term, we define $t^{\mathfrak{A}}[\alpha]$, an element of A, to be the result of replacing all instances of relation, function and constant symbols with their interprations in $\mathfrak{A}$, and variables with their assigned values via α.

Semantics (cont'd)

Definition (Basic Semantic Definition.)

Let $\mathfrak{A}$ be an $\mathcal{L}$-structure, $t_0, \ldots, t_n$ $\mathcal{L}$-terms, R a relation symbol, α an $\mathfrak{A}$-assignment, φ and ψ $\mathcal{L}$-formulas, and x a variable. Then

- $\mathfrak{A} \models (t_0 = t_1)[\alpha]$, if $t_0^{\mathfrak{A}}[\alpha] = t_1^{\mathfrak{A}}[\alpha]$,
- $\mathfrak{A} \models R(t_1, \ldots, t_n)[\alpha]$, if $(t_1^{\mathfrak{A}}[\alpha], \ldots, t_n^{\mathfrak{A}}[\alpha]) \in R^{\mathfrak{A}}$,
- $\mathfrak{A} \models \neg\varphi[\alpha]$, if it is not the case that $\mathfrak{A} \models \varphi[\alpha]$,
- $\mathfrak{A} \models (\varphi \vee \psi)[\alpha]$, if $\mathfrak{A} \models \varphi[\alpha]$ or $\mathfrak{A} \models \psi[\alpha]$,
- $\mathfrak{A} \models \forall x\varphi[\alpha]$, if for every $a \in A$, $\mathfrak{A} \models \varphi[\alpha(a/x)]$ (where $\alpha(a/x)$ agrees with α on all of Vb except x, and assigns x to a).

- $\wedge$, $\rightarrow$, $\leftrightarrow$ and $\exists$ can be defined in terms of $\neg$, $\vee$ and $\forall$.
- Whenever $\mathfrak{A} \models \varphi[\alpha]$, we say that $\mathfrak{A}$ *models* or *satisfies* φ at α.

Semantics: Examples

- If φ is an $\mathcal{L}$-sentence, then $\mathfrak{A}$ models φ at *some* assignment if and only if $\mathfrak{A}$ models φ at *every* assignment. Thus, we can omit assignments when discussing sentences.
- In the language of groups, let φ be the sentence given by $\forall x((x \times e = x) \wedge (e \times x = x))$. Then if $\mathfrak{G}$ is a group, we know that $\mathfrak{G} \models \varphi$.
- However, if we take ψ to be the sentence $\forall x \forall y(x \times y = y \times x)$, then a group $\mathfrak{G}$ will model ψ if and only if $\mathfrak{G}$ is abelian.
- In the language of rings, let θ be the formula $\exists y((x \times y = 1) \wedge (y \times x = 1))$. If $\mathfrak{R}$ is a ring, then $\mathfrak{R} \models \theta[\alpha]$ if and only if α assigns x to a unit in R.

Substructures and Homomorphisms

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. We say that $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if:

- $A \subseteq B$,
- for each n-ary $R \in \mathrm{Rs}_{\mathcal{L}}$, $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^n$,
- for each n-ary $f \in \mathrm{Fs}_{\mathcal{L}}$, $f^{\mathfrak{A}} = f^{\mathfrak{B}}|_{A^n}$, and
- for each $c \in \mathrm{Cs}_{\mathcal{L}}$, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.

Equivalenlty, $\mathfrak{A} \subseteq \mathfrak{B}$ provided $A \subseteq B$, and for every *quantifier-free* $\mathcal{L}$-formula φ, and $\mathfrak{A}$-assignment α, $\mathfrak{A} \models \varphi[\alpha]$ if and only if $\mathfrak{B} \models \varphi[\alpha]$.

Substructures and Homomorphisms (cont'd)

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. A map $\eta : A \to B$ is called a *homomorphism* if:

- for each n-ary $R \in \mathrm{Rs}_{\mathcal{L}}$, and $\vec{a} \in A^n$, $\vec{a} \in R^{\mathfrak{A}}$ if and only if $\eta(\vec{a}) \in R^{\mathfrak{B}}$,
- for each n-ary $f \in \mathrm{Fs}_{\mathcal{L}}$, and $\vec{a} \in A^n$, $\eta(f^{\mathfrak{A}}(\vec{a})) = f^{\mathfrak{B}}(\eta(\vec{a}))$, and
- for each $c \in \mathrm{Cs}_{\mathcal{L}}$, $\eta(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.

If η is injective, then we say η is an *embedding*. If η is bijective, then we say η is an *isomorphism* and write $\mathfrak{A} \simeq \mathfrak{B}$.

- Likewise, homomorphisms are exactly the maps which preserve quantifier-free forumlas not containing $=$.

Elementary Equivalence and Substructures

Definition

If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures such that for every $\mathcal{L}$-sentence φ, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{B} \models \varphi$, then we say that $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent*, written $\mathfrak{A} \equiv \mathfrak{B}$.

Definition

If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures such that $\mathfrak{A} \subseteq \mathfrak{B}$ and for every $\mathcal{L}$-formula φ and $\mathfrak{A}$-assignment α, $\mathfrak{A} \models \varphi[\alpha]$ if and only if $\mathfrak{B} \models \varphi[\alpha]$, then we say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$, written $\mathfrak{A} \preceq \mathfrak{B}$.

- Note that if $\mathfrak{A} \preceq \mathfrak{B}$, then $\mathfrak{A} \equiv \mathfrak{B}$.

Elementary Equiv. and Substructures: Examples

- In the language of rings, consider the field of real numbers $\mathfrak{R} = \langle \mathbb{R}; +, \times, -, 0, 1 \rangle$. If we let A be the set of all real *algebraic numbers* (i.e. roots of integer polynomials), and restrict our interpretations to A, we obtain an elementary substructure $\mathfrak{A} = \langle A; +, \times, -, 0, 1 \rangle$ of $\mathfrak{R}$.
- In the language of partial orders, take the real numbers with their usual ordering, $\mathfrak{R}_< = \langle \mathbb{R}; < \rangle$. Then $\mathfrak{Q}_< = \langle \mathbb{Q}; < \rangle$ is an elementary substructure of $\mathfrak{R}_<$.
- In particular, the last example shows that as $\mathcal{L}_<$-structures, $\mathfrak{R}_< \equiv \mathfrak{Q}_<$. Yet, it is well-known that $\mathbb{R}$ has the property that *every non-empty subset which is bounded above has a least-upper bound (supremum)*, and $\mathbb{Q}$ does not have this property. This shows that this property cannot be expressed as a *first-order* $\mathcal{L}_<$-sentence.

Important Theorems

- In order to get a further taste of the subject, we will survey a few of the core theorems in model theory. The proofs of these results are outside of the scope of this presentation, but can be found in standard texts in mathematical logic (such as Hinman, or Bell and Machover).

Compactness

Theorem (Compactness Theorem)

Let Φ be an $\mathcal{L}$-theory. If every finite subset of Φ has a model, then Φ has a model.

- This theorem, which suggests underlying topological considerations, greatly simplifies the question of when a given theory has a model.

Cardinality

Definition

We define the *cardinality* of a language $\mathcal{L}$ to be the cardinality of the set of all symbols in $\mathcal{L}$.

Definition

We define the *cardinality* of an $\mathcal{L}$-structure $\mathfrak{A}$ to be the cardinality of the underlying universe A.

- Recall that $\aleph_0$ is the cardinality of the set of all natural numbers; the countable cardinal.

Löwenheim-Skolem Theorems

Theorem (Downwards Löwenheim-Skolem Theorem)

If $\mathcal{L}$ is a language of cardinality at most $\kappa \geq \aleph_0$, $\mathfrak{B}$ an $\mathcal{L}$-structure, and $X \subseteq B$ with $|X| \leq \kappa$, then there is an $\mathcal{L}$-structure $\mathfrak{A}$ of cardinality $\leq \kappa$, such that $X \subseteq A$, and $\mathfrak{A} \preceq \mathfrak{B}$.

Theorem (Upwards Löwenheim-Skolem Theorem)

If $\mathcal{L}$ is a language of cardinality at most $\kappa \geq \aleph_0$, and $\mathfrak{A}$ an infinite $\mathcal{L}$-structure of cardinality $\leq \kappa$, then for every cardinal $\lambda \geq \kappa$, there is an $\mathcal{L}$-structure $\mathfrak{B}$ of cardinality λ such that $\mathfrak{A} \preceq \mathfrak{B}$.

Löwenheim-Skolem Theorems (cont'd)

The following is an important special case of the Downwards Löwenheim-Skolem Theorem:

Corollary (Countable Downwards Löwenheim-Skolem Theorem)

If $\mathcal{L}$ is a countable language, $\mathfrak{B}$ an $\mathcal{L}$-structure, and X any countable subset of $\mathfrak{B}$, then there is a countable $\mathcal{L}$-structure $\mathfrak{A}$ such that $X \subseteq A$ and $\mathfrak{A} \preceq \mathfrak{B}$.

Löwenheim-Skolem Theorems: Examples

- *Skolem's Paradox*: In the language of set theory, supposing that the ZFC axioms are consistent, there is a standard model of set theory. Such a model will satisfy a sentence which says "there is an uncountable set". But by the Downwards Löwenheim-Skolem Theorem, there is a countable model of set theory which will also satisfy this sentence.

Categoricity

Definition

Let Φ be an $\mathcal{L}$-theory. We say the Φ is *categorical* if for any models $\mathfrak{A}$ and $\mathfrak{B}$ of Φ, $\mathfrak{A} \simeq \mathfrak{B}$.
For any cardinal κ, we say that Φ is κ-*categorical* if for any models $\mathfrak{A}$ and $\mathfrak{B}$ of Φ, both with cardinality κ, $\mathfrak{A} \simeq \mathfrak{B}$.

- In the language of groups, if Grp is the set of axioms of group theory, then Grp is p-categorical for every prime p.
- We want to exhibit a (non-trivial) example of an $\aleph_0$-categorical theory.

The Back-and-Forth Property

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures, and $X \subseteq A$. A function $\eta : X \to B$ is called a *partial embedding* if for every quantifier-free formula φ, with n free variables, and every $a_1, \ldots, a_n \in X$,

$$\mathfrak{A} \models \varphi[a_1, \ldots, a_n] \quad \text{if and only if} \quad \mathfrak{B} \models \varphi[\eta(a_1), \ldots, \eta(a_n)].$$

The Back-and-Forth Property (cont'd)

Definition

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. We say that a family, $\mathcal{E}$, of partial embeddings from $\mathfrak{A}$ to $\mathfrak{B}$, has the *back-and-forth property* if:

- For every $\eta \in \mathcal{E}$, and every $a \in A$, there is $\zeta \in \mathcal{E}$ such that ζ extends η and $a \in \mathrm{dom}(\zeta)$. (*Forth*)
- For every $\eta \in \mathcal{E}$, and every $b \in B$, there is $\zeta \in \mathcal{E}$ such that ζ extends η and $b \in \mathrm{ran}(\zeta)$. (*Back*)

- If η is an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, then $\{\eta\}$ has the back-and-forth property.
- In *some* cases, a family of partial embeddings with the back-and-forth property can be used to build an isomorphism.

Dense Linear Orders

Definition

In the language of partial orders, $\mathcal{L}_<$, the theory generated by the following set of sentences, denoted by T_{DLO}, is called the *theory of dense linear orders*:

- $\exists x \exists y \neg(x = y)$ (at least two elements),
- $\forall x \neg(x < x)$ (irreflexivity),
- $\forall x \forall y \forall z(((x < y) \wedge (y < z)) \rightarrow (x < z))$ (transitivity),
- $\forall x \forall y(\neg(x = y) \rightarrow ((x < y) \vee (y < x)))$ (totality and antisymmetry),
- $\forall x \forall y((x < y) \rightarrow \exists z((x < z) \wedge (z < y)))$ (density).

- If we add the sentences $\neg\exists x(\forall y((x < y) \vee (x = y))$ and $\neg\exists x(\forall y((y < x) \vee (y = x))$, we obtain the *theory of dense linear orders without endpoints*, $T_{(\text{DLO})}$.

$T_{(\mathrm{DLO})}$ and $\aleph_0$-categoricity

Theorem

$T_{(DLO)}$ is $\aleph_0$-categorical, i.e. any two countable dense linear orders without endpoints are isomorphic.

- This result is originally due to Cantor in the late 19th century, however, the machinery and terminology of model theory was only developed decades later.
- In order to prove this, we will construct a sequence of partial embeddings with the back-and-forth property, from which an isomorphism can be obtained.

$T_{(\mathrm{DLO})}$ and $\aleph_0$-categoricity (cont'd)

Proof.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable models of $T_{(\mathrm{DLO})}$, with $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ enumerations of A and B respectively.
Observe that for $X \subseteq A$, $f : X \to B$ is a partial embedding if:
for every $a, a' \in X$, $a < a'$ in $\mathfrak{A}$ if and only if $f(a) < f(a')$ in $\mathfrak{B}$.
Set η_0 to be the *empty function*.
Suppose that we have defined a partial embedding η_k from $\mathfrak{A}$ to $\mathfrak{B}$, with finite domain and range.
We split up our recursive construction into two cases:
Case 1 *(forth)*. k is even. Let m be the least index of an element of A which is not in the domain of η_k.
We need to extend η_k to a partial embedding which includes a_m in its domain; to do this, we must ensure that there is a corresponding b in the range which plays the same role relative to the ordering as a_m.

$T_{(\mathrm{DLO})}$ and $\aleph_0$-categoricity (cont'd)

(cont'd).

There are only three possibilities:
i) If a_m is greater than every element of $\mathrm{dom}(\eta_k)$, then since $\mathfrak{B}$ is a DLO without endpoints, we can find $b \in B \setminus \mathrm{ran}(\eta_k)$, such that b is greater than every element of $\mathrm{ran}(\eta_k)$. Extend η_k to η_{k+1} by setting $\eta_{k+1}(a_m) = b$.
ii) Similarly if a_m is less than every element of $\mathrm{dom}(\eta_k)$.
iii) If $a < a_m < a'$, with $a, a' \in \mathrm{dom}(\eta_k)$, and no other elements of $\mathrm{dom}(\eta_k)$ between them, then since η_k is a partial embedding, $f(a) < f(a')$. $\mathfrak{B}$ has the density property, so we can find $b \in B \setminus \mathrm{ran}(\eta_k)$ such that $f(a) < b < f(a')$. Extend η_k to η_{k+1} by setting $\eta_{k+1}(a_m) = b$.
Case 2 *(back)*. Suppose k is odd, and let m be the least index of an element of B which is not in the range of η_k. This is done similarly.
It follows (easily) that $\eta = \bigcup_{n\in\omega} \eta_n$ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. □

More on the Back-and-Forth Property

A similar proof shows that:

Theorem

If $\mathfrak{A}$ and $\mathfrak{B}$ are countable $\mathcal{L}$-structures with the back-and-forth property, then $\mathfrak{A} \simeq \mathfrak{B}$.

In the case of arbitrary structures, an isomorphism is too much to hope for, but since sentences are finite objects, we still obtain:

Theorem

If $\mathfrak{A}$ and $\mathfrak{B}$ are $\mathcal{L}$-structures with the back-and-forth property, then $\mathfrak{A} \equiv \mathfrak{B}$.

Summary

- Model Theory is the study of mathematical structures, the languages in which they are discussed, and the sentences and formulas which they satisfy.
- This general setting provides natural generalizations of the concepts of homomorphism and substructure from algebra.
- We can strengthen these notions to consider when structures are simialr with respect to the sentences or formulas they satisfy.
- This leads to the notion of categoricity, and we have seen one of the first important examples of an $\aleph_0$-categorical theory; the theory of dense linear orders without endpoints.
- If you would like a .pdf version of these slides, please feel free to email me at `ibs24@cornell.edu`, or see me any time during CUMC 2011.

References

- Bell, J. L. and Machover, M. *A Course in Mathematical Logic*, North Holland, Amsterdam, 1977.
- Hinman, Peter G. *Fundamentals of Mathematical Logic*, A K Peters, Wellesley, MA, 2005.
- Hodges, Wilfrid. *Model Theory*, Encyclopedia of Mathematics, Vol. 42, Cambridge Univ. Press, 1993.
- Marker, David. *Model Theory: An Introduction*, Graduate Texts in Mathematics, Springer, New York, 2002.